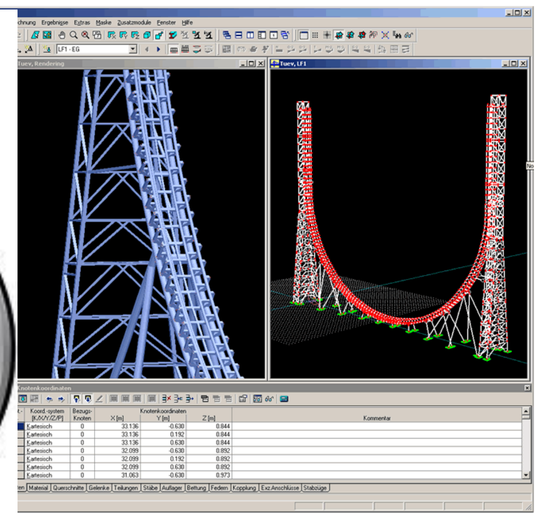
Lecture 3
Complexity Analysis

EECS 281: Data Structures & Algorithms

Complexity Analysis: Overview

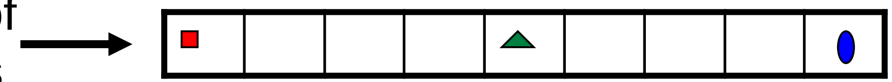
Data Structures & Algorithms

Complexity Analysis

- What is it?
  - Given an algorithm and input size *n*,
    how many steps are needed?
  - Each step should take ***O*(1)** time
  - As input size grows, how does number of steps change?
    - Focus is on TREND
- How do we measure complexity?
  - Express the rate of growth as a function *f*(*n*)
  - Use the big-O notation
- Why is this important?
  - Tells how well an algorithm scales to larger inputs
  - Given two algorithms, we can compare performance before implementation

Metrics of Algorithm Complexity

Array of
n items



Best-case: 1 comparison
Worst-case: *n* comparisons
Average-case: *n*/2 comparisons

Using a linear search over *n* items,
how many comparisons will it take to find item *x*?

- Best-Case
  - Least number of comparisons required, given ideal input
  - Analysis performed over inputs of a given size
  - Example: Data is found in the first place you look
- Worst-Case
  - Most number of comparisons required, given hard input
  - Analysis performed over inputs of a given size
  - Example: Data is found in the last place you could possibly look
- Average-Case
  - Average number of comparisons required, given any input
  - Average performed over all possible inputs of a given size

What Affects Runtime?

- The algorithm
- Implementation details
  - Skills of the programmer
- CPU Speed / Memory Speed
- Compiler (Options used)

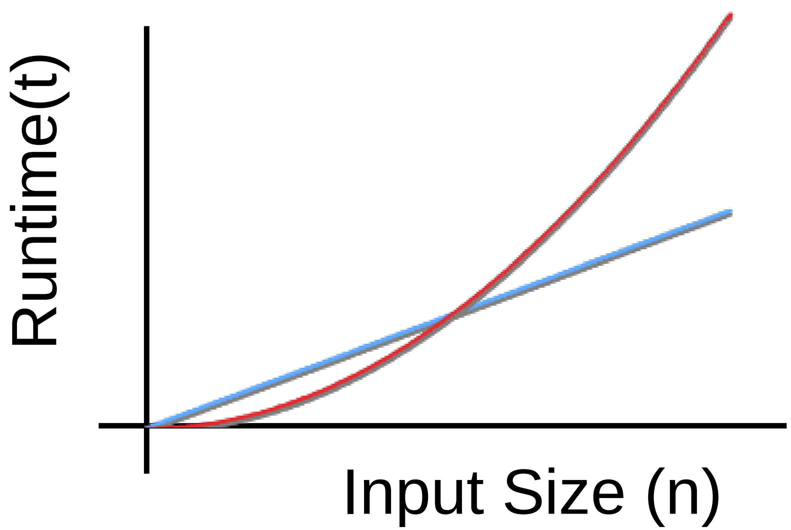
  `g++ -g3` (for debu**g**ging, highest level of information)

  `g++ -O3` (**O**ptimization level **3** for speed)
- Other programs running in parallel

- Amount of data processed (Input size)

Input Size versus Runtime

- Rate of growth independent of most factors
  - CPU speed, compiler, etc.
- Does doubling input size mean doubling runtime?
- Will a “fast” algorithm still be “fast” on large inputs?



