BME/EECS 516 - Solution Homework # 1

Problem # 1

Test for linearity.
Linear if: $S\{\alpha f_1(x,y) + \beta f_2(x,y)\} = \alpha S\{f_1(x,y)\} + \beta S\{f_2(x,y)\}$ $\quad \forall\ f_1, f_2, \alpha, \beta$

Test for space-invariance.
Invariant if: $S\{f(x-\xi , y-\eta)\} = S\{f(x,y)\}\big|_{x = x - \xi ,\, y = y - \eta}$ $\quad \forall\ f, \xi, \eta$

Impulse response.
$h(x, y ; \xi, \eta) = S\{\delta(x - \xi , y - \eta)\}$

I.

(a) $S\{\alpha f_1(x,y) + \beta f_2(x,y)\} = \alpha f_1(ax,ay) + \beta f_2(ax,ay)$
$= \alpha S\{f_1(x,y)\} + \beta S\{f_2(x,y)\}$
$\Rightarrow$ Linear

(b) $S\{f(x-\xi , y-\eta)\} = f(ax-\xi , ay-\eta) \neq f(ax-a\xi,ay-a\eta) = S\{f(x, y)\}\big|_{x = x - \xi ,\, y = y - \eta}$
$\Rightarrow$ Space Variant

(c) $h(x, y ; \xi, \eta) = \delta(ax - \xi , ay - \eta) = \underline{\underline{\dfrac{1}{a^2}\delta\left(x - \dfrac{\xi}{a}, y - \dfrac{\eta}{a}\right)}}$

(d) N/A

II.

(a) $S\{\alpha f_1(x,y) + \beta f_2(x,y)\} = \alpha f_1(x-a,y-b) + \beta f_2(x-a,y-b)$
$= \alpha S\{f_1(x,y)\} + \beta S\{f_2(x,y)\}$
$\Rightarrow$ Linear

(b) $S\{f(x-\xi , y-\eta)\} = f(x-\xi-a , y-\eta-b) = f(x-a,y-b)\big|_{x = x - \xi ,\, y = y - \eta}$
$= S\{f(x, y)\}\big|_{x = x - \xi ,\, y = y - \eta} \Rightarrow$ Space Invariant

(c) $h(x, y ; \xi, \eta) = \underline{\delta(x - \xi - a , y - \eta - b)}$ -or- $h(x, y) = \underline{\delta(x - a , y - b)}$

(d) $G(u,v) = \underline{\exp(-i2\pi(ua+vb))F(u,v)}$

III.

(a) $S\{\alpha f_1(x,y) + \beta f_2(x,y) \} = \exp(\alpha f_1(x,y) + \beta f_2(x,y)) \neq \alpha \exp(f_1(x,y)) + \beta \exp(f_2(x,y))$

$\Rightarrow$ Non-linear

(b) $S\{f(x-\xi , y-\eta)\} = \exp(f(x - \xi, y - \eta)) = \exp(f(x, y))\big|_{x = x - \xi ,\, y = y - \eta}$
$= S\{f(x, y)\}\big|_{x = x - \xi ,\, y = y - \eta} \Rightarrow$ Space Invariant

(c) N/A

(d) N/A

IV.

(a) $S\{\alpha f_1(x,y) + \beta f_2(x,y)\}$

$= \frac{1}{2} (\alpha f_1(x-a,y-b) + \beta f_2(x-a,y-b) + \alpha f_1(x+a,y+b) + \beta f_2(x+a,y+b))$

$= \alpha \frac{1}{2} (f_1(x-a,y-b) + f_1(x+a,y+b)) + \beta \frac{1}{2} (f_2(x-a,y-b) + f_2(x+a,y+b))$

$= \alpha S\{f_1(x,y)\} + \beta S\{f_2(x,y)\}$

$\Rightarrow$ Linear

(b) $S\{f(x-\xi , y-\eta)\} = \frac{1}{2}(f(x-\xi-a , y-\eta-b) + f(x-\xi+a , y-\eta+b))$

$= \frac{1}{2}(f(x-a,y-b) + f(x+a,y+b)) \big|_{x = x - \xi , y = y - \eta}$

$= S\{f(x, y)\} \big|_{x = x - \xi , y = y - \eta} \Rightarrow$ Space Invariant

(c) $h(x, y ; \xi, \eta) = \frac{1}{2}(\delta(x - \xi - a , y - \eta - b) + \delta(x - \xi - a , y - \eta - b))$

