Computed Tomography

BME/EECS 516

CT lecture #2

Announcements

- MRI Project due Tues, Nov 21
- HW #6 (x-ray), Due Tues, Nov 28
- US and MRI Demos – Tues, Nov 21
  - Meet at entrance to Gerstacker Building at 3pm.

Last Lecture

- CT introduction + history + basics
  - Line integral
  - Radon transform
- Central Section Theorem

Computed Tomography

- **Computed tomography (CT)**: digital projective geometry processing is used to generate a 3D image of an object from a large series of x-ray 2D projection taken around a single axis of rotation.

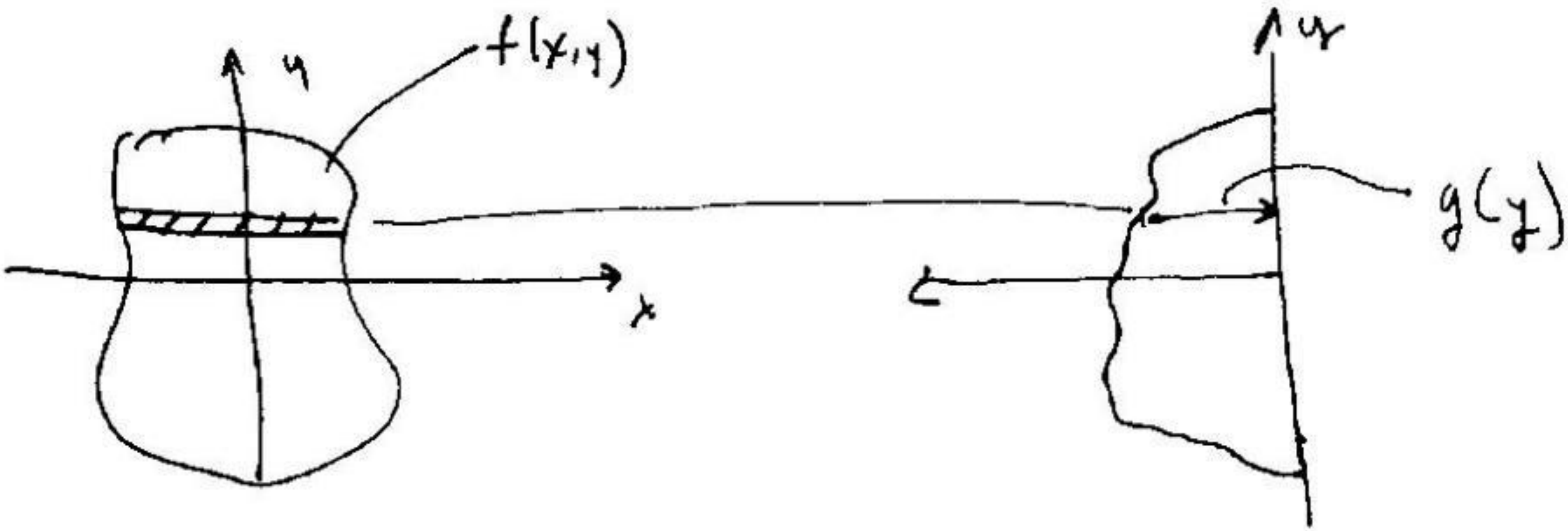
Computed Tomography

- Define the "line integral" through *f* in the *x* direction

$$g(y) = \ln \frac{I_0}{I_d(y)} = \int \mu(x,y)dx = \int f(x,y)dx$$

- Goal: solve f(x,y) from g(y) contain spatial attenuation coefficient info

Computed Tomography

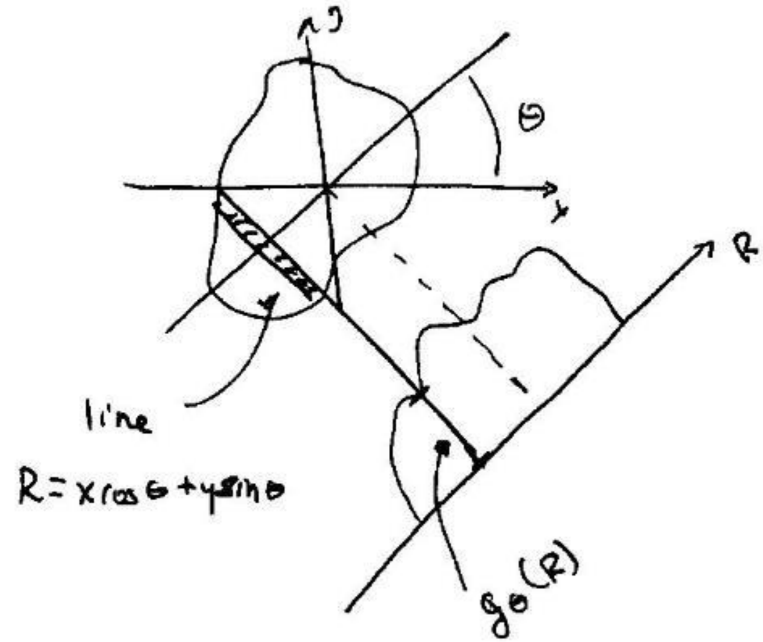


- Sampling at y = R

$$g(R) = \iint f(x, y)\delta(y - R)dxdy = \int f(x, R)dx$$

Computed Tomography

Line integral at an arbitrary angle, θ



When $R = x\cos\theta + y\sin\theta$:

