**Q1.1** We can use the following property (#11) to do this question

$$\mathcal{Z}[k\ x(k)] = -z\ \frac{d}{dz}X(z)$$

Note that

$$\mathcal{Z}[1] = \frac{1}{1 - z^{-1}}$$

$$\left(\frac{z}{z-1}\right)' = \frac{(z-1)-z}{(z-1)^2} = -\frac{1}{(z-1)^2}$$

$$Z\{k\} = -z\left(\frac{z}{z-1}\right)' = \frac{z}{(z-1)^2}$$

$$\left(\frac{z}{(z-1)^2}\right)' = \frac{(z-1)^2 - 2z(z-1)}{(z-1)^4} = -\frac{z+1}{(z-1)^3}$$

$$Z\{k^2\} = -z\left(\frac{z}{(z-1)^2}\right)' = \frac{z(z+1)}{(z-1)^3} = \frac{z^2+z}{(z-1)^3}$$

$$\left(\frac{z^2+z}{(z-1)^3}\right)' = \frac{(2z+1)(z-1)^3 - 3(z-1)^2(z^2+z)}{(z-1)^6}$$

$$= \frac{(2z+1)(z-1) - 3(z^2+z)}{(z-1)^4}$$

$$= -\frac{z^2+4z+1}{(z-1)^4}$$

$$Z\{k^3\} = -z\left(-\frac{z^2+4z+1}{(z-1)^4}\right) = \frac{z^3+4z^2+z}{(z-1)^4}$$

1.~~1~~(a)

**Q1.2**

We can use the Complex Translation property of $\mathcal{Z}$ transform

$$\mathcal{Z}\left[x(t)e^{-at}\right] = X(ze^{aT})$$

as follows:

$$\begin{aligned}
\mathcal{Z}\left[t^2\right] &= \frac{T^2 z^{-1}(1+z^{-1})}{(1-z^{-1})^3} \\
\mathcal{Z}\left[t^2 e^{-at}\right] &= \frac{T^2 (ze^{aT})^{-1}\left(1+(ze^{aT})^{-1}\right)}{(1-(ze^{aT})^{-1})^3} \\
&= \frac{T^2 z^{-1}e^{-aT}(1+z^{-1}e^{-aT})}{(1-z^{-1}e^{-aT})^3}
\end{aligned}$$

We can also use #21 (Table 2-2) property of $\mathcal{Z}$ transform:

$$\frac{\partial}{\partial a}x(t,a) = \frac{\partial}{\partial a}X(z,a)$$

Namely,

$$\mathcal{Z}\left[e^{-at}\right] = \sum_{k=0}^{\infty} e^{-akT}z^{-k} = \frac{1}{1-z^{-1}e^{-aT}}$$

and

$$\mathcal{Z}\left[-te^{-at}\right] = \mathcal{Z}\left[\frac{\partial}{\partial a}e^{-at}\right] = \frac{\partial}{\partial a}\mathcal{Z}\left[e^{-at}\right] = \frac{\partial}{\partial a}\frac{1}{1-z^{-1}e^{-aT}}$$

Repeating the above one more time, we have

$$\mathcal{Z}\left[t^2e^{-at}\right] = \mathcal{Z}\left[\frac{\partial^2}{\partial a^2}e^{-at}\right] = \frac{\partial^2}{\partial a^2}\mathcal{Z}\left[e^{-at}\right] = \frac{\partial^2}{\partial a^2}\frac{1}{1-z^{-1}e^{-aT}}$$

Evaluating the above differentiations, we have

$$\mathcal{Z}\left[t^2e^{-at}\right] \;=\; \frac{T^2z^{-1}e^{-aT}(1+z^{-1}e^{-aT})}{(1-z^{-1}e^{-aT})^3}$$

**Q1.3**

$$\begin{aligned} x(k) &= 9\ k\ 2^{k-1} - 2^k + 3, \qquad k \geq 0 \\ &= \frac{9}{2}\ k\ 2^k - 2^k + 3 \end{aligned}$$