How do we measure input size?

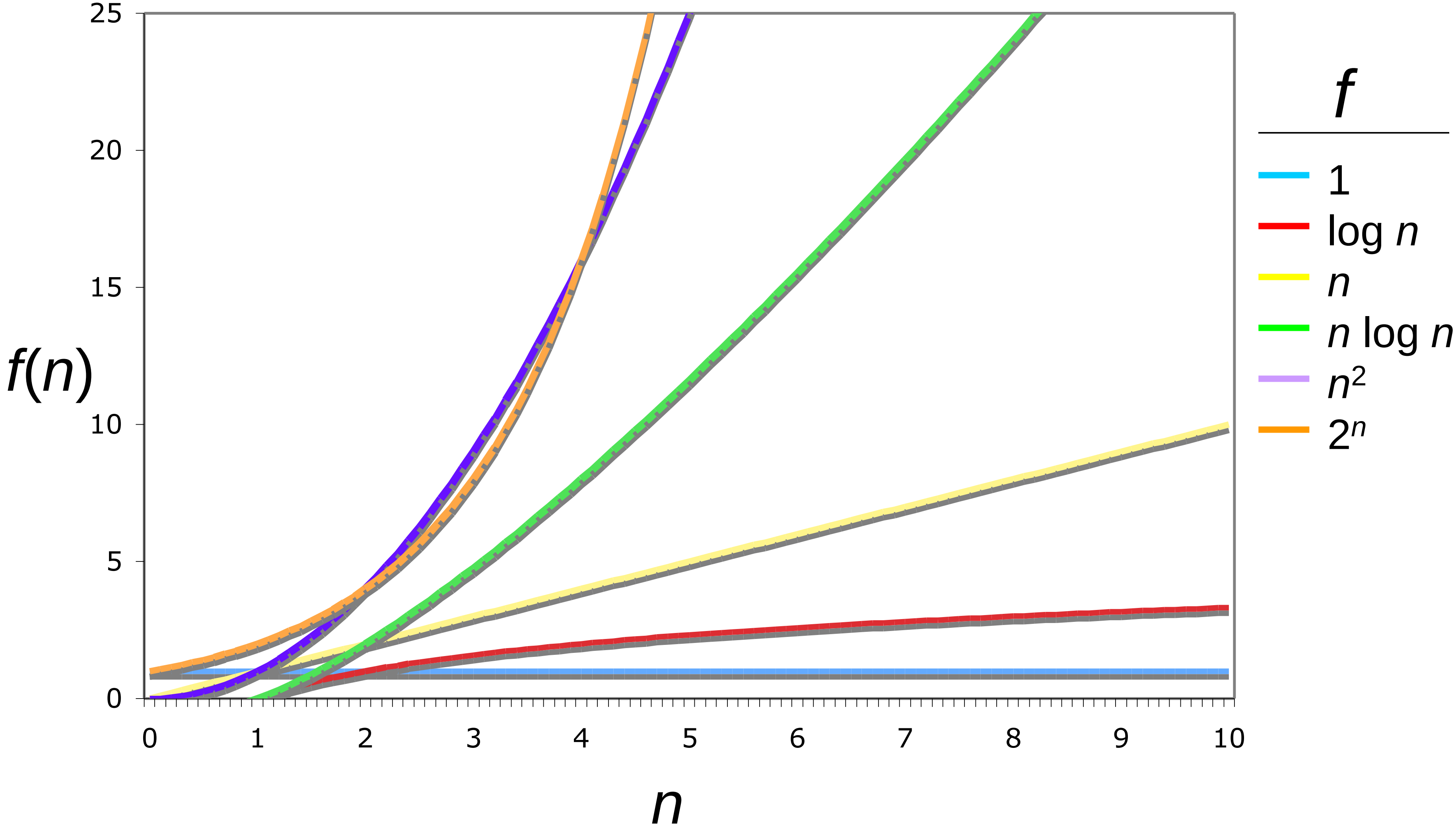
Measuring & Using Input Size

- Number of bits
  - In an `int`, a `double`? (32? 64?)
- Number of items: what counts as an item?
  - Array of integers? One integer? One digit? …
  - One string? Several strings? A char?
- Notation and terminology
  - *n* Input size
  - *f(n)* Maximum number of steps taken by an algorithm when input has size *n* ("*f* of *n*")
  - ***O(f(n))*** Complexity class of *f*(*n*) ("Big-O of *f* of *n*")

