1. Cluster Means Updating:

Proof: Given the distortion function:

$$SSE(z,c) = \sum_{i=1}^{n} ||x_i - c_{zi}||_2^2$$

Then for a single cluster, say cluster p, where p could be any integer between 1 and k in the scenario of this question, with its current center denoted as c, the above then equation becomes:

$$SSE = \sum_{x \epsilon p} ||x - c||^2$$

That is just the sum of distance between the points to the center point assigned to them. Obviously, the above equation is minimized when $c = mean(p)$. Hence, statement is proved. We can further reinforce the proof by evaluating the cost function for the case when $c \neq mean(p)$, that is:

$For\ any\ set\ p \subset R^d\ and\ any\ z \in R^d,$

$$Cost(p;c) = cost\big(p; mean(p)\big) + |p| \cdot \big||c - mean(p)|\big|^2 \quad (1)$$

Clearly, $|p| \cdot ||z - mean(p)||^2$ is positive. Hence, $c = mean(p)$ must be satisfied to minimize the cost function.

2. Cluster Assignments Updating:

Proof: Let $c_1^{(t)}, \dots, c_k^{(t)},\ p_1^{(t)}, \dots,\ p_k^{(t)}$ denote the center and clusters at the start of the t^{th} iteration of k-means. By assigning each point to its closet cluster, we obtain the following:

$$Cost\Big(\ p_1^{(t+1)}, \dots,\ p_k^{(t+1)}; c_1^{(t)}, \dots, c_k^{(t)}\Big) \leq Cost\Big(\ p_1^{(t)}, \dots,\ p_k^{(t)}; c_1^{(t)}, \dots, c_k^{(t)}\Big) \quad (2)$$

Combines equations (1) and (2), we have:

$$Cost\Big(\ p_1^{(t+1)}, \dots,\ p_k^{(t+1)}; c_1^{(t+1)}, \dots, c_k^{(t+1)}\Big) \leq Cost\Big(\ p_1^{(t+1)}, \dots,\ p_k^{(t+1)}; c_1^{(t)}, \dots, c_k^{(t)}\Big)$$

which is basically how k-means algorithm works fundamentally.