Hence

$$\begin{aligned} X(z) &= \mathcal{Z}\left[x(k)\right] = \sum_{k=0}^{\infty} x(k) z^{-k} \\ &= \frac{9}{2} \sum_{k=0}^{\infty} k\ 2^k z^{-k} - \sum_{k=0}^{\infty} 2^k z^{-k} + 3 \sum_{k=0}^{\infty} z^{-k} \\ &= \frac{9}{2} \sum_{k=0}^{\infty} k (2z^{-1})^k - \sum_{k=0}^{\infty} (2z^{-1})^k + 3 \sum_{k=0}^{\infty} (z^{-1})^k \end{aligned}$$

Noting that

$$\mathcal{Z}[k] = \frac{z^{-1}}{(1-z^{-1})^2}, \qquad \mathcal{Z}[1] = \frac{1}{(1-z^{-1})}$$

we have

$$\begin{aligned} X(z) &= \frac{9}{2} \left. \frac{z^{-1}}{(1-z^{-1})^2} \right|_{z^{-1} \to 2z^{-1}} - \left. \frac{1}{1-z^{-1}} \right|_{z^{-1} \to 2z^{-1}} + \frac{3}{1-z^{-1}} \\ &= \frac{2 + z^{-2}}{(1-2z^{-1})^2 (1-z^{-1})} \end{aligned}$$

**Q1.4** From Lecture notes

$$\mathcal{Z}\left[\sum_{h=0}^{k} x(h)\right] = \frac{1}{1-z^{-1}}\mathcal{Z}\left[x(k)\right]$$

Hence, for

$$y(k) = \sum_{h=0}^{k} a^h$$

we have

$$\mathcal{Z}\left[y(k)\right] = \frac{1}{1-z^{-1}}\mathcal{Z}\left[a^k\right] = \frac{1}{(1-z^{-1})(1-az^{-1})}$$

**Q1.5** With $T = 1$, we have

$$x(0) = x(1) = x(2) = 0, \quad x(3) = \frac{1}{3}, \quad x(4) = \frac{2}{3},$$

and

$$x(k) = 1 \text{ for all } k \geq 5$$

Hence

$$\begin{aligned}
\mathcal{Z}\left[x(k)\right] &= x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} \\
&\quad + x(5)z^{-5} + x(6)z^{-6} + \cdots \\
&= 0 + 0z^{-1} + 0z^{-2} + \frac{1}{3}z^{-3} + \frac{2}{3}z^{-4} + z^{-5} + z^{-6} + \cdots \\
&= \frac{1}{3}z^{-3} + \frac{2}{3}z^{-4} + z^{-5}\left[1 + z^{-1} + z^{-2} + z^{-3} + \cdots\right] \\
&= \frac{1}{3}z^{-3} + \frac{2}{3}z^{-4} + \frac{z^{-5}}{1 - z^{-1}} \\
&= \frac{z^{-3}(1 + z^{-1} + z^{-2})}{3(1 - z^{-1})}
\end{aligned}$$

**Q2.1**

$$X(z) = \frac{1 + 2z + 3z^2 + 4z^3 + 5z^4}{z^4} = 5 + 4z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}$$

Hence

$$x(0) = 5, \quad x(1) = 4, \quad x(2) = 3, \quad x(3) = 2, \quad x(4) = 1,$$

and

$$x(k) = 0 \quad \text{for} \quad k \geq 5$$

**Q2.2**

$$X(z) = \frac{z^{-1}(0.5 - z^{-1})}{(1 - 0.5z^{-1})(1 - 0.8z^{-1})^2} = \frac{z(0.5z - 1)}{(z - 0.5)(z - 0.8)^2}$$

$$\frac{X(z)}{z} = \frac{(0.5z - 1)}{(z - 0.5)(z - 0.8)^2} = \frac{-25/3}{z - 0.5} + \frac{25/3}{z - 0.8} + \frac{-2}{(z - 0.8)^2}$$

$$X(z) = \frac{-25/3}{1 - 0.5z^{-1}} + \frac{25/3}{1 - 0.8z^{-1}} + \frac{-2z^{-1}}{(1 - 0.8z^{-1})^2}$$

