Date.

Question1. (a) $\|x-y\|^2 = \langle x-y, x-y\rangle = \langle x,x\rangle + \langle y,y\rangle - 2\langle x,y\rangle$

$$= \int_{-\infty}^{+\infty} x^2(t)\,dt + c^2T - 2c\int_0^T x(t)\,dt = h(c)$$

$$\frac{dh(c)}{dc} = 2cT - 2\int_0^T x(t)\,dt$$

$$= 2T\left(c - \frac{1}{T}\int_0^T x(t)\,dt\right)$$

当 $c < \frac{1}{T}\int_0^T x(t)\,dt$, $\frac{dh}{dc} < 0$; 当 $c > \frac{1}{T}\int_0^T x(t)\,dt$, $\frac{dh}{dc} > 0$

$$c = \frac{1}{T}\int_0^T x(t)\,dt$$

$$y(t) = \begin{cases} \frac{1}{T}\int_0^T x(t)\,dt, & t \in [0,T) \\ 0, & \text{otherwise} \end{cases}$$

(b) let $e(t) = \begin{cases} 1, & t\in[0,T) \\ 0, & \text{otherwise} \end{cases}$. it is easy to verify that it is a basis of V.

Assume $y(t) = k\,e(t) = \begin{cases} k, & t\in[0,T) \\ 0, & \text{otherwise} \end{cases}$. we have $\langle x-y, e\rangle = 0$, according to the definition of the orthogonal projection.

$$0 = \langle x-y, e\rangle = \langle x, e\rangle - \langle y, e\rangle$$

$$= \int_{-\infty}^{+\infty} x(t)e(t)\,dt - \int_{-\infty}^{+\infty} y(t)e(t)\,dt$$

$$= \int_0^T x(t)\,dt - kT$$

$$\Rightarrow k = \frac{1}{T}\int_0^T x(t)\,dt$$

$$y(t) = \begin{cases} \frac{1}{T}\int_0^T x(t)\,dt, & t\in[0,T) \\ 0, & \text{otherwise} \end{cases}$$

(c) I prefer the method in (b).

Question2. (a) $\varphi_{-1}(t) = \begin{cases} -t, & t \in [-1,0] \\ 0, & \text{otherwise} \end{cases}$

$\varphi_0(t) = \begin{cases} t+1, & t \in [-1,0] \\ -t+1, & t \in [0,1] \\ 0, & \text{otherwise} \end{cases}$

$\varphi_1(t) = \begin{cases} t, & t \in [0,1] \\ 0, & \text{otherwise} \end{cases}$

$f(t) = \begin{cases} k_1 t + b, & t \in [-1,0] \\ k_2 t + b, & t \in [0,1] \\ 0, & \text{otherwise} \end{cases}$

Assume $f(t) = \alpha \varphi_{-1}(t) + \beta \varphi_0(t) + \gamma \varphi_1(t)$

$f(-1) = \alpha \varphi_{-1}(-1) + \beta \varphi_0(-1) + \gamma \varphi_1(-1) = \alpha = b - k_1$

$f(0) = \alpha \varphi_{-1}(0) + \beta \varphi_0(0) + \gamma \varphi_1(0) = \beta = b$

$f(1) = \alpha \varphi_{-1}(1) + \beta \varphi_0(1) + \gamma \varphi_1(1) = \gamma = b + k_1$

Then we can verify that

$(b-k_1)\varphi_{-1}(t) + b\varphi_0(t) + (b+k_1)\varphi_1(t) = \begin{cases} k_1 t + b, & t \in [-1,0] \\ k_2 t + b, & t \in [0,1] \\ 0, & \text{otherwise} \end{cases} = f(t).$

(b) $\langle \varphi_{-1}, \varphi_1 \rangle = \int_{-1}^{1} \varphi_{-1}(t) \varphi_1(t)\, dt = 0$

$\langle \varphi_{-1}, \varphi_0 \rangle = \int_{-1}^{1} \varphi_{-1}(t) \varphi_0(t)\, dt = \int_{-1}^{0} (-t)\cdot(t+1)\, dt = \frac{1}{6} \neq 0$

$\langle \varphi_0, \varphi_1 \rangle = \int_{-1}^{1} \varphi_0(t) \varphi_1(t)\, dt = \int_{0}^{1} (-t+1) t\, dt = \frac{1}{6} \neq 0$

φ_{-1}, φ_1 are orthogonal.

(c) Since $V_1 = \mathrm{Span}\{\varphi_{-1}, \varphi_1\}$, $\hat{\varphi}_0 = \varphi_0 - P_{V_1}(\varphi_0)$,

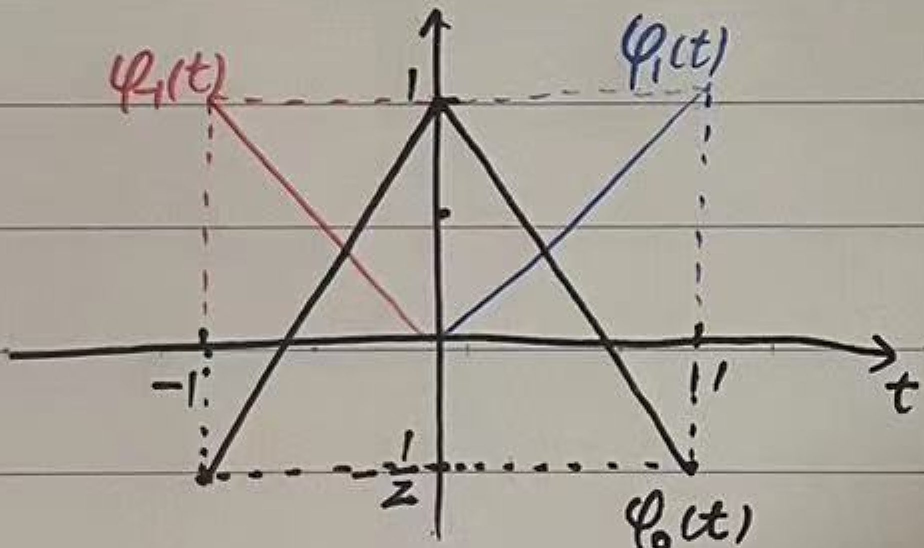
$$\hat{\varphi}_0 = \varphi_0 + \alpha\varphi_{-1} + \beta\varphi_1, \ \langle \hat{\varphi}_0, \varphi_{-1}\rangle = \langle \hat{\varphi}_0, \varphi_1\rangle = 0$$

