Optional Homework

Proof:
(1) By the definition of independent set of vertices, $\emptyset \in I$.

(2) If $A \subset B, B \in I, B = \{b_1, b_2, ..., b_{|B|}\}$, then there are $|B|$ pairwise-disjoint paths $p_1, p_2, ..., p_{|B|}$ such that each path p_i connects v^* and b_i. Since $A \subset B$, let $A = \{b_{i_1}, b_{i_2}, ..., b_{i_{|A|}}\}$. $p_{i_1}, p_{i_2}, ..., p_{i_{|A|}}$ are $|A|$ pairwise-disjoint paths such that each p_{i_t} connects v^* and b_{i_t}. Therefore, $A \in I$.

(3) It's obvious that if the property is proved in the condition that $|B| = |A| + 1$, then the property is true for any $|B| > |A|$. Assume $|B| = |A| + 1$. Write A, B as $A = \{a_1, a_2, ..., a_{|A|}\}, B = \{b_1, b_2, ..., b_{|B|}\}$. Write the pair-disjoint paths as $p_1, p_2, ..., p_{|A|}$ and $q_1, q_2, ..., q_{|B|}$, respectively. If there is an integer j such that there is no intersection point of p_i and q_j for any i, then we have found an x satisfies $A \cup x \in I$. If there is at least one intersection point of p_i and q_j for any j, note the last intersection point of p_i and q_j as w_{ij}. Here, the last intersection point of p_i and q_j means it has the longest length from v^* among all intersection points of p_i and q_j. Then for every p_i, note the point w_{ij} which has the longest length from v^* among others as x_i (if there isn't an w_{ij} on p_i, then note a_i as x_i.)

a) If there aren't two point x_{i_1} and x_{i_2} satisfy they are on the same path q_j, i.e. for any i, there exists k, such that $x_i = w_{ik}$. Then we can construct new paths: for any i, $p_i^{'}$: v^* to w_{ik} through q_k, w_{ik} to a_i through p_i. It's obvious these new paths are $|A|$ pairwise-disjoint paths in this case.

b) If there are at least two point $x_{i_1}, x_{i_2}, ...$ satisfy they are on the same path q_j, then $x_{i_1} = w_{i_1 j}, x_{i_2} = w_{i_2 j}$,... Write the length from v^* to $w_{i_m j}$ as $d(w_{i_m j})$. Assume $d(w_{i_1 j}) < d(w_{i_2 j}) < ...$ We should assign a new $x_{i_t}^{'}$ such that for any $t > 2$, which has the second longest length from v^* among others. If it causes new condition that there are at least two point $x_{i_1}, x_{i_2}, ...$ satisfy they are on the same path q_j, then we should assign a new $x_{i_t}^{'}$ such that for any $t > 2$, which has the third longest length from v^* among others. If it causes new condition that there are at least two point $x_{i_1}, x_{i_2}, ...$ satisfy they are on the same path q_j over and over again, then change first/second to third/fourth/etc. If there isn't any w_{ij} nearer v^* than x_{i_t}, then new $x_{i_t}^{'}$ cannot be found in this way, in this case just let the new path $p_i^{'} = p_i$. After doing t times of new assingment can we find out all of these new $x_{i_t}^{'}$ (if we can) and then we can construct $|A|$ new paths such that they are pairwise-disjoint like the method of a). t is finite because the graph is determined and the adjustments adjust only close to v^*.

Then by Pigeonhole Principle in combinatorics, for the set of $x_i^{'}$ only has n elements, there exists $b_k \in B$ such that the intersection points of q_k and $p_i(\forall i)$ w_{ik} has the shorter length than $x_i^{'}$. It's easy to see that $p_i^{'}$ does not intersect q_k. So $A \cup \{b_k\} \in I$.