Winter 2017 MATH 15910 Section 55

HW6 Solution

Richard

1. Ex 3.6.24

(iii) An arbitrary union of open sets in $\mathbb{R}$ is open in $\mathbb{R}$.

Proof.

Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a collection of open sets.

WTS: $\bigcup_{\lambda\in\Lambda} U_\lambda$ is open in $\mathbb{R}$.

Let $x \in \bigcup_{\lambda\in\Lambda} U_\lambda$. Then $\exists\, \lambda\in\Lambda$ s.t. $x \in U_\lambda$.

$\because U_\lambda$ open

$\therefore \exists\, \varepsilon_\lambda > 0$ s.t. $x \in B(x, \varepsilon_\lambda) = (x-\varepsilon_\lambda, x+\varepsilon_\lambda) \subseteq U_\lambda \subseteq \bigcup_{\lambda\in\Lambda} U_\lambda$

Given our choice of x is arbitrary in $\bigcup_{\lambda\in\Lambda} U_\lambda$, we have thus proved the thm

(iv) ~~If~~ A finite intersection of open sets in $\mathbb{R}$ is open in $\mathbb{R}$.

Proof.

Let $\{U_i\}_{i\in[n]=\{1,2,\dots,n\}}$ be a finite collection of open sets

WTS: $\bigcap_{i\in[n]} U_i$ is open

① $\bigcap U_i = \emptyset$. Open ✓

② $\bigcap U_i \neq \emptyset$.

Let $x \in \bigcap U_i$.

$\Rightarrow x \in U_1,\ x \in U_2, \dots, x \in U_n$

$\therefore \exists\, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n > 0$ s.t.

$B(x,\varepsilon_1), B(x,\varepsilon_2), \dots, B(x,\varepsilon_n) \subseteq \bigcap U_i$

where $B(x,\varepsilon_i) = (x-\varepsilon_i, x+\varepsilon_i)$.

Let $\varepsilon = \min_{i\in[n]}\{\varepsilon_i\}$

$\therefore x \in (x-\varepsilon, x+\varepsilon) \subseteq \bigcap U_i$

$\therefore \dots$

(vi) Show that an arbitrary intersection of closed sets in $\mathbb{R}$ is a closed set in $\mathbb{R}$.

Proof Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a collection of closed sets in $\mathbb{R}$.

WTS: $\bigcap_{\lambda\in\Lambda} U_\lambda$ is closed in $\mathbb{R}$

We know $\{U_\lambda{}^c\}_{\lambda\in\Lambda}$ is a collection of open sets in $\mathbb{R}$.

$\Rightarrow \bigcup_{\lambda\in\Lambda} (U_\lambda{}^c)$ is open in $\mathbb{R}$

(we just proved!)

$\Rightarrow \left(\bigcap_{\lambda\in\Lambda} U_\lambda\right)^c$ is open in $\mathbb{R}$

(De'Morgan's laws)

$\Rightarrow \bigcap_{\lambda\in\Lambda} U_\lambda$ is closed in $\mathbb{R}$.

(viii) Show that a finite union of ~~closed~~ closed sets in $\mathbb{R}$ is a closed sets in $\mathbb{R}$.

Proof. Similar to (vi)

Think (very important)

~~what if~~ why in some questions it's "arbitrary," but in others it's "finite"?

2. Ex 3.6.25

A subset of $\mathbb{R}$ is closed iff it contains all its acc pts.

Proof Let $A \subseteq \mathbb{R}$.

$\boxed{\Rightarrow}$ $\because$ A is closed

$\therefore A^c$ is open

Let p be an acc pt of A.

Suppose for contradiction $p \notin A$.

$\therefore p \in A^c$

$\therefore \exists \varepsilon > 0$ s.t. $\underset{\substack{\| \\ (p-\varepsilon,\, p+\varepsilon)}}{B(p,\varepsilon)} \subseteq A^c$

$\because$ p is acc pt of A

$\therefore \exists y \in A,\ y \neq p,\ y \in B(p,\varepsilon) \subseteq A^c$

$\therefore y \in A \cap A^c = \emptyset$

$\Rightarrow\Leftarrow$

$\boxed{\Leftarrow}$ Suppose not.

$\therefore$ A not closed

$\therefore A^c$ not open

$\therefore \exists y \in A^c$ s.t. $\forall \varepsilon > 0,\ \underset{\substack{\| \\ (y-\varepsilon,\, y+\varepsilon)}}{B(y,\varepsilon)} \not\subseteq A^c$

$\Rightarrow \forall \varepsilon > 0, \exists z \in B(y,\varepsilon)$ s.t. $z \in A,\ z \neq y$

$\Rightarrow y$ is acc pt of A

But $y \notin A$ $\Rightarrow\Leftarrow$

. Ex 3.8.5

$z = a + bi$

Identify z w/ pt $(a,b) \in \mathbb{R}^2$.

Abs value of z is equal to distance of pt (a,b) from $(0,0)$.

Proof.

$$|z| = (z\bar{z})^{\frac{1}{2}}$$
$$= \sqrt{(a+bi)(a-bi)}$$
$$= \sqrt{a^2+b^2}$$

Distance from (a,b) to $(0,0)$ equals $\sqrt{a^2+b^2}$ as well.

