Q1. (a) $X[k]=\begin{cases}\sum_{n=0}^{3} x[n]e^{-\frac{j2\pi nk}{4}}, & k=0,1,2,3\\ 0, & \text{otherwise}\end{cases}$

$=\begin{cases}\sum_{n=0}^{3} \cos\frac{n\pi}{2} e^{-\frac{j2\pi nk}{4}}, & k=0,1,2,3\\ 0, & \text{otherwise}\end{cases}$

$=\begin{cases}2, & k=1,3\\ 0, & \text{otherwise}\end{cases}$

(b) $H[k]=\begin{cases}\sum_{n=0}^{3} h(n) e^{-\frac{j2\pi nk}{4}}, & k=0,1,2,3\\ 0, & \text{otherwise}\end{cases}$

$=\begin{cases}\sum_{n=0}^{3} 2^n e^{-\frac{j\pi nk}{2}}, & k=0,1,2,3\\ 0, & \text{otherwise}\end{cases}$

$=\begin{cases}15, & k=0\\ -3+6i, & k=1\\ -5, & k=2\\ -3-6i, & k=3\\ 0, & \text{otherwise}\end{cases}$

(c) $y[n]=x[n] \circledast_4 h[n]$

$=\sum_{m=0}^{3} x[(m)_4]h[(n-m)_4]$

$=h[(n)_4]-h[(n-2)_4]$

$=\begin{cases}-3, & n=0\\ -6, & n=1\\ 3, & n=2\\ 6, & n=4\\ 0, & \text{otherwise}\end{cases}$ (Calculate directly by definition)

(d) $Y[k]=X[k]\cdot H[k]=\begin{cases}6(-1+2i), & k=1\\ 6(-1-2i), & k=3\\ 0, & \text{otherwise}\end{cases}$

$y[n]=\frac{1}{4}\sum_{k=0}^{3} Y[k]e^{\frac{j2\pi nk}{4}}$

$=\frac{6}{4}\left((-1+2i)e^{\frac{j\pi n}{2}}+(-1-2i)e^{\frac{j\pi 3n}{2}}\right)$

$=\begin{cases}-3, & n=0\\ -6, & n=1\\ 3, & n=2\\ 6, & n=3\\ 0, & \text{otherwise}\end{cases}$

(Calculate by inverse DFT)

Q2. (a) Let $m-k=t$. Verify $\sum_{n=-\frac{N}{2}}^{\frac{N}{2}} W_{N+1}^{-nt} = (N+1)\sum_{r=-\infty}^{+\infty} \delta(t-r(N+1))$.

$$RHS = \begin{cases} N+1, & t=r(N+1), r\in\mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$$

If $t=r(N+1), r\in\mathbb{Z}$, $LHS = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} 1 = N+1 = RHS$,

If $\forall r\in\mathbb{Z}, t\neq r(N+1)$, $LHS = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} W_{N+1}^{-tn} = W_{N+1}^{t\cdot\frac{N}{2}} \sum_{n=0}^{N} W_{N+1}^{-tn} = W_{N+1}^{t\cdot\frac{N}{2}} \frac{1-W_{N+1}^{-t(N+1)}}{1-W_{N+1}^{-t}} = 0 = RHS$. □.

(b) The corresponding inverse DFT is

$$f_n = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} F_k \cdot W_{N+1}^{nk}.$$

The proof is as below:

$$\sum_{k=-\frac{N}{2}}^{\frac{N}{2}} F_k W_{N+1}^{kr} = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \left(\frac{1}{N+1}\sum_{n=-\frac{N}{2}}^{\frac{N}{2}} f_n W_{N+1}^{-nk}\right) W_{N+1}^{kr}$$

$$= \frac{1}{N+1}\sum_{k=-\frac{N}{2}}^{\frac{N}{2}}\sum_{n=-\frac{N}{2}}^{\frac{N}{2}} f_n W_{N+1}^{-k(n-r)}$$

$$= \frac{1}{N+1}\sum_{n=-\frac{N}{2}}^{\frac{N}{2}} f_n \left(\sum_{k=-\frac{N}{2}}^{\frac{N}{2}} W_{N+1}^{-k(n-r)}\right)$$

$$= \frac{1}{N+1}\sum_{n=-\frac{N}{2}}^{\frac{N}{2}} f_n \left((N+1)\sum_{t=-\infty}^{+\infty} \delta[n-r-t(N+1)]\right)$$

$$= \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} f_n \sum_{t=-\infty}^{+\infty} \delta[n-r-t(N+1)]$$

$$= \sum_{t=-\infty}^{+\infty}\sum_{n=-\frac{N}{2}}^{\frac{N}{2}} f_n \cdot \delta[n-r-t(N+1)]$$

$$= \sum_{t=-\infty}^{+\infty}\sum_{\substack{n=-\frac{N}{2} \\ -\frac{N}{2}\le r+t(N+1)\le\frac{N}{2}}}^{\frac{N}{2}} f_{r+t(N+1)} = f_r \quad \text{for } r=-\frac{N}{2}:\frac{N}{2}. \quad \square.$$

(c) If x_n is the endpoint, $x_n = \frac{nA}{N+1} = \pm\frac{A}{2} \Rightarrow n = \pm\frac{1}{2}(N+1)$.

