Winter 2017 MATH 15910 Section 55

HW3 Solution

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Ex 1.7.25

Find example that

$f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2)$

Suppose $A_1 = \{a\}$, $A_2 = \{b\}$ where $b \neq a$

Define $f : A \to B$ by

$f(x) = c \quad \forall x \in A$, where $c \in B$.

Then, $A_1 \cap A_2 = \emptyset$

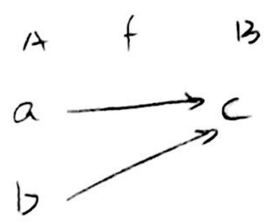
$\Rightarrow f(A_1 \cap A_2) = \emptyset$

However, $f(A_1) = f(A_2) = c$

$\Rightarrow f(A_1) \cap f(A_2) = c$

$\therefore f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2)$.

Picture



Ex 1.7.38 first part

$A_1, A_2 \subseteq X$. $\exists$ bijection b/w $A_1 \times A_2$ and ϕ the set of all functions $f : \{1,2\} \to X$ s.t. $f(1) \in A_1$, $f(2) \in A_2$.

Proof.

Note we need to

① construct a function

② show it's surjective (onto)

③ show it's injective (one-to-one)

Let $\mathcal{F} = \{ f : \{1,2\} \to X \mid f(1) \in A_1$ and $f(2) \in A_2 \}$.

We define function $\varphi : A_1 \times A_2 \to \mathcal{F}$ as follows:

$\forall (a,b) \in A_1 \times A_2$
$\varphi(a,b) = f$ where

$f : \{1,2\} \to X$ and $f(1) = a$, and $f(2) = b$.

① Why is φ a function?

Let $(a,b) \in A_1 \times A_2$. Then $\exists f_1 : \{1,2\} \to X$ s.t. $f_1(1) = a$, $f_1(2) = b$. f_1 is unique b/c otherwise, either $f_1(1) \neq a$ or $f_1(2) \neq b$ (contradicting def of f_1)

② why is φ a surjection?

Let $f \in \mathcal{F}$. Then $f: \{1,2\} \to X$ where $f(1) = m$, $f(2) = n$ for some $m \in A_1 \subseteq X$, $n \in A_2 \subseteq X$.

Consider the element $(m,n) \in A_1 \times A_2$ (given $m \in A_1$, $n \in A_2$). Then by def of φ, $\varphi(m,n) = f$.

We have thus proved $\mathcal{F} \subseteq \varphi(A_1 \times A_2)$

By def of function, $\varphi(A_1 \times A_2) \subseteq \mathcal{F}$

$\Rightarrow \varphi(A_1 \times A_2) = \mathcal{F}$.

By def of surjection in Sally's book, φ is surjective.

③ why is φ an injection?

Let $(a_1, a_2), (b_1, b_2) \in A_1 \times A_2$,

s.t. $\varphi(a_1, a_2) = \varphi(b_1, b_2) \in \mathcal{F}$

Then, $(\varphi(a_1,a_2))(1) = a_1$

$(\varphi(a_1,a_2))(2) = a_2$

$(\varphi(b_1,b_2))(1) = b_1$

$(\varphi(b_1,b_2))(2) = b_2$.

$\because \varphi(a_1,a_2) = \varphi(b_1,b_2)$ (two functions equal)

$\therefore (\varphi(a_1,a_2))(x) = (\varphi(b_1,b_2))(x)$ $\forall x \in \{1,2\}$.

$\therefore a_1 = b_1$, $a_2 = b_2$

$\therefore \varphi$ is injective

$\therefore \varphi$ is bijective

<u>Ex 1.8.26 (i)</u>

If $A_1, A_2, \ldots, A_n$ are countable, then $A_1 \times A_2 \times \cdots \times A_n$ is countable.

<u>Sketch</u> By simple induction on $n \in \mathbb{N}$.

<u>Note</u> why is this claim true?

<u>Claim</u>: If A_1, A_2 are countable, then $A_1 \times A_2$ is countable.

