$\xrightarrow{r(t)} \bigcirc \rightarrow e(t) \xrightarrow{ADC} e[k] \xrightarrow{\text{computer}} w[k] \xrightarrow{DAC} u(t) \ ..$

$$X(z) = Z\{x(k)\} = \sum_{k=0}^{\infty} x(k) z^{-k} \qquad k = 0, 1, 2 \ldots$$

for $x(t) = u(t)$ $\quad x(k) = u(k)$ $\quad Z\{x(k)\} = Z\{u(k)\} = \sum_{k=0} u(k) z^{-k} = \sum_{0}^{\infty} z^{-k}$

$\Sigma = 1 + z^{-1} + \cdots$

$z^{-1}\Sigma = z^{-1} + \cdots + z^{-\infty}$

$(1 - z^{-1})\Sigma = 1$

$\Sigma = \dfrac{1}{1 - z^{-1}}$

① $x(t) = t$ for $t > 0$

$\overset{dis}{\Rightarrow} x(k) = k$ for $k > 0$

$Z\{x(k)\} = Z\{k\} = \sum_{k=0}^{\infty} k \cdot z^{-k}$

② $x(k) = k = k \cdot u(t)$ $\qquad X(z) = Z\{x(k)\} = \sum z^{-k} \cdot k \cdot u(k)$

$Z\{u(t)\} = U(z) = \sum z^{-k} \cdot u(k)$ $\qquad = -z \cdot \dfrac{dU(z)}{dz}$

$\dfrac{dU(z)}{dz} = -k \sum z^{-k-1} u(k)$

$= -k z^{-1} \sum z^{-k} u(k) = -k z^{-1} U(z)$ $\qquad U(z) = \dfrac{z}{z-1}$

$$\therefore \ Z\{k \cdot x(k)\} = -z \cdot \frac{dX(z)}{dz}$$

$= -z \dfrac{(z-1) - z}{(z-1)^2} = X(z) = \dfrac{z^{-1}}{(1 - z^{-1})^2}$

③ $x(t) = a^{t} \xrightarrow{dis} x(k) = a^{k}$

$X(z) = Z\{x(k)\} = \sum_{0}^{\infty} a^{k} z^{-k} = \sum_{0}^{\infty} 1 \cdot (a^{-1} \cdot z)^{-k}$ for $k = 0$

$= U(z) \Big|_{z = a^{-1}z} = \dfrac{1}{1 - a z^{-1}}$

④ $x(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0, & t < 0 \end{cases}$

$x(k) = e^{-ak}$ $\quad k \geq 0$ $\qquad Z\{x(k)\} = \sum_{0}^{\infty} z^{-k} \cdot e^{-ak}$

$= \sum 1 \cdot (z \cdot e^{a})^{-k}$

$= U(z) \Big|_{z = ze^{a}}$

$= \dfrac{1}{1 - e^{-a} z^{-1}}$

⑤ $x(t) = \sin\omega t \quad t \geq 0 \qquad x(k) = \sin\omega k \qquad k \geq 0$

$e^{j\omega t} = \cos\omega t + j\sin\omega t$

$\therefore \sin(\omega k) = \left(\dfrac{e^{j\omega k} - e^{-j\omega k}}{2j}\right)$

$$Z\{\sin(\omega k)\} = Z\left\{\frac{e^{j\omega k} - e^{-j\omega k}}{2j}\right\} = \frac{\dfrac{1}{1-e^{j\omega}z^{-1}} - \dfrac{1}{1-e^{-j\omega}z^{-1}}}{2j}$$

$$\frac{(e^{j\omega} - e^{-j\omega})z^{-1}}{(1-e^{j\omega}z^{-1})(1-e^{-j\omega}z^{-1})}$$

$$1 + z^{-2} - (e^{j\omega} + e^{-j\omega})z^{-1}$$

$$\frac{\sin\omega \; z^{-1}}{(1+z^{-2} - 2\cos\omega\, z^{-1})}$$

⑥ $x(t) = \cos(\omega t)$

Similar with sin

$$\frac{1 - \cos\omega\; z^{-1}}{1+z^{-2} - 2\cos\omega\; z^{-1}}$$

Ex 2.1 $\quad X(s) = \dfrac{1}{s(s+1)} = \dfrac{1}{s} - \dfrac{1}{s+1}$

$\Leftrightarrow u(t) - e^{-t} = x(t)$

$x(k) = u(k) - e^{-k}$

$$Z\{x(k)\} = \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-1}z^{-1}}$$

Properties:

① $\mathcal{Z}\{a x(k)\} = a \mathcal{Z}\{x(k)\} = a X(z)$

P: $\sum z^{-k} a x(k) = a \sum z^{-k} x(k) = a \mathcal{Z}\{x(k)\}$

② $\mathcal{Z}\{f(k) + g(k)\} = \mathcal{Z}\{f(k)\} + \mathcal{Z}\{g(k)\} = F(z) + G(z)$

P: $\sum z^{-k} (f(k) + g(k)) = \sum \left(z^{-k} f(k) + z^{-k} g(k) \right)$

$= \sum z^{-k} f(k) + \sum z^{-k} g(k) = \mathcal{Z}\{f(k)\} + \mathcal{Z}\{g(k)\}$

③ $\mathcal{Z}\{a^k x(k)\} = X(a^{-1} z)$

P: $\mathcal{Z}\{a^k x(k)\} = \sum z^{-k} a^k x(k) = \sum (a^{-1} z)^{-k} \cdot x(k)$

$= X(z) \big|_{z = a^{-1} z} = X(a^{-1} z)$

④ $\mathcal{Z}\{x(k-n)\} = z^{-n} X(z)$

P: $\mathcal{Z}\{x(k-n)\} = \sum_{k=0}^{\infty} z^{-k} x(k-n)$

$= x(0-n) z^{-0} + x(1-n) z^{-1} + \cdots$

for $k<0$ $x(k) = 0$ $\therefore \mathcal{Z}\{\ \} = \sum_{k=n}^{\infty} z^{-k} x(k-n)$

let $k-n = m$ $\mathcal{Z}\{\} = \sum_{m=0}^{\infty} z^{-m} x(m) \cdot z^{-n} = z^{-n} \sum z^{-m} x(m)$