$$g_\theta(R) = \iint f(x,y)\,\delta(x\cos\theta + y\sin\theta - R)\,dx\,dy$$

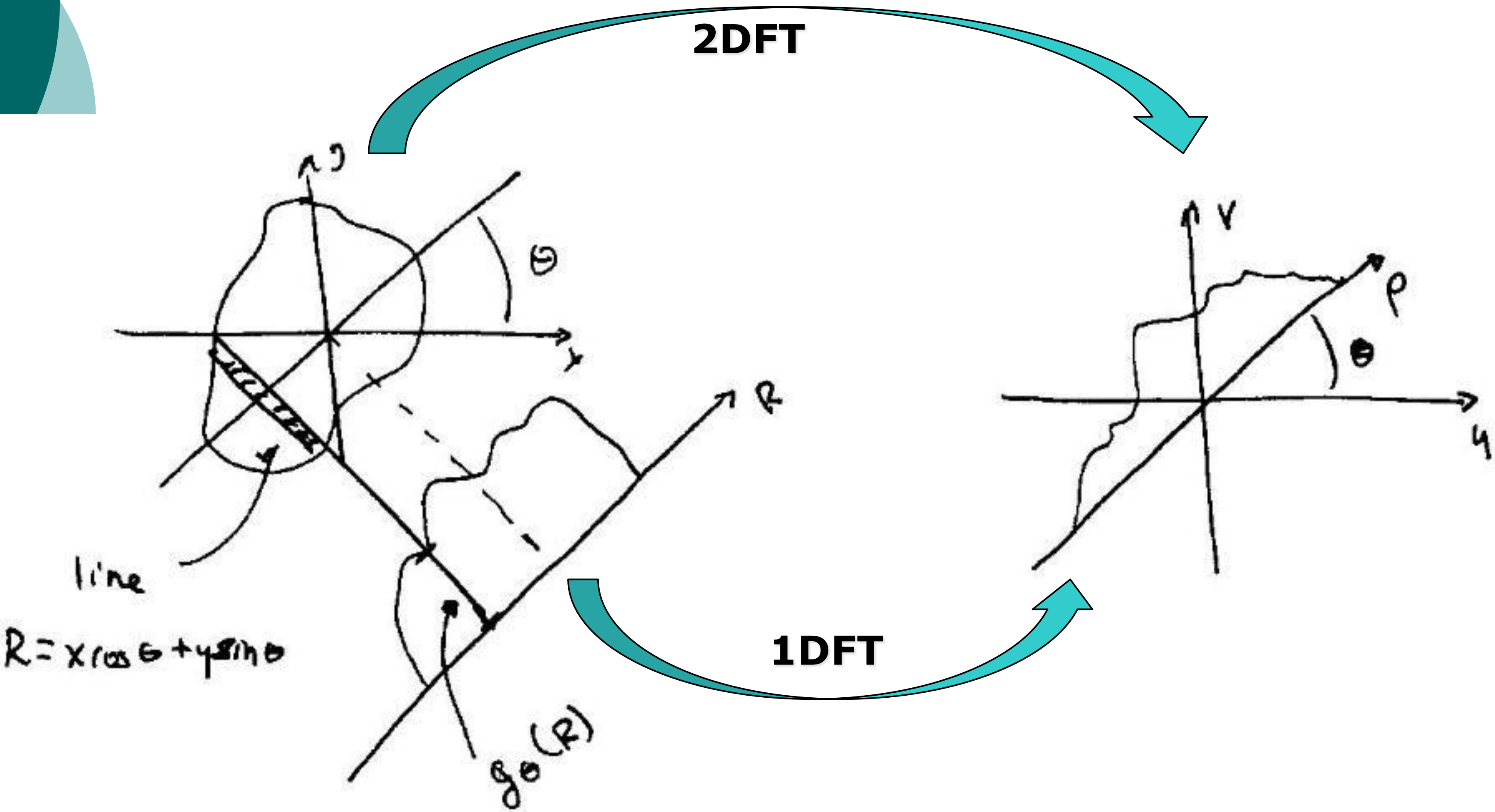
$g_\theta(R)$ is known as the ***Radon transform*** of $f(x,y)$.

Central Section Theorem

The central section theorem:

- The 1D FT of a projection $g_\theta(R)$ is the 2D FT of $f(x,y)$ evaluated at angle θ.

Central Section Theorem



$$G_\theta(\rho) = F_{1D,R}\{g_\theta(R)\} = F_{2D}\{f(x,y)\}_{u=\rho\cos\theta, v=\rho\sin\theta}$$

Central Section Theorem

- 2D FT of $f(x,y)$

$$F(u,v) = \iint f(x,y)\exp(-i2\pi(ux+vy))dxdy$$

- 1D FT of $g_\theta(R)$

$$G_\theta(\rho) = F_{1D(R)}\{g_\theta(R)\}$$

$$= \iint f(x,y)\exp(-i2\pi(\rho\cos\theta x + \rho\sin\theta y))dxdy$$

- Compare the two - same when $u = \rho\cos\theta$, $v = \rho\sin\theta$ (polar coords.)

Central Section Theorem

- 2D FT in Polar Coordinates.
  - $u = \rho\cos\theta$, $v = \rho\sin\theta$
  - A line in ρ at angle θ
- Thus:

$$G_\theta(\rho) = F_{1D(R)}\{g_\theta(R)\}$$

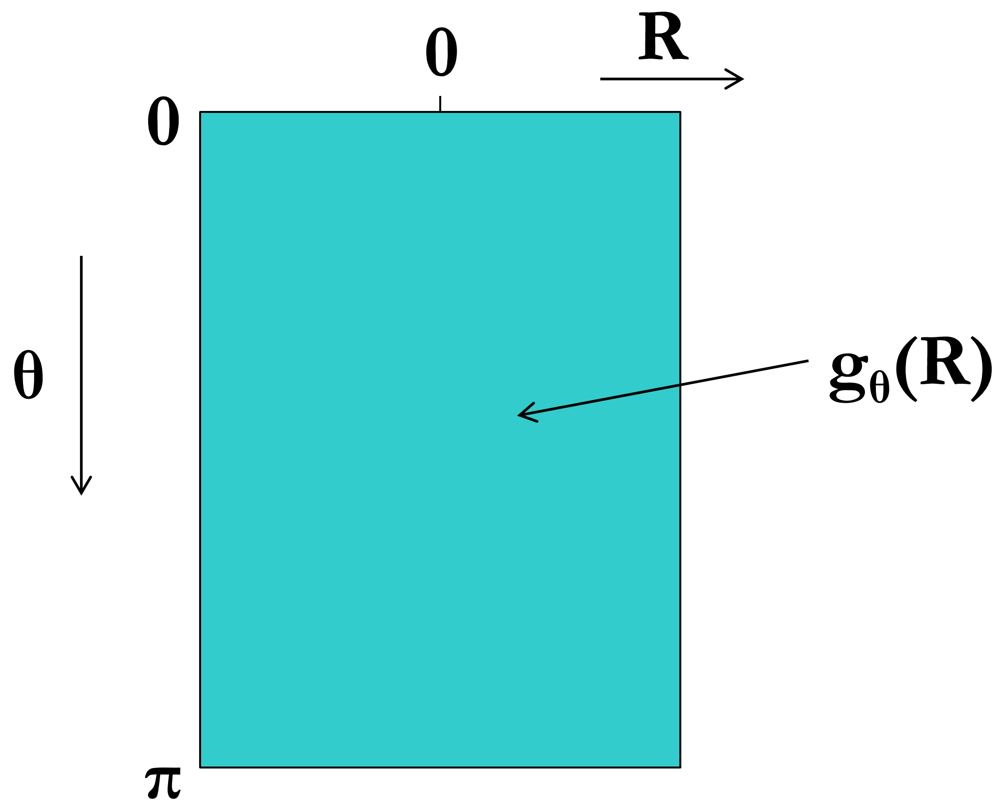
$$G_\theta(\rho) = F(u,v)\big|_{u=\rho\cos\theta, v=\rho\sin\theta} = F(\rho,\theta)$$

Computed Tomography

- Projections → Images
- Image reconstruction: Inverse problems
- We will present several approaches

Sinograms

- Sinpgrams - $g_\theta(R)$ data matrix for many different angles θ



Sinograms

- Example – point projection at (x_0,y_0)

$$g_\theta(R) = \iint \delta(x - x_0)\delta(y - y_0)\delta(x\cos\theta + y\sin\theta - R)\,dx\,dy$$