-or- $h(x, y) = \frac{1}{2}(\delta(x - a , y - b) + \delta(x + a , y + b))$

(d) $G(u,v) = \cos(2\pi(ua+vb))F(u,v)$

Problem # 2

(a) $\text{rect}\,(ax-b) = \text{rect}\left(a\left(x - \frac{b}{a}\right)\right)\cdot 1$

$g_x(x) = \text{rect}\left(a\left(x - \frac{b}{a}\right)\right)$

$g_y(y) = 1$

The 2D FT is

$$\mathcal{F}_{1D,\,x}\left\{\text{rect}\left(a\left(x - \frac{b}{a}\right)\right)\right\}\cdot\ \mathcal{F}_{1D,\,y}\{1\} = \boxed{\frac{1}{|a|}\text{sinc}\left(\frac{u}{a}\right)e^{-j2\pi\frac{b}{a}u}\cdot\delta(v)}$$

(b) $\text{rect}(x-a)\cdot\text{sinc}(y-by) \xleftrightarrow{\text{FT}} \boxed{\text{sinc}(u)e^{-j2\pi a u}\cdot\frac{1}{|b|}\text{rect}\left(\frac{v}{b}\right)}$

(c) $\text{circ}(r)\,\delta(x) = \text{circ}\left(\sqrt{x^2 + y^2}\right)\delta(x)$

$= \text{circ}\left(\sqrt{0 + y^2}\right)\delta(x)$ Product property of delta function

$= \mathrm{circ}(|\,y\,|)\ \delta(x)$

$$= \boxed{\mathrm{rect}\left(\frac{y}{2}\right)\delta(x) \xleftrightarrow{\ FT\ } 1\cdot 2\,\mathrm{sinc}(2v)}$$

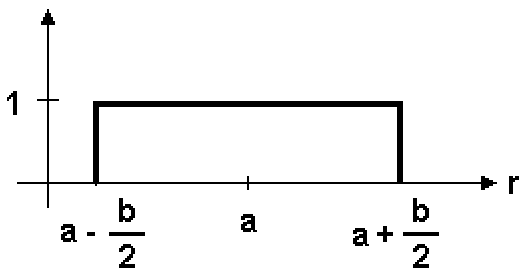
(d) $\delta(r-r_o)$

$$G_R(\rho) = 2\pi\int_0^\infty r\,\delta(r-r_o)\,J_0(2\pi\,r\,\rho)\,dr = \boxed{2\pi\,r_o\,J_0(2\pi\,r_o\,\rho)}$$

(e) $\mathrm{rect}\left(\frac{r-a}{b}\right)$ $\qquad a > b$

$$\mathrm{rect}\left(\frac{r-a}{b}\right) = \mathrm{circ}\left(\frac{r}{r_2}\right) - \mathrm{circ}\left(\frac{r}{r_1}\right)$$

where $r_2 = a + \frac{b}{2}$, and $r_1 = a - \frac{b}{2}$



$G_R(\rho) =$

$$r_2^2\ 4\ \mathrm{Jinc}(2\,r_2\ \rho) - r_1^2\ 4\ \mathrm{Jinc}(2\,r_1\ \rho)$$

$$\boxed{G_R(\rho) = 4\left(a+\frac{b}{2}\right)^2 \mathrm{Jinc}\left(2\left(a+\frac{b}{2}\right)\rho\right) - 4\left(a-\frac{b}{2}\right)^2 \mathrm{Jinc}\left(2\left(a-\frac{b}{2}\right)\rho\right)}$$

Where $\mathrm{Jinc}(x) = \frac{J_1(\pi\,x)}{2x}$

(f) If $G(\rho)$ is FT of $g_R(r)$

$$\mathcal{F}\{\ g_R(ar)\ \} = 2\pi\int_0^\infty r\,g_R(ar)\,J_0(2\pi\,r\,\rho)\,dr \qquad a > 0$$