$= z^{-n} X(z)$

⑤ $Z\{x(k+n)\} = z^n\left[X(z) - \sum_{i=0}^{n-1} z^{-i} x(i)\right]$

P: $Z\{x(k+n)\} = \sum_{k=0}^{\infty} x(k+n) * z^{-k}$

$= x(n) \cancel{z^0} + x(n+1) z + x(n+2) z^2 + \cdots$

$\cancel{a} z^n \cancel{x} \cdot Z\{\} = z^n x(n) + z^{n+1} x(n+1) + \cdots$

$$= \sum_{k=0}^{\infty} z^{-k} x(k) - \sum_{k=0}^{n-1} z^{-k} x(k)$$

$$= X(z) - \sum_{k=0}^{n-1} z^{-k} x(k)$$

$$Z\{\} = z^n\left(X(z) - \sum_{k=0}^{n-1} z^{-k} x(k)\right)$$

Ex 2-2: $\cancel{Z}$ Z of $1(t-T)$, $1(t-4T)$

$\xrightarrow{dis}$ $x(k) = 1(k-1) = z^{-1} Z\{1\}$

$$= \frac{z^{-1}}{1-z^{-1}}$$

$\xrightarrow{dis}$ $x(k) = 1(k-4) = \dfrac{z^{-4}}{1-z^{-1}}$

Ex 2-3. $f(k) = a^{k-1}$ $k = 1, 2, 3 \cdots$

$g(m) = f(m+1) = a^m$

$Z\{g(k)\} = Z\{a^k\} = \dfrac{1}{1-az^{-1}}$

$Z\{f(k)\} = Z\{g(k-1)\} = \dfrac{z^{-1}}{1-az^{-1}}$

$Y(z) = \dfrac{X(z)}{1-z^{-1}}$

Ex 2-4. $x(h)$ $h = 0, 1, 2 \ldots$

$y(k) = \sum_{h=0}^{k} x(h)$ $k = 0, 1, 2, \ldots$

$y(k) = \sum_{h=0}^{k} x(h)$

$Z\{y(k)\} = \sum_{k=0}^{\infty} z^{-k} \cancel{y(k)} \sum_{h=0}^{k} x(h)$

$X(z) + z^{-1} Y(z)$

$= \sum_{k=0}^{\infty} z^{-k} (x(0) + x(1) + \cdots + x(k))$ //

$= Y(z)$

$= \sum_{k=0}^{\infty} \left[z^{-k} x(k) + z^{-(k-1)} \underbrace{(x(0) \cdots + x(k-1))}_{y(k-1)} \cdot z^{-1} \right]$

⑥ $Z\{e^{-at}x(t)\} = X(e^{a}z)$

P: $z\{e^{-ak}x(k)\} = \sum z^{-k}(e^{a})^{-k}\cdot x(k) = \sum (e^{a}z)^{-k}\cdot x(k)$

Ex 2-5: $e^{-at}\sin(wt) = \left.\dfrac{z^{-1}\sin w}{1+z^{-2}-2z^{-1}\cos w}\right|_{z=e^{a}z}$

$$\frac{e^{-aT}z^{-1}\sin w}{1+e^{-2aT}z^{-2}-2e^{-aT}z^{-1}\cos w}$$

Ex 2-6 Z of te^{-at}

$x(k) = k\cdot e^{-ak} \quad k\geq 0$

m1) $y(k) = k \quad Y(z) = \dfrac{z^{-1}}{(1-z^{-1})^2}$

$x(k) = e^{-ak}\cdot y(k) \quad X(z) = Y(e^{a}z) = Y(z)|_{z=e^{a}z} = \dfrac{e^{-a}z^{-1}}{(1-e^{-a}z^{-1})^2}$

m2) $y(k) = e^{-ak} = \dfrac{1}{1-e^{-a}z^{-1}}$

$x(k) = k\cdot y(k) = -z\cdot\dfrac{dY(z)}{d(z)} = \dfrac{z\cdot e^{-a}z^{-2}}{(1-e^{-a}z^{-1})^2} = \dfrac{e^{-a}z^{-1}}{(1-e^{-a}z^{-1})^2}$

Initial Value: if ① $Z\{x(k)\} = X(z)$ and ② $\lim\limits_{z\to\infty} X(z)$ exist. Then $\underset{= x(t)|_{t=0}}{x(0)} = \lim\limits_{z\to\infty} X(z)$ proof

Final Value: ∃ ① if $x(k) = 0$ for $k<0$ and ② $X(z) = Z\{x(k)\}$ and ③ all poles of $X(z)$ lie inside unit circle except 1 proof

Then $\lim\limits_{k\to\infty} x(k) = x(\infty) = \lim\limits_{z\to 1}(1-z^{-1})\cdot X(z)$

prove of Final Value.

$$x(\infty) = \sum_{k=0}^{\infty} [x(k) - x(k-1)]$$

$$Z\{x(k)\} - Z\{x(k-1)\} = Z\{x(k)\} - z^{-1} Z\{x(k)\}$$

$$\sum x(k) z^{-k} - \sum x(k-1) z^{-k}$$

Ex 2-8 $\quad x(\infty)$ of $\dfrac{1}{1-z^{-1}} - \dfrac{1}{1-e^{-a}z^{-1}}$

m1: poles: $z=1$, e^{-a} in unit circle

~~x(z)~~ $X(z) = Z\{x(t)\}$

$$\therefore \quad x(\infty) = \lim_{z \to 1} \left(1 - \frac{1-z^{-1}}{1-e^{-a}z^{-1}} \right)$$

$$= 1$$

m2: $x(k) = u(k) - e^{-ak}$

$$\lim_{k \to \infty} = 1 - 0 = 1$$

Inverse Z trans

1) long division

~~X(Z)~~ Ex 2-10.

$$X(z) = \frac{10z+5}{(z-1)(z-0.2)}$$

$$\frac{10z^{-1} + 5z^{-2}}{1 - 1.2z^{-1} + 0.2z^{-2}} \quad \frac{10z+5}{z^2 - 1.2z + 0.2}$$

$x(0) = 1$

$y(0) = 0$

$y(1) = \cdots y(0) = \cdots x(0)$

1

$$\begin{array}{r|l} & 10z^{-1} + 17z^{-2}, \\ 1 - 1.2z^{-1} + 0.2z^{-2} & 10z^{-1} + 5z^{-2} \\ & 10z^{-1} - 12z^{-2} + 2z^{-3} \\ \hline & 17z^{-2} \mp 2z^{-3} \end{array}$$

$x(0) = 0 \quad x(1) = 10 \quad x(2) = 17$

$y(k) - \cdots y(k-1) + \cdots y(k-2) = -\cdots x(k-1) + \cdots x(k-2)$

② Difference Equation Approach

$z^2 Y(z) - 1.5327 z Y(z) + 0.6607 Y(z) = 0.4673z - 0.3393$

$$Y(z) = \frac{0.4673z - 0.3393}{z^2 - 1.5327z + 0.6607}$$

Y ~~[illegible]~~ $1.5327 z^{-1} Y(z) + 0.6607 z^{-2} Y(z) = \cdots z^{-1} \cdots z^{-2}$

③ PFE. Pg. 59.