4. Ex 3.9.5

iv) ~~Same as in~~

Similar to problem 1

- except we need to use the B notation as we are not working in $\mathbb{R}^1$.

vi) Infinite intersection of open sets in $\mathbb{C}$ need not be an open set in $\mathbb{C}$

$$U_n = \{ z \in \mathbb{C} \mid |z| < \tfrac{1}{n} \}$$

But $\bigcap_{n=1}^{\infty} U_n = \{0\}$ not open.

Or

$$V_n = \{ z = a + bi \mid a \in (-\tfrac{1}{n}, \tfrac{1}{n}),\ b = 0 \}$$

$$= (-\tfrac{1}{n}, \tfrac{1}{n})$$

5. $[0,1]$ is compact

Proof.

Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of $[0,1]$.

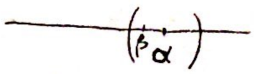
Let $A = \{x\in[0,1] \mid [0,x]$ can be covered by finitely many U_λ's$\}$.

Let $\alpha = \sup A$.
[least upper bound, supremum

Suppose $[0,1]$ is not compact.
$\Rightarrow \alpha < 1$.

Note We cannot claim $\alpha\in A$ at this point. If $A=[0,\alpha)$, for example, $\sup A = \alpha$ as well. It takes some work to prove $\alpha\in A$. Or we can do this:

Now,
Let U_0 be an open set containing α



$$\therefore \exists\, \varepsilon>0 \text{ s.t. } \underbrace{B(\alpha,\varepsilon)}_{=(\alpha-\varepsilon,\,\alpha+\varepsilon)} \subseteq U_0.$$

Let $\beta\in(\alpha-\varepsilon,\alpha)\cap[0,\alpha) \Rightarrow \beta\in A$

Given $[0,\beta]$ can be covered by finitely many U_λ's. Say the cover is called $\mathcal{A}$.

Then $[0,\alpha+\frac{\varepsilon}{2}]$ is covered by $\mathcal{A}\cup U_0$, which is finite.

$\Rightarrow \alpha+\frac{\varepsilon}{2}\in A$

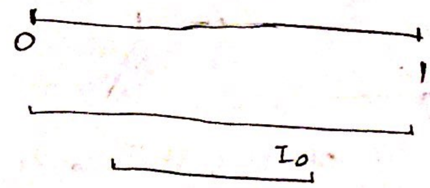
$\Rightarrow\Leftarrow$ (contradicting $\alpha=\sup A$)

$\therefore [0,1]$ is compact

Alternative proof

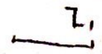
Suppose not.

$\therefore \exists$ closed-interval ("1-cell"), I_0 and an open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of I_0 which does not have a finite cover.



Let $\operatorname{diam}(I_1) \le \dfrac{\operatorname{diam}(I_0)}{2^1}$

s.t. $I_1\subseteq I_0$, and I_1 cannot be



covered by finitely many U_α's.

We then get $I_0\supset I_1\supset I_2\supset\cdots\supset I_n \supset\cdots$

s.t. each I_n cannot be covered by finitely many U_α's, and

$\operatorname{diam}(I_n)\le \dfrac{\operatorname{diam}(I_0)}{2^n}$.

Btw,

Def $diam(E) = \sup\{d(x,y) \mid x, y \in E\}$.

Now, we can prove

$\exists z$ s.t. $z \in \bigcap_{n=0}^{\infty} I_n$.

Since $z \in I_0$, $\exists \alpha \in A$ s.t. $z \in U_\alpha$.

$\therefore \exists \varepsilon > 0$, s.t. $\underbrace{B(z, \varepsilon)}_{= (z-\varepsilon, z+\varepsilon)} \subseteq U_\alpha$

For n large enough, $I_n \subseteq \underbrace{B(z, \varepsilon)}_{= (z-\varepsilon, z+\varepsilon)} \subseteq U_\alpha$.

$\Rightarrow I_n$ is covered by a single U_α

$\Rightarrow\Leftarrow$ Abound.

$\therefore [0,1]$ compact.

<u>Fun fact</u>

Def A k-cell is a set of the form $[a_1, b_1] \times [a_2, b_2] \times \dots \times [a_k, b_k]$ where $a_i, b_i \in \mathbb{R}$ $\forall i \in [k] = \{1, \dots, k\}$

<u>Thm</u>

Any k-cell is compact

<u>Pf of Thm</u>

Very similar to the proof above.

6. If $z = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$ for $k \in \mathbb{Z}$, and $0 \le k \le n-1$, then $z^n = 1$.

<u>Proof</u>

Refer to Def 3.9.22

$z = \cos\frac{2k\pi}{n} + \sin\frac{2k\pi}{n} \cdot i$

$\Rightarrow z = e^{\frac{2k\pi i}{n}}$

$\Rightarrow z^n = e^{2k\pi i} = \cos(2k\pi) + \sin(2k\pi) \cdot i$

$\because k \in \mathbb{Z}$

$\therefore z^n = 1 + 0 \cdot i = 1$