$$\begin{aligned} 0 = \langle \hat{\varphi}_0, \varphi_{-1}\rangle &= \langle \varphi_0 + \alpha\varphi_{-1} + \beta\varphi_1, \varphi_{-1}\rangle \\ &= \langle \varphi_0, \varphi_{-1}\rangle + \alpha\langle \varphi_{-1}, \varphi_{-1}\rangle \\ &= \frac{1}{6} + \alpha\int_{-1}^{0} t^2\,dt = \frac{1}{6} + \frac{1}{3}\alpha \Rightarrow \alpha = -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} 0 = \langle \hat{\varphi}_0, \varphi_{\text{[redacted]}}\rangle &= \langle \varphi_0 + \alpha\varphi_{-1} + \beta\varphi_1, \varphi_{\text{[redacted]}}\rangle \\ &= \langle \varphi_0, \varphi_1\rangle + \beta\langle \varphi_1, \varphi_1\rangle \\ &= \frac{1}{6} + \beta\int_0^1 t^2\,dt = \frac{1}{6} + \frac{1}{3}\beta \Rightarrow \beta = -\frac{1}{2} \end{aligned}$$

$$\hat{\varphi}_0 = \varphi_0 - \frac{1}{2}\varphi_1 - \frac{1}{2}\varphi_{-1} = \begin{cases} \frac{3}{2}t + 1, & t \in [-1, 0] \\ -\frac{3}{2}t + 1, & t \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

Then $\{\varphi_{-1}, \hat{\varphi}_0, \varphi_1\}$ pairwise orthogonal.



value	-1	0	1
φ_{-1}	1	0	0
φ_0	$-\frac{1}{2}$	1	$-\frac{1}{2}$
φ_1	0	0	1

(d) $f(t) = a\varphi_{-1} + b\varphi_1 + c\hat{\varphi}_0.$

$$\|g(t)-f(t)\|^2 = \langle g(t),g(t)\rangle + \langle f(t),f(t)\rangle + -2\langle g(t),f(t)\rangle$$

$$= a^2\langle\varphi_{-1},\varphi_{-1}\rangle + b^2\langle\varphi_1,\varphi_1\rangle + c^2\langle\hat{\varphi}_0,\hat{\varphi}_0\rangle - 2a\langle g(t),\varphi_{-1}\rangle - 2b\langle g(t),\varphi_1\rangle - 2c\langle g(t),\hat{\varphi}_0\rangle + \frac{5}{3}$$

$$\langle\varphi_{-1},\varphi_{-1}\rangle = \int_{-1}^{0}(-t)^2dt = \frac{1}{3}$$

$$\langle\varphi_1,\varphi_1\rangle = \frac{1}{3}$$

$$\langle\hat{\varphi}_0,\hat{\varphi}_0\rangle = 2\int_0^1 (1-\tfrac{3}{2}t)^2dt = \frac{1}{2}.$$

$$\langle g(t),\varphi_{-1}\rangle = \int_{-1}^{0} -t\,dt = \frac{1}{2}$$

$$\langle g(t),\varphi_1\rangle = \int_0^{\frac{2}{3}} t\,dt = \frac{2}{9}$$

$$\langle g(t),\hat{\varphi}_0\rangle = \int_{-1}^{0}(\tfrac{3}{2}t+1)dt + \int_0^{\frac{2}{3}}(1-\tfrac{3}{2}t)dt = \frac{1}{4}+\frac{1}{3} = \frac{7}{12}$$

$$\|g(t)-f(t)\|^2 = \frac{1}{3}a^2 + \frac{1}{3}b^2 + \frac{1}{2}c^2 + \cancel{\frac{1}{2}c^2} - a - \frac{4}{9}b - \frac{7}{6}c + \frac{5}{3}$$

$$= \frac{1}{3}(a-\frac{3}{2})^2 - \frac{1}{3}\times(\frac{3}{4})^2 + \frac{1}{3}(b-\frac{2}{3})^2 - \frac{1}{3}\times(\frac{1}{3})^2 + \frac{1}{2}(c-\frac{7}{6})^2 - \frac{1}{2}\times(\frac{7}{6})^2 + \frac{5}{3}$$

当 $\|g(t)-f(t)\|$ 取最小值时，$a=\frac{3}{2}$，$b=\frac{2}{3}$，$c=\frac{7}{6}$

$f(-1)=\frac{11}{12}$

$f(0)=\frac{7}{6}$

$f(1)=\frac{1}{12}$.

$$f(x) = \frac{3}{2}\varphi_{-1} + \frac{2}{3}\varphi_1 + \frac{7}{6}\hat{\varphi}_0$$

$$= \begin{cases} \frac{1}{4}x+\frac{7}{6}, & x\in[-1,0] \\ -\frac{13}{12}x+\frac{7}{6}, & x\in[0,1] \end{cases}$$