$$X(z) = \frac{z^2+z+2}{(z-1)(z^2-z+1)} = \frac{4}{z-1} + \frac{-3z+2}{z^2-z+1}$$

$$\frac{A}{z-1} + \frac{Bz+C}{z^2-z+1} = \frac{4z^{-1}}{1-z^{-1}} + \frac{-3z^{-1}+2z^{-2}}{1-z^{-1}+z^{-2}}$$

$$-\frac{3}{2} + \frac{1}{2}z^{-1} \qquad + \frac{1}{2}z^{-1}$$

$$Az^2 - Az + A + Bz^2 + (C-B)z - C \qquad \frac{\sqrt{3}}{2} \qquad \frac{\sqrt{3}}{3}$$

$$(A+B)z^2 \qquad 1\,z^2$$
$$+ \qquad +$$
$$(C-B-A)z \qquad 1\,z$$
$$+ \qquad +$$
$$A-C \qquad 2$$

$$= z^{-1}\left(\frac{4}{1-z^{-1}} + \frac{-3+2z^{-1}}{1-z^{-1}+z^{-2}}\right)$$

$$\frac{-3+2z^{-1}}{1-z^{-1}+z^{-2}} = Y(z)$$

$\cos w = \frac{1}{2}$

$\sin w = \frac{\sqrt{3}}{2}$

$A - C = 2$

$A - C + B = -1$

$A + B = 1$

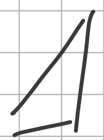
$A = 4$

$B = -3$

$C = 2$

$\cancel{z X(z)} = X(z) = z^{-1} Y(z) \Leftrightarrow y(k-1)$

$Y(z) \Leftrightarrow y(k)$



$$y(k) = 4 \cdot 1(k) - 3\cos\frac{\pi}{3}(k \cdot 14\ \frac{\sqrt{3}}{3}\sin\frac{\pi}{3}k$$

$$y(k-1) = 4 \cdot 1(k-1) + \cdots$$

$$\overline{\qquad\qquad}$$
$$1 - z^{-1} + z^{-2} \qquad -\frac{3}{2}$$

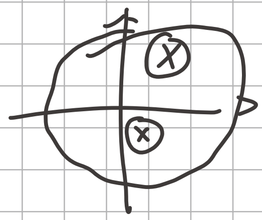
$$\frac{X(z)}{z} = \frac{z^2+z+2}{z(z-1)(z^2-z+1)}$$

$u(0) = 1$

$1(k=0) = 1$

④ Inversion Integral

$$x(k) = \mathcal{Z}^{-1}\{ \cancel{z} X(z) \} = \frac{1}{j2\pi} \oint X(z)\, z^{k-1}\, dz$$



$$x(k) = K_1 + K_2 + \cdots + K_m$$

$$= \sum_{i=1}^{m} [\text{residue of } X(z) z^{k-1} \text{ at pole } z = z_i \text{ of } X(z) z^{k-1}]$$

if $z = z_i$ is simple pole of $X(z) z^{k-1}$, then

$$K_i = \lim_{z \to z_i} [(z - z_i) \cdot X(z) z^{k-1}]$$

if $z = z_j$ is order q multiple pole of $X(z) z^{k-1}$

$$K_j = \frac{1}{(q-1)!} \lim_{z \to z_q} \frac{d^{q-1}}{dz^{q-1}} [(z - z_j)^q X(z) z^{k-1}]$$

Ex 2-16

$$X = \frac{z(1 - \cancel{a} e^{-a})}{(z-1)(z - e^{-\cancel{aT}})}$$

$$X(z) z^{k-1} = \frac{z^k (1 - e^{-a})}{(z-1)(z - \cancel{a} e^{-a})}$$

poles: $1, e^{-a}$ simples.

$$\therefore K_{z=1} = \lim_{z \to 1} \cancel{(z-1)} \frac{z^k (1 - e^{-a})}{\cancel{(z-1)}(z - e^{-a})} = 1$$

$$K_{z=e^{-a}} = \lim_{z \to e^{-a}} \cancel{(z - e^{-a})} \frac{z^k (1 - e^{-a})}{z - 1} = -e^{-ak}$$

$$\therefore x(k) = 1 - e^{-ak}$$

Ex more $X(z) = \dfrac{z^3}{(z-\frac{1}{2})^3 (z-\frac{1}{3})}$

Inverse Integral M:

$$X(z) z^{k-1} = \frac{z^{k+2}}{(z-\frac{1}{2})^3 (z-\frac{1}{3})}$$

$\frac{1}{2}, \frac{1}{3}$ poles.

$$K|_{z=\frac{1}{3}} = \lim_{z \to \frac{1}{3}} \cancel{(z-\tfrac{1}{3})} \cdot \frac{z^{k+2}}{(z-\frac{1}{2})^3} = \frac{(\frac{1}{3})^{k+2}}{(\frac{1}{6})^3} = \frac{\frac{1}{9}(\frac{1}{3})^k}{\frac{1}{216}}$$

$$(k+1) z^{k+2} - \frac{(k+2)}{3} z^{k+1}$$

$$\begin{array}{c} 36 \\ \times \frac{1}{216} \end{array}$$

$$K|_{z=\frac{1}{2}} = \lim_{z \to \frac{1}{2}} \frac{1}{2!} \cdot \frac{d^2}{dz^2} \left[\frac{z^{k+2}}{\cancel{(z-\frac{1}{2})^3} \cdot (z-\frac{1}{3})} \right] \cancel{(z-\tfrac{1}{2})^3}$$

$$(k+2)(z^{k+2} - \tfrac{1}{3} z^{k+1})$$

$$\left(\frac{z^{k+2}}{z-\frac{1}{3}} \right)' = \frac{(k+2) z^{k+1} (z-\frac{1}{3}) - z^{k+2}}{(z-\frac{1}{3})^2}$$

$$\frac{(k+1) z^{k+2} - \frac{(k+2)}{3} z^{k+1}}{z^2 - \frac{2}{3} z + \frac{1}{9}}$$

$$\left[(k+2)(k+1) z^{k+1} - \frac{(k+2)(k+1)}{3} z^k \right] (\ )$$

$$\frac{- (2z - \frac{2}{3}) \left[(k+1) z^{k+2} - \frac{k+2}{3} z^{k+1} \right]}{2 (z-\frac{1}{3})^4}$$

$$\frac{}{2 \cdot (\frac{1}{6})^4}$$

① Simple z_i pole

$$k = \lim_{z \to z_i} (z - z_i) \cdot X(z) \cdot z^{k-1}$$

② multiple q th z_j pole.

$$k = \lim_{z \to z_j} \frac{1}{(q-1)!} \cdot \frac{d^{q-1}}{dz^{q-1}} \left[(z - z_j)^q \cdot X(z) z^{k-1} \right]$$

$\cancel{x(2) + 3x(1) + 2x(0)} = 0.$

Ex 2-1]