<u>Sketch of Pf</u>

~~Consider~~

Suppose $A_1 = \{a_{11}, a_{12}, \ldots\}$

$A_2 = \{a_{21}, a_{22}, \ldots\}$

(Indexes can be like this bc A_1, A_2 are countable)

Consider

$B_j \subseteq A_1 \times A_2$,

$B_j = \{(a_{1j}, a_{2k}) \mid k \in \mathbb{N}\}$

for a fixed $j \in \mathbb{N}$.

That is,

$B_1 = \{(a_{11}, a_{21}), (a_{11}, a_{22}), (a_{11}, a_{23}), \cdots\}$

$B_j = \{(a_{1j}, a_{21}), (a_{1j}, a_{22}), \cdots\}$.
where $j \in \mathbb{N}$.

Then, we can prove

$$\underset{\substack{\uparrow \\ \text{bijective}}}{A_2 \sim B_1}, \quad A_2 \sim B_2, \cdots$$

$A_2 \sim B_j, \ \forall j \in \mathbb{N}$.

$\because A_2 \sim \mathbb{N}$

$\therefore B_j \sim \mathbb{N}, \ \forall j \in \mathbb{N}$

By Facts 1.8.25 (3),

$A_1 \times A_2 = \bigcup_{j \in \mathbb{N}} B_j$ is countable.

Ex 1.8.27

If A is any set (including $\emptyset$), there is no bijection b/w A and $P(A)$.

Proof.

If $A = \emptyset$, $P(A) = \{\emptyset\}$

($|A| = 0$, $|P(A)| = 1$, can be proven that $\nexists$ bijection b/w $\emptyset$ and $P(\emptyset)$)

If $A \neq \emptyset$. Suppose for contradiction $\exists$ bijective function b/w A and $P(A)$ and for every $a \in A$, $\exists$ an associated subset $\varphi(a)$. (*)

Let $B = \{x \in A \mid x \notin \varphi(x)\} \subseteq A$

Then $\exists\, b \in A$ s.t. $\underbrace{\varphi(b)}_{\text{set}} = \underbrace{B}_{\text{set}}$ by (*)

$\because \varphi(b) = B$

If $b \in B$, then by def, $b \notin \varphi(b)$
$\Rightarrow b \notin B$.
$\Rightarrow\Leftarrow$

If $b \notin B$ ($b \in A \setminus B$), then $b \in \varphi(b)$
$\Rightarrow b \in B$
$\Rightarrow\Leftarrow$

Contradiction!

Therefore, - —

Thm 1.8.32 The set of all real numbers $\in [0,1]$ is not countable

(Note: $\forall a_m \in \mathbb{R}$ s.t. $a_m \in [0,1]$, a_m can be represented by decimal expansion $0.a_{m1}a_{m2}\cdots a_{mn}\cdots$, note $0.999\cdots = 1$)

Proof Suppose ~~it is~~ they are countable, (for contradiction) We can then list all real numbers $\in [0,1]$ as follows

$$a_1 = 0.a_{11}a_{12}\cdots a_{1n}\cdots$$
$$a_2 = 0.a_{21}a_{22}\cdots a_{2n}\cdots$$
$$\vdots$$
$$a_m = 0.a_{m1}a_{m2}\cdots a_{mn}\cdots$$

where $m \in \mathbb{N}$ (given a_i's are countable)

Note no a_m's terminate in all 9's except for $1 = 0.999\cdots$

let $b = 0.b_1b_2\cdots b_n\cdots$

where $b_j = \begin{cases} 0, & \text{if } a_{jj} \neq 0 \\ 1, & \text{if } a_{jj} = 0 \end{cases}$

We know $b = a_k$ for some $k \in \mathbb{N}$, as $b \in [0,1]$

The k^{th} decimal digit of b is b_k.

The k^{th} decimal digit of a_k is a_{kk}.

But by our def of b,

$b_k \neq a_{kk} \Rightarrow b \neq a_k$

$\Rightarrow\Leftarrow$

contradicting $b = a_k$.

Thus, - - - not countable.

Practice

Let E be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7.

Is E countable?

Text