Exercise 5

Zhiwei Ying

1.

a) Here is the truth table of $(p \to q) \land (p \to r)$ and $p \to (q \land r)$.

p	q	r	$p \to q$	$p \to r$	$(p \to q) \land (p \to r)$	$q \land r$	$p \to (q \land r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	F	F	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	F	T
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

Thus, $(p \to q) \land (p \to r)$ and $p \to (q \land r)$ are logically equivalent.

b) Here is the truth table of $(p \to r) \land (q \to r)$ and $(p \lor q) \to r$.

p	q	r	$p \to r$	$q \to r$	$(p \to r) \land (q \to r)$	$p \lor q$	$(p \lor q) \to r)$
T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	F
T	F	T	T	T	T	T	T
T	F	F	F	T	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	F
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

Thus, $(p \to r) \land (q \to r)$ and $(p \lor q) \to r)$ are logically equivalent.

c) Here is the truth table of $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$.

p	q	r	$p \to q$	$p \to r$	$(p \to q) \vee (p \to r)$	$q \vee r$	$p \to (q \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	F	T

Thus, $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$ are logically equivalent.

2. Let $[[p]]_J$ =F, $[[q]]_J$ =T, $[[r]]_J$ =F. Then $[[(p \to q) \to r]]_J$ =F, $[[p \to (q \to r)]]_J$ =T. Thus $(p \to q) \to r$ and $p \to (q \to r)$ are not logically equivalent.

3. Let $[[p]]_J$ =T, $[[q]]_J$ =F, $[[r]]_J$ =F. Then $[[(p \wedge q) \to r]]_J$ =T, $[[(p \to r) \wedge (q \to r)]]_J$ =F. Thus $(p \wedge q) \to r$ and $(p \to r) \wedge (q \to r)$ are not logically equivalent.

4.

a) $\forall x \in \mathbb{N}, \exists y = x + 1,\ s.t.\ y \in \mathbb{N}$. In this case, $x < x + 1 = y$, so $\mathcal{J}_1(R, x, y) = $ T, which means $[[\forall x.\exists y.R(x, y)]]_{\mathcal{J}_1} = $ T.

b) $\because\ \mathcal{J}_2(x) = 0\ \therefore\ \forall y \in \mathbb{N},\ y \geq x$. Thus by definition of $\mathcal{J}_2$, $\mathcal{J}_2(R, x, y) = $ F. So $[[\exists y.R(x, y)]]_{\mathcal{J}_2} = $ F.

c) When $x = 0$, from b), we know $[[\exists y.R(x, y)]]_{\mathcal{J}_3} = $ F. So $[[\forall x.\exists y, R(x, y)]]_{\mathcal{J}_3} = $ F.

d) $\forall x \in \mathbb{N}, \exists y = x + 1,\ s.t.\ y \in \mathbb{N}$. In this case, if $z \in \mathbb{N}$, z either greater than or equal to y or less than or equal to x, which means $R(x, z) \wedge R(z, y) = $ F. So $[[\forall x.\forall y.(R(x, y) \to \exists z.(R(x, z) \wedge R(z, y)))]]_{\mathcal{J}_1} = $ F.

e) $\forall x.\forall y.\mathcal{J}_4(R, x, y) = $ T iff $x < y$, $\mathcal{J}_4(R, x, y) = $ F iff $x \geq y$.

Case 1. $x \geq y$, $[[R(x, y)]]_{\mathcal{J}_4} = $ F.

Whatever the true value of $(\exists z.\ (R(x, z) \wedge R(z, y)))$ is, $[[R(x, y) \to (\exists z.\ (R(x, z) \wedge R(z, y)))]]_{\mathcal{J}_4} = $ T.

Case 2. $x < y$, $[[R(x, y)]]_{\mathcal{J}_4} = $ T.

$\exists z = (x + y)/2.\ s.t.\ z > x, z < y$, so $[[\exists z.\ (R(x, z) \wedge R(z, y))]]_{\mathcal{J}_4} = $ T.

Thus $[[\forall x.\ \forall y.\ R(x, y) \to (\exists z.\ (R(x, z) \wedge R(z, y)))]]_{\mathcal{J}_4} = $ T.

5.

a) $\because\ \forall x.\ \forall y.\ \mathcal{J}_1(f)(x, y) = \mathcal{J}_1(f)(y, x) = x + y, \therefore\ [[\forall x.\forall y.\ R(f(x, y), f(x, y))]]_{\mathcal{J}_1}$ =T.

b) $\because\ \forall x.\ \forall y.\ \mathcal{J}_2(f)(x, y) = \mathcal{J}_2(f)(y, x) = x * y, \therefore\ [[\forall x.\forall y.\ R(f(x, y), f(x, y))]]_{\mathcal{J}_2}$ =T.

c) $\because\ \forall x.\ \forall y.\ \mathcal{J}_3(f)(x, y) = \mathcal{J}_2(f)(y, x) = x \wedge y, \therefore\ [[\forall x.\forall y.\ R(f(x, y), f(x, y))]]_{\mathcal{J}_3}$ =T.

d) If $\mathcal{J}_4(f, a, b) = a - b,\ \ \mathcal{J}_4(R, a, b)$ =T $iff\ a = b$. Then $\forall x.\ \forall y.\ \mathcal{J}_4(f)(x, y) = x - y.\ \mathcal{J}_4(f)(y, x) = y - x$. When $x \neq y,\ \ x - y \neq y - x$. Thus,$[[\forall x.\forall y.\ R(f(x, y), f(x, y))]]_{\mathcal{J}_4}$ =F. So $\forall x.\forall y.\ R(f(x, y), f(x, y))$ is not vavid.