$x(2) + 3x(1) + \cancel{2x(0)} = 0$

$x(2) = -3$

$x(k+2) + 3x(k+1) + 2x(k) = 0$ $\quad$ $x(0) = 0 \quad x(1) = 1$

$$z^2 \left[X(z) - x(1) z^{-1} - x(0) \right] + 3z \left[X(z) - x(0) \right] + 2X(z) = 0$$

$$(z^2 + 3z + 2) X(z) = z x(1) + z^2 x(0) + 3z x(0)$$

$$= z$$

$$X(z) = \frac{z}{(z+1)(z+2)}$$

$$\frac{\cancel{z} X(z)}{\cancel{z} z} = \cancel{(z+1)} \frac{1}{z+1} - \frac{1}{z+2}$$

$$x(z) = \frac{1}{1+z^{-1}} - \frac{1}{1+2z^{-1}}$$

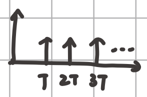
$$= -1^{k}(k) - (-2)^k \cdot 1(k)$$

ADC Model

$$\xrightarrow[X(s)]{x(t)} \ \delta_T \ \xrightarrow[X^*(s)]{x^*(t)}$$

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

$$\delta_T(t) = \sum_{k=0}^{\infty} \delta(t - kT)$$



$$x^*(t) = \sum_{k=0}^{\infty} x(kT) \cdot \delta(t - kT)$$

$$= x(0)\,\delta(t) + x(T)\,\delta(t-T) + \cdots$$

strength of each pulse $= x(kT)$

$$X^*(s) = \mathcal{L}\{x^*(t)\} = x(0)\,\mathcal{L}\{\delta(t)\} + x(T)\,\mathcal{L}\{\delta(t-T)\} + \cdots$$

$$= x(0)\,e^0 + x(T)\,e^{-Ts} + x(2T)\,e^{-2Ts} + \cdots x(kT)\,e^{-kTs}$$

$$= \sum_{k=0}^{\infty} x(kT)\,e^{-kTs} = \sum x(kT)\,(e^{Ts})^{-k}$$

let $e^{Ts} = z \iff s = \frac{1}{T}\ln z$

$$X^*(s)\Big|_{s=\frac{1}{T}\ln z} = \sum x(kT)\,z^{-k} = X(z)$$

$$= X^*\left(\tfrac{1}{T}\ln z\right) = X(z)$$

Ex 3-1 $\quad x(t) = e^{-t} \quad X(s) = \mathcal{L}\{x(t)\} = \frac{1}{s+1}$

$$x^*(t) = \sum x(kT)\,\delta(t - kT) = \sum e^{-kT}\,\delta(t-kT)$$

$$X^*(s) = \sum e^{-kT}\,e^{-kTs} = \sum_k e^{-kT(1+s)} = 1 + e^{-(1+s)T} + e^{-2(1+s)T}$$

$$= \frac{1}{1 - e^{-(1+s)T}} = \frac{1}{1 - e^{-T}e^{-sT}}$$

$$x(k) = e^{-kT}$$

$$X(z) = \sum e^{-kT} z^{-k} = \sum 1 * e^{-kT} z^{-k} = \frac{1}{1 - e^{-T}z^{-1}}$$

or let $s = \frac{1}{T}\ln z$ $\quad X^*(s)\Big|_{s=\frac{1}{T}\ln z} = \dfrac{1}{1 - e^{-T}z^{-1}}$

DAC Model

Data - Hold is process that generate continuous - time signal $h(t)$ from $x(kT)$

$$h(kT+\tau) = a_n \tau^n + a_{n-1}\tau^{n-1} + \cdots + a_1\tau + a_0$$

$(n+1)$ a coefficients ↑ ∴ need $n+1$ data points :

$$x((k-n)T) \ldots x(kT)$$

must have $h(kT) = x(kT)$ ∴ $a_0 = x(kT)$

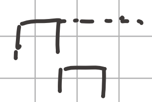
$$h(kT+\tau) = a_n\tau^n + \cdots + a_1\tau + x(kT)$$

for zero-order hold. $n=0$

∴ $h(kT+\tau) = x(kT)$, $0 \le \tau \le T$

∴ for $k=0$ from $0-T$ interval of h is $x(0)$

Unless otherwise stated, assume zero-order hold.
ZOH



∴ for ZOH , $h_1(t) = x(0)\cdot[1(t) - 1(t-T)] + x(T)\cdot \cdots\cdots$

$$= \sum_k x(kT)\cdot\{1(t-kT) - 1[t-(k+1)T]\}$$

$$H_1(s) = \mathcal{L}\{h(t)\} = \sum x(kT)\cdot \mathcal{L}\{(\ldots) - [\;]\}$$

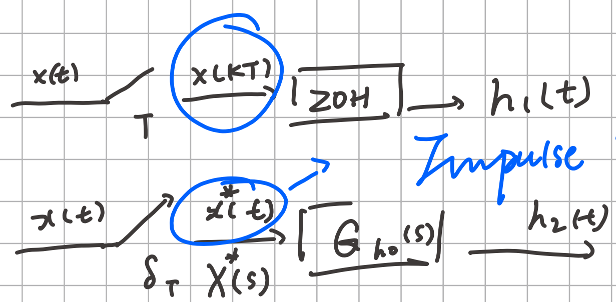
$$= \sum x(kT)\cdot \frac{e^{-kTs} - e^{-(k+1)Ts}}{s}$$

$$= \frac{1-e^{-Ts}}{s}\cdot \sum_k x(kT)\, e^{-kTs}$$

$$= \frac{1-e^{-Ts}}{s} \overset{*}{X}(s)$$

real sampler

Impulse Sampler



$$x^*(t) = \sum_k x(kT)\,\delta(t-kT)$$

$$\overset{*}{X}(s) = \sum_k x(kT)\, e^{-kTs}$$

$$H_2(s) = G_{ho}(s)\cdot \overset{*}{X}(s) = \sum_k x(kT)\cdot e^{-kTs} G_{ho}(s) = H_1(s)$$

$$\therefore\; G_{ho}(s) = \frac{1-e^{-Ts}}{s}$$

therefor ZOH with real Sampler can be replaced by impulse sampler and transfer function

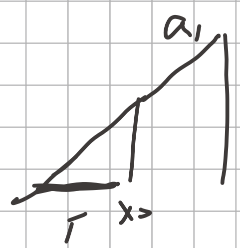
$$\frac{1-e^{-Ts}}{s}$$

For ~~‡~~ first-order hold $G_{h_1}(s) = \left(\frac{1-e^{-Ts}}{s}\right)^2 \cdot \frac{Ts+1}{T}$

