Problem 1

Assume the first customer arrives after $T-s$ arrives at $T-s+t_1$, the second one arrives at $T-s+t_1+t_2$. Then Joe achieves his goal iff $t_1<s$ and $t_1+t_2>s$.

$$\begin{aligned}
f(s)=\mathbf{Pr}\left[t_1<s\wedge t_1+t_2>s\right] &= \int_0^s \lambda e^{-\lambda t_1}\mathbf{Pr}\left[t_1+t_2>s\right]dt_1\\
&= \int_0^s \lambda e^{-\lambda t_1}\mathbf{Pr}\left[t_2>s-t_1\right]dt_1\\
&= \int_0^s \lambda e^{-\lambda t_1}e^{-\lambda(s-t_1)}dt_1\\
&= \lambda\int_0^s e^{-\lambda s}dt_1\\
&= \lambda s e^{-\lambda s}.
\end{aligned}$$

$$f'(s)=\lambda e^{-\lambda s}(1-\lambda s)=0\Rightarrow s=\frac{1}{\lambda},\text{Max Probability}=f(s)=\frac{1}{e}.$$

Problem 2

1.

$$\begin{aligned}
\mathbf{Pr}\left[X=\lambda+k\right] &= \frac{\lambda^{\lambda+k}}{(\lambda+k)!}e^{-\lambda},\\
\mathbf{Pr}\left[X=\lambda-k-1\right] &= \frac{\lambda^{\lambda-k-1}}{(\lambda-k-1)!}e^{-\lambda}.
\end{aligned}$$

$$\begin{aligned}
\frac{\mathbf{Pr}\left[X=\lambda+k\right]}{\mathbf{Pr}\left[X=\lambda-k-1\right]} &= \frac{(\lambda-k-1)!}{(\lambda+k)!}\lambda^{2k+1}\\
&= \frac{1}{\lambda+k}\frac{1}{\lambda+k-1}\cdots\frac{1}{\lambda-k}\lambda^{2k+1}\\
&= \frac{\lambda^2}{\lambda^2-k^2}\frac{\lambda^2}{\lambda^2-(k-1)^2}\cdots\frac{\lambda^2}{\lambda^2-1^2}\geq 1(=1\text{ only when }k=0).
\end{aligned}$$

$$\begin{aligned}
1=\sum_{x=0}^{\infty}\mathbf{Pr}\left[X=x\right] &= \sum_{x=0}^{\lambda-1}\mathbf{Pr}\left[X=x\right]+\mathbf{Pr}\left[X\geq\lambda\right]\\
&\leq \sum_{x=\lambda}^{2\lambda-1}\mathbf{Pr}\left[X=x\right]+\mathbf{Pr}\left[X\geq\lambda\right]\\
&\leq \sum_{x=\lambda}^{\infty}\mathbf{Pr}\left[X=x\right]+\mathbf{Pr}\left[X\geq\lambda\right]=2\mathbf{Pr}\left[X\geq\lambda\right].
\end{aligned}$$

2. Note that $\sum_{i=1}^n Y_i\sim\texttt{Pois}(m)$, so by 1., we have $\mathbf{Pr}\left[\sum_{i=1}^n Y_i\geq m\right]\geq\frac{1}{2}$.

$$
\begin{aligned}
\mathbf{E}\left[f(Y_1,\ldots,Y_n)\right] &= \sum_{k=0}^{\infty}\mathbf{E}\left[f(Y_1,\ldots,Y_n)|\sum_{i=1}^{n}Y_i=k\right]\mathbf{Pr}\left[\sum_{i=1}^{n}Y_i=k\right] \\
&\geq \sum_{k=m}^{\infty}\mathbf{E}\left[f(X_1,\ldots,X_n)|\sum_{i=1}^{n}X_i=k\right]\mathbf{Pr}\left[\sum_{i=1}^{n}Y_i=k\right] \\
&\geq \sum_{k=m}^{\infty}\mathbf{E}\left[f(X_1,\ldots,X_n)|\sum_{i=1}^{n}X_i=m\right]\mathbf{Pr}\left[\sum_{i=1}^{n}Y_i=k\right] \\
&= \mathbf{E}\left[f(X_1,\ldots,X_n)\right]\sum_{k=m}^{\infty}\mathbf{Pr}\left[\sum_{i=1}^{n}Y_i=k\right] \\
&= \mathbf{E}\left[f(X_1,\ldots,X_n)\right]\mathbf{Pr}\left[\sum_{i=1}^{n}Y_i\geq m\right] \\
&\geq \frac{1}{2}\mathbf{E}\left[f(X_1,\ldots,X_n)\right].
\end{aligned}
$$

