Exercise Sheet 11

Discrete Mathematics, 2021.11.2

1. Show that the equation
$$f(m,n) = 2^m(2n+1) - 1$$
defines a one-to-one correspondence between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.

2. We know that $F(x) = 1/3 + x/3$ is an injection from $[0,1]$ to $(0,1)$, and $G(x) = x$ is an injection from $(0,1)$ to $[0,1]$. Please construct a bijection from $[0,1]$ to $(0,1)$ according to the proof of Berstein's Theorem.

3. We proved in class that if $F : A \to B$ is a function, then $R = \{(a,b) \mid F(a) = F(b)\} \subseteq A \times A$ is an equivalence relations. Now, you need to prove the reverse direction: if $R \subseteq A \times A$ is an equivalence relation on A, then there exists a set B and a function $F : A \to B$ such that $R = \{(a,b) \mid F(a) = F(b)\}$.

   a) Let $B = \{[a]_R \mid a \in A\}$ be the partition defined by R. Prove that $F = \{(a,[a]_R) \mid a \in A\} \subseteq A \times B$ is a function.

   b) Prove that this F defined above is a surjection from A to B.

   c) Prove that $R = \{(a,b) \mid F(a) = F(b)\}$.

4. Suppose R_1 and R_2 are equivalence relations on A_1 and A_2, respectively.

   a) Let $R \subseteq (A_1 \times A_2) \times (A_1 \times A_2)$ be the relation that $\{((a_1,a_2),(b_1,b_2)) \mid a_1 R_1 b_1 \wedge a_2 R_2 b_2\}$. Prove R is an equivalence relation on $A_1 \times A_2$.

   b) Let $B_1 \subseteq \mathcal{P}(A_1)$, $B_2 \subseteq \mathcal{P}(A_2)$ and $B \subseteq \mathcal{P}(A_1 \times A_2)$ be corresponding partitions of R_1, R_2 and R. Prove that $B_1 \times B_2 \approx B$.

5. (Optional homework. 1 additional point.) Suppose R is a binary relation on A. Prove that $R^+ = (R^+)^+$. **(选做题可以不做)**

6. (Optional homework. 1 additional point.) Suppose S and R are two binary relations on A. Prove that if $R \subseteq S \subseteq R^+$ then $S^+ = R^+$. **(选做题可以不做)**