$h(kT+T) = a_1T + a_0 = a_1T + x(kT)$

$\because h(kT) = x(kT)$

$h((k-1)T) = x((k-1)T)$



$\therefore h(kT-T) = -a_1T + x(kT) = x(kT-T)$

$\therefore a_1 = \frac{x(kT) - x[(k-1)T]}{T}$

$\int te^{-t}\,dt$

$\therefore h(kT+T) = \cancel{a_1}\frac{[\quad]}{T}T + x(kT)$

for $x(t) = 1$

$h(t) = (1+\frac{t}{T})(1(t) - 1(t-T))$

$+ 1(t-1)$

$= 1(t) + 1(t)\frac{t}{T} - \frac{t}{T}1(t-T)$

$= 1(t) + \frac{t}{T} - \frac{t-T}{T}\cancel{(t-)}\frac{1}{T}1(t-T)$

$H(s) = \mathcal{L}\{h\} = \frac{1}{s} + \frac{1}{Ts^2} - \frac{1}{Ts^2}e^{-Ts}$

$-\frac{1}{s}e^{-Ts}$

$= \frac{1-e^{-Ts}}{s} + \frac{1-e^{-Ts}}{Ts^2}$

$= (1-e^{-Ts}) \cdot \frac{Ts+1}{Ts^2} = X^*(s) \cdot G_{h_1}(s)$

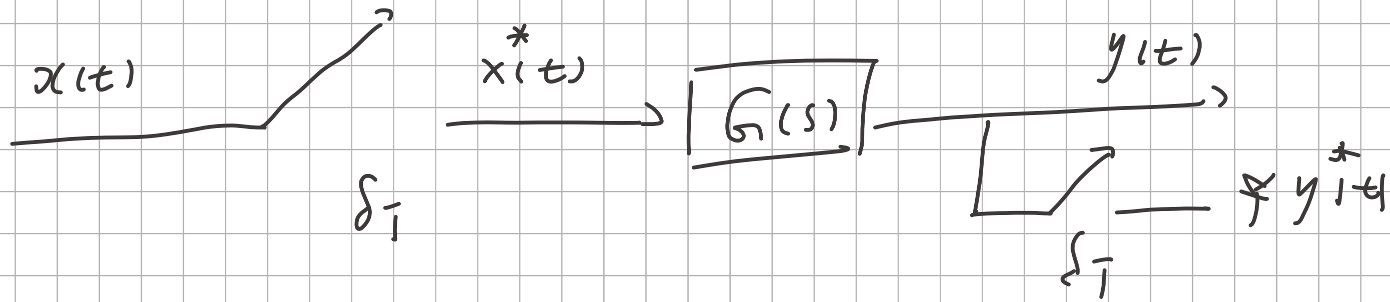
Proof

$X^*(s) = \mathcal{L}\{\sum x(kT) \cdot \delta(t-kT)\}$

$= \sum x(kT)\, \mathcal{L}\,\delta(t-kT) = \sum x(kT)e^{-kTs}$

$x(kT) = 1 \quad \therefore X^*(s) = \frac{1}{1-e^{-Ts}}$

$\therefore G_{h_1}(s) = \left(\frac{1-e^{-Ts}}{s}\right)^2 \frac{Ts+1}{T}$



$Y(z) = G(z)X(z)$

$$\begin{cases} Y(s) = G(s)X^*(s) \\ Y^*(s) = [G(s)X^*(s)]^* \end{cases}$$

$Y(z) = Y^*(s)\big|_{s=\frac{1}{T}\ln z}$,

so as $G(z)$, $X(z)$

$\cancel{Y(z)}$

$\therefore Y(z) = G(z)X(z)$

$\Rightarrow Y^*(s) = G^*(s)X^*(s)$

$$[G(s)X^*(s)]^* = G^*(s)X^*(s)$$

$x(t) = 1(t)$ $\quad \cancel{x(s)}$ $x^*(t) = \sum 1 \cdot \delta(t-kT)$

$X^*(s) = \mathcal{L}\{x^*(t)\} = \sum \mathcal{L}\{\delta(t-kT)\} = \sum^{\infty} e^{-kTs} = \frac{1}{1-e^{-Ts}}$

$Y^*(s) = [G(s)X^*(s)]^* = Y(z)$

$= G(z)X(z) = \frac{1}{1-e^{-T}z^{-1}} \cdot \frac{1}{1-z^{-1}} = \frac{1}{1-e^{-T}e^{-Ts}} \cdot \frac{1}{1-e^{-Ts}}$

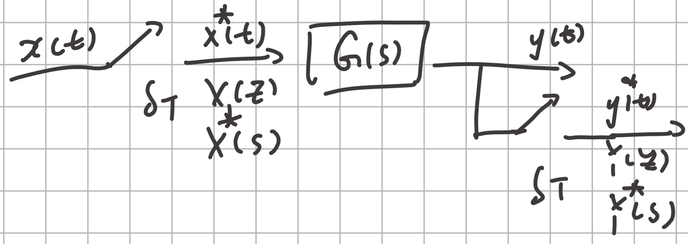
$G(s) = \frac{1}{s+1}$ $\quad g(t) = e^{-t}$ $\quad e^{-kT}$ $\quad G(z)$ $= \frac{1}{1-e^{-T}z^{-1}}$

$G^*(s) = \frac{1}{1-e^{-T}e^{-Ts}}$

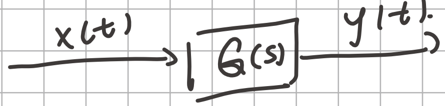
$z = e^{Ts}$ $\quad \frac{1}{T}\ln z$ $\quad = \frac{1}{(1-e^{-T}e^{-Ts})(1-e^{-Ts})}$

e^{Ts}

I



II



for I : $Y(s) = X^*(s) \cdot G(s)$

$Y^*(s) = X^*(s)\, G^*(s) = Y(z) = X(z)\, G(z)$

for II : $Y(s) = X(s)\, G(s)$ Without Sampler

$Y^*(s) = [X(s) G(s)]^* = [GX(s)]^*$

$Y(z) = [GX(s)]^* \big|_{s=\frac{1}{T}\ln z} = \mathcal{Z}\{GX(s)\} = GX(z) \neq G(z)X(z)$

Ex 3-3 $\quad G(s) = \frac{1}{s+a} \quad g(t) = e^{-at} \quad g(kT) = e^{-akT}$

$G(z) = \frac{1}{1-e^{aT}z^{-1}}$

$Y(z) = \mathcal{Z}\{G(s)X(s)\}$

$X(s) = \mathcal{L}\{1(t)\} = \frac{1}{s}$

$Y(s) = G(s)X(s) = \frac{1}{s(s+a)}$

$Y(z) = \mathcal{Z}\{GX(s)\} = \mathcal{Z}\left\{\frac{1}{s(s+a)}\right\} = \frac{1}{a}\mathcal{Z}\left\{\frac{1}{s} - \frac{1}{s+a}\right\}$

$= \frac{1}{a}\left(\frac{1}{1-z^{-1}} - \frac{1}{1-e^{-aT}z^{-1}}\right)$

$(1-z^{-1})\,\mathcal{Z}\left\{\frac{1}{s^2(s+1)}\right\} \qquad \frac{-s+1}{s^2}$

$-\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1}$

$\frac{As+B}{s^2} + \frac{C}{s+1}$

$-\frac{z^{-1}}{1-z^{-1}} + \frac{Tz^{-2}}{(1-z^{-1})^2} \qquad Cs^2 + As^2 + (A+B)s + B = 1$