Hence

$$x(k) = -\frac{25}{3}(0.5)^k + \frac{25}{3}(0.8)^k - 2k(0.8)^{k-1}, \quad k = 0,\ 1,\ 2,\ 3,\ \ldots$$

**Q2.3**

$$x(0) = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{z^{-1}}{(1-z^{-1})(1+1.3z^{-1}+0.4z^{-2})}$$

$$= \frac{0}{(1-0)(1+0)} = 0$$

Before applying the Final Value Theorem, need to check the poles of $X(z)$:

$$z = 1, \quad z = -0.5, \quad z = -0.8$$

As there is one pole at $z = 1$ and all the other poles have absolute values of less than 1, hence we can apply the Final Value Theorem.

$$x(\infty) = \lim_{z\to 1}(1-z^{-1})X(z) = \lim_{z\to 1} \frac{z^{-1}}{1+1.3z^{-1}+0.4z^{-2}} = \frac{1}{2.7}$$

Indeed,

$$X(z) = \frac{z^{-1}}{(1-z^{-1})(1+1.3z^{-1}+0.4z^{-2})}$$

$$= \frac{1}{2.7}\left[\frac{1}{1-z^{-1}} + \frac{3}{1+0.5z^{-1}} + \frac{-4}{1+0.8z^{-1}}\right]$$

Hence

$$x(k) = \frac{1}{2.7}\left[1 + 3(-0.5)^k - 4(-0.8)^k\right], \qquad k \geq 0$$

Obviously

$$x(0) = \frac{1}{2.7}(1+3-4) = 0$$

$$x(\infty) = \frac{1}{2.7}(1+0+0) = \frac{1}{2.7}$$

2.3

**Q2.4** Inversion Integral Method

$$X(z) = \frac{1+z^{-1}-z^{-2}}{1-z^{-1}} = \frac{z^2+z-1}{z(z-1)}$$

$$X(z)z^{k-1} = \frac{(z^2+z-1)z^{k-1}}{z(z-1)}$$

For $k=0$,

$$X(z)z^{k-1} = \frac{z^2+z-1}{z^2(z-1)}$$

and

$$\begin{aligned}
x(0) &= \left[\text{Residual of } \frac{z^2+z-1}{z^2(z-1)} \text{ at the simple pole } z=1\right] \\
&+ \left[\text{Residual of } \frac{z^2+z-1}{z^2(z-1)} \text{ at the double pole } z=0\right] \\
&= \lim_{z\to 1}.\left[(z-1)\times\frac{z^2+z-1}{z^2(z-1)}\right] \\
&+ \frac{1}{(2-1)!}\lim_{z\to 0}\frac{d^{2-1}}{dz^{2-1}}\left[z^2\times\frac{z^2+z-1}{z^2(z-1)}\right] \\
&= 1+\lim_{z\to 0}\frac{(2z+1)(z-1)-(z^2+z-1)}{(z-1)^2} \\
&= 1+0 = 1
\end{aligned}$$

For $k = 1$,

$$X(z)z^{k-1} = \frac{z^2 + z - 1}{z(z-1)}$$

and

$$x(1) = \left[ \text{Residual of } \frac{z^2+z-1}{z(z-1)} \text{ at the simple pole } z = 1 \right]$$

$$+ \left[ \text{Residual of } \frac{z^2+z-1}{z(z-1)} \text{ at the simple pole } z = 0 \right]$$

$$= \lim_{z \to 1} \left[ (z-1) \times \frac{z^2+z-1}{z(z-1)} \right] + \lim_{z \to 0} \left[ z \times \frac{z^2+z-1}{z(z-1)} \right]$$

$$= 1 + 1 = 2$$

For $k \geq 2$,

$$X(z)z^{k-1} = \frac{(z^2+z-1)z^{k-2}}{(z-1)}$$

Note: Only one pole at $z = 1$. No more pole at $z = 0$ !

$$x(k) = \left[ \text{Residual of } \frac{(z^2+z-1)z^{k-2}}{(z-1)} \text{ at the simple pole } z = 1 \right]$$

$$= \lim_{z \to 1} \left[ (z-1) \times \frac{(z^2+z-1)z^{k-2}}{(z-1)} \right] = 1$$