3. Assume X_i be the variables counting the number of students whose birthday is on the i's day of a year ($i=1,2,\ldots,365$). Then we have $\sum_{i=1}^{365}X_i=105$, and for each X_i, $X_i\sim\texttt{Binom}(105,\frac{1}{365})$. Define $f(X_1,X_2,\ldots,X_{365})\triangleq\mathbb{1}[\exists X_i,X_i\geq 5]$, and then $\mathbf{E}\left[f(X_1,X_2,\ldots,X_{365})\right]=\mathbf{Pr}\left[\exists X_i,X_i\geq 5\right]$. Define $Y_i\sim\texttt{Pois}(\frac{105}{365})$ with the condition that $\sum_{i=1}^{365}Y_i=105$. Similarly, let $f(Y_1,Y_2,\ldots,Y_{365})\triangleq\mathbb{1}[\exists Y_i,Y_i\geq 5]$, and then $\mathbf{E}\left[f(Y_1,Y_2,\ldots,Y_{365})\right]=\mathbf{Pr}\left[\exists Y_i,Y_i\geq 5\right]$. We have

$$
\begin{aligned}
\mathbf{Pr}\left[\exists Y_i,Y_i\geq 5\right] &= 1-\mathbf{Pr}\left[\forall Y_i,Y_i\leq 4\right] \\
&= 1-\prod_{i=1}^{365}\mathbf{Pr}\left[Y_i\leq 4\right] \\
&= 1-(e^{-\frac{105}{365}}(\frac{(\frac{105}{365})^0}{0!}+\frac{\frac{105}{365}}{1!}+\frac{(\frac{105}{365})^2}{2!}+\frac{(\frac{105}{365})^3}{3!}+\frac{(\frac{105}{365})^4}{4!}))^{365} \\
&\approx 0.4707853\% < 0.5\%.
\end{aligned}
$$

It is trivial that $\mathbf{E}\left[f(X_1,\ldots,X_{365})\right]$ is monotonically increasing in m. Thus, by the conclusion in 2., $\mathbf{E}\left[f(X_1,\ldots,X_{365})\right]\leq 2\mathbf{E}\left[f(Y_1,\ldots,Y_{365})\right]<1\%$.

Problem 3

(a) We have:

1. If $t=0$, $c^{-\frac{1}{2}}W(ct)=0$.
2. It holds the property called **independent increments** obviously.
3. **Stationary increments**:

$$\forall s,t>0, c^{-\frac{1}{2}}W(c(s+t))-c^{-\frac{1}{2}}W(cs)=c^{-\frac{1}{2}}(W(c(s+t))-W(cs))\sim\mathcal{N}(0,(c^{-\frac{1}{2}})^2ct).$$

4. $c^{-\frac{1}{2}}W(ct)$ is continuous almost surely.

Therefore, $c^{-\frac{1}{2}}W(ct)$ is also a standard Brownian motion.

(b) We have:

1. If $t=0$, $X(t)=W(c+t)-W(c)=0$.
2. **Independent increments**: $\forall 0\leq t_1\leq t_2\leq t_3\leq t_4$, $X(t_1)-X(t_0)=W(c+t_1)-W(c+t_0)$, $X(t_3)-X(t_2)=W(c+t_3)-W(c+t_2)$, $cov(X(t_1)-X(t_0),X(t_3)-X(t_2))=cov(W(c+t_1)-W(c+t_0),W(c+t_3)-W(c+t_2))$, then it's clear that $cov(W(c+t_1)-W(c+t_0),W(c+t_3)-W(c+t_2))=0$ if taking the

covariance apart.

3. **Stationary increments**:

$$\forall s,t>0, X(s+t)-X(s)=W(c+s+t)-W(c+s)\sim\mathcal{N}(0,t).$$

4. $X(t)$ is continuous almost surely.

Therefore, $X(t)$ is also a standard Brownian motion.

Because $\forall s\geq 0, 0\leq t\leq c$,

$$\begin{aligned} cov(X(s),W(t)) &= cov(W(c+s)-W(c+t),W(t)) \\ &= cov(W(c+s),W(t))+cov(W(c+t),W(t))=0, \end{aligned}$$

$\{X(t): t\geq 0\}$ is independent of $\{W(t): 0\leq t\leq c\}$.