Common Orders of Functions

Notation	Name
$O(1)$	Constant
$O(\log n)$	Logarithmic
$O(n)$	Linear
$O(n \log n)$	Loglinear, Linearithmic
$O(n^2)$	Quadratic
$O(n^3)$, $O(n^4)$, …	Polynomial
$O(c^n)$	Exponential
$O(n!)$	Factorial
$O(2^{2^n})$	Doubly Exponential

Graphing $f(n)$ Runtimes

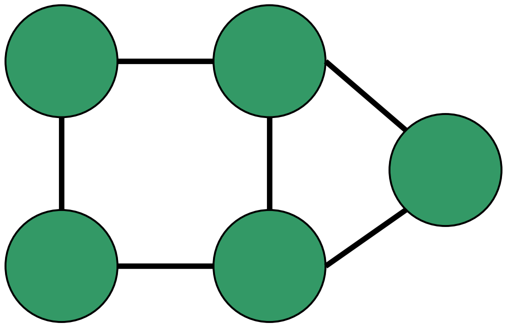


Input Size Example

Graph $G = (V, E)$:

V = 5 Vertices

E = 6 Edges



What should we measure?

- Vertices?
- Edges?
- Vertices and Edges?

When in doubt, measure input size in bits

Use V and E to determine which contributes more to the total number of steps

- Big-O examples: $E \log V$, EV, $V^2 \log E$

From Analysis to Application

- Algorithm comparisons are independent of hardware, compilers and implementation tweaks
- Predict which algorithms will eventually be faster
  - *For large enough inputs*
  - $\boldsymbol{O(n^2)}$ time algorithms will take longer than $\boldsymbol{O(n)}$ algorithms
- Constants can often be ignored because they do not affect asymptotic comparisons

Complexity Analysis: Overview

Data Structures & Algorithms

Complexity Analysis:
Counting Steps

Data Structures & Algorithms

Q: What counts as one step in a program?

A: Primitive operations

- Variable assignment
- Arithmetic operation
- Comparison
- Array indexing or pointer reference
- Function call (not counting the data)
- Function return (not counting the data)

Runtime of 1 step is independent on input

Counting Steps: `for` Loop

- The basic form of a for-loop:

```
for (initialization; test; update)
```

- The initialization is performed once (1)
- The test is performed every time the body of the loop runs, plus once for when the loop ends (n + 1)
- The update is performed every time the body of the loop runs (n)
- Total is **2*n* + 2** steps (used a lot next slide)

Counting Steps: Polynomial

```
1  int func1(int n) {
2    int sum = 0;
3    for (int i = 0; i < n; ++i) {
4        sum += i;
5    }  // for
6    return sum;
7  }  // func1()
```

```
1
2  1 step
3  2n + 2 steps
4    1 step (loop n times)
5
6  1 step
7
```

Total steps: $3n + 4$

```
8  int func2(int n) {
9    int sum = 0;
10   for (int i = 0; i < n; ++i) {
11       for (int j = 0; j < n; ++j)
12           ++sum;
13   }  // for i
14   for (int k = 0; k < n; ++k) {
15       --sum;
16   }  // for
17   return sum;
18 }  // func2()
```

```
8
9  1 step
10 2n + 2 steps
11   2n + 2 steps (inside loop)
12     1 step (inside 2 loops)
13
14 2n + 2 steps
15   1 step (inside loop)
16
17 1 step
18
```