Hence,

$$x(0) = 1, \quad x(1) = 2, \quad x(k) = 1 \text{ for } k \geq 2$$

Note: We can also use the Final Value Theorem to verify

$$x(\infty) = \lim_{z \to 1} (1 - z^{-1}) X(z) = 1$$

Of course, given the choice, we would rather do the following:

$$
\begin{aligned}
X(z) &= \frac{1+z^{-1}-z^{-2}}{1-z^{-1}} = \frac{1-z^{-1}+2z^{-1}-2z^{-2}+z^{-2}}{1-z^{-1}} \\
&= 1+2z^{-1}+\frac{z^{-2}}{1-z^{-1}}
\end{aligned}
$$

Hence

$$
x(k) = \delta_0(k) + 2\delta_0(k-1) + 1(k-2)
$$

**Q2.5** (B-2-12)

$$\begin{aligned}
X(z) &= \frac{z^{-3}}{(1-z^{-1})(1-0.2z^{-1})} \\
&= \frac{1}{z(z-1)(z-0.2)} \\
&= \frac{5}{z} + \frac{1.25}{z-1} + \frac{-6.25}{z-0.2} \\
&= 5z^{-1} + 1.25 \times \frac{z^{-1}}{1-z^{-1}} - 6.25 \times \frac{z^{-1}}{1-0.2z^{-1}} \\
&= z^{-1}\left[5 + \frac{1.25}{1-z^{-1}} - \frac{6.25}{1-0.2z^{-1}}\right]
\end{aligned}$$

Hence

$$\begin{aligned}
x(k) &= \left[5\delta_0(k) + 1.25\,1(k) - 6.25(0.2)^k 1(k)\right]\Big|_{k\to k-1} \\
&= 5\delta_0(k-1) + 1.25\,1(k-1) - 6.25(0.2)^{k-1} 1(k-1)
\end{aligned}$$

i.e.,

$$x(k) = 0 \quad \text{for} \quad k = 0, 1, 2$$

$$x(k) = 1.25(1 - 0.2^{k-2}) \quad \text{for} \quad k = 3, 4, 5, \cdots$$

However, the following will not work:

$$X(z) = \frac{z^{-3}}{(1-z^{-1})(1-0.2z^{-1})} = \frac{A}{1-z^{-1}} + \frac{B}{1-0.2z^{-1}}$$

as $X(z)$ above is not strictly proper in $z^{-1}$.

Instead, we must write

$$X(z) = \frac{z^{-2}\,z^{-1}}{(1-z^{-1})(1-0.2z^{-1})} = \frac{Az^{-2}}{1-z^{-1}} + \frac{Bz^{-2}}{1-0.2z^{-1}}$$

$$\Longrightarrow \quad z^{-3} = Az^{-2}(1-0.2z^{-1}) + Bz^{-2}(1-z^{-1})$$

$$\Longrightarrow \qquad A = 1.25, \quad B = -1.25$$

Hence,

$$x(k) = \left[1.25 - 1.25(0.2)^k\right] 1(k) \Big|_{k \to k-2} = 1.25\left[1-(0.2)^{k-2}\right] 1(k-2)$$

Q3.1.

Firstly, we determine the initial values of $x(k)$.

When $k=-2$, we have

$$x(0)-2x(-1)+x(-2)=\delta(-2)=0$$

$$\Rightarrow x(0)=0$$

When $k=-1$,

$$x(1)-2x(0)+x(-1)=\delta(-1)=0$$

$$\Rightarrow x(1)=0$$

Then taking z-transform gives

$$z^2X(z)-z^2x(0)-zx(1)$$
$$-2[zX(z)-zx(0)]+X(z)=1$$

$$z^2 X(z) - 2z X(z) + X(z) = 1$$

$$(z^2 - 2z + 1) X(z) = 1$$

$$X(z) = \frac{1}{(z-1)^2}$$

$$= \frac{z^{-2}}{(1-z^{-1})^2}$$

$$= \frac{(1-z^{-1})^2 - (1-z^{-1}) + z^{-1}}{(1-z^{-1})^2}$$

$$= 1 - \frac{1}{(1-z^{-1})} + \frac{z^{-1}}{(1-z^{-1})^2}$$

$$\therefore \quad X(k) = \delta(k) - 1(k) + k,$$

$$k = 0, 1, 2, \ldots$$

3.2

Q3.2

(1) Use the z-Transform Table after doing partial fraction expansion

(2) Method Based on Inverse Laplace Transform of *X(s)*.