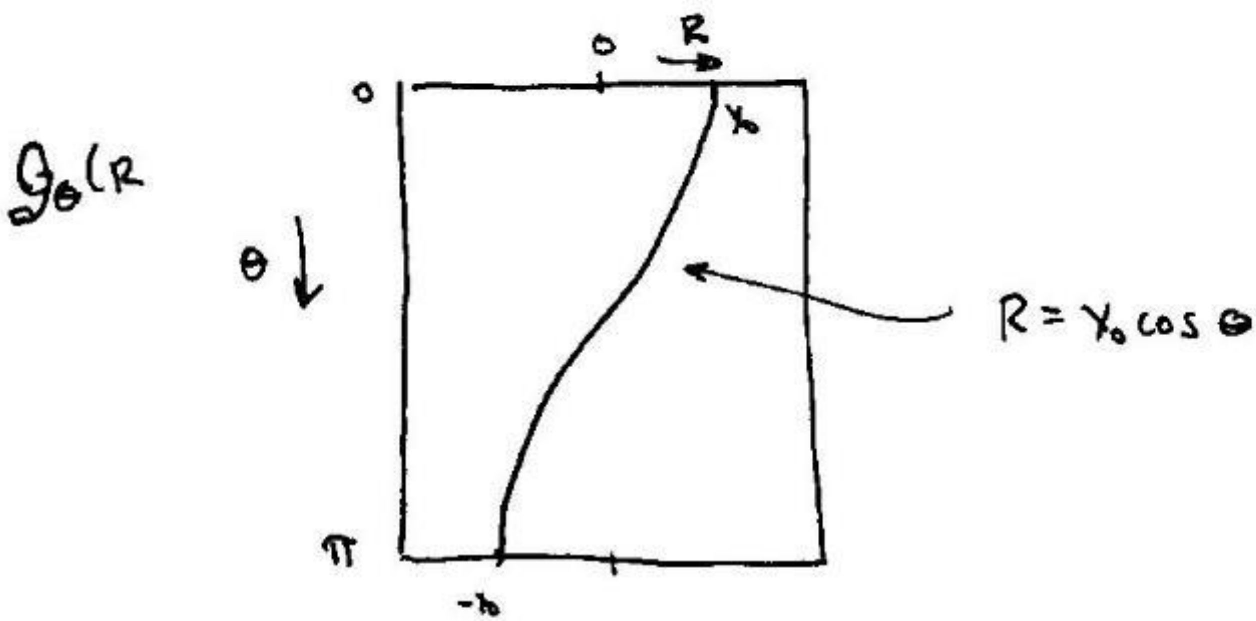
$$= \delta(x_0\cos\theta + y_0\sin\theta - R)$$

- Letting $y_0 = 0$, then:

$$g_\theta(R) = \delta(x_0\cos\theta - R)$$

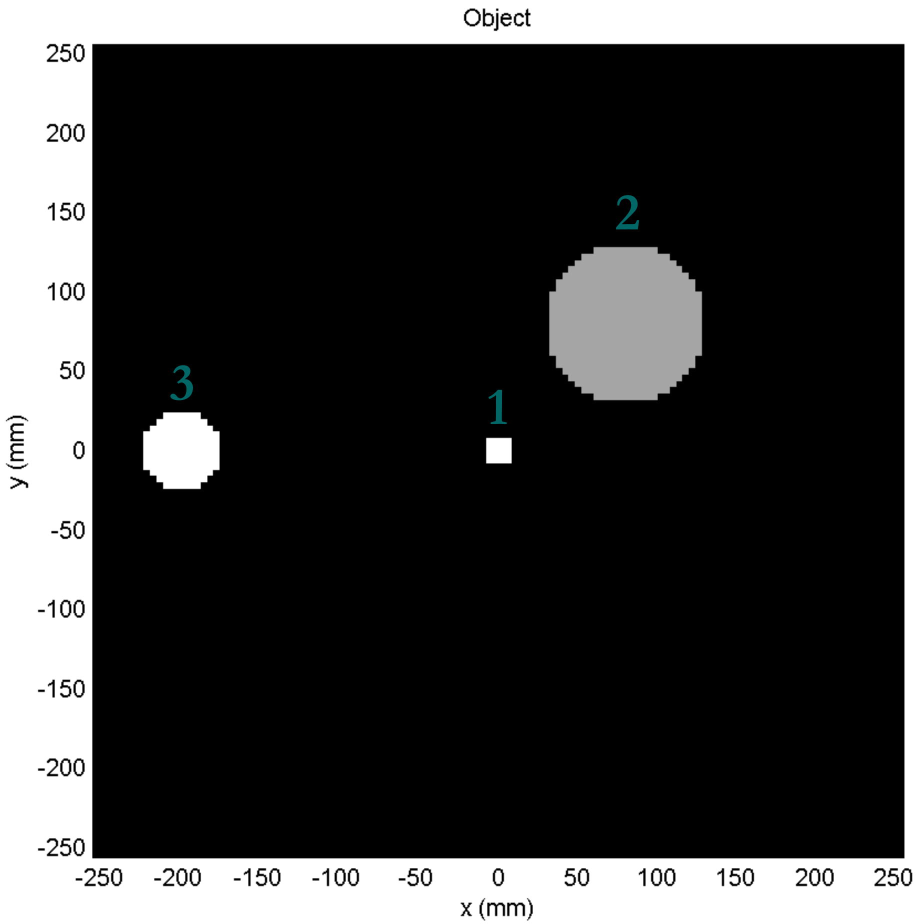
Sinograms

- A point traces out a sinusoid in the *R*-*θ* space --thus the name sinogram.



