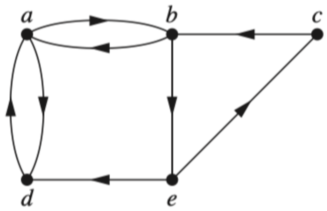
Exercise Sheet 14

Discrete Mathematics, 2021.11.23

1. ([R], Page 666, Exercise 31) Suppose there is an integer k such that every man on a desert island is willing to marry exactly k of the women on the island and every woman on the island is willing to marry exactly k of the men. Also, suppose that a man is willing to marry a woman if and only if she is willing to marry him. Show that it is possible to match the men and women on the island so that everyone is matched with someone that they are willing to marry.

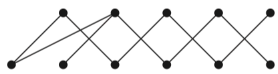
2. ([R], Page 666, Exercise 32) In this exercise we prove a theorem of Öystein Ore. Suppose that $G = (V, E)$ is a bipartite graph with bipartition (V_1, V_2) and that $A \subseteq V_1$. Show that the maximum number of vertices of V_1 that are the endpoints of a matching of G equals $|V_1| - \max_{A \subseteq V_1} \operatorname{def}(A)$, where $\operatorname{def}(A) = |A| - |N(A)|$. (Here, def (A) is called the deficiency of A.) [Hint: Form a larger graph by adding $\max_{A \subseteq V_1} \operatorname{def}(A)$ new vertices to V_2 and connect all of them to the vertices of V_1.]

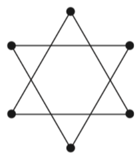
3. ([R], Page 689, Exercise 2(a)(b)(d)) Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths? （只需给出答案无需证明）
   $a) a, b, e, c, b$
   $b) a, d, a, d, a$
   $d) a, b, e, c, b, d, a$



4. ([R], Page 689, 6) How many connected components does each of the graphs in Exercises 3-5 have? For each graph find each of its connected components. （只需给出答案无需证明）

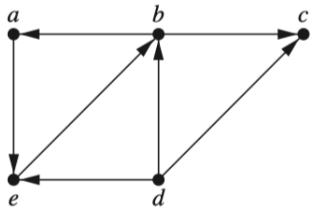
3.


4.


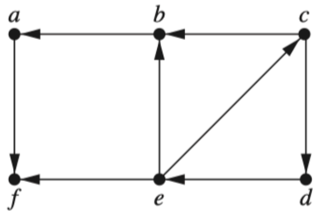
5.


5. ([R], Page 690, 14(a)(b)(c)) Find the strongly connected components of each of these graphs. （只需给出答案无需证明）

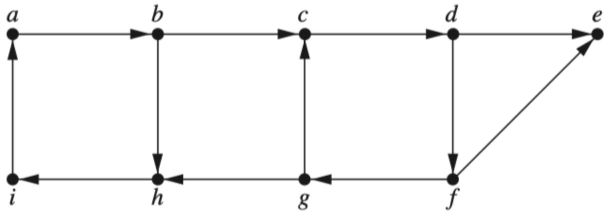
a)



b)



c)



6. ([R], Page 692, Exercise 63) Show that a simple graph G is bipartite if and only if it has no circuits with an odd number of edges.

7. Suppose $G = (V, E)$ is a subgraph of $G' = (V, E \cup \{e_0\})$ ($e_0 \notin E$ is an additional edge) and both G and G' are undirected graphs. Prove: if e_0 connects u and v but u is not connected to v in G, then $[u]_{\mathrm{conn}(G')} = [v]_{\mathrm{conn}(G')} = [u]_{\mathrm{conn}(G)} \cup [v]_{\mathrm{conn}(G)}$.

8. **(Optional Homework, 2 additional points)**

   Matriods. A finite matroid is an ordered pair $(E, \mathcal{I})$, where E is a finite set and $\mathcal{I} \subseteq 2^E$ is a collection of subsets of E such that

   - $\emptyset \in \mathcal{I}$, and
   - for any sets $A \subseteq B \subseteq E$, if $B \in \mathcal{I}$ then $A \in \mathcal{I}$, and
   - for any sets $A, B \in \mathcal{I}$, if $|A| < |B|$ then there exists $x \in B \backslash A$ such that $A \cup \{x\} \in \mathcal{I}$.

   Disjoint Paths. Let $G = (V, E)$ be a(n undirected) simple graph. Given a path $\rho = x_0, e_1, \ldots, x_{n-1}, e_n, x_n$ in G, we denote by $\mathit{Edges}(\rho)$ the set of all edges appearing in ρ, i.e., $\mathit{Edges}(\rho) := \{e_0, \ldots, e_n\}$. We say that two paths ρ, ρ' in G are disjoint if $\mathit{Edges}(\rho) \cap \mathit{Edges}(\rho') = \emptyset$.

   Independent Sets of Vertices. Let $G = (V, E)$ be a(n undirected) connected simple graph such that $|V| \geq 2$. We assume a designated vertex v^* called the source vertex. A set $U \subseteq V \backslash \{v^*\}$ of vertices is called independent if $U = \{u_1, \ldots, u_k\}$ (k can be zero) and there are k pairwise-disjoint paths $\rho_1, \ldots, \rho_k$ such that each path ρ_i connects v^* and u_i (as endpoints).

   Define $\mathcal{I}$ to be the set of all independent sets of vertices. Show that $(V, \mathcal{I})$ is a finite matroid.
   (选做题可以不做)