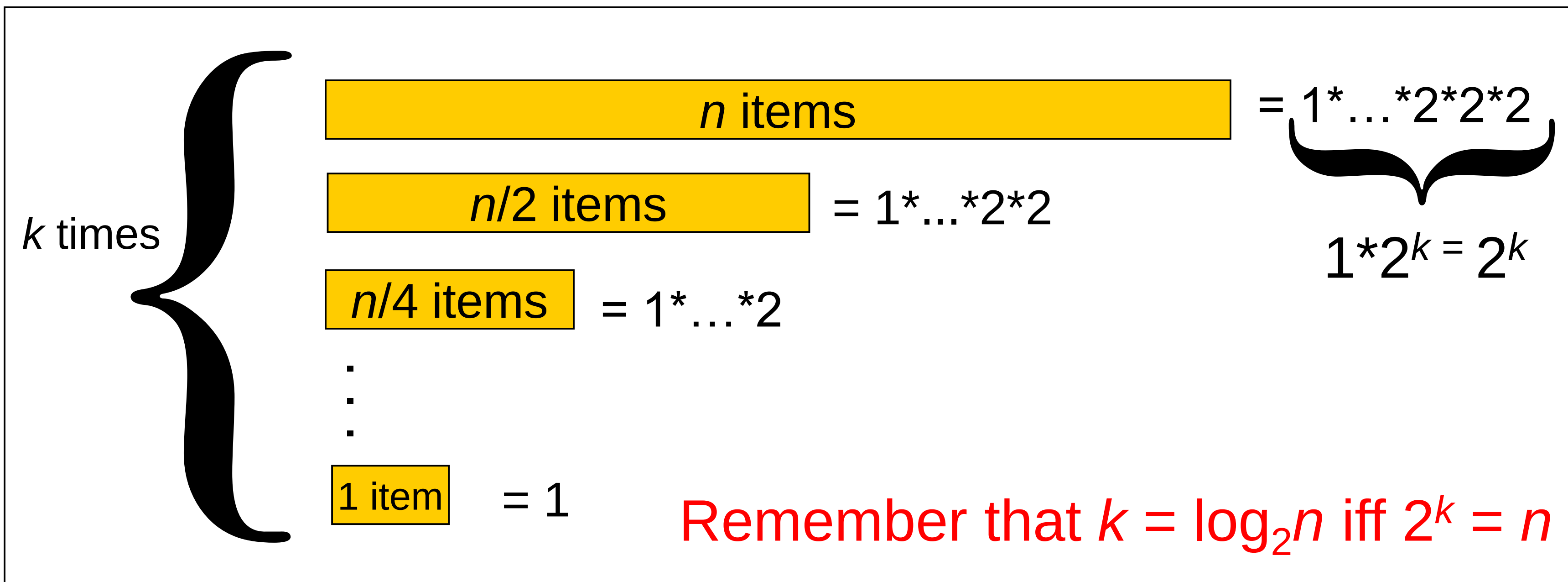
Total steps: $3n^2 + 7n + 6$

Counting Steps: Logarithmic

```
1  int func3(int n) {
2    int sum = 0;
3    for (int i = n; i > 1; i /= 2)
4        sum += i;
5
6    return sum;
7  }  // func3()
```

```
1
2  1 step
3  (2 * ~log n) + 2 steps
4    1 step (loop ~log n times)
5
6  1 step
7
```

Total steps: (3 * ~log *n*) + 4 = *O*(log *n*)



Examples of *O*(log *n*) Time

```
uint32_t logB(uint32_t n) {
  // find binary log, round up
  uint32_t r = 0;
  while (n > 1) {
    n /= 2
    r++;
  }  // while
  return r;
}  // logB()
```

```
int *bsearch(int *lo, int *hi, int val) {
  // find position of val between lo,hi
  while (hi >= lo) {
    int *mid = lo + (hi - lo) / 2;
    if (*mid == val)
      return mid;
    else if (*mid > val)
      hi = mid - 1;
    else
      lo = mid + 1;
  }  // while
  return nullptr;
}  // bsearch()
```

Algorithm Exercise

How many multiplications, if size = ***n*?**

```
// REQUIRES: in and out are arrays with size elements
// MODIFIES: out
// EFFECTS: out[i] = in[0] *...* in[i-1] * in[i+1] *...* in[size-1]
void f(int *out, const int *in, int size) {
  for (int i = 0; i < size; ++i) {
    out[i] = 1;
    for (int j = 0; j < size; ++j) {
      if (i != j)
        out[i] *= in[j];
    }  // for j
  }  // for i
}  // f()
```

in[0]	**in[1]**	**in[2]**
3	1	2

out[0]	**out[1]**	**out[2]**
2	6	3

Algorithm Exercise

How many multiplications and divisions, if size = ***n*?**

```
void f(int *out, const int *in, int size) {
  int product = 1;
  for (int i = 0; i < size; ++i)
    product *= in[i];

  for(int i = 0; i < size; ++i)
    out[i] = product / in[i];
}  // f()
```

