Q1. 对于一组对应的 r 和 z, $CDF_r(r)=CDF_z(z)$

$$CDF_r(r)=\frac{(2+2(1-r))r}{2}=(2-r)r$$

$$CDF_z(z)=\begin{cases}\frac{1}{2}z\cdot 4z, & 0\le z\le 0.5\\ \frac{(4+4(1-z))z}{2}-1, & 0.5\le z\le 1\end{cases}=\begin{cases}2z^2, & 0\le z\le 0.5\\ 4z-2z^2-1, & 0.5\le z\le 1\end{cases}$$

故对于 $0\le z\le 0.5$, $2z^2=(2-r)r$, $\Rightarrow z=\sqrt{\frac{-r^2+2r}{2}}$, $r\in[0,\frac{2-\sqrt{2}}{2}]$

对于 $0.5\le z\le 1$, $4z-2z^2-1=(2-r)r \Rightarrow z=\frac{\sqrt{2}}{2}r+1-\frac{\sqrt{2}}{2}$, $r\in[\frac{2-\sqrt{2}}{2},1]$

Q2. (a) 假设图片为 $m\times n$ 的尺寸, 设 $f_{a,b}(x,y)=\begin{cases}f(a,b), & x=a, y=b\\ 0, & \text{otherwise}\end{cases}$.

则 $f(x,y)=\sum_{a,b}f_{a,b}(x,y)$, $w(x,y)*f(x,y)=w(x,y)*\sum_{a,b}f_{a,b}(x,y)$

$$=\sum_{a,b}w(x,y)*f_{a,b}(x,y)$$

下面证明, $\forall a,b$, $w(x,y)*f_{a,b}(x,y)$ 的像素值的和为0.

事实上, $w(x,y)*f_{a,b}(x,y)=\left(\sum_{x,y}w(x,y)\right)f(a,b)=0$

故当 spatial filter mask $w(x,y)$ 的系数和为0时, 卷积得到的图像像素和为0.

(b) 可以将(b)中的 $w(x,y)$ 理解成(a)中的翻折, 类似也有对于每个只有单像素值的子图片 correlation后的像素和为0, 故当 $w(x,y)$ 的系数和为0时, correlation后得到的图像像素和也为0.

Date.

3、二维傅立叶变换表达式：$\mathcal{F}(f(x,y))=F(u,v)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(x,y)e^{-2\pi j(ux+vy)}dxdy$

二维卷积表达式：　$h(x,y)=f(x,y)*h(x,y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(x',y')h(x-x',y-y')dx'dy'$

故有二维卷积的傅立叶变换表达式为：

$$\mathcal{F}(f(x,y)*h(x,y))=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(x',y')h(x-x',y-y')dx'dy'\right)e^{-2\pi j(ux+vy)}dxdy$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}h(x-x',y-y')e^{-2\pi j(ux+vy)}dxdy\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(x',y')dx'dy'\right)$$

$$=H(u,v)\cdot\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(x',y')e^{-2\pi j(ux'+vy')}dx'dy'\right)=H(u,v)\cdot F(u,v)$$

而反方向的证明只需使用二维傅立叶逆变换表达式，$\mathcal{F}^{-1}(F(u,v))=f(x,y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F(u,v)e^{2\pi j(ux+vy)}dudv$.

故有二维卷积的傅立叶逆变换表达式为：

$$\mathcal{F}^{-1}(F(u,v)*H(u,v))=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F(u',v')H(u-u',v-v')du'dv'\right)e^{2\pi j(ux+vy)}dudv$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}H(u-u',v-v')e^{2\pi j(ux+vy)}dudv\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F(u',v')du'dv'\right)$$

$$=h(x,y)\cdot\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F(u',v')e^{2\pi j(u'x+v'y)}du'dv'\right)=h(x,y)\cdot f(x,y)\quad\square$$

4. The 3×3 spatial mask can be written as $\begin{array}{|c|c|c|}\hline 0 & \frac{1}{4} & 0 \\ \hline \frac{1}{4} & 0 & \frac{1}{4} \\ \hline 0 & \frac{1}{4} & 0 \\ \hline\end{array}$.

(a) After applying this mask, we can get

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 0&0&0&0&0&0&0\\ \hline 0&1&0&0&0&0&0\\ \hline 0&1&1&0&0&0&0\\ \hline 0&0&1&1&0&0&0\\ \hline 0&0&0&1&1&1&0\\ \hline 0&0&1&1&0&0&0\\ \hline 0&1&1&0&0&0&0\\ \hline 0&1&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0\\ \hline\end{array} \Rightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline 0&\frac{1}{4}&0&0&0&0&0\\ \hline \frac{1}{4}&\frac{1}{4}&\frac{1}{2}&0&0&0&0\\ \hline \frac{1}{4}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&0&0&0\\ \hline 0&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{4}&0\\ \hline 0&0&\frac{3}{4}&\frac{3}{4}&\frac{1}{2}&\frac{1}{4}&\frac{1}{4}\\ \hline 0&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{4}&0\\ \hline \frac{1}{4}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&0&0&0\\ \hline \frac{1}{4}&\frac{1}{4}&\frac{1}{2}&0&0&0&0\\ \hline 0&\frac{1}{4}&0&0&0&0&0\\ \hline\end{array}$$

(b) In fact, we can write the spatial mask as a function $g(x,y)=\frac{1}{4}(f(x,y+1)+f(x+1,y)+f(x-1,y)+f(x,y-1))$

Applying discrete fourier transform on both sides, we have,

$$G(u,v)=\frac{1}{4}\left(e^{j\frac{2\pi v}{N}}+e^{j\frac{2\pi u}{M}}+e^{-j\frac{2\pi v}{N}}+e^{-j\frac{2\pi u}{M}}\right)F(u,v)=H(u,v)\cdot F(u,v)$$

Thus, $H(u,v)=\frac{1}{2}\left(\cos\frac{2\pi u}{M}+\cos\frac{2\pi v}{N}\right)$ (where $M=7$, $N=9$)

$=\frac{1}{2}\left(\cos\frac{2\pi u}{7}+\cos\frac{2\pi v}{9}\right)$, which is the filter transfer function in the frequency domain.