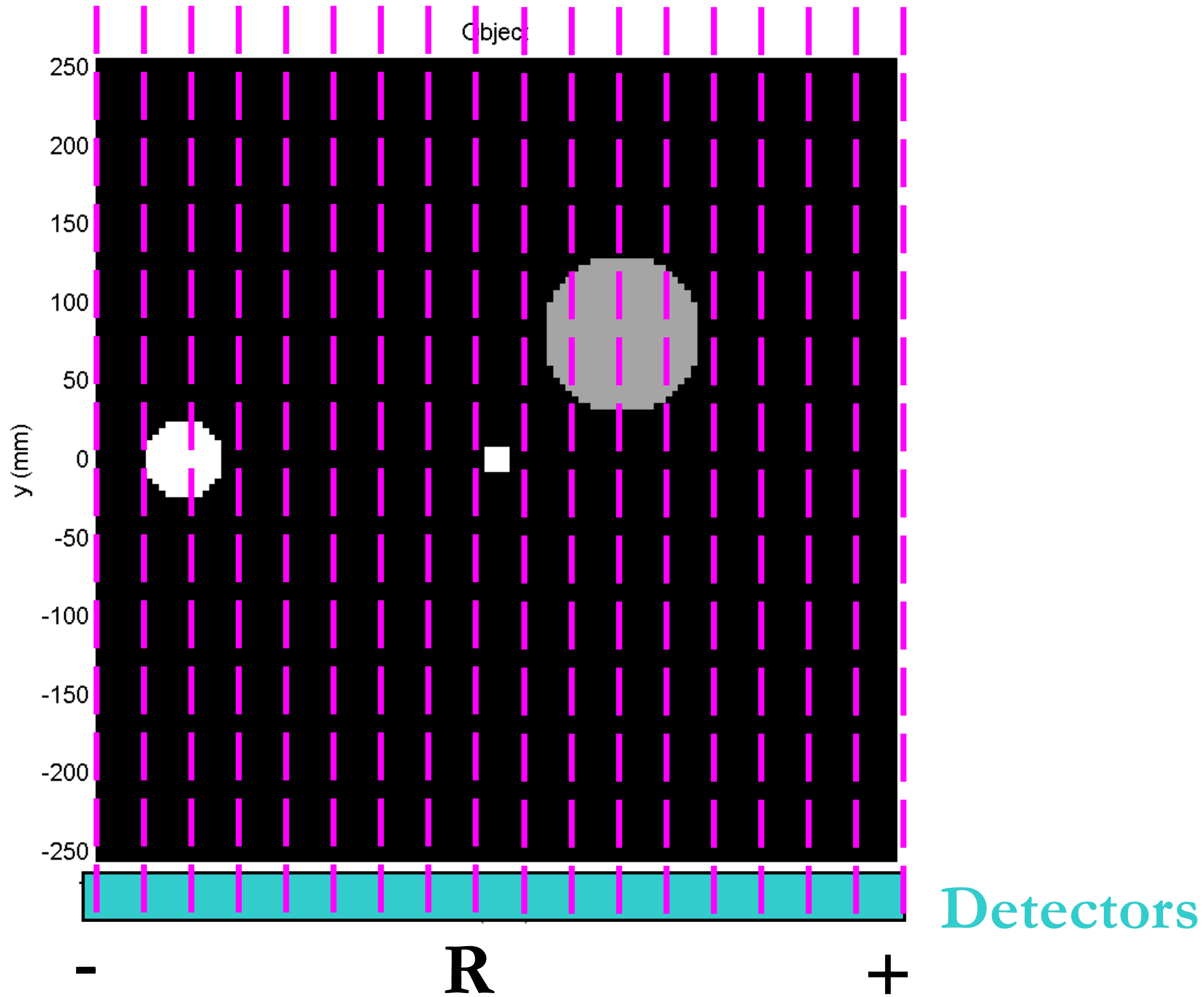
Sinograms

- Objects



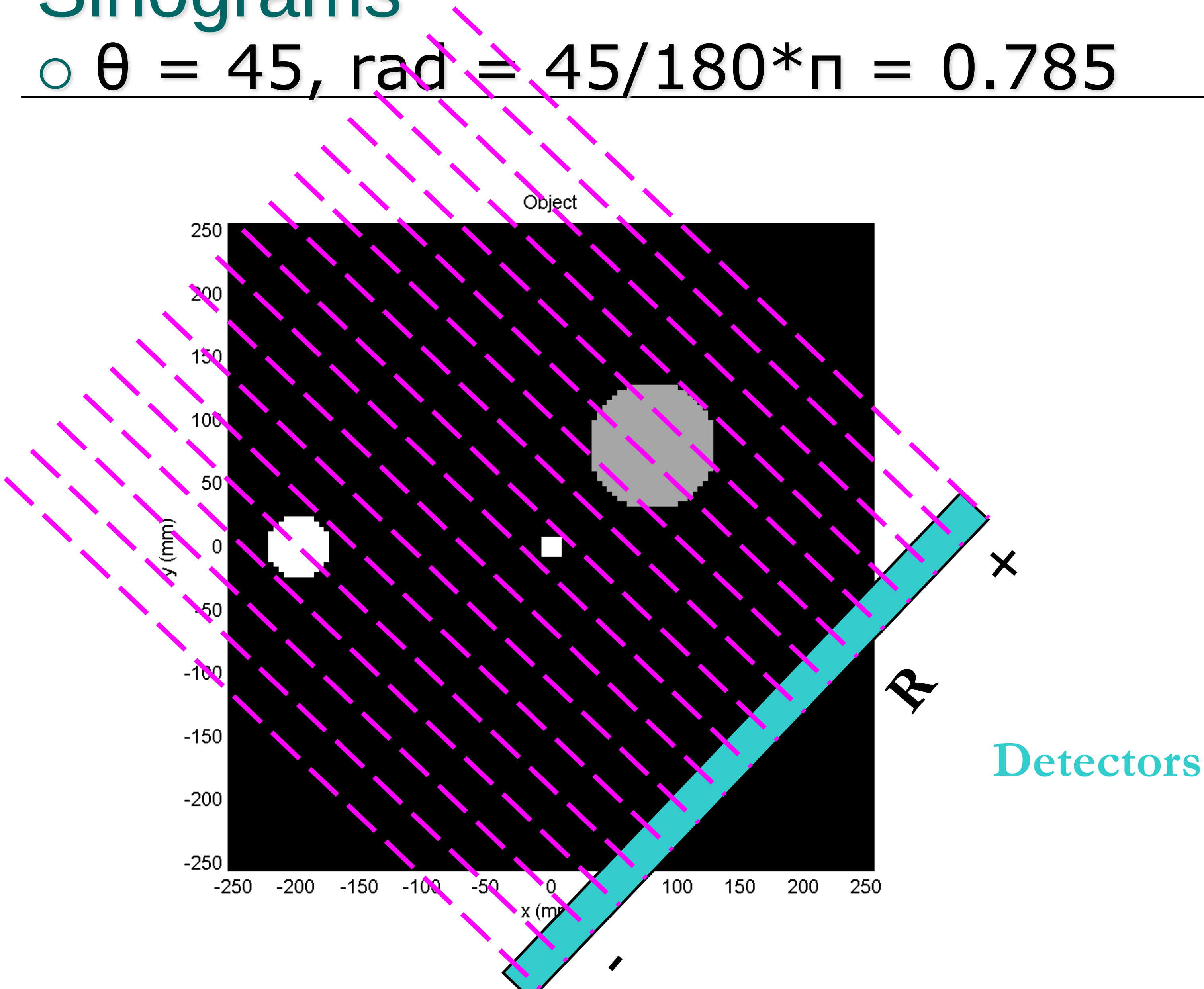
Sinograms

- θ = 0, get project

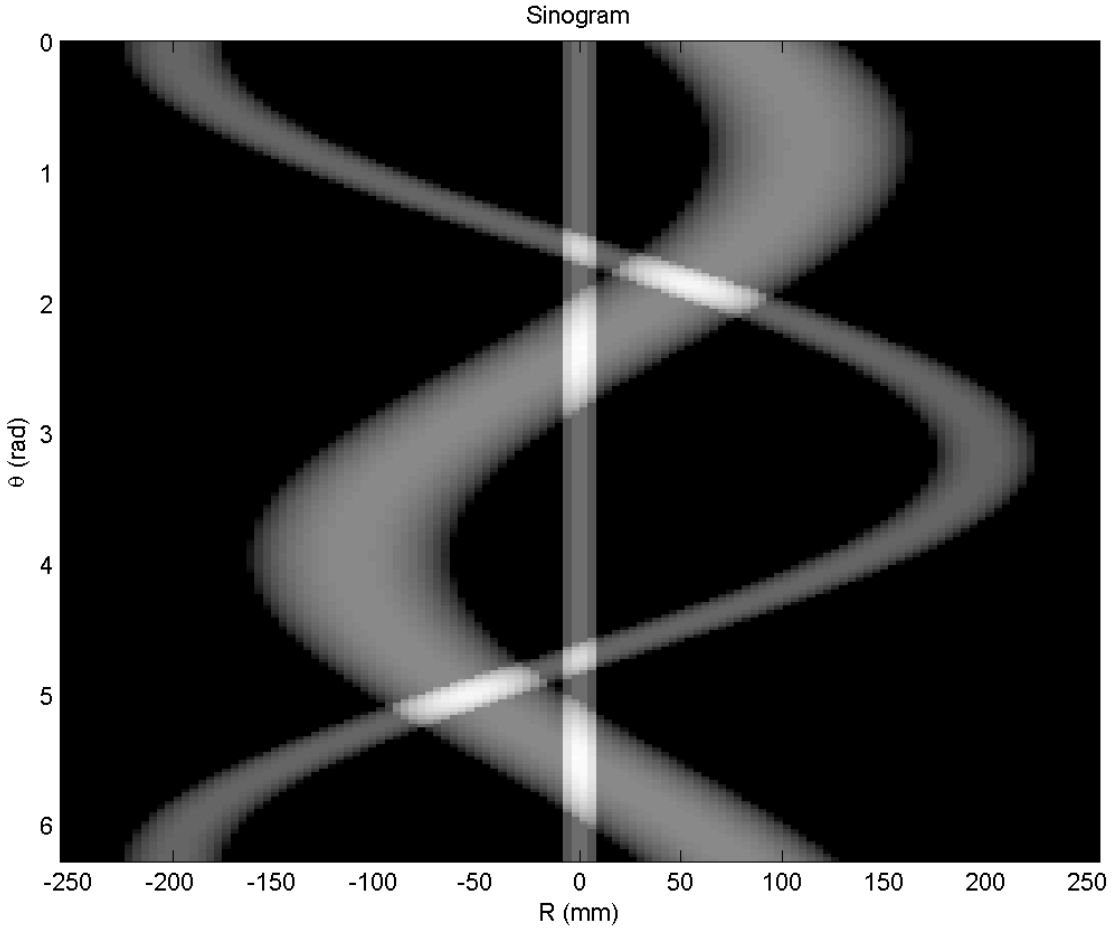


Sinograms

- θ = 45, rad = 45/180*π = 0.785



Sinograms



Sinogram
0
1
2
3
4
5
6
θ (rad)
-250
-200
-150
-100
-50
0
50
100
150
200
250
R (mm)

Sinograms

- The maximum deviation describes an object's distance from the origin

- The point of peak deviation describes the angular location of object.

- Three objects above, from smallest to largest are located at
  - $(r, \theta) = (0,0)$
  - $(r, \theta) = (113, \pi/4)$
  - $(r, \theta) = (-200,0)\ or\ (200, \pi)$.
- Symmetry: $\rho(r, \theta) = \rho(-r, \theta + \pi)$

Question?

Sinograms

Reconstruction Methods

1. Direct Fourier Interpolation Method
2. Backprojection-Filtering Method
3. Direct Fourier Superposition and Filtering Method
4. Filtered Backprojection Method
5. Algebraic Reconstruction Technique (ART)

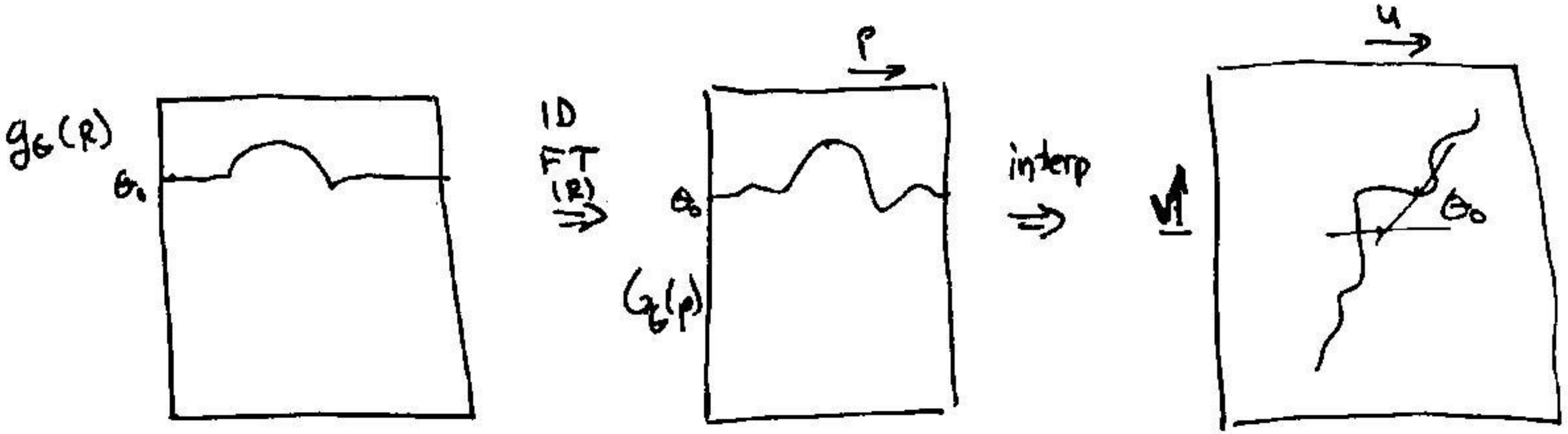
Reconstruction Methods

1. **Direct Fourier Interpolation Method – CT project**
2. **Backprojection-Filtering Method – Introduce backprojection concept**
3. **Direct Fourier Superposition and Filtering Method**
4. **Filtered Backprojection Method – CT project**
5. **Algebraic Reconstruction Technique (ART) – conventional method**

1. Direct Fourier Interpolation Method

- Direct use of the Central Section Theorem. Steps:
  1. 1D FT each of the projections:
     $$F_{1D}\left\{g_\theta(R)\right\} = G_\theta(\rho) = F(\rho,\theta)$$
  2. Interpolate $F(\rho,\theta)$ to $\hat{F}(u,v)$ (polar to rectangular coordinates – e.g. Matlab functions interp2 or griddata)
  3. Inverse 2D FT

1. Direct Fourier Interpolation Method



Can use interp2 or griddata in Matlab.

1. Direct Fourier Interpolation Method

- Simple concepts but have drawbacks:
  - Two-dimensional inverse FT is required for each projection line.
  - Involves interpolations and coordinate transformations.
    - Errors associated with interpolation in Fourier space
  - Complex implementation for practical systems

Question?

