Sep 17th, 2019 Zheng Yuan

F - statistic :

- H_0: Model M_α: $\beta_\alpha = (\beta_0, \beta_{\alpha_1}, \cdots\cdots \beta_{\alpha_{p-d}})$

$\beta_\alpha \subset \beta := (\beta_0, \beta_1, \beta_2, \cdots\cdots, \beta_p)$

- H_1: Model M_λ: $\beta_i \neq 0$, $i = 0, 1, 2, \cdots, p$, i.e., $\beta_\lambda = \beta$

p = # of variables in unconstrained model

$M_\alpha \subset M_\lambda$ and M_α possesses $(p+1-d)$ variables

$\dim(M_\lambda) - \dim(M_\alpha) = d$ = # of constraints imposed

Then
$$F_{d,\, n-p-1} = \frac{(RSS_\alpha - RSS_\lambda)/d}{RSS_\lambda/(n-p-1)}$$
$$= \frac{n-p-1}{d} \cdot \frac{RSS_\alpha - RSS_\lambda}{RSS_\lambda}$$
$$= \frac{n-p-1}{RSS_\lambda} \cdot \frac{RSS_\alpha - RSS_\lambda}{d}.$$
$$= \frac{n-p-1}{\hat{\sigma}_\lambda^2} \cdot (\hat{\sigma}_\alpha^2 - \hat{\sigma}_\lambda^2) \cdot \frac{1}{d}$$

Recall "New Test Statistic": $\frac{n}{\hat{\sigma}_\lambda^2}(\hat{\Gamma}_{\alpha,n} - \hat{\Gamma}_{\lambda,n}) - 2d$

$$F_{d,\,n-p-1} = \frac{n-p-1}{\hat{\sigma}_\lambda^2}\left(\hat{\sigma}_\alpha - \hat{\sigma}_\lambda^2\right)\cdot\frac{1}{d}$$

$$NTS = \frac{n}{\hat{\sigma}_\lambda^2}\left(\hat{\Gamma}_{\alpha,n} - \hat{\Gamma}_{\lambda,n}\right) - 2d$$

As $n \to \infty$ $\quad \dfrac{n-p-1}{\hat{\sigma}_\lambda^2} \Big/ \dfrac{n}{\hat{\sigma}_\lambda^2} = \dfrac{n-p-1}{n} \to 1.$

As I prove before

$$\hat{\Gamma}_{\alpha,n} = \frac{1}{n}RSS_\alpha + \frac{1}{n}\sum_i\left[2h_{i\alpha} + O(h_{i\alpha}^2)\right]r_{i\alpha}^2$$

[$r_{i\alpha} = y_i - x_{i\alpha}^T\hat{\beta}_\alpha$ $\quad h_{i\alpha}$: the ith diagonal element of projection matrix P_α]

$$\hat{\Gamma}_{\alpha,n} = \hat{\sigma}_\alpha^2 + \frac{2}{n}\sum_i h_{i\alpha}r_{i\alpha}^2 + \frac{1}{n}\sum_i O(h_{i\alpha}^2)\,r_{i\alpha}^2$$

Similarly, $\hat{\Gamma}_{\lambda,n} = \hat{\sigma}_\lambda^2 + \frac{2}{n}\sum_i h_{i\lambda}r_{i\alpha}^2 + \frac{1}{n}\sum_i O(h_{i\lambda}^2)\,r_{i\alpha}^2$

$$\Rightarrow \left(\hat{\Gamma}_{\alpha,n} - \hat{\Gamma}_{\lambda,n}\right) - \left(\hat{\sigma}_\alpha^2 - \hat{\sigma}_\lambda^2\right) = \frac{2}{n}\sum_i\left(h_{i\alpha} - h_{i\lambda}\right)r_{i\alpha}^2 + \frac{1}{n}\sum_i O(h_{i\alpha}^2)\,r_{i\alpha}^2 - \frac{1}{n}\sum_i O(h_{i\lambda}^2)\,r_{i\alpha}^2$$

Under the condition $\lim_{n\to\infty} \max_{i\le n} h_{i\alpha} = 0$

$$(\hat{\Gamma}_{\alpha,n} - \hat{\Gamma}_{\lambda,n}) - (\hat{\sigma}^2_{\alpha} - \hat{\sigma}^2_{\lambda}) = o\left(\frac{1}{n}\right) \longrightarrow 0 \quad \text{as } n\to\infty$$

So the only difference between $\bar{F}$-statistic and NTS is:

$$\bar{F}_{d,\,n-p-1} = \frac{n-p-1}{\hat{\sigma}^2_{\lambda}} (\hat{\sigma}^2_{\alpha} - \hat{\sigma}^2_{\lambda}) \cdot \frac{1}{d}$$

$$NTS = \frac{n}{\hat{\sigma}^2_{\lambda}} (\hat{\Gamma}_{\alpha,n} - \hat{\Gamma}_{\lambda,n}) - 2d.$$