(c) Let $c=\frac{1}{2}$, we have $\mathbf{Pr}\left[W(1)>0\mid W(1/2)>0\right]=\mathbf{Pr}\left[W(1)<2W(1/2)\mid W(1/2)>0\right]$

$$\begin{aligned} 2\mathbf{Pr}\left[W(1)>0\mid W(1/2)>0\right] &= \mathbf{Pr}\left[W(1)>0\mid W(1/2)>0\right]+\mathbf{Pr}\left[W(1)<2W(1/2)\mid W(1/2)>0\right] \\ &= 1+\mathbf{Pr}\left[0<W(1)<2W(1/2)\mid W(1/2)>0\right] \\ &= 1+\mathbf{Pr}\left[0<W(1)<2W(1/2)\right] \\ &= 1+\mathbf{Pr}\left[0<W(1/2)+X(1/2)<2W(1/2)\right] \\ &= 1+\mathbf{Pr}\left[-W(1/2)<X(1/2)<W(1/2)\right] \\ &= 1+\mathbf{Pr}\left[-\sqrt{2}W(1/2)<\sqrt{2}X(1/2)<\sqrt{2}W(1/2)\right] \\ &= 1+\Phi(\sqrt{2}W(1/2))-\Phi(-\sqrt{2}W(1/2))\ (\text{by (b)}) \\ &= 1+\Phi(\sqrt{2}W(1/2))-(1-\Phi(\sqrt{2}W(1/2))) \\ &= 2\Phi(\sqrt{2}W(1/2)) \end{aligned}$$

Thus, $\mathbf{Pr}\left[W(1)>0\mid W(1/2)>0\right]=\Phi(\sqrt{2}W(1/2))$.

Problem 4

(a) $W(t)\sim\mathcal{N}(0,t)$, $\frac{W(t)}{\sqrt{t}}\sim\mathcal{N}(0,1)$, and $\xi\sim\mathcal{N}(0,1)$.

$$\begin{aligned} \mathbf{Pr}\left[X(t)\leq\delta\right] &= \mathbf{Pr}\left[\mu t+\sigma W(t)\leq\delta\right] \\ &= \mathbf{Pr}\left[\frac{W(t)}{\sqrt{t}}\leq\frac{\delta-\mu\cdot t}{\sigma\sqrt{t}}\right] \\ &= \mathbf{Pr}\left[\xi\leq\frac{\delta-\mu\cdot t}{\sigma\sqrt{t}}\right]. \end{aligned}$$

(b)

$$\begin{aligned} \mathbf{E}\left[T\right] &= \mathbf{E}\left[\int_0^\infty \mathbb{1}[X(t)\in[0,\delta]]\mathrm{d}t\right] \\ &= \int_0^\infty \mathbf{Pr}\left[X(t)\in[0,\delta]\right]\mathrm{d}t \\ &= \int_0^\infty \mathbf{Pr}\left[-\frac{\mu\sqrt{t}}{\sigma}\leq\frac{W(t)}{\sqrt{t}}\leq\frac{-\mu t+\delta}{\sigma\sqrt{t}}\right]\mathrm{d}t \\ &= \int_0^\infty \mathbf{Pr}\left[-\frac{\mu\sqrt{t}}{\sigma}\leq\xi\leq\frac{-\mu t+\delta}{\sigma\sqrt{t}}\right]\mathrm{d}t \\ &= \int_0^\infty \mathbf{Pr}\left[\frac{\mu t-\delta}{\sigma\sqrt{t}}\leq\xi\leq\frac{\mu\sqrt{t}}{\sigma}\right]\mathrm{d}t. \end{aligned}$$

(c) Solve the equation $\frac{\mu t-\delta}{\sigma\sqrt{t}} = \xi$. We can get $f(\delta,\xi) = t = \frac{\xi^2\sigma^2+2\mu\delta+\xi\sigma\sqrt{\xi^2\sigma^2+4\mu\delta}}{2\mu^2}$.

(d)

$$\mathbf{E}\left[f(\delta,\xi)\right] = \frac{\mathbf{E}\left[\xi^2\sigma^2 + 2\mu\delta + \xi\sigma\sqrt{\xi^2\sigma^2 + 4\mu\delta}\right]}{2\mu^2} = \frac{\sigma^2\mathbf{E}\left[\xi^2\right] + 2\mu\delta}{2\mu^2} = \frac{\sigma^2\mathbf{Var}\left[\xi\right] + 2\mu\delta}{2\mu^2} = \frac{\sigma^2 + 2\mu\delta}{2\mu^2}.$$

(e)

$$\begin{aligned}
\mathbf{E}\left[T\right] &= \int_0^\infty \mathbf{Pr}\left[\frac{\mu t - \delta}{\sigma\sqrt{t}} \le \xi \le \frac{\mu\sqrt{t}}{\sigma}\right]\mathrm{d}t \\
&= \int_0^\infty \mathbf{Pr}\left[f(0,\xi) \le t \le f(\delta,\xi)\right]\mathrm{d}t \\
&= \mathbf{E}\left[f(\delta,\xi) - f(0,\xi)\right] \\
&= \mathbf{E}\left[f(\delta,\xi)\right] - \mathbf{E}\left[f(0,\xi)\right] \\
&= \frac{\delta}{\mu}.
\end{aligned}$$