Complexity Analysis: Counting Steps

Data Structures & Algorithms

Complexity Analysis:
Big-O Notation

Data Structures & Algorithms

Big-O Definition

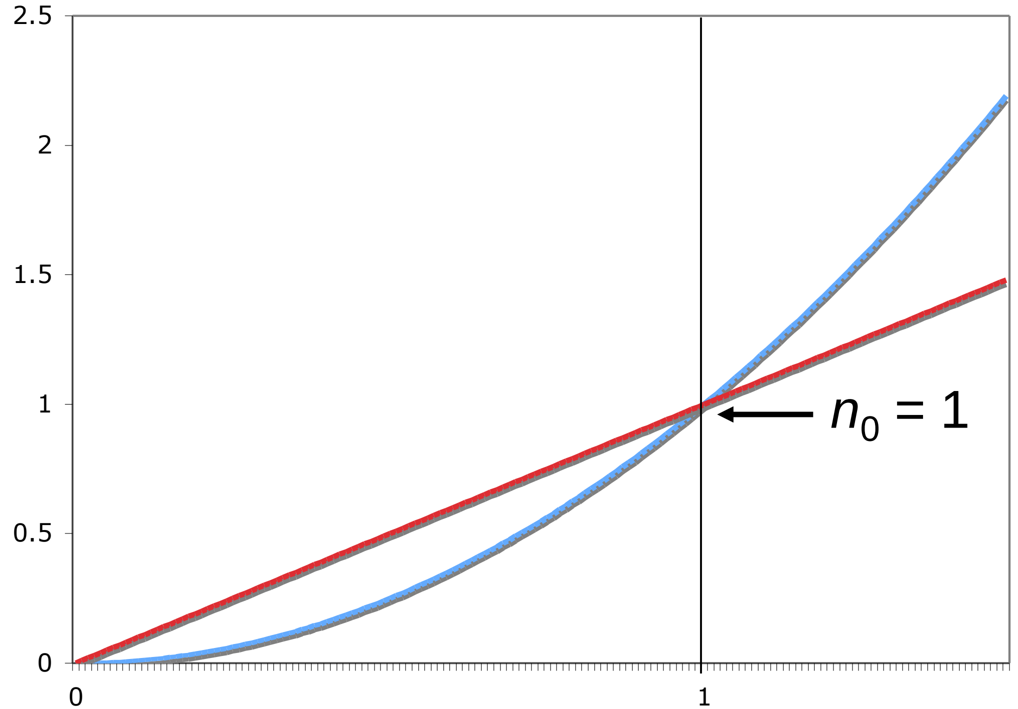
$f(n) = O(g(n))$ if and only if there are constants

$$\left.\begin{array}{l} c \;\; > 0 \\ n_0 \geq 0 \end{array}\right\} \text{ such that } f(n) \leq c * g(n) \text{ whenever } n \geq n_0$$

Is $n = O(n^2)$?

$f(n) = n$

$g(n) = n^2$



Big-O: Sufficient (but not necessary) Condition

If $\left[\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = d < \infty\right]$ then $f(n)$ is $O(g(n))$

$\boxed{\log_2 n = O(2\text{n})?}$

$f(n) = \log_2 n$

$g(n) = 2n$

$$\lim_{n\to\infty}\left(\frac{\log n}{2n}\right)$$: $\frac{\infty}{\infty}$

$$\lim_{n\to\infty}\left(\frac{1}{2n}\right)$$: Use L'Hôpital's Rule

$0 = d < \infty$

: $\log_2 n = O(2n)$ ✓

$\boxed{\sin\left(\frac{n}{100}\right) = O(100)?}$

$f(n) = \sin\left(\frac{n}{100}\right)$

$g(n) = 100$

$$\lim_{n\to\infty}\left(\frac{\sin\left(\frac{n}{100}\right)}{100}\right)$$: Condition does not hold, but it is true that $f(n) = O(g(n))$