However, $2|N$, $2\nmid N+1$, n is an integer, which contradicts each other!

So this method of sampling don't include either endpoint of the interval $[-\frac{A}{2}, \frac{A}{2}]$.

$$X_{\pm\frac{N}{2}} = \frac{\pm\frac{N}{2}A}{N+1} = \pm\frac{N}{2(N+1)}A$$

$$\lim_{N\to\infty} X_{\pm\frac{N}{2}} = \lim_{N\to\infty} \pm\frac{N}{2(N+1)}A = \pm\frac{1}{2}A.$$

Q3. $\bar{F}_k = \sum_{n=-3}^{4} \bar{f}_n e^{-\frac{j2\pi nk}{8}}$

$$= -\sum_{n=-3}^{-1} e^{-\frac{j2\pi nk}{8}} + \sum_{n=1}^{4} e^{-\frac{j2\pi nk}{8}}$$

$$= \begin{cases} 1, & k=0 \\ -1-(2\sqrt{2}+2)i, & k=1 \\ 1, & k=2 \\ -1-(2\sqrt{2}-2)i, & k=3 \\ 1, & k=4 \\ -1+(2\sqrt{2}+2)i, & k=-1 \\ 1, & k=-2 \\ -1+(2\sqrt{2}-2)i, & k=-3 \end{cases}$$

So DFT is imaginary but not odd (because of its definition.)

(It's not purely imaginary.) (定义域不对称显然不奇)

Then if the input sequence is sampled at the points $x_n = \frac{n}{8}$, $n=-4:4$,

which means $\bar{f}_n = \{-1,-1,-1,-1,0,1,1,1,1\}$, the DFT is odd and purely imaginary.

Now, new DFT can be calculated as below:

$$\bar{F}_k = \sum_{n=-4}^{4} \bar{f}_n e^{-\frac{j2\pi nk}{8}}$$

$$= -\sum_{n=-4}^{-1} e^{-j\frac{2\pi nk}{8}} + \sum_{n=1}^{4} e^{-j\frac{2\pi nk}{8}} = \sum_{n=1}^{4}\left(e^{-j\frac{2\pi nk}{8}} - e^{-j\frac{2\pi(-n)k}{8}}\right) \Rightarrow \text{It's purely imaginary.}$$

$$\bar{F}_{-k} = -\sum_{n=-4}^{-1} e^{-j\frac{2\pi n(-k)}{8}} + \sum_{n=1}^{4} e^{-j\frac{2\pi n(-k)}{8}}$$

$$= -\sum_{n=-4}^{-1} e^{-\frac{j2\pi(-n)k}{8}} + \sum_{n=1}^{4} e^{-j\frac{2\pi(-n)k}{8}}$$

$$= -\sum_{n=1}^{4} e^{-j\frac{2\pi nk}{8}} + \sum_{n=-4}^{-1} e^{-j\frac{2\pi nk}{8}} = -\bar{F}_k \Rightarrow \text{It's odd.}$$

Q4. (a) The maximum possible number of non-zero values in the linear convolution of $x[n]$ and $h[n]$ is $50+10-1=59$.

(b) As what we learned in class, $x[n] \circledN y[n] = \sum_{r=-\infty}^{+\infty} z[n+rN]$,

where $z[n] = x[n] * y[n]$.

So if we let $y[n] = x[n] * h[n]$, $y[n]$ is non-zero only if $0 \le n \le 58$.

$$10 = x[n] \circled{50} h[n] = \sum_{r=-\infty}^{+\infty} y[n+50r] = y[n] + y[n+50], \quad 0 \le n \le 49$$

We also have $y[n]=5, \quad 0 \le n \le 4$

$\Rightarrow \quad y[50] = y[51] = \cdots = y[54] = 5$

for $n \ge 9$ and $n \le 49$,

$$y[n] + y[n+50] = y[n] = 10$$

So all we can get are:

$$y[n] = \begin{cases} 5, & 0 \le n \le 4 \\ 10, & 9 \le n \le 49 \\ 5, & 50 \le n \le 54 \\ \text{uncertain value}, & 5 \le n \le 8,\ 55 \le n \le 58. \end{cases}$$