$+ \frac{z^{-1}}{1-e^{-T}z^{-1}} \qquad A+C=0 \quad C=1$

$A+B=0 \quad A=-1$

$B=1$

Ex 3-5 a) $U(s) = G(s)X^*(s)$

$U^*(s) = G^*(s) X^*(s)$

$Y(s) = U^*(s)\, H(s) = G^*(s) X^*(s)\, H(s)$

$Y^*(s) = U^*(s)\, H^*(s) = \cdots$

$\therefore \cancel{Y(s)}\ \frac{Y(z)}{X(z)} = U^*(s)\, {}^*H(s) = G(z)\, H(z)$

$= Z\left\{\frac{1}{s+a}\right\} \cdot Z\left\{\frac{1}{s+b}\right\}$

$\frac{1}{(1-e^{-aT}z^{-1})(1-e^{-bT}z^{-1})}$

$Y(s) = X^*(s)\, G(s)\, H(s) = X^*(s)\, GH(s)$

$Y^*(s) = GH^*(s)\, X^*(s)$

$Y(z) = X(z) \cdot Z\{GH(s)\}$

$Z\left\{\frac{1}{(s+a)(s+b)}\right\}$

$\frac{1}{b-a}\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$

$E(s) = R(s) - C(s)H(s)$

$C(s) = E^*(s)\, G(s)$

$E(s) = R(s) - E^*(s)\, G(s)\, H(s)$

$E^*(s) = R^*(s) - E^*(s)\, GH^*(s)$

$E^*(s) = \frac{R^*(s)}{1 + GH^*(s)}$

$C^*(s) = E^*(s)\, G^*(s) = \frac{G^*(s)\, R^*(s)}{1 + GH^*(s)}$

$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + \cancel{Z\{GH(s)\}}\, GH(z)}$

$E(s) = R(s) - C^*(s)H(s)$

$E^*(s) = R^*(s) - C^*(s)H^*(s)$

$C(s) = E^*(s)\, G(s)$

$C^*(s) = E^*(s)\, G^*(s)$

$= G^*(s)R^*(s) - G\cdots\cdots$

$\frac{C(z)}{R(z)} = G(z) - G(z)\frac{C(z)}{R(z)}H(z)$

$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)H(z)}$

$E(s) = R(s) - C(s)H(s)$

$F(s) = G_1(s)E^*(s)$

$C(s) = F^*(s)G_2(s)$

$E(s) = R(s) - F^*(s)G_2(s)H(s)$

$= R(s) - G_1^*(s)E^*(s)[G_2H(s)]$

$E^*(s) = R^*(s) - G_1^*E^*(G_2H)^*$

$E^* = \frac{R^*}{1+G_1^*(G_2H)^*}$

$C^* = G_1^*E^*G_2^*$

$\frac{C(z)}{R(z)} = \frac{G_1(z)G_2(z)}{1+G_1(z)G_2H(z)}$

$-\frac{1}{s+1} + \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3}$

$\times \frac{1}{1-z^{-1}} \quad - \frac{1}{1-e^{-1}z^{-1}}$

$E(s) = G_1R(s) - CH(s)\cdot G_1(s)$

$C(s) = E^*(s)G_2(s)$

$E(s) = G_1R(s) - E^*(s)G_1G_2H(s)$

$E^*(s) = [G_1R(s)]^* - E^*(s)[G_1G_2H(s)]^*$

$E^*(s) = \frac{[G_1R(s)]^*}{1+[G_1G_2H(s)]^*}$

$C^*(s) = E^*(s)G_2^*(s)$

$C(z) = \frac{G_2(z)G_1R(z)}{1+G_1G_2H(z)}$

$E(s) = R(s) - C(s)$

$C(s) = E^*(s)G_D^*(s)\left[\frac{1-e^{-Ts}}{s}\right]G_p(s)$

$C(s) = GR(s) - C^*(s)GH(s)$

$E(s) = R(s) - E^*(s) \cdots\cdots$

$C^*(s) = [GR(s)]^* - C^*(s)[GH(s)]^*$

$E^*(s) = R^*(s) - E^*(s)G_D^*(s)\left[\frac{\cdots}{s}G_p(s)\right]^*$

$C(z) = \frac{GR(z)}{1+GH(z)}$

$E^*(s) = \frac{R^*(s)}{1+G_D^*(s)[\ \ ]^*}$

$\frac{A}{s+1} + \frac{Bs^2+Cs+D}{s^3} = \frac{1}{\cdots}$

$C^*(s) = R^*(s)\left(\frac{G_D^*(s)\cdots}{1+G_D\cdots}\right)$

$Bs^3 + (B+C)s^2 + (C+D)s + D + As^3 = 1$

digital

$\frac{C(z)}{R(z)} = \frac{G_D(z)}{1+G_D(z)}$

$(1-z^{-1})Z\{\cdots(s)\}\ Z\left(\frac{G(s)}{s}\right)$

PID

$G(D) = K_p + \frac{K_I}{1-z^{-1}} + (1-z^{-1})K_D$

$G_h(s)\cdot G_p(s) = G(s) = \frac{1-e^{-Ts}}{s}G_p(s)$

for $T=1$ $\quad G_p = \frac{1}{s(s+1)} \quad \frac{1-e^{-s}}{s^2(s+1)}$

$C=-1$, $B=1$, $A=-1$

$Z\{G(s)\} = (1-z^{-1})Z\left\{\frac{1}{s^2(s+1)}\right\}$

$A+B=0$

$B+C=0$

$C+D=0$

$D=1$

$$\mathcal{Z}\left\{ \cancel{0}\cdot\frac{1-e^{-Ts}}{s} X(s) \right\} = (1-z^{-1})\, \mathcal{Z}\left\{ \frac{X(s)}{s} \right\}$$

Th: $\quad X_1(s) = \dfrac{X(s)}{s}$

$$\Rightarrow \mathcal{Z}\left\{ (1-e^{-Ts}) X_1(s) \right\}$$

$$= \mathcal{Z}\{X_1(s)\} - \mathcal{Z}\{e^{-Ts} X_1(s)\}$$

$$= \mathcal{Z}\{x_1(t)\} - \mathcal{Z}\{x_1(t-T)\}$$

$$= \cancel{1-z}\,\mathcal{Z}\{\} - z^{-1}\,\mathcal{Z}\{x_1(t)\}$$

$$= (1-z^{-1})\, X_1(z)$$

$$= (1-z^{-1})\,\mathcal{Z}\left\{ \frac{X(s)}{s} \right\} = \mathcal{Z}\left\{ \frac{1-e^{-Ts}}{s} X(s) \right\}$$

$$\cancel{\mathcal{Z}}\ Y(z) = G(z)\, X(z)$$

proof: convolution

Q3.1 Solve the following difference equation:

$$x(k+2)-2x(k+1)+x(k)=\delta(k),\ x(k)=0 \text{ for } k<0,$$

where $\delta(k)$ is a unit impulse function. [Ans: $x(k)=\delta(k)-1(k)+k,\ k=0,1,2,.....$]

