

Exercise Sheet 10

Discrete Mathematics, 2021.10.28

1. ([R], Page 617, Exercise 54) Suppose that R_1 and R_2 are equivalence relations on a set A . Let P_1 and P_2 be the partitions that correspond to R_1 and R_2 , respectively. Show that $R_1 \subseteq R_2$ if and only if P_1 is a refinement of P_2 .

Refinement: A partition P_1 is called a refinement of the partition P_2 if every set in P_1 is a subset of one of the sets in P_2 .

2. ([R], Page 583, Exercise 58) Let R be a symmetric relation. Show that R^n is symmetric for all positive integers n .
3. Prove that for any relations R, S , we have that

$$R \circ S \subseteq S \Leftrightarrow (\bigcup_{n=1}^{\infty} R^n) \circ S \subseteq S .$$

4. For any binary relation $R \subseteq A \times A$, we call R a **preorder** on A if R is reflexive on A and transitive, and we call R a **partial order** on A if R is reflexive on A , antisymmetric and transitive.
- a) Suppose $R = \{(a, b) \in \mathbb{R} \times \mathbb{R} \mid \text{there is no integer } n \text{ such that } b < n \leq a\}$. Prove that R is a preorder on \mathbb{R} but R is not a partial order on \mathbb{R} .
- b) Suppose $R \subseteq A \times A$ is a preorder. Prove that $R^+ = R$ (where R^+ represents the transitive closure of R).
- c) Suppose $R \subseteq A \times A$ is a preorder. Prove that $R \cap R^{-1}$ is an equivalence relation on A .
- d) Suppose $R \subseteq A \times A$ is a preorder. Let

$$B = \{[a]_{R \cap R^{-1}} \mid a \in A\}$$

and

$$S = \{([a]_{R \cap R^{-1}}, [b]_{R \cap R^{-1}}) \mid aRb\} \subseteq B \times B.$$

Prove that S is a partial order on B .