Direct Fourier Interpolation Method

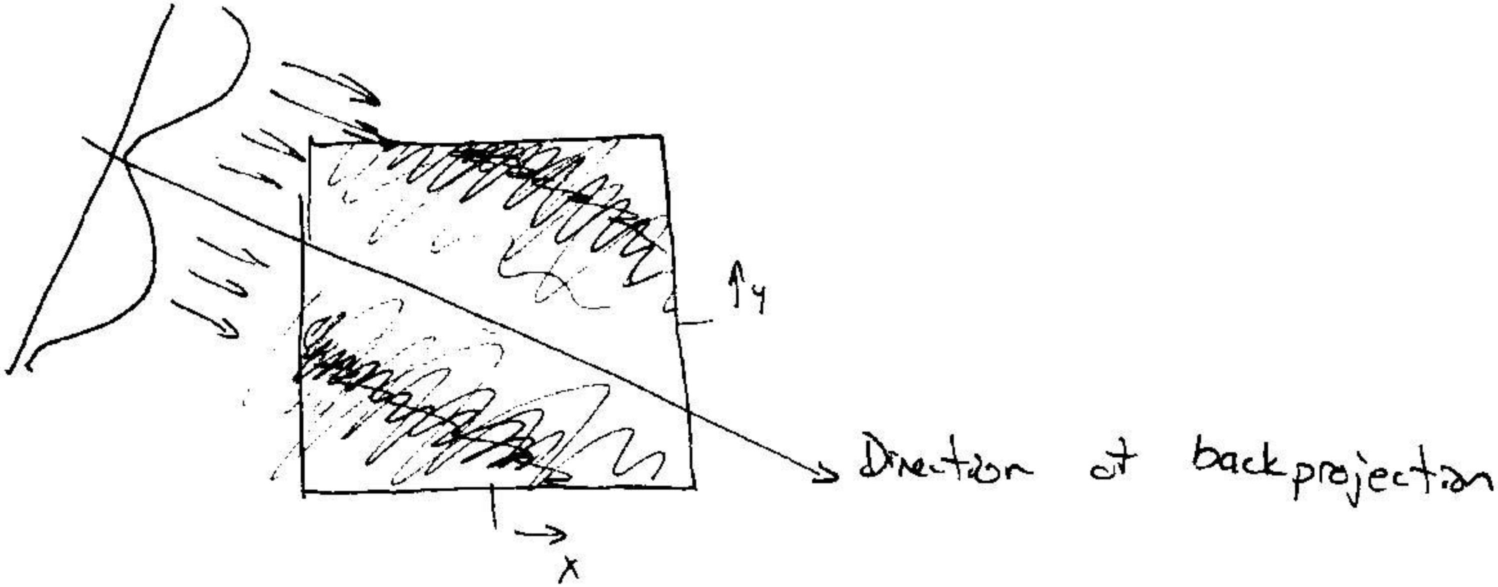
Backprojection

Backprojection

- Alternative reconstruction method
- Based on the same general principles but having distinct computational advantages
- Introduce backprojection

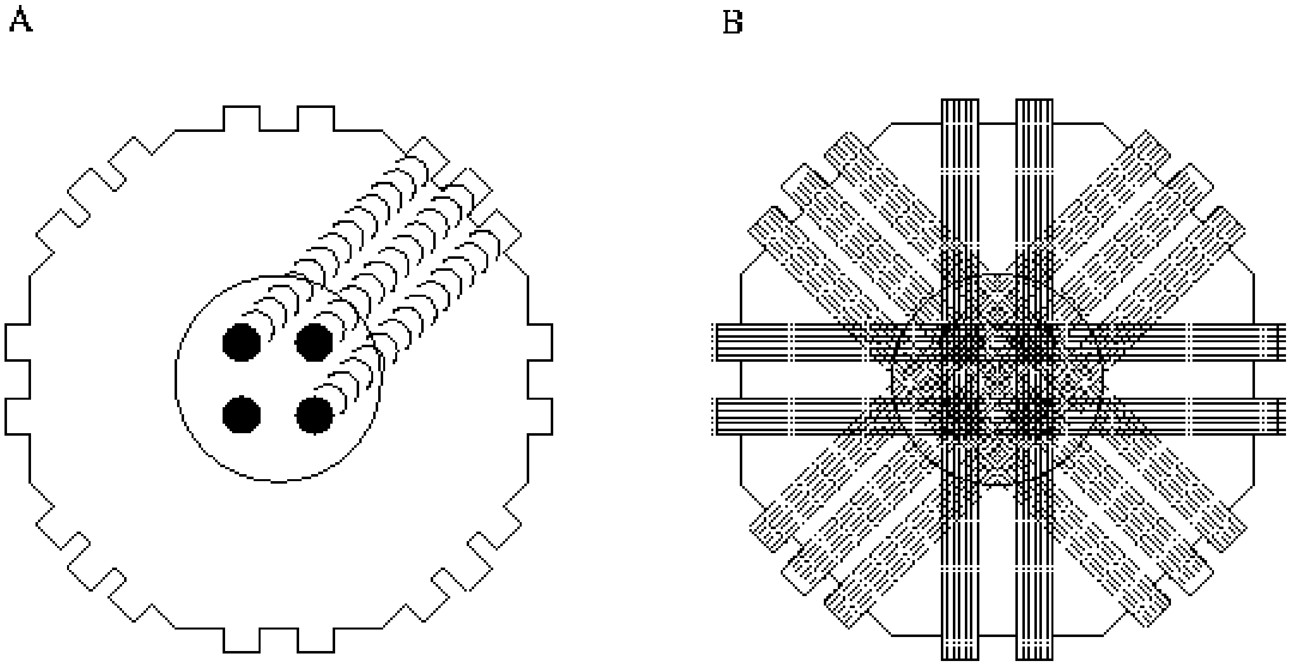
Backprojection

- Backprojection – the measurements obtained at each projection are projected back along the same line (hence "backproject) - "Smear" the projection data back across the object space



Backprojection

Illustration of back projection



Projection Backprojection

http://www.owlnet.rice.edu/~elec539/Projects97/cult/node2.html

Backprojection

- Backprojection of a single measured projection at angle θ

$$b_\theta(x,y) = \int_{-\infty}^{\infty} g_\theta(R)\delta(x\cos\theta + y\sin\theta - R)\,dR$$

- The total backprojected image is the integral (sum) of this over all angles:

$$f_b(x,y) = \int_0^{\pi} b_\theta(x,y)\,d\theta$$

$$= \int_0^{\pi}\int_{-\infty}^{\infty} g_\theta(R)\delta(x\cos\theta + y\sin\theta - R)\,dR\,d\theta$$

Backprojection

- Now consider:

$$g_\theta(R) = F^{-1}_{1D,\rho}\{F(\rho,\theta)\} = \int_{-\infty}^{\infty} F(\rho,\theta)\exp(i2\pi\rho R)\,d\rho$$

- The backprojected image can be written as:

$$f_b(x,y) = \int_{-\infty}^{\infty}\int_{0}^{\pi}\int_{-\infty}^{\infty} F(\rho,\theta)\delta(x\cos\theta + y\sin\theta - R)\exp(i2\pi\rho R)\,dR\,d\theta\,d\rho$$

$$= \int_{0}^{\pi}\int_{-\infty}^{\infty} F(\rho,\theta)\exp(i2\pi\rho(x\cos\theta + y\sin\theta))\,d\theta\,d\rho$$

Backprojection

$$f_b(x,y) = \int_0^{\pi} \int_{-\infty}^{\infty} F(\rho,\theta) \exp(i2\pi\rho(x\cos\theta + y\sin\theta)) d\theta d\rho$$

- Comparing to inverse 2D FT of F(ρ,θ)

$$f\ (x,y) = \int_0^{2\pi} \int_0^{\infty} F(\rho,\theta) \exp(i2\pi\rho(x\cos\theta + y\sin\theta)) \rho d\theta d\rho$$

- Differences:
  - limits of integration
  - $d\theta d\rho$ vs. $\rho d\theta d\rho$

