Exercise Sheet 5

Discrete Mathematics, 2021.9.30

1. (P35, Ex.22-24, [R])
   a) Show that $(p \to q) \land (p \to r)$ and $p \to (q \land r)$ are logically equivalent.
   b) Show that $(p \to r) \land (q \to r)$ and $(p \lor q) \to r$ are logically equivalent.
   c) Show that $(p \to q) \lor (p \to r)$ and $p \to (q \lor r)$ are logically equivalent.

2. (P35, Ex.31, [R]) Show that $(p \to q) \to r$ and $p \to (q \to r)$ are not logically equivalent.

3. (P35, Ex.32, [R]) Show that $(p \land q) \to r$ and $(p \to r) \land (q \to r)$ are not logically equivalent.

4. Consider the first order language with symbol set $S = \{R\}$ in which R represents a binary predicate.

   a) Let $\mathcal{J}_1$ be an S-interpretation such that
      - the domain in $\mathcal{J}_1$ is $\mathbb{N}$,
      - $\mathcal{J}_1(R, a, b) = \mathbf{T}$ if and only if $a < b$.

      Prove that $[\![\ \forall x.\ \exists y.\ R(x, y)\ ]\!]_{\mathcal{J}_1} = \mathbf{T}$

   b) Let $\mathcal{J}_2$ be an S-interpretation such that
      - the domain in $\mathcal{J}_2$ is $\mathbb{N}$,
      - $\mathcal{J}_2(R, a, b) = \mathbf{T}$ if and only if $a > b$.
      - $\mathcal{J}_2(x) = 0$

      Prove that $[\![\ \exists y.\ R(x, y)\ ]\!]_{\mathcal{J}_2} = \mathbf{F}$

   c) Let $\mathcal{J}_3$ be an S-interpretation such that
      - the domain in $\mathcal{J}_3$ is $\mathbb{N}$,
      - $\mathcal{J}_3(R, a, b) = \mathbf{T}$ if and only if $a > b$.

      Prove that $[\![\ \forall x.\ \exists y.\ R(x, y)\ ]\!]_{\mathcal{J}_3} = \mathbf{F}$

   d) Prove that $[\![\ \forall x.\ \forall y.\ (R(x, y) \to \exists z.\ (R(x, z) \land R(z, y)))\ ]\!]_{\mathcal{J}_1} = \mathbf{F}$.

   e) Let $\mathcal{J}_4$ be an S-interpretation such that
      - the domain in $\mathcal{J}_4$ is $\mathbb{Q}$ (rational numbers, 有理数集),
      - $\mathcal{J}_4(R, a, b) = \mathbf{T}$ if and only if $a < b$.

      Prove that $[\![\ \forall x.\ \forall y.\ (R(x, y) \to \exists z.\ (R(x, z) \land R(z, y)))\ ]\!]_{\mathcal{J}_4} = \mathbf{T}$.

5. Consider the first order language with symbol set $S = \{f, R\}$ in which f represents a binary function and R represents a binary predicate.

   a) Let $\mathcal{J}_1$ be an S-interpretation such that
      - the domain in $\mathcal{J}_1$ is $\mathbb{N}$,
      - $\mathcal{J}_1(f, a, b) = a + b$,
      - $\mathcal{J}_1(R, a, b) = \mathbf{T}$ if and only if $a = b$.

      Prove that $[\![\ \forall x.\ \forall y.\ R(f(x, y), f(y, x))\ ]\!]_{\mathcal{J}_1} = \mathbf{T}$.

   b) Let $\mathcal{J}_2$ be an S-interpretation such that
      - the domain in $\mathcal{J}_2$ is $\mathbb{N}$,
      - $\mathcal{J}_2(f, a, b) = a * b$,

- $\mathcal{J}_2(R, a, b) = \mathbf{T}$ if and only if $a = b$.

Prove that $[\![\; \forall x.\ \forall y.\ R(f(x, y), f(y, x)) \;]\!]_{\mathcal{J}_2} = \mathbf{T}$.

c) Let $\mathcal{J}_3$ be an S-interpretation such that

- the domain in $\mathcal{J}_3$ is $\{\mathbf{T}, \mathbf{F}\}$,
- $\mathcal{J}_3(f, a, b) = [\![\wedge]\!]\,(a, b)$,
- $\mathcal{J}_3(R, a, b) = \mathbf{T}$ if and only if $a = b$.

Prove that $[\![\; \forall x.\ \forall y.\ R(f(x, y), f(y, x)) \;]\!]_{\mathcal{J}_3} = \mathbf{T}$.

d) Prove that $\forall x.\ \forall y.\ R(f(x, y), f(y, x))$ is not valid. (Remark: a proposition ϕ is valid iff. $[\![\phi]\!]_{\mathcal{J}} = \mathbf{T}$ for every possible interpretation $\mathcal{J}$.)