Exercise Sheet 9

Discrete Mathematics, 2021.10.26

1. ([R], Page 581, Exercise 6(c)) Determine whether the relation R on the set of all real numbers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if
   c) $x - y$ is a rational number.

2. ([R], Page 581, Exercise 7(b)(f)) Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if
   b) $xy \geq 1$
   f) x and y are both negative or both nonnegative.

3. ([R], Page 582, Exercise 36) Exercises 36 deal with these relations on the set of real numbers:
   $R_1 = \{(a, b) \in \mathbb{R}^2 | a > b\}$, the "greater than " relation,
   $R_2 = \{(a, b) \in \mathbb{R}^2 | a \geq b\}$, the "greater than or equal to" relation,
   $R_3 = \{(a, b) \in \mathbb{R}^2 | a < b\}$, the "less than" relation,
   $R_4 = \{(a, b) \in \mathbb{R}^2 | a \leq b\}$, the "less than or equal to" relation
   $R_5 = \{(a, b) \in \mathbb{R}^2 | a = b\}$, the "equal to " relation,
   $R_6 = \{(a, b) \in \mathbb{R}^2 | a \neq b\}$, the "unequal to " relation and
   $\mathbb{R} \times \mathbb{R}$, all pairs of real numbers. Find:
   a) $R_1 \circ R_1$
   b) $R_1 \circ R_2$
   c) $R_1 \circ R_3$
   e) $R_1 \circ R_5$
   f) $R_1 \circ R_6$
   g) $R_2 \circ R_3$
   h) $R_3 \circ R_3$
   You should answer these questions using R_1, R_2, ..., R_6 and/or $\mathbb{R} \times \mathbb{R}$ whenever it is possible.

4. Prove that the composition operator $\circ$ is associative over relations, i.e. $(R \circ S) \circ T = R \circ (S \circ T)$.

5. Prove or disprove that, if both R_1 and R_2 are equivalence relations on A, then $R_1 \cup R_2$ is also an equivalence relation on A.

6. Prove or disprove that, if both R_1 and R_2 are equivalence relations on A, then $R_1 \cap R_2$ is also an equivalence relation on A.

7. ([R], Page 617, Exercise 57) Consider the equivalence relation $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x - y \text{ is an integer }\}$.
   a) What is the equivalence class of 1 for this equivalence relation?
   b) What is the equivalence class of 1/2 for this equivalence relation?

8. Consider the equivalence relation $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x - y \text{ is an integer }\}$.

   - a) Prove that
   $$S_2 = \{([a]_R, [b]_R) \mid a, b \in \mathbb{R} \ \wedge \ a - b = 1/2\} \cup \{([a]_R, [a]_R) \mid a \in \mathbb{R}\}$$
   is an equivalence relation on the set of all equivalence classes of R.
   - b) Prove that
   $$S_3 = \{([a]_R, [b]_R) \mid a, b \in \mathbb{R} \ \wedge \ |a - b| = 1/3\} \cup \{([a]_R, [a]_R) \mid a \in \mathbb{R}\}$$
   is an equivalence relation on the set of all equivalence classes of R.

- c) Prove that

$$S_4 = \{([a]_R, [b]_R) \mid a, b \in \mathbb{R} \ \wedge \ |a - b| = 1/4\} \cup \{([a]_R, [a]_R) \mid a \in \mathbb{R}\}$$

  is not an equivalence relation.