Backprojection

- To address difference #1: limit of integration

$$F(\rho,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp(-i2\pi(\rho\cos\theta x + \rho\sin\theta y))dxdy$$

$$F(-\rho,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp(-i2\pi(-\rho\cos\theta x - \rho\sin\theta y))dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp(-i2\pi(\rho\cos(\theta+\pi)x + \rho\sin(\theta+\pi)y))dxdy$$

$$= F(\rho,\theta+\pi)$$

Backprojection

$$f_b(x,y) = \int_0^{\pi} \int_{-\infty}^{\infty} F(\rho,\theta)\exp(i2\pi\rho(x\cos\theta + y\sin\theta))d\rho d\theta$$

$$= \int_0^{2\pi} \int_0^{\infty} F(\rho,\theta)\exp(i2\pi\rho(x\cos\theta + y\sin\theta))d\rho d\theta$$

Address difference #1

$$= \int_0^{2\pi} \int_0^{\infty} \frac{F(\rho,\theta)}{\rho}\exp(i2\pi\rho(x\cos\theta + y\sin\theta))\rho d\rho d\theta$$

$$= F_{2D}^{-1}\left\{\frac{F(\rho,\theta)}{\rho}\right\}$$

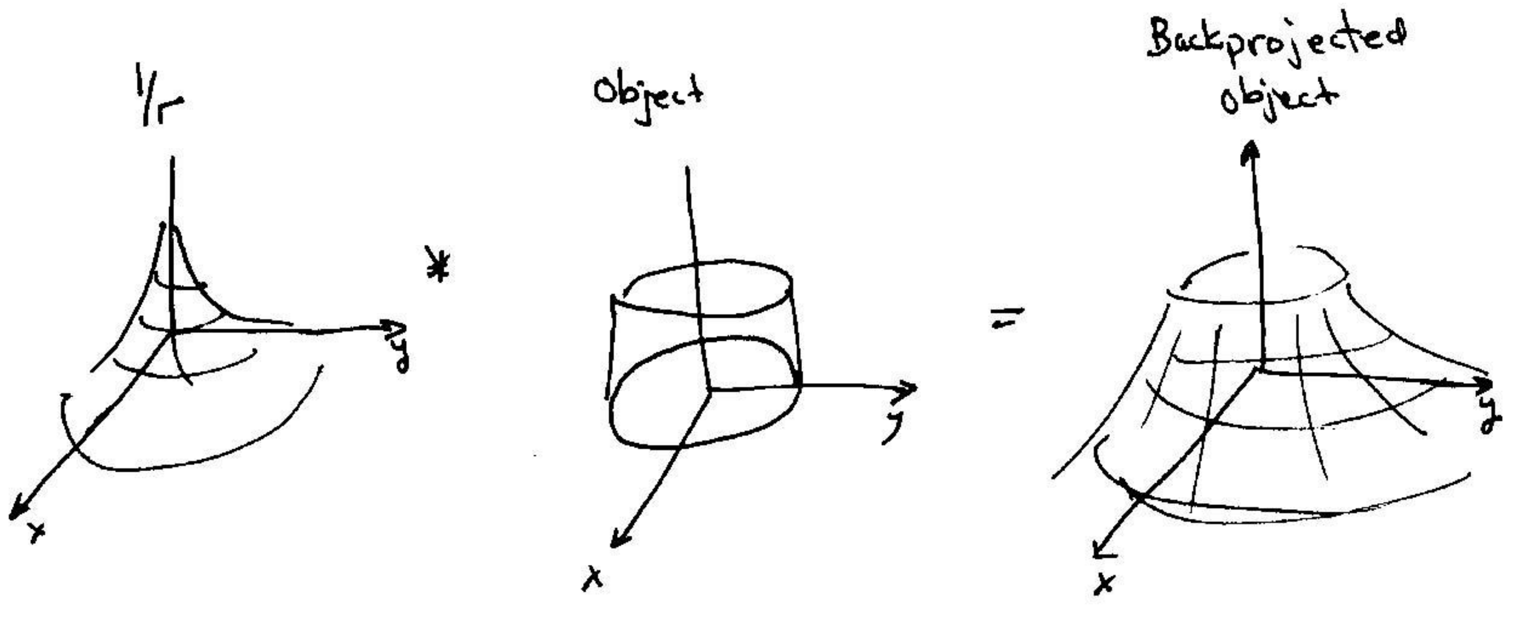
$$= f(x,y) ** F_{2D}^{-1}\left\{\frac{1}{\rho}\right\}$$

$$= f(x,y) ** \frac{1}{r}$$

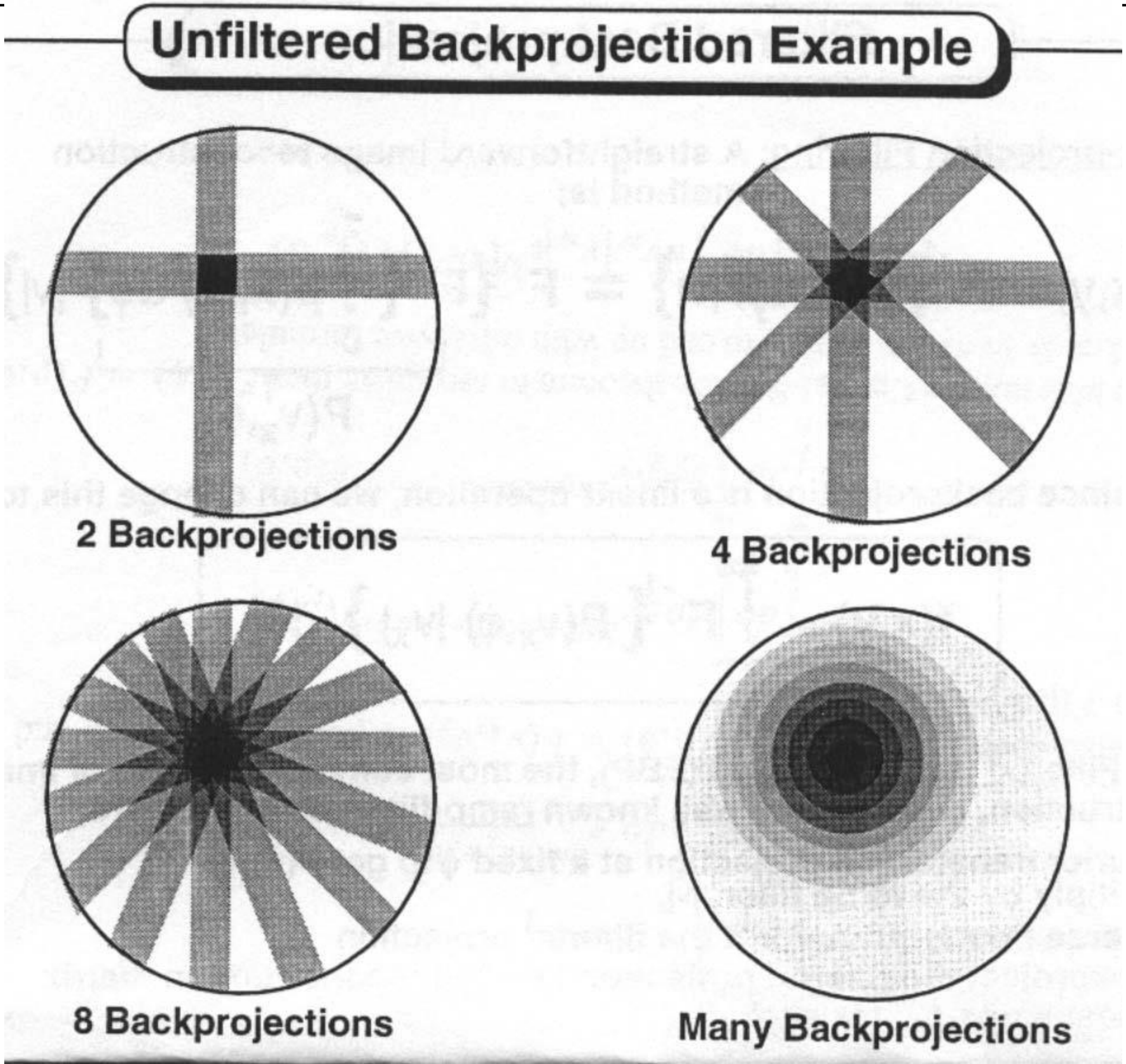
Address difference #2

Backprojection

- Backprojected image is equal to the desired image convolved with a 1/*r* blurring function



