Exercise 3

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1.

Proof of a)

Here is the truth table of $p \to (q \to p)$.

p	q	$q \to p$	$p \to (q \to p)$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	T

Thus, $p \to (q \to p)$ is a tautology. *Q.E.D.*

Proof of b)

Here is the truth table of $(p \to q \to r) \to (p \to q) \to (p \to r)$.

p	q	r	$p \to q$	$p \to r$	$q \to r$	$p \to q \to r$	$(p \to q) \to (p \to r)$	$(p \to q \to r) \to (p \to q) \to (p \to r)$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	T	T	T	T
T	F	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

Thus, $(p \to q \to r) \to (p \to q) \to (p \to r)$ is a tautology. *Q.E.D.*

Proof of c)

Here is the truth table of $p \to q \to r$.

p	q	r	$q \to r$	$p \to q \to r$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

p	q	r	$p \wedge q$	$(p \wedge q) \to r$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

Thus, $p \to q \to r$ and $(p \wedge q) \to r$ are logically equivalent. $Q.E.D.$

2.

Proof of a)

If $[[\phi]]_J = T$, then $[[\phi|\phi]]_J = F$, $[[\neg\phi]]_J = F$.

If $[[\phi]]_J = F$, then $[[\phi|\phi]]_J = T$, $[[\neg\phi]]_J = T$.

Thus for any J, $\phi|\phi$ and $\neg\phi$ are logically equivalent. $Q.E.D.$

Proof of b)

From a), we know that $(\phi|\psi)|(\phi|\psi) \equiv \neg(\phi|\psi)$.

Here are the truth tables of $\neg(\phi|\psi)$ and $\phi \wedge \psi$.

ϕ	ψ	$\phi \wedge \psi$
T	T	T
T	F	F
F	T	F
F	F	F

ϕ	ψ	$\phi|\psi$	$\neg(\phi|\psi)$
T	T	F	T
T	F	T	F
F	T	T	F
F	F	T	F

Thus, we know that $(\phi|\psi)|(\phi|\psi)$ and $\phi \wedge \psi$ are logically equivalent. $Q.E.D.$

Proof of c)

We know that every compound proposition is logically equivalent to a compound proposition involving only $\neg$ and $\wedge$.

From a), b), we can find that $(\phi|\psi)|(\phi|\psi)$ and $\phi \wedge \psi$ are logically equivalent, $\phi|\phi$ and $\neg\phi$ are logically equivalent. Thus, every compound proposition is logically equivalent to a compound proposition involving only |, which means | itself forms a functionally complete collection of logical operators. $Q.E.D.$