[Solution of Homework 1]

Probability Space of Tossing Coins

Let us construct the probability space of tossing an infinite sequence of independent fair coins. Let $\Omega = \{0,1\}^*$. We can write each $\omega \in \Omega$ as an infinite sequence $\omega = (\omega_1, \omega_2, \ldots)$ where $\omega_i \in \{0,1\}$.

1. Let $n \in \mathbb{N}$. For every $s = (s_1, \ldots, s_n) \in \{0,1\}^n$, let

$$C_s = \{\omega \in \Omega \mid \omega_1 = s_1, \ldots, \omega_n = s_n\}.$$

Prove that for every $n \in \mathbb{N}$, the collection $\{C_s\}_{s \in \{0,1\}^n}$ forms a partition of Ω.
Proof.
For any $\omega \in \Omega$, there is exactly one $s = (\omega_1, \omega_2, \ldots, \omega_n) \in \{0,1\}^n$ such that $\omega \in C_s$. Therefore, $\cup_{s \in \{0,1\}^n} C_s = \Omega$ and $C_{s_1} \cap C_{s_2} = \varnothing$ for any $s_1 \neq s_2 \in \{0,1\}^n$, which is to say $\{C_s\}_{s \in \{0,1\}^n}$ forms a partition of Ω.

1. Let $\mathcal{F}_n$ be the σ-algebra generated by $\{C_s\}_{s \in \{0,1\}^n}$ (that is, the minimal σ-algebra containing sets in $\{C_s\}_{s \in \{0,1\}^n}$). Note that $\mathcal{F}_n$ is called the σ-algebra of *tossing n coins*. Prove that there exists a bijection between $\mathcal{F}_n$ and $2^{\{0,1\}^n}$.
   Proof.
   We construct a map $f : \mathcal{F}_n \to 2^{\{0,1\}^n}$ for any $A \in \mathcal{F}_n$:
   $f(A) = \cup_{\omega \in A} \{(\omega_1, \omega_2, \ldots, \omega_n)\}$.
   For any $S = \{s^1, \ldots, s^k\} \in 2^{\{0,1\}^n}$, $f(\cup_{i=1}^k C_{s^i}) = S$. So f is surjective.
   Since both $\mathcal{F}_n$ and $2^{\{0,1\}^n}$ are of size 2^{2^n}, we can infer that f is a bijection between $\mathcal{F}_n$ and $2^{\{0,1\}^n}$.

2. Prove that $\mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \ldots$ is increasing. The collection $\{\mathcal{F}_n\}_{n \geq 1}$ is called a *filtration*.
   Proof.
   We will prove $\mathcal{F}_n \subsetneq \mathcal{F}_{n+1}$ for every n.
   Let $f : \mathcal{F}_n \to 2^{\{0,1\}^n}$ be the bijection defined in Problem 2. For any $A \in \mathcal{F}_n$, we write it as $\cup_{s \in f(A)} C_s$. For any $s = (s_1, s_2, \ldots, s_n) \in \{0,1\}^n$, we write $C_s = C_{(s_1, s_2, \ldots, s_n, 0)} \cup C_{(s_1, s_2, \ldots, s_n, 1)}$. Therefore, for any $A \in \mathcal{F}_n$, $A = \cup_{s \in f(A)} (C_{(s_1, s_2, \ldots, s_n, 0)} \cup C_{(s_1, s_2, \ldots, s_n, 1)}) \in \mathcal{F}_{n+1}$, which implies $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. It is obvious that $\mathcal{F}_n \neq \mathcal{F}_{n+1}$ (For example, $C_{(s_1, s_2, \ldots, s_{n+1})} \in \mathcal{F}_{n+1}$ but not in $\mathcal{F}_n$), so $\mathcal{F}_n \subsetneq \mathcal{F}_{n+1}$.

3. Let $\mathcal{F}_\infty = \bigcup_{n \geq 1} \mathcal{F}_n$ [1]. Prove that $\mathcal{F}_\infty$ is an algebra [2] (not necessarily a σ-algebra) and $\mathcal{F}_\infty \neq 2^\Omega$.
   Proof.
   For any $A \in \mathcal{F}_\infty$, there exists i such that $A \in \mathcal{F}_i$, so $A^c \in \mathcal{F}_i \subset \mathcal{F}_\infty$. For any $A, B \in \mathcal{F}_\infty$, there exist i, j such that $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$, so $A \cup B \in \mathcal{F}_{\max\{i,j\}} \subset \mathcal{F}_\infty$. Therefore, $\mathcal{F}_\infty$ is an algebra.
   2^Ω is not countable. $\mathcal{F}_n$ is countable for any n, so $\mathcal{F}_\infty$ is also countable. Therefore, $\mathcal{F}_\infty \neq 2^\Omega$. (In the following problem, we will show that any $\omega \in \Omega$, $\{\omega\} \in 2^\Omega \setminus \mathcal{F}_\infty$.)
   Anti-cheating problem: what's the difference between algebra and σ-algebra ?