$x(1) - 2x(0) + 0$
$= 0$

$x(0) = 0$

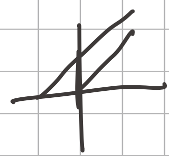
$x(1) = 0$

$$z^2 \left(X(z) - x(1) z^{-1} - x(0) \right) - 2z \left(X(z) - x(0) \right) + X(z) = 1$$

$$(z^2 - 2z + 1) X(z) = 1$$

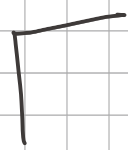
$$X(z) = \frac{1}{z^2-2z+1} = \frac{1}{(z-1)^2} = \frac{1}{z^2-2z+1}$$

$$= 1 - \frac{z^2}{z^2-2z+1} + \frac{2z}{z^2-2z+1}$$



$$= 1 - \frac{\cancel{z}\,1-z^{-1}}{\cancel{z}(1-z^{-1})^2} + \frac{z^{-1}}{(1-z^{-1})^2}$$

$$= 1 - \frac{1}{1-z^{-1}} + \frac{z^{-1}}{(1-z^{-1})^2}$$



$$= \delta(k) - 1(k) + \cancel{3}k$$

$k=0$ $\quad 0$

$k=1$ $\quad 0$

$$\frac{1}{(z-1)^2} \qquad z^{-1} \qquad \frac{z^{-1}}{\cancel{z}(1-z^{-1})^2}$$

$\Downarrow$

$(k-1)\,u(k-1)$

$k=2$ $\quad 1$

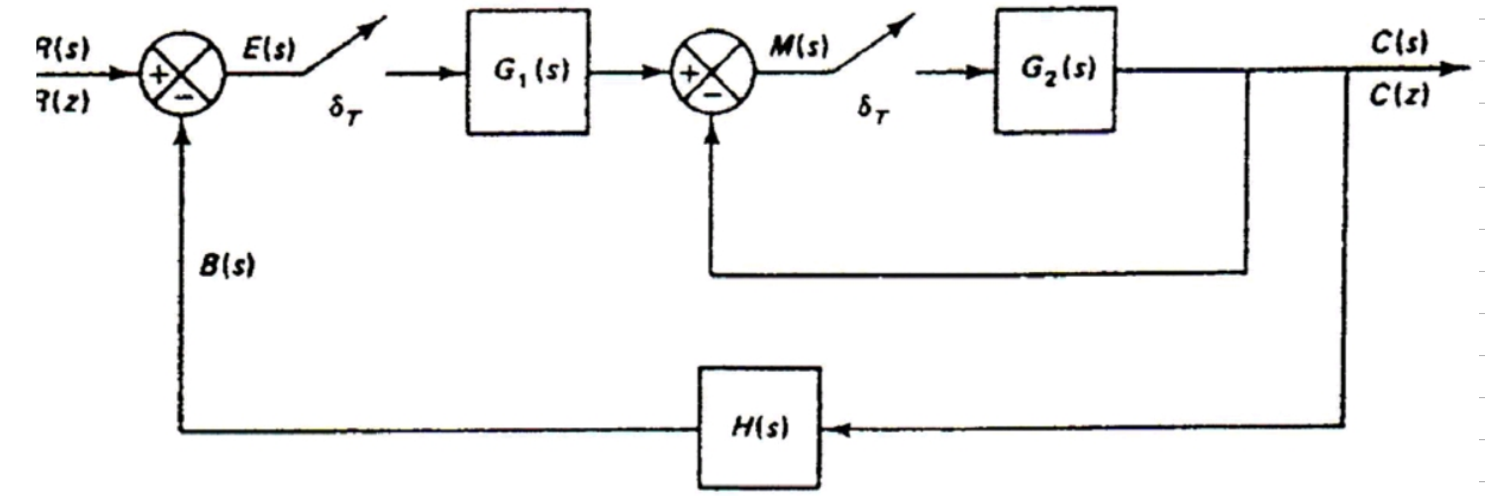
2

3

$\therefore$ for $Y(z) = z^{-1} X(z)$ $\quad k=0 \quad 0$

$y(k) = x(k-1) \cancel{1}(k-1)$ $\quad k=1 \quad 0$

$k=2 \quad 1$



$$C^*(s) = M^*(s)\, G_2^*(s)$$

$$= \frac{E^*(s)\, G_1^*(s)\, G_2^*(s)}{1 + G_2^*(s)}$$

$$= \frac{R^*(s)\,(\cancel{1+G_2^*(s)})(G_1G_2)}{(\cancel{1+G_2^*(s)})(1+G_2+G_1 \ldots)}$$

$$C(s) = M^*(s)\, G_2(s)$$

$$M(s) = E^*(s)\, G_1(s) - C(s)$$

$$E(s) = R(s) - C(s)\, H(s)$$

$$M(s) = E^*(s)\, G_1(s) - M^*(s)\, G_2(s)$$

$$M^*(s) = E^*(s)\, G_1^*(s) - M^*(s)\, G_2^*(s)$$

$$M^*(s) = \frac{E^*(s)\, G_1^*(s)}{1 + G_2^*(s)}$$

#1

#2

$$E(s) = R(s) - M^*(s)\, G_2(s)\, H(s)$$

$$E^*(s) = R^*(s) - M^*(s)\, G_2H^*(s)$$

$$E^*(s) = R^*(s) - E^*(s) \cdot \frac{G_1^*(s)\, G_2H^*(s)}{1 + G_2^*(s)}$$

$$E^*(s) = \frac{R^*(s)\,(1 + G_2^*(s))}{1 + G_2^*(s) + G_1^*(s)\, G_2H^*(s)}$$

$$\therefore \frac{C(z)}{R(z)} = \frac{G_1(z)\, G_2(z)}{1 + G_2(z) + G_1(z)\, G_2H(z)}$$

$$X(z) = \frac{z^{-1}(0.5 - z^{-1})}{(1-0.5z^{-1})(1-0.8z^{-1})^2}$$

$$\frac{z(0.5z-1)}{(z-0.5)(z-0.8)^2}$$

$Z^{-1}\{X(z)\}$

① PFE. $\frac{X(z)}{z} = \frac{z^{-2}(0.5-z^{-1})}{(1-0.5z^{-1})(1-0.8z^{-1})^2}$

$$= \frac{0.5z-1}{(z-0.5)(z-0.8)^2}$$

$= -\frac{25}{3}(0.5)^k + \frac{25}{3}(0.8)^k - 2(0.8)^{k-1}k$

$$\frac{A}{(z-0.5)} + \frac{B}{z-0.8} + \frac{C}{(z-0.8)^2}$$

$Az^2 - 1.6Az + 0.64A$

$+ \quad Bz^2 - 1.3Bz + 0.4B$ $\quad \frac{X}{z} = -\frac{25}{3}\frac{1}{1-0.5z^{-1}}$

$+ \quad Cz - 0.5C$ $\quad + \frac{25}{3}\frac{1}{1-0.8z^{-1}}$

$= \quad 0.5z - 1$ $\quad -2\frac{0.8z^{-1}}{(1-0.8z^{-1})^2}$

$1.28A + 0.8B - C = -2$

$0.64A + 0.4B - 0.5C = -1$

$-1.6A - 1.3B + C = 0.5$

$A + B = 0.$ $\quad A = -\frac{25}{3}$ $\quad C = -2$

$+0.32A + 0.5B = 1.5$

$0.18B = 1.5$

$B = \frac{25}{3}$

0.09

$0.3 \times 0.64 \times 0.5(k+1)\, 0.8^{k-1}$

$0.8^k \times 0.3$

$0.5z^{k+1} - z^k \times 0.8z^{k-1}$

$$\frac{\cancel{z^k(0.5z-1)}}{z-0.5}$$

$z^k - kz^{k-1}$

$0.5(k+1)z^k$

$(z-0.5)$ $\quad -0.5z^{k+1} + z^k$

$(z-0.5)^2$

② Inverse Intergral

$$X(z)z^{k-1} = \frac{z^k(0.5z-1)}{(z-0.5)(z-0.8)^2}$$

poles: 0.5, 0.8

for 0.5 $\quad K_1 = \lim_{z \to 0.5} \cancel{(z-0.5)}\frac{z^k(0.5z-1)}{(z-0.8)^2}$

$$= \frac{-0.75\, 0.5^k}{0.09}$$

for 0.8 $\quad K_2 = \frac{1}{(2-1)!}\frac{d^1}{dz}\lim_{z \to 0.8}\frac{d^1}{dz}(z-0.8)^2 \cdots$