Change of variables: $r' = ar$

$$\mathcal{F}\{\ g_R(ar)\ \} = 2\pi\int_0^\infty \frac{r'}{a}\,g_R(r')\,J_0(2\pi\,\frac{r'}{a}\,\rho)\,\frac{dr'}{|a|}$$

$$\mathcal{F}\{\ g_R(ar)\ \} = \frac{1}{a^2}\,2\pi\int_0^\infty r'\,g_R(r')\,J_0(2\pi\,r'\,\frac{\rho}{a})\,dr'$$

$$\mathcal{F}\{\ g_R(ar)\ \} = \boxed{\frac{1}{a^2}\,G\left(\frac{\rho}{a}\right)}$$

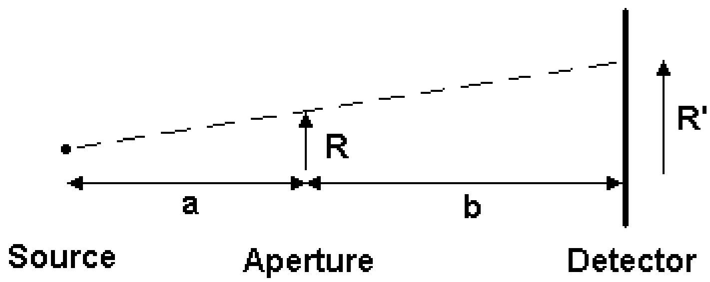
(g) Using the above result:

$$\mathcal{F}\{\ \exp(-\pi(r/a)^2)\ \} = \boxed{a^2\,\exp(-\pi(a\rho)2)}$$

Problem # 3

From book (Eq. 2.16): $I_2(x_2, y_2) = \frac{1}{M^2} I_1\left(\frac{x_2}{M}, \frac{y_2}{M}\right) ** h(x_2, y_2)$

Where $h(x_2,y_2)$ is the impulse response in the output plane.



$$R' = \left(\frac{a+b}{a}\right) \cdot R$$

So, an impulse at the source will appear as a circ of radius R' at the output plane

$$\Rightarrow \; h(x_2, y_2) = C \cdot \mathrm{rect}\left(\frac{x_2\, a}{d \cdot (a+b)}\right) \cdot \mathrm{rect}\left(\frac{y_2\, a}{d \cdot (a+b)}\right)$$

where C is an arbitrary scaling constant.

so

$I_2(u,v) = I_1(Mu,Mv) \cdot H(u,v)$

$$\boxed{I_2(u,v) = C \cdot I_1(Mu,Mv) \cdot \left(d\left(\frac{a+b}{a}\right)\right)^2 \mathrm{sinc}\left(2d\left(\frac{a+b}{a}\right)u\right) \mathrm{sinc}\left(2d\left(\frac{a+b}{a}\right)v\right)}$$

Problem # 4

(a) $\mathcal{F}\left\{ f(x,y)\, \delta(x-a, y-b)\right\} = \iint f(x,y)\, \delta(x-a, y-b) e^{-i2\pi(xu+yv)} dx\, dy$

$= f(a,b)\, e^{-i2\pi(au+bv)}$ by sifting property

$\mathcal{F}\left\{ f(a,b)\, \delta(x-a, y-b)\right\} = f(a,b)\, \mathcal{F}\left\{\delta(x-a, y-b)\right\} = f(a,b)\, e^{-i2\pi(au+bv)}$

proof by uniqueness of F.T.

(b) $f(x,y) ** \delta(x-a, y-b) = \iint f(\xi,\eta)\, \delta(x-a-\xi, y-a-\eta)\, d\xi\, d\eta = f(x-a, y-b)$

proof by sifting property

(c) $\mathcal{F}\left\{\delta(ax, by)\right\} = \iint \delta(ax, by) e^{-i2\pi(xu+yv)} dx\, dy$

