Exercise Sheet 12

Discrete Mathematics, 2021.11.09

1. ([R], Page 172, Exercise 2(a)(b)(d)(e)) Determine whether each of these sets is countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

   a) the integers greater than 10

   b) the odd negative integers

   d) the real numbers between 0 and 2

   e) the set $A \times Z^+$ where $A = \{2, 3\}$

2. Prove that $[0, 1) \times [0, 1) \approx [0, 1)$.

3. Prove that for any sets A, B and C, $(A \to (B \to C)) \approx (B \to (A \to C))$.

4. Prove that for any sets A, B and C, $(A \to B \times C) \approx (A \to B) \times (A \to C)$.

5. Prove that the set of all binary relations on $\mathbb{R}$ is equinumerous to the set of all functions from $\mathbb{R}$ into $\mathbb{R}$.

6. We call f a choice function of S if $f : S \to \bigcup S$ and $f(X) \in X$ for any $X \in S$. Thus, the axiom of choice actually says: if $\emptyset \notin S$ then S has at least a choice function. Now, please determine whether f is a choice function of S in the following examples. (If yes, you only need to say yes. If no, briefly explain why.)

   a) $S = \{\{0\}, \{1\}, \{2\}\}$, $f = \{(\{0\}, 0), (\{1\}, 1), (\{2\}, 2)\}$

   b) $S = \{\{0\}, \{1\}, \{2\}\}$, $f = \{(0, 0), (1, 1), (2, 2)\}$

   c) $S = \emptyset$, $f = \emptyset$

   d) $S = \mathcal{P}(\mathbb{N})$, $f = \{(\{m \in \mathbb{N} \mid m \geq n\}, n) \mid n \in \mathbb{N}\}$

7. Suppose R is an equivalence relation on A. Prove that there exists an injection from $\{[a]_R \mid a \in A\}$ into A, based on the axiom of choice.