Unfiltered Backprojection



Backprojection Summary

1. Backprojection of a single measured projection at angle θ

$$b_\theta(x,y) = \int_{-\infty}^{\infty} g_\theta(R)\delta(x\cos\theta + y\sin\theta - R)\,dR$$

2. Total backprojected image - integral (sum) of this over all angles:

$$f_b(x,y) = \int_0^{\pi} b_\theta(x,y)\,d\theta$$

3. Backprojected image is equal to the desired image convolved with 1/*r*

$$F_{2D}\{f_b(x,y)\} = \frac{F(r,q)}{r}, \quad f_b(x,y) = f(x,y) ** \frac{1}{r}$$

Backprojection

- Image reconstruction – remove blurring function to recover $f(x,y)$ from $f_b(x,y)$ (or $b(x,y)$)

- Methods
  - Backprojection-filtered Method
  - Filtered-backprojection Method

Questions?

Backprojection

Backprojection Filtering Method

2. Backprojection-Filtering Method

- Steps:
  1. Backproject all projections, $g_\theta(R)$ to get $f_b(x,y)$
  2. Forward 2D FT to get $\dfrac{F(\rho,\theta)}{\rho}$
  3. Filter with ρ (or $|\rho|$) to get $F(\rho,\theta)$.
  4. Inverse 2D FT to get $\widehat{f}(x,y)$

2. Backprojection-Filtering Method

- Interpolation happens in image domain (better for artifacts)
- The big problem is that the blur $\left(\frac{1}{r}\right)$ extends over a long distance and you need accurately backproject over an extended region
- Presented for comparison (doesn’t really get used)

Questions?

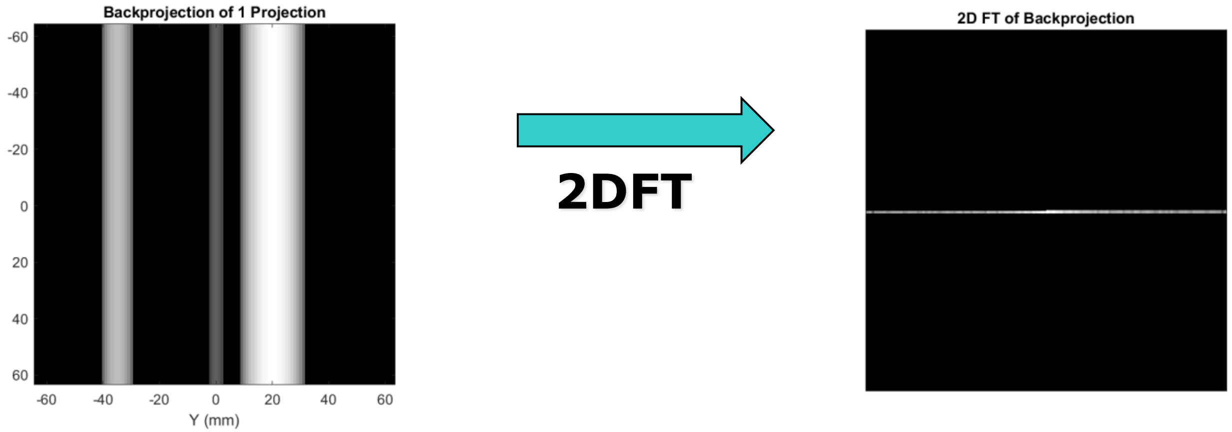
Backprojection-Filtering Method

Fourier Superposition and Filtering

3. Fourier Superposition-Filtering

- Steps:
  1. 1D FT all projections, $g_\theta(r)$, to get $G_\theta(\rho)$.
  2. Add $G_\theta(\rho)$ to the 2D FT space to $\hat{F}(\rho, \theta)$ along the line at θ for all θ. (Note: that this is equivalent to backprojection at all θ)
  3. Filter with $|\rho|$ to get $F(\rho, \theta)$.
  4. Inverse 2D FT to get $\hat{f}(x, y)$.

3. Fourier Superposition-Filtering



- Interpolation happens in Fourier image domain (artifacts)
- Can filter before or after superposition (before is usually better)
- This is equivalent to a method called convolution gridding

Questions?

Fourier Superposition - Filtering Method

Filtered Backprojection Method

4. Filtered Backprojection Method

- Solution to eliminate 2D transforms:
  - Still use backprojection
  - Undo the 1/r blur without:
    - 2D transforms and
    - Backprojection over extended space
- **Idea: Reverse the order of backprojection and filtering**

4. Filtered Backprojection Method

Steps:

1. Filter the backprojection with a |ρ| filter (ramp filter)
   - Fourier method:

$$F_{1D}\left\{g'_q(R)\right\} = \left|r\right| F_{2D}(r,q)$$

$$g'_q(R) = F_{1D}^{-1}\left\{\left|r\right| F_{1D}\left\{g_q(R)\right\}\right\} = F_{1D}^{-1}\left\{\left|r\right| F_{2D}(r,q)\right\}$$

   - Convolution method:

$$g'_\theta(R) = g_\theta(R) * c(R), \text{ where } \; c(R) \approx F_{1D}^{-1}\left\{\left|\rho\right|\right\}$$

2. Backproject for all angles to get $\widehat{f}(x,y)$

4. Filtered Backprojection Method

- Putting it all together:

$$\hat{f}(x,y) = \int_0^{\pi} \int_{-\infty}^{\infty} F_{1D}^{-1}\left\{ |\rho| F_{1D}\left\{ g_\theta(R) \right\} \right\} \delta(x\cos\theta + y\sin\theta - R)\, dR\, d\theta$$

$$= \int_0^{\pi} \int_{-\infty}^{\infty} F_{1D}^{-1}\left\{ |\rho| F(\rho,\theta) \right\} \delta(x\cos\theta + y\sin\theta - R)\, dR\, d\theta$$

$$= \int_{-\infty}^{\infty} \int_0^{\pi} \int_{-\infty}^{\infty} |\rho| F(\rho,\theta) \exp(i2\pi\rho R)\, \delta(x\cos\theta + y\sin\theta - R)\, d\rho\, dR\, d\theta$$

$$= \int_0^{\pi} \int_{-\infty}^{\infty} F(\rho,\theta) \exp(i2\pi\rho(x\cos\theta + y\sin\theta))\, |\rho|\, d\rho\, d\theta$$