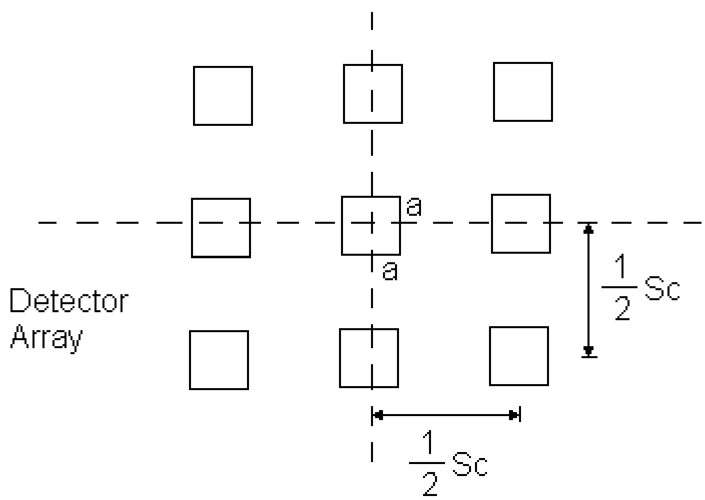
Change of variables: x' = ax, y' = ay

$$\mathcal{F}\left\{\delta(ax, by)\right\} = \iint \delta(x', y') e^{-i2\pi(x'\frac{u}{a}+y'\frac{v}{b})} \frac{dx'}{|a|}\frac{dy'}{|b|} = \frac{1}{|ab|}\cdot 1 \quad \text{by sifting property}$$

$$\mathcal{F}\left\{\frac{1}{|ab|}\delta(x,y)\right\} = \frac{1}{|ab|}\mathcal{F}\left\{\delta(x,y)\right\} = \frac{1}{|ab|}$$

proof by uniqueness of F.T.

Problem # 5



(a)
Take detector element at some particular sample location ($nX=x_o, mX=y_o$)

Sample value will be sum of image over detector element = $\displaystyle\int_{x_0-\frac{a}{2}}^{x_0+\frac{a}{2}} \int_{y_0-\frac{a}{2}}^{y_0+\frac{a}{2}} g(x,y)dxdy$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,\mathrm{rect}\left(\frac{x-x_0}{a}, \frac{y-y_0}{a}\right) dx\, dy$$

We will model the sampling process by the convolution of g(x,y) with $\text{rect}\left(\frac{x}{a},\frac{y}{a}\right)$ sifted out by a delta function at (x_o,y_o)

$$\left[g(x,y) ** \text{rect}\left(\frac{x}{a},\frac{y}{b}\right)\right] \cdot \delta(x-x_0, y-y_0) =$$

$$= \iint g(\xi,\eta)\,\text{rect}\left(\frac{x-\xi}{a},\frac{y-\eta}{a}\right) d\xi\, d\eta \cdot \delta(x-x_0, y-y_0)$$

$$= \iint g(\xi,\eta)\,\text{rect}\left(\frac{x_0-\xi}{a},\frac{y_0-\eta}{a}\right) d\xi\, d\eta \cdot \delta(x-x_0, y-y_0) \qquad \text{by product/sifting property}$$

$$= \iint g(\xi,\eta)\,\text{rect}\left(\frac{\xi - x_0}{a},\frac{\eta - y_0}{a}\right) d\xi\, d\eta \cdot \delta(x-x_0, y-y_0)$$

= [Sample value at (x_o,y_o)]$\cdot\delta(x- x_o,y-y_o)$

(b)
So the sampled object in a CCD camera can be written as

$$g_s(x,y) = \left[g(x,y) ** \text{rect}\left(\frac{x}{a},\frac{y}{a}\right)\right] \cdot \text{comb}(2\,Sc\,x, 2\,Sc\,y)$$

The F.T. is

$$G_s(u,v) = \frac{a^2}{(2\,Sc)^2}\left[G(u,v)\cdot \text{sinc}(au,av)\right] ** \text{comb}\left(\frac{u}{2\,Sc},\frac{v}{2\,Sc}\right)$$

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a^2 G(u-2\,Sc\,n, v-2\,Sc\,m)\,\text{sinc}(a(u-2\,Sc\,n), a(v-2\,Sc\,m))$$

We have $a^2 G(u,v)\,\text{sinc}(au,av)$ replicated many times with spacing 2 Sc. Since G(u,v) has no frequency components greater than Sc there is no aliasing. Therefore, we will select out the central lobe using $\text{rect}\left(\frac{u}{2\,Sc},\frac{v}{2\,Sc}\right)$ and we also need to undo the effects of the blurring by a rect.

$$\boxed{H(u,v) = \frac{1}{a^2\,\text{sinc}(au)\,\text{sinc}(av)} \cdot \text{rect}\left(\frac{u}{2\,Sc},\frac{v}{2\,Sc}\right)}$$