4. Let $\mathcal{B}(\Omega) \triangleq \sigma(\mathcal{F}_\infty)$ be the minimal σ-algebra containing $\mathcal{F}_\infty$. Prove that for any $\omega \in \Omega$, it holds that $\{\omega\} \in \mathcal{B}(\Omega) \setminus \mathcal{F}_\infty$.

   Proof.

   For any $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$, $\{\omega\} \in 2^\Omega$. However, there doesn't exist i such that $\{\omega\} \in \mathcal{F}_i$, hence $\{\omega\} \notin \mathcal{F}_\infty$.
   For any $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$ and n, $\omega \in C_{(\omega_1, \omega_2, \ldots, \omega_n)}$. Therefore, $\{\omega\} = \cap_{n \geq 1} C_{(\omega_1, \omega_2, \ldots, \omega_n)} \in \mathcal{B}(\Omega)$.

   (Notes that we use the union operation to define the σ-algebra, but for any $A_i \in \mathcal{F}, i \in \mathbb{N}$, we have $\overline{\cup_i \bar{A}_i} = \cap_i A_i \in \mathcal{F}$ because $\bar{A}_i \in \mathcal{F}$.)

5. Prove that for every $A \in \mathcal{F}_\infty$, there exist some $n \in \mathbb{N}$ and $s_1, \ldots, s_k \in \{0,1\}^n$ such that $A = C_{s_1} \cup \cdots \cup C_{s_k}$. Although the choice of n might not be unique, prove that the value $\frac{k}{2^n}$ only depends on A.

   Proof.

   There exists n such that $A \in \mathcal{F}_n$. Let $f_n : \mathcal{F}_n \to 2^{\{0,1\}^n}$ be the bijection defined in Problem 2. Let $f_n(A) = \left\{s^1, s^2, \ldots, s^{k_n}\right\}$. Then $A = C_{s^1} \cup \cdots \cup C_{s^{k_n}}$.
   Let n be the minimum index such that $A \in \mathcal{F}_n$. We can also find a set $S' = \{(s_1, s_2, \cdots, s_n, 0), (s_1, s_2, \cdots, s_n, 1) | s = (s_1, s_2, \ldots, s_n) \in f_n(A)\} \in 2^{\{0,1\}^{n+1}}$ such that $A = \cup_{s' \in S'} C_{s'}$ and $k_{n+1} := |S'| = 2k_n$. Therefore, $\frac{k_{n+1}}{2^{n+1}} = \frac{2k_n}{2^{n+1}} = \frac{k_n}{2^n}$. Applying this procedure inductively, we obtain that for any $i > n$, the value $\frac{k_i}{2^i} = \frac{k_n}{2^n}$, which is to say that $\frac{k}{2^n}$ only depends on A.

6. Prove that there exists a unique probability measure $P : \mathcal{B}(\Omega) \to [0,1]$ satisfying for every $A \in \mathcal{F}_\infty$, $P(A) = \frac{k}{2^n}$ where k and n are defined in the last question. (You can use the Carathéodory's extension theorem)

   Proof.

   We define a measure μ on F_∞ that $\mu(A) = \frac{k}{2^n}$ where k and n are defined in the last question:

   a. $\mu(C_s) = \frac{1}{2^n}$ for $s \in \{0,1\}^n$.
   b. $\mu(A) = \sum_{i=1}^k P(C_{s^k}) = \frac{k}{2^n}$ for $A = \cup_{i=1}^k C_{s^k} \in \mathcal{F}_n$.
      For any disjoint sets $A_1, A_2, \cdots \in \mathcal{F}_\infty$ such that $\cup_{n \geq 1} A_n \in \mathcal{F}_\infty$, assuming $A_i = \cup_{s \in S_i} C_s$, we obtain that $A = \cup_{s \in S_i, i \in \mathbb{N}^+} C_s$.
      Therefore,
      $\mu(\cup_{n \geq 1} A_n) = \sum_{n \geq 1} \mu(A_n)$,
      and it is obvious that $\mu(\Omega) = \sum_{s \in 2^{\{0,1\}^n}} \mu(C_s) = 1$.
      And then we extend the measure μ on $\mathcal{F}_\infty$ to a measure on $\mathcal{B}(\Omega)$ by *Carathéodory Extension Theorem*. There exists a unique measure $P : \mathcal{B}(\Omega) \to [0,1]$ such that $P(A) = \mu(A)$ for any $A \in \mathcal{F}_\infty$. Since $P(\Omega) = \mu(\Omega) = 1$, P is a probability measure.

Then $(\Omega, \mathcal{B}(\Omega), P)$ is our probability space for tossing coins, and it is isomorphic to the Lebesgue measure on $[0,1]$.