$I:2$

$g(k)$: $k a^{k-1} = a^{k-1} y(k)$

$y(k) = k \iff Y(z) = \sum y(k) z^{-k} = \frac{z^{-1}}{(1-z^{-1})^2}$

$G(z) = a^{-1} \sum a^{k} y(k) z^{-k}$

$= a^{-1} \sum y(k) (a^{-1}z)^{-k}$

$= a^{-1} Y(z)\big|_{z = a^{-1}z} = a^{-1} \cdot \frac{a z^{-1}}{(1-az^{-1})^2} = \frac{z^{-1}}{(1-az^{-1})^2}$

$$k a^{k} = \frac{a z^{-1}}{(1-az^{-1})^2}$$

$$\Downarrow$$

$$\frac{z^{-1}}{(1-az^{-1})^2} = \frac{k a^k}{a} = k a^{k-1}$$

$$D(z) = K_p + \frac{K_I}{1-z^{-1}} + (1-z^{-1}) K_D$$

$$= \frac{K_I + (1-z^{-1}) K_p + (1-z^{-1})^2 K_D}{1-z^{-1}}$$

$$= \frac{K_D z^{-2} - (2K_D + K_p) z^{-1} + K_p + K_I + K_D}{1-z^{-1}}$$

digital PID

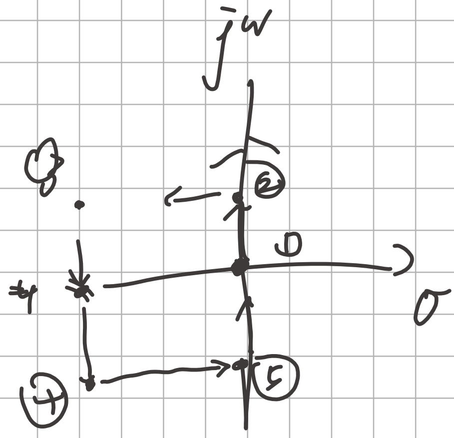
$$\frac{C(z)}{P(z)} = \cancel{\frac{1+G_D G}{G_D G}} \rightarrow \frac{G_D(z) G(z)}{1+G_D(z) G(z)}$$

$z = e^{Ts}$

$\cancel{e} : s = \sigma + j\omega$

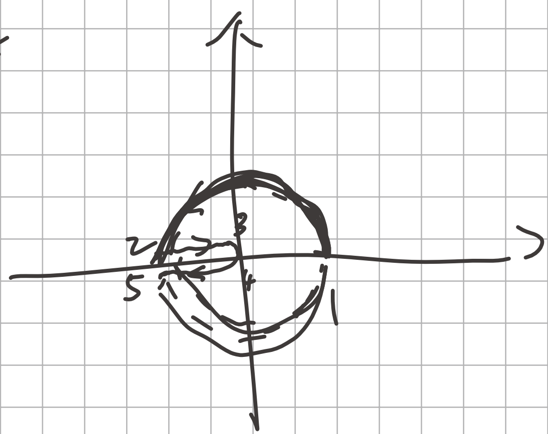
$z = e^{T\sigma + jT\omega}$

$= e^{T\sigma} \cdot \cancel{e^{jT\omega}} \angle T\omega$



$1 \to 2 \to 3 \to 4 \to 5$

z



July Test:

①	a_0	a_1	$\cdots$	a_n
r①	a_n	a_{n-1}	$\cdots$	a_0
②	b_{n-1}	b_{n-2}	$\cdots$	b_0
r②	b_0	b_1	$\cdots$	b_{n-1}

$\vdots$

$$b_{n-1} = \begin{vmatrix} a_0 & a_n \\ a_n & a_0 \end{vmatrix}$$

$$b_{n-2} = \begin{vmatrix} a_0 & a_{n-1} \\ a_n & a_1 \end{vmatrix}$$

$$b_{n-3} = \begin{vmatrix} a_0 & a_{n-2} \\ a_n & a_2 \end{vmatrix}$$

Routh Stability Criterion [for bilinear Transfor]

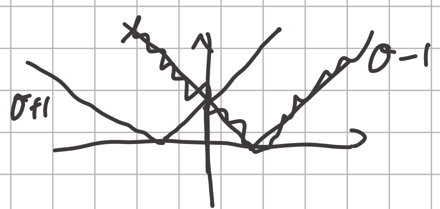
z bilinear transformation

$$z = \frac{w+1}{w-1} \quad \text{or} \quad w = \frac{z+1}{z-1}$$

let $w = \sigma \pm j\omega$

$$|z| = \left|\frac{w+1}{w-1}\right| = \left|\frac{\sigma \pm j\omega + 1}{\sigma \pm j\omega - 1}\right| < 1 \iff \frac{(\sigma+1)^2 + \omega^2}{(\sigma-1)^2 + \omega^2} < 1$$

$$\iff (\sigma+1)^2 + \omega^2 < (\sigma-1)^2 + \omega^2 \iff (\sigma+1)^2 < (\sigma-1)^2 \iff \sigma < 0$$



a	b	c
d	e	

$$f = -\frac{1}{d}\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

Ex: $p(z) = z^4 - 1.2z^3 + 0.07z^2 + 0.3z - 0.08$

bilinear: $p'(w) = w^4 + 14.67w^3 + 59.78w^2 + 81.33w + 21$

w^4	1	59.78	21
w^3	14.67	81.33	
w^2	$-\frac{1}{14.67}\begin{vmatrix} 1 & 59.78 \\ 14.67 & 81.33 \end{vmatrix}$	$-\frac{1}{81.33}\begin{vmatrix} & \\ & \end{vmatrix}$	
$\vdots$			
w^0			

Jury does not reveal the number of unstable roots, But Routh test does

FOH $\left(\frac{1-e^{-Ts}}{s}\right)^2 \frac{Ts+1}{T}$

ZOH $\frac{1-e^{-Ts}}{s}$

$$\mathcal{Z}\left\{1-e^{-Ts}X(s)\right\} = (1-z^{-1})\,\mathcal{Z}\left\{\frac{X(s)}{s}\right\}$$

$$\frac{1}{(q-1)!}\lim_{z \to z_j}\frac{d^{q-1}}{dz^{q-1}}\left((z-z_j)\cdot X(z)z^{k-1}\right) = k_i$$