Winter 2017 MATH 15910 HW 1 Solution

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Ex 1.2.4

$\emptyset \subseteq A$.

Proof.

$\emptyset \subseteq A$ if and only if for all $x \in \emptyset$, $x \in A$ (*). However, there is no element in $\emptyset$, which means that for all x, the statement $x \in \emptyset$ is false. Then $x \in A$.

Statement (*) is a vacuous truth. (Check Wiki if interested!)

Thus, $\emptyset \subseteq A$.

The exercise can also be proven by contrapositive.

Ex

Interesting exercise:

Does empty set have a subset?

If yes, how many subsets does $\emptyset$ have? Why?

Ex 1.3.9

(ii) $A \cup (B \cup C) = (A \cup B) \cup C$

Proof.

Let $x \in A \cup (B \cup C)$.

Then $x \in A \cup (B \cup C)$

iff (if and only if)

$x \in A$ or $x \in B \cup C$

iff $x \in A$ or ($x \in B$ or $x \in C$).

iff ($x \in A$ or $x \in B$) or $x \in C$

iff $x \in A \cup B$ or $x \in C$

iff $x \in (A \cup B) \cup C$.

Thus, $A \cup (B \cup C) = (A \cup B) \cup C$.

Note: To prove $A = B$, we need

$\forall x \in A, x \in B$

and

$\forall x \in B, x \in A$.

Here, we use iff ($\Leftrightarrow$) to solve this issue.

(iii) $A \Delta (B \Delta C) = (A \Delta B) \Delta C$

Outline of proof.

For arbitrary set A, B,

$$A \Delta B = (A \setminus B) \cup (B \setminus A)$$
$$= (A \cap B^c) \cup (B \cap A^c) \quad (*)$$
$$= (A^c \cup B^c) \cap (A \cup B) \quad (**)$$

$$A \Delta (B \Delta C) = (A \cap (B \Delta C)^c) \cup (A^c \cap (B \Delta C))$$

$$= (A \cap (B \Delta C)^c) \cup (A^c \cap ((B \cap C^c) \cup (B^c \cap C)))$$

$$= \underline{(A \cap (B \Delta C)^c)} \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$$

Now we'll work on

$$(A \cap (B \Delta C)^c)$$
$$= A \cap ((B^c \cup C^c) \cap (B \cup C))^c$$
$$= A \cap ((B^c \cup C^c)^c \cup (B \cup C)^c) \quad \text{(De'Morgan's)}$$
$$= A \cap ((B \cap C) \cup (B^c \cap C^c)) \quad \text{(De'Morgan's)}$$
$$= (A \cap B \cap C) \cup (A \cap B^c \cap C^c)$$

Therefore,

$$A \Delta (B \Delta C) = (A \cap B \cap C) \cup (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C) \quad (***)$$

After going through similar steps, we can find that $(A \Delta B) \Delta C$ equals the same thing as in $(***)$.

Here I apologize for using slightly different notations for the complement of a set

(iv) $A \cup \emptyset = A$

Proof.

$A \cup \emptyset = \{x \mid x \in A \text{ or } x \in \emptyset\}$

$\because$ $x \in \emptyset$ is false, that is,

there is no element in $\emptyset$,

$\therefore$ we have

$A \cup \emptyset = \{x \mid x \in A\} = A$.

OR

Let $x \in A \cup \emptyset$.

• $x \in A \cup \emptyset$

iff $x \in A$ or $x \in \emptyset$

iff $x \in A$

$\Rightarrow A \cup \emptyset = A$.

(xvi)

$(A \cap B)^c = A^c \cup B^c$

Proof.

Assuming our "universe" is X.

$(A \cap B)^c = X \setminus (A \cap B)$. (by def)

$A^c \cup B^c = (X \setminus A) \cup (X \setminus B)$. (by def)

We only need to show

$X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.

Let $x \in X \setminus (A \cap B)$.

We have

$x \in X \setminus (A \cap B)$

iff $x \in X$ and $x \notin \underline{A \cap B = \{u \mid u \in A \text{ and } u \in B\}}$

iff $x \in X$ and $(x \notin A$ or $x \notin B)$

iff $(x \in X$ and $x \notin A)$ or $(x \in X$ and $x \notin B)$

iff $x \in X \setminus A$ or $x \in X \setminus B$

iff $x \in (X \setminus A) \cup (X \setminus B)$

$\therefore X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$

$\therefore (A \cap B)^c = A^c \cup B^c$

Ex 1.4.5

$A \times \emptyset = \emptyset \times A = \emptyset$

Sketch of proof.

① We first prove $A \times \emptyset = \emptyset$

$A \times \emptyset = \{(a,b) \mid a \in A \text{ and } b \in \emptyset\}$

However, $\nexists\, b \in \emptyset$.

$\Rightarrow \nexists\, (a,b) \in A \times \emptyset$.

$\Rightarrow A \times \emptyset = \emptyset$

② Similarly, we can prove

$\emptyset \times A = \emptyset$

Therefore, $A \times \emptyset = \emptyset \times A = \emptyset$

Ex 1.4.6

Suppose $A \neq \emptyset$, $B \neq \emptyset$. Show $A \times B = B \times A$ iff $A = B$.

Proof

$\boxed{\Rightarrow}$ Know: $A \times B = B \times A$, $A, B \neq \emptyset$

WTS: $A = B$

Suppose not. Then $A \neq B$. WLOG (w/o loss of generality) suppose

$\exists p \in A$ s.t. $p \notin B$.

Let $q \in B$. Consider point (p, q). Now,

$\because p \in A$, $q \in B$

$\therefore (p, q) \in A \times B$.

$\because p \notin B$

$\therefore (q, p) \notin B \times A$

$\therefore A \times B \neq B \times A$

$\Rightarrow\Leftarrow$ (contradicting our assumption)

$\boxed{\Leftarrow}$ Know: $A = B$, $A, B \neq \emptyset$

WTS: $A \times B = B \times A$

$\because A = B$

$\therefore A \times B = A \times A$

$B \times A = A \times A$

We only need to show $A \times A = A \times A$.

This can be done by using definition of $A \times A = \{(a, b) \mid a \in A, b \in A\}$

Ex 1.4.9

$\#(A_1 \times A_2 \times \cdots \times A_n) = (\# A_1)(\# A_2) \cdots (\# A_n)$
$= k_1 k_2 \cdots k_n$

Proof

By induction.

Base case: $n = 1$

$\#(A_1) = (\# A_1) = k_1$ ✓

Inductive Hypothesis (IH):

$\#(A_1 \times A_2 \times \cdots \times A_n) = (\# A_1)(\# A_2) \cdots (\# A_n)$

Consider

$\#(A_1 \times A_2 \times \cdots \times A_n \times A_{n+1})$

$= (\#(A_1 \times A_2 \times \cdots \times A_n))(\# A_{n+1})$

(by textbook P8-9)

$= \underbrace{(\# A_1)(\# A_2) \cdots (\# A_n)}_{\text{by I.H.}} (\# A_{n+1})$

We're done! (by the Principles of Mathematical Induction)

Ex ~~4~~ 1.11

(i) $X = \{1\}$ Subset of X? $\emptyset$, $\{1\}$

(ii) $X = \{1, 2\}$ Subset of X? $\emptyset$, $\{1\}$, $\{2\}$, $\{1, 2\}$

Why? Look at definition for subset.