8. Formalize $X \sim \texttt{Geom}(1/2)$ in this probability space.

   Solution.

   For any $\omega \in \Omega$, $X(\omega) := \min\{i \in \mathbb{N} | \omega_i = 1\}$.

Conditional Expectation

(In this problem, all random variables take *discrete* value)

1. Let X be a random variable and $f:\mathbb{R}\to\mathbb{R}$ be a measurable function (that is, for every borel set $A\in\mathscr{R}$, $f^{-1}(A)\in\mathscr{R}$). We usually use $f(X)$ to denote the random variable: $\omega\in\Omega\mapsto f(X(\omega))\in\mathbb{R}$. Prove that $f(X)$ is $\sigma(X)$-measurable.
   Proof.
   For any Borel set $B\subseteq\mathbb{R}$, $(f\circ X)^{-1}(B)=X^{-1}(f^{-1}(B))\in\sigma(X)$.
2. Let Y,Y' be two random variables such that $\sigma(Y)=\sigma(Y')$. Prove that $\mathbf{E}[X\mid Y]=\mathbf{E}[X\mid Y']$.
   Proof.
   It suffices to show that $Y^{-1}(Y(\omega))=Y'^{-1}(Y'(\omega))$. Since $\sigma(Y)=\sigma(Y')$, if there exists $\omega\in\Omega$ such that $Y^{-1}(Y(\omega))\neq Y'^{-1}(Y'(\omega))$, $Y^{-1}(Y(\omega))\cap Y'^{-1}(Y'(\omega))\subseteq\sigma(Y)$, contradicting to the definition of $Y^{-1}(Y(\omega))$ and $Y'^{-1}(Y'(\omega))$. Therefore, $\mathbf{E}\left[X|Y^{-1}(Y(\omega))\right]=\mathbf{E}\left[X|Y'^{-1}(Y'(\omega))\right]$

3. (The coarser always wins) Let $\mathcal{F}_1,\mathcal{F}_2$ be two σ-algebra such that $\mathcal{F}_1\subseteq\mathcal{F}_2$ and $X:\Omega\to\mathbb{R}$ be a random variable. Prove that $\mathbf{E}[\mathbf{E}[X\mid\mathcal{F}_1]\mid\mathcal{F}_2]=\mathbf{E}[\mathbf{E}[X\mid\mathcal{F}_2]\mid\mathcal{F}_1]=\mathbf{E}[X\mid\mathcal{F}_1]$.
   Proof.
   Let $P:\mathcal{B}(\Omega)\to[0,1]$ be a probability measure.Since for any $A\in\mathcal{F}_1\subset\mathcal{F}_2$,
   $\sum_{\omega\in A}\mathbf{E}[X|\mathcal{F}_1(\omega)]P(\omega)=\sum_{\omega\in A}X(\omega)P(\omega)=\sum_{\omega\in A}\mathbf{E}[X|\mathcal{F}_2(\omega)]P(\omega)$,

thus

$$\sum_{\omega\in A}\mathbf{E}[\mathbf{E}[X\mid\mathcal{F}_2]\mid\mathcal{F}_1(\omega)]P(\omega)=\sum_{\omega\in A}\mathbf{E}[X\mid\mathcal{F}_2(\omega)]P(\omega)=\sum_{\omega\in A}\mathbf{E}[X|\mathcal{F}_1(\omega)]P(\omega)=\sum_{\omega\in A}\mathbf{E}[\mathbf{E}[X\mid\mathcal{F}_1]\mid\mathcal{F}_1(\omega)]P(\omega).$$

According to the definition of conditional expectation, we know if X is $\mathcal{F}$-measurable, then $\mathbf{E}[X|\mathcal{F}]=X$.
Therefore, $\mathbf{E}[\mathbf{E}[X\mid\mathcal{F}_2]\mid\mathcal{F}_1]=\mathbf{E}[\mathbf{E}[X\mid\mathcal{F}_1]\mid\mathcal{F}_1]=\mathbf{E}[X\mid\mathcal{F}_1]$.

$\mathbf{E}[X\mid\mathcal{F}_1]$ is $\mathcal{F}_1$-measurable, therefore $\mathcal{F}_2$-measurble, so $\mathbf{E}[\mathbf{E}[X\mid\mathcal{F}_1]\mid\mathcal{F}_2]=\mathbf{E}[X\mid\mathcal{F}_1]$.

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1. $x\in\bigcup_{n\geq1}\mathcal{F}_n \iff \exists n\geq1: x\in\mathcal{F}_n$. ↩

2. A set $\mathcal{F}$ is an algebra if for every $A,B\in\mathcal{F}$, it holds $A^c\in\mathcal{F}$ and $A\cup B\in\mathcal{F}$. ↩