$$X(s) = \frac{s+3}{(s+1)(s+2)}$$

$$= \frac{2}{(s+1)} - \frac{1}{(s+2)}$$

The inverse Laplace transform of this equation gives

$$x(t) = 2e^{-t} - e^{-2t}$$

Hence

$$X(z) = \frac{2}{(1-e^{-T}z^{-1})} - \frac{1}{(1-e^{-2T}z^{-1})}$$

$$= \frac{1+e^{-T}(1-2e^{-T})z^{-1}}{(1-e^{-T}z^{-1})(1-2e^{-2T}z^{-1})}$$

3.3

From Figure 3.2 we obtain

$$C(s) = G(s)E^*(s)$$
$$E(s) = R(s) - H_2(s)M^*(s)$$
$$M(s) = H_1(S)G(S)E^*(s)$$

By taking the starred Laplace transforms of the preceding equations, we have

$$C^*(s) = G^*(s)E^*(s)$$
$$E^*(s) = R^*(s) - H_2{}^*(s)M^*(s)$$
$$M^*(s) = [GH_1(s)]^*E^*(s)$$

Hence

$$E^*(s) = R^*(s) - H_2{}^*(s)\,[GH_1(s)]\,^*E^*(s)$$

or

$$E^*(s) = \frac{R^*(s)}{1 + H_2^*(s)[GH_1(s)]^*}$$

$$C^*(s) = \frac{G^*(s)R^*(s)}{1 + H_2^*(s)[GH_1(s)]^*}$$

3.5

and

$$\frac{C^*(s)}{R^*(s)} = \frac{G^*(s)}{1 + H_2^*(s)[GH_1(s)]^*}$$

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + H_2(z)[GH_1(z)]}$$

**Q4.1** (B-3-17)

$$C(s) = G_2(s)M^*(s) \tag{1}$$

$$M(s) = G_1(s)E^*(s) - C(s) \tag{2}$$

$$E(s) = R(s) - H(s)C(s) \tag{3}$$

Substituting (1) into (2) and (3) yields

$$M(s) = G_1(s)E^*(s) - G_2(s)M^*(s) \tag{4}$$

$$E(s) = R(s) - H(s)G_2(s)M^*(s) \tag{5}$$

Now apply the $*$ transformation on (4) and (5)

$$M^*(s) = G_1^*(s)E^*(s) - G_2^*(s)M^*(s) \tag{6}$$

$$E^*(s) = R^*(s) - (HG_2)^*(s)M^*(s) \tag{7}$$

Substituting (7) into (6) gives

$$M^*(s) = G_1^*(s)\left[R^*(s) - (HG_2)^*(s)M^*(s)\right] - G_2^*(s)M^*(s)$$

hence

$$M^*(s) = \frac{G_1^*(s)R^*(s)}{1 + G_1^*(s)(HG_2)^*(s) + G_2^*(s)}$$

Also

$$C(s) \;=\; G_2(s)M^*(s) \;=\; \frac{G_2(s)G_1^*(s)R^*(s)}{1 + G_1^*(s)(HG_2)^*(s) + G_2^*(s)}$$

and

$$C^*(s) \;=\; \frac{G_2^*(s)G_1^*(s)R^*(s)}{1 + G_1^*(s)(HG_2)^*(s) + G_2^*(s)}$$

$$C(z) \;=\; \frac{G_2(z)G_1(z)R(z)}{1 + G_1(z)HG_2(z) + G_2(z)}$$

**Q3.2** (B-3-18)

$$C(s) \;=\; \left(\frac{K}{s}\,\frac{1-e^{-Ts}}{s}\right) E^*(s)$$

$$E(s) \;=\; R(s) - C(s) \;=\; R(s) - \left(\frac{K}{s}\,\frac{1-e^{-Ts}}{s}\right) E^*(s)$$