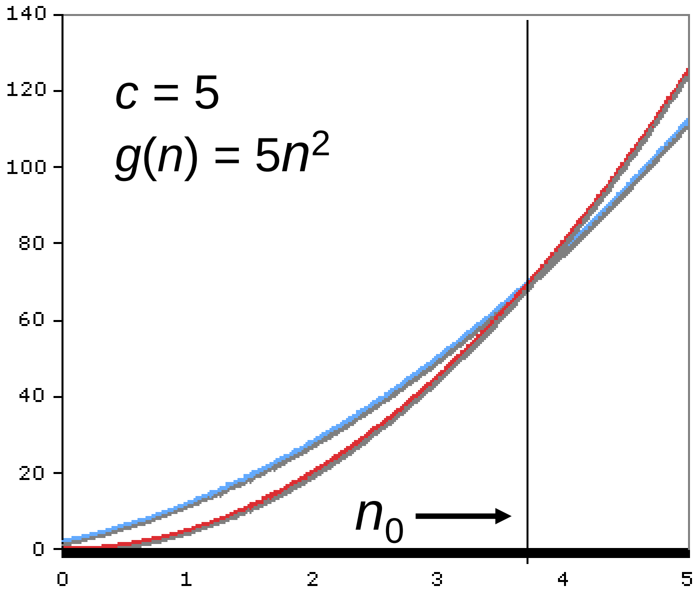
Big-O: Can We Drop Constants?

$$3n^2 + 7n + 42 = O(n^2)?$$

$f(n) = 3n^2 + 7n + 42$

$g(n) = n^2$

Definition	Sufficient Condition
$c > 0, n_0$ ³ 0 such that $f(n)$ £ c ×$g(n)$ whenever n ³ n_0	$\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = d < \infty$



$$\lim_{n\to\infty}\left(\frac{3n^2+7n+42}{n^2}\right) = \cdots$$

$$\lim_{n\to\infty}\left(\frac{6n+7}{2n}\right) = \cdots$$

$$\lim_{n\to\infty}\left(\frac{6}{2}\right) = 3 < \infty \checkmark$$

Rules of Thumb

1. Lower-order terms can be ignored (because of n_0)
   - $n^2 + n + 1 = O(n^2)$
   - $n^2 + \log(n) + 1 = O(n^2)$
2. Coefficient of the highest-order term can be ignored (because of c)
   - $3n^2 + 7n + 42 = O(n^2)$

Log Identities

Identity	Example
$\log_a(xy) = \log_a x + \log_a y$	$\log_2 (3*4)$
$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$	$\log_2 (4/3)$
$\log_a(x^r) = r \log_a x$	$\log_2 x^3$
$\log_a\left(\frac{1}{x}\right) = -\log_a x$	$\log_2 1/3$
$\log_a x = \frac{\log x}{\log a} = \frac{\ln x}{\ln a}$	$\log_2 1024$
$\log_a a = 1$ $\log_a 1 = 0$	

Power Identities

Identity	Example
$a^{(m+n)} = a^m \cdot a^n$	2^{2+3}
$a^{(m-n)} = \frac{a^m}{a^n}$	2^{3-2}
$(a^m)^n = a^{mn}$	$(2^2)^3$
$a^{-n} = \frac{1}{a^n}$	2^{-4}
$a^{-1} = \frac{1}{a}$ $a^0 = 1$ $a^1 = a$	

Exercise

True or False?

- $10^{100} = O(1)$
- $3n^4 + 45n^3 = O(n^4)$
- $3^n = O(2^n)$
- $2^n = O(3^n)$
- $45 \log(n) + 45n = O(\log(n))$
- $\log(n^2) = O(\log(n))$
- $[\log(n)]^2 = O(\log(n))$

Can you?

