Winter 2017 MATH 15910 Section 55

Exam 2 Solution

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1. $S = \{(-1)^n + \frac{2}{n} \mid n \in \mathbb{N}\}$

1) Least upper bound of S (lub S, sup S, ..)

2) Greatest lower bound of S (glb S, inf S, ...)

When n is odd, we get set

$S_{odd} = \{1, -\frac{1}{3}, -\frac{3}{5}, -\frac{5}{7}, \dots\}$

When n is even, we get set

$S_{even} = \{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots\}$

Suppose n is odd, use a_n to denote $(-1)^n + \frac{2}{n}$.

$a_n > a_{n+2}$ given $\frac{2}{n} > \frac{2}{n+2}$

We have the following observation

① 1 is the maximal element of S_{odd}:

$\forall b \in S_{odd}$, $b \le 1 < 2$.

② $-1 \le b$, $\forall b \in S_{odd}$

Suppose n is even, use a_n to denote $(-1)^n + \frac{2}{n}$.

$a_n > a_{n+2}$ given $\frac{2}{n} > \frac{2}{n+2}$.

We have the following observation

③ 2 is the max element of S_{even}

$\forall b \in S_{even}$, $b \le 2$.

④ $\forall b \in S_{even}$, $b > 1$, given

$b = (-1)^n + \frac{2}{n} = 1 + \frac{2}{n} > 1$, $\forall n$ even natural numbers

Now, from ① ③, 2 is an upper bound of S. Let x be an upper bound of S. Then $x \ge 2$.

$\Rightarrow$ $\sup S = 2$

From ② ④, -1 is lower bound of S. Let x be an arbitrary lower bound of S.

$-1 \quad x$

Suppose for contradiction $x > -1$.

Then $\exists n \in \mathbb{N}$ s.t. $\frac{2}{n} < x - (-1) = x + 1$

$\Rightarrow \exists b \in S$ s.t. $b = \frac{2}{n} + (-1) < x$

$\Rightarrow\Leftarrow$ (contradiction)

$\therefore x \le -1$

$\therefore \inf S = -1$.

2. 1) Cauchy sequence

Def A sequence $(p_n) \subset \underline{X}$ is a

can use $\mathbb{R}$, in our course

Cauchy sequence if $\forall \varepsilon > 0$,

$\exists N \in \mathbb{N}$ s.t. $\forall m, n \geq N$,

$d(p_m, p_n) < \varepsilon$.

or $|p_m - p_n| < \varepsilon$.

(please refer to def on textbook)

2) Sequentially compact set

Def K is sequentially compact

if and only if any sequence

(in def, we can use "if" as well b/c. ($\Rightarrow$) this direction is implied by the nature of def)

of points of K has a subsequence

(can write any infinite sequence in K)

which converges to an element of K.

3) Compact Set (in $\mathbb{R}$) if we assume $\mathbb{R}$; it can be any topological space X

Def Let $A \subset \mathbb{R}$. A is compact set if any open cover of A has a finite subcover. i.e. if $\{U_i\}_{i\in I}$ is an open cover of A,

$\exists \alpha_1, \dots, \alpha_n$ s.t. $A \subset U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$.

(Refer to Def 3.6.29).

3. 1) Infinite union of closed sets in $\mathbb{R}$ is not necessarily a closed set in $\mathbb{R}$.

$E_n = \{\frac{1}{n}\}$

$\bigcup^{\infty} E_n = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$

Note A open $\not\Leftrightarrow$ A not closed

A not open $\not\Leftrightarrow$ A closed

Use Ex 3.6.25 to prove $\bigcup^{\infty} E_n$ is not closed.

2) Open interval $(0,1)$ not compact.

Let $\mathcal{U} = \{U_n\}_{n\in\mathbb{N}}$ where $U_n = (\frac{1}{n}, 7)$.

Check ① U_n open $\forall n \in \mathbb{N}$

② $\forall x \in (0,1)$, $x \in \bigcup_{n\in\mathbb{N}} U_n$

$\Rightarrow \mathcal{U}$ is open cover for $(0,1)$

See next page

Suppose for contradiction that $(0,1)$ is compact.

Then $\exists$ finite subcover; that is, $\exists\ i_1, i_2, \cdots, i_n \in \mathbb{N}$ s.t. $\{U_{i_k}\}_{k\in[n]}$ is the finite subcover of $(0,1)$. $(0,1) \subset \bigcup_{k\in[n]} U_{i_k}$

$= U_{i_1} \cup \cdots \cup U_{i_n}$.

(Note: $[n] = \{1, 2, \cdots, n\}$)

Therefore, $(0,1) \subset \bigcup_{k\in[n]} (\frac{1}{i_k}, 7)$

$= (p, 7)$ $(*)$

where $p = \min_{k\in[n]} \{\frac{1}{i_k}\}$.

However, $\frac{p}{2} \in (0,1)$ but $\frac{p}{2} \notin (p, 7)$.

$\Rightarrow\Leftarrow$

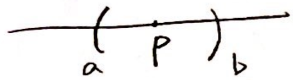
(This contradicts w/ $(*)$)

By def of compactness, $(0,1)$ is not compact.

4. Let S be a dense subset of $\mathbb{R}$. Prove that the set of all acc pts of S is equal to $\mathbb{R}$.

Proof Suppose not.

$\exists\ p \in \mathbb{R}$ s.t. p is not acc pt of S.



then $\exists$ neighborhood $(a,b) \ni p$ s.t. (a,b) contains only finitely many points of S. Call these points $r_1, r_2, \cdots, r_n$, s.t. $r_1 < r_2 < \cdots < r_n$.

