# Exercise Sheet 7

Discrete Mathematics, 2020.10.13

1. Show that for any sets $A$ and $B$
$$\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$$

2. Show that $A \times \bigcup B = \bigcup\{A \times X | X \in B\}$

3. Here is a proof of $C \cap (A \cup B) = (C \cap A) \cup (C \cap B)$. Show its corresponding first order logic proof.

   Proof.

   $$\begin{array}{ll} \text{For any } x, & \\ x \in C \cap (A \cup B) & \iff \\ x \in C \wedge (x \in A \vee x \in B) & \iff \\ (x \in C \wedge x \in A) \vee (x \in C \wedge x \in B) & \iff \\ x \in (C \cap A) \cup (C \cap B) & \end{array}$$

   Qed.

4. Use the language of ZF to state empty set's uniqueness and prove your statement. **Note**: you are only allow to use the predicates $\in$ and $=$ in your statement in this task.

5. We define a set $A$ to be "$\in$-well-ordered" if for every nonempty subset $x$ of $A$, $x$ must have a $\in$-least element $y$. In other words,
$$A \text{ is } \in\text{-well-ordered} \quad \leftrightarrow \quad \forall x.\ (x \subseteq A \wedge \neg x = \emptyset \rightarrow \exists y.(y \in x \wedge \forall z.(z \in x \rightarrow y \in z \vee y = z)))$$
   Prove that:

   a) $0$ is $\in$-well-ordered

   b) if $n$ is $\in$-well-ordered, then $n \cup \{n\}$ is $\in$-well-ordered.

6. Prove that there exists at most one "smallest inductive set", i.e. if

   - Inductive($u$)
   - $\forall x.$ (Inductive($x$) $\rightarrow u \subseteq x$)
   - Inductive($v$)
   - $\forall x.$ (Inductive($x$) $\rightarrow v \subseteq x$)

   then $u = v$.

7. Suppose $u$ and $v$ are two inductive sets. Prove that $u \cap v$ is also inductive.

8. Suppose $u$ is an inductive set.

   a) Prove that $\{x \in u \mid \forall v.\ (v \subseteq u \wedge \text{Inductive}(v) \rightarrow x \in v)\}$ is also inductive.

   b) Prove that $\{x \in u \mid \forall v.\ (v \subseteq u \wedge \text{Inductive}(v) \rightarrow x \in v)\}$ is the smallest inductive subset of $u$.

9. Prove there exists at least one "smallest inductive set". **Hint**: you can use the conclusions above.

10. **Note: in this task, you will prove the induction principle of natural numbers.** Suppose $X$ is a set such that

- $\emptyset \in X$,
- for any $n \in \mathbb{N}$, $n \in X$ implies $n \cup \{n\} \in X$ .

Prove that

a) $\mathbb{N} \cap X$ is an inductive set.

b) for any $n \in \mathbb{N}$, $n \in X$ always holds.