Find $f(n)$ and $g(n)$,

such that $f(n) \neq O(g(n))$

and $g(n) \neq O(f(n))$

Big-O, Big-Theta, and Big-Omega

	Big-O (O)	Big-Theta (Θ)	Big-Omega (Ω)
Defines	Asymptotic upper bound	Asymptotic tight bound	Asymptotic lower bound
Definition	$f(n) = O(g(n))$ if and only if there exists an integer n_0 and a real number c such that for all $n \geq n_0$, $f(n) \leq c \cdot g(n)$	$f(n) = \Theta(g(n))$ if and only if there exists an integer n_0 and real constants c_1 and c_2 such that for all $n \geq n_0$: $c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$	$f(n) = \Omega(g(n))$ if and only if there exists an integer n_0 and a real number c such that for all $n \geq n_0$, $f(n) \geq c \cdot g(n)$
Mathematical Definition	$n0 Î Z, $c Î R : " n ³ n0, f(n) £ c ×g(n)	$\Theta(f(n)) = O(f(n)) \bigcap \Omega(f(n))$	$n0 Î Z, $c Î R : " n ³ n0, f(n) ³ c ×g(n)
$f_1(n)=2n+1$	$\boldsymbol{O(n)}$ or $\boldsymbol{O(n^2)}$ or $\boldsymbol{O(n^3)}$…	Q(n)	$\boldsymbol{\Omega(n)}$ or $\boldsymbol{\Omega(1)}$
$f_2(n)=n^2+n+5$	$\boldsymbol{O(n^2)}$ or $\boldsymbol{O(n^3)}$ …	Q(n^2)	$\boldsymbol{\Omega(n^2)}$ or $\boldsymbol{\Omega(n)}$ or $\boldsymbol{\Omega(1)}$

Complexity Analysis:
Big-O Notation

Data Structures & Algorithms

Complexity Analysis: Amortized Complexity

Data Structures & Algorithms

Amortized Complexity

- A type of worst-case complexity
- Used when the work/time profile is “spiky” (sometimes it is very expensive, but most times it is a small expense)
- Analysis performed over a sequence of operations covering of a given range
  - The sequence selected includes expensive and cheap operations

Amortized Complexity

- Considers the average cost of one operation <u>over a sequence of operations</u>
  - Best/Worst/Average-case only consider operations independently
  - <u>Different from average-case complexity</u>!
- Average case complexity is ONE event that is representative of all possible events
- Amortized complexity is an average of a sequence of events that ALL occur
- Key to understanding expandable arrays and STL vectors, priority queues, and hash tables

Cell Phone Bill Example

- Pay \$100 once per month, each call and text has no added cost (unlimited plan)
- If you make 1000 calls/texts during the month, each one effectively costs \$0.10
  - This is the “sequence of events that all occur”
- The rate at which money leaves your pocket is very “spiky”
- But each call or text appears to have basically a constant cost: the *amortized* cost per text is ***O*(1)**

Common Amortized Complexity

Analyze the asymptotic runtime complexity of the push operation for a stack implemented using an array/vector

Method	Implementation
`push(object)`	1. If needed, allocate a bigger array and copy data 2. Add new element at `top_ptr`, increment `top_ptr`

Exercise

Analyze the asymptotic runtime complexity of the push operation for a stack implemented using an array/vector

Amortized ***Θ*(1)**

- Assume vector is filled with ***n*** elements
- Double vector size (1 + ***Θ*(*n*)**) steps
- ***Θ*(1)** push ***n*** times until full
- Amortized cost: $\dfrac{(1 + \Theta(n)) + n_{*}\Theta(1)}{n\,\text{push operations}} = \Theta(1)$

Example (drawn in lecture)

Container Growth Options

1. Constant Growth
   - When container fills, increase size by c
   - Amortized cost: $\dfrac{(1 + \Theta(n)) + c * \Theta(1)}{c \text{ push operations}} = \Theta(n)$
   - Amortized linear
2. Linear Growth
   - When container fills, increase size by n
   - Amortized cost: $\dfrac{(1 + \Theta(n)) + n * \Theta(1)}{n \text{ push operations}} = \Theta(1)$
   - Amortized constant

Complexity Analysis: Amortized Complexity

Data Structures & Algorithms

Complexity Analysis: Balance Exercise

Data Structures & Algorithms

Exercise

- You have n billiard balls. All have equal weight, except for one which is heavier. Find the heavy ball using only a balance.
- Describe an $\boldsymbol{O(n^2)}$ algorithm
- Describe an $\boldsymbol{O(n)}$ algorithm
- Describe an $\boldsymbol{O(\log n)}$ algorithm
- Describe another $\boldsymbol{O(\log n)}$ algorithm

Two $O(\log n)$ solutions

- Two groups: $\log_2(n) = \boldsymbol{O(\log_3 n)}$
- Three groups: $\log_3(n) = \boldsymbol{O(\log_2 n)}$
- True or False? Why?

Complexity Analysis: Balance Exercise

Data Structures & Algorithms