However, $\nexists\ r \in (r_1, r_2) \cap S$ s.t. $r_1 < r < r_2$.

This contradicts Def 3.2.7.

Thus, ...

5. $\{a_n = \frac{(-1)^n}{n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence

Proof. Let $\varepsilon > 0$.

Claim: $|a_n - a_{n+1}| \geq |a_n - a_{n+k}|$ $\forall k \in \mathbb{N}$ $(*)$

Pf of claim

By induction on $k \in \mathbb{N}$.

Base case

$k = 1$ ✓

I.H. $(*)$ true for some $k \in \mathbb{N}$

① if n odd

1) if $n+k$ odd

$a_n < 0$, $a_{n+1} > 0$,

$a_{n+k} < 0$, $a_{n+k+1} > 0$, $a_{n+1} > a_{n+k+1} > 0$

$\therefore |a_n - a_{n+1}| = |a_n| + |a_{n+1}|$

$> |a_n| + |a_{n+k+1}|$

$= |a_n - a_{n+k+1}|$

2) if $n+k$ even, $a_{n+k+1} < 0$

$|a_n - a_{n+1}| > |a_n| > |a_n - a_{n+k+1}|$

② if n even

Similar.

$|a_n - a_{n+1}| = \left| \frac{(-1)^n}{n} - \frac{(-1)^{n+1}}{n+1} \right|$

$= \begin{cases} \left| \frac{1}{n} + \frac{1}{n+1} \right|, & \text{if } n \text{ even} \\ \left| \frac{-1}{n} - \frac{1}{n+1} \right|, & \text{if } n \text{ odd} \end{cases}$

$= \frac{1}{n} + \frac{1}{n+1}$

$= \frac{2n+1}{n(n+1)}$

Let $|a_n - a_{n+1}| < \varepsilon$, where $\varepsilon > 0$,

$\varepsilon n^2 + (\varepsilon - 2)n - 1 > 0$.

$\therefore \exists N \in \mathbb{N}$ s.t. if $n \geq N$, then $|a_n - a_{n+1}| < \varepsilon$. (can solve for N)

Let $m, n \geq N$, WLOG $m \geq n$

$|a_m - a_n| \leq |a_n - a_{n+1}| < \varepsilon$.

By def of Cauchy, ...

6. Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Proof. Call the sequence (a_n).

Let $K = \{k_i\}_{i \in I}$ s.t.

① $k_i \in \mathbb{N}$

② if $n \geq k_i$, then $a_n \leq a_{k_i}$

Note that I is the index set.

Case 1 $\exists$ infinite k_i's. Let $I = \mathbb{N}$.

Rearrange K, if necessary, s.t.

$k_1 < k_2 < \cdots < k_n < \cdots$

Consider subsequence (a_{k_n})

$k_r > k_s \Rightarrow k_r \geq k_s \Rightarrow a_{k_r} \leq a_{k_s}$

$\therefore a_{k_1} \geq a_{k_2} \geq \cdots \geq a_{k_n} \geq \cdots$

$\because (a_{k_n})$ monotone decreasing, bounded

$\therefore (a_{k_n})$ is convergent

Case 2 Only n k_i's. $n>0$. Let $I = [n]$

Rearrange K s.t. $k_1 < k_2 < \cdots < k_n$

Let $s_0 = k_n$.

$s_1 = k_n + 1$

Then $\exists$ s_2 s.t. $s_2 > s_1$ and $a_{s_2} > a_{s_1}$.

(Otherwise, $s_2 \in I \Rightarrow \Leftarrow$)

Then $\exists$ s_3 s.t. $s_3 > s_2$ and $a_{s_3} > a_{s_2}$.

...

Get $(a_{s_k})_{k \in \mathbb{N} \cup \{0\}}$.

Monotone increasing, bounded.

$\Rightarrow$ convergent

Case 3 $\nexists$ k_i's. $I = \emptyset$.

Take $s_1 \in \mathbb{N}$.

Then $\exists$ s_2 s.t. $s_2 > s_1$ and $a_{s_2} > a_{s_1}$

(Otherwise, $I \neq \emptyset$)

Then $\exists$ s_3 s.t. $s_3 > s_2$ and $a_{s_3} > a_{s_2}$

(Otherwise, $I \neq \emptyset$)

$\vdots$

Get $(a_{s_k})_{k \in \mathbb{N}}$

which is monotone increasing, bounded

$\Rightarrow$ convergent

Reminder

<u>Thm</u> (Monotone convergent Thm)

If (s_n) monotone, then

(s_n) converges $\Leftrightarrow$ (s_n) bounded

<u>Proof.</u>

$\boxed{\Rightarrow}$ Suppose $s_n \to s \in \mathbb{R}$.

Then $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$\forall n \geq N$, $|s_n - s| < \varepsilon$.

Take $\varepsilon = 1$. Let $M = \max\{\max_{1 \leq i \leq n} |s_i|, |s| + 1\}$.

Then $\forall n \in \mathbb{N}$, if $n \geq N$, we have

$|s_n| \leq |s_n - s| + |s| < 1 + s$.

And if $1 \leq i \leq N$, $|s_i| \leq M$ as well.

$\boxed{\Leftarrow}$ WLOG suppose (s_n) monotone increasing

Let $s = \sup E$ where $E = \{s_n | n \in \mathbb{N}\}$.

$\therefore \forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $s_N > s - \varepsilon$

and $\forall n \geq N$, $s_n \geq s_N > s - \varepsilon$.

$\Rightarrow s - s_n < \varepsilon$.

$\because s \geq s_n \; \forall n \geq N$

$\therefore |s - s_n| < \varepsilon$

$\Rightarrow s_n \to s$ as $n \to \infty$.