Thus

$$E^*(s) = R^*(s) - C^*(s) \;=\; R^*(s) - \left(\frac{K}{s}\,\frac{1-e^{-Ts}}{s}\right)^* E^*(s)$$

and

$$E^*(s) = \frac{R^*(s)}{1+\left(\dfrac{K}{s}\,\dfrac{1-e^{-Ts}}{s}\right)^*}$$

Hence

$$C(s) \;=\; \frac{\dfrac{K}{s}\,\dfrac{1-e^{-Ts}}{s}}{1+\left(\dfrac{K}{s}\,\dfrac{1-e^{-Ts}}{s}\right)^*}\,R^*(s)$$

and

$$C^*(s) \;=\; \frac{\left(\dfrac{K}{s}\,\dfrac{1-e^{-Ts}}{s}\right)^*}{1+\left(\dfrac{K}{s}\,\dfrac{1-e^{-Ts}}{s}\right)^*}\,R^*(s)$$

$$C(z) \;=\; \frac{\mathcal{Z}\left[\dfrac{K}{s}\,\dfrac{1-e^{-Ts}}{s}\right]}{1+\mathcal{Z}\left[\dfrac{K}{s}\,\dfrac{1-e^{-Ts}}{s}\right]}\,R(z)$$

Since $R(z) = \dfrac{1}{1-z^{-1}}$ and

$$\mathcal{Z}\left[\frac{1-e^{-Ts}}{s}\frac{K}{s}\right] = (1-z^{-1})\mathcal{Z}\left[\frac{K}{s^2}\right]$$

$$= (1-z^{-1}) \times \frac{KTz^{-1}}{(1-z^{-1})^2}$$

$$= \frac{Kz^{-1}}{1-z^{-1}}$$

where $T = 1$s is used, we have

$$C(z) = \frac{Kz^{-1}}{1+(K-1)z^{-1}}\frac{1}{1-z^{-1}}$$

If $K \neq 0$, we have

$$C(z) = \frac{-1}{1+(K-1)z^{-1}} + \frac{1}{1-z^{-1}}$$

and

$$c(k) = 1-(1-K)^k, \qquad k = 0, 1, 2, \cdots$$

Note that $c(k)$ is convergent if $|1-K| < 1$, i.e., if $0 < K < 2$.
If $K > 2$ or $K < 0$), $c(k) \to \infty$ as $k \to \infty$ (i.e., divergent).

If $K = 2$, $c(k)$ oscillates between 0 and 2, and will not converge to a single value.

If $K = 0$, we have $C(z) = 0$, hence, $c(k) = 0$ for all $k \geq 0$

(i.e., convergent).

4.3

From Figure 4.3, we obtain

$$M(z) = \frac{K_I}{1 - z^{-1}}[R(z) - C(z)]$$
$$-[K_p + K_D(1 - z^{-1})]C(z)$$

$$= \frac{K_I}{1 - z^{-1}}R(z)$$
$$-[\frac{K_I}{1-z^{-1}} + K_p \quad + K_D(1-z^{-1})]C(z)$$

$$C(z) = G(z)M(z)$$
$$= G(z)\frac{K_I}{1 - z^{-1}}R(z)$$
$$-G(z)[\frac{K_I}{1-z^{-1}} + K_p \quad + K_D(1-z^{-1})]C(z)$$

4.5

$$\{1+G(z)[\frac{K_I}{1\text{-}z^{-1}}+K_p \quad +K_D(1\text{-}z^{-1})]\}C(z)$$
$$=G(z)\frac{K_I}{1-z^{-1}}R(z)$$

$$\frac{C(z)}{R(z)}=\frac{G(z)\frac{K_I}{1-z^{-1}}}{1+G(z)[\frac{K_I}{1\text{-}z^{-1}}+K_p \quad +K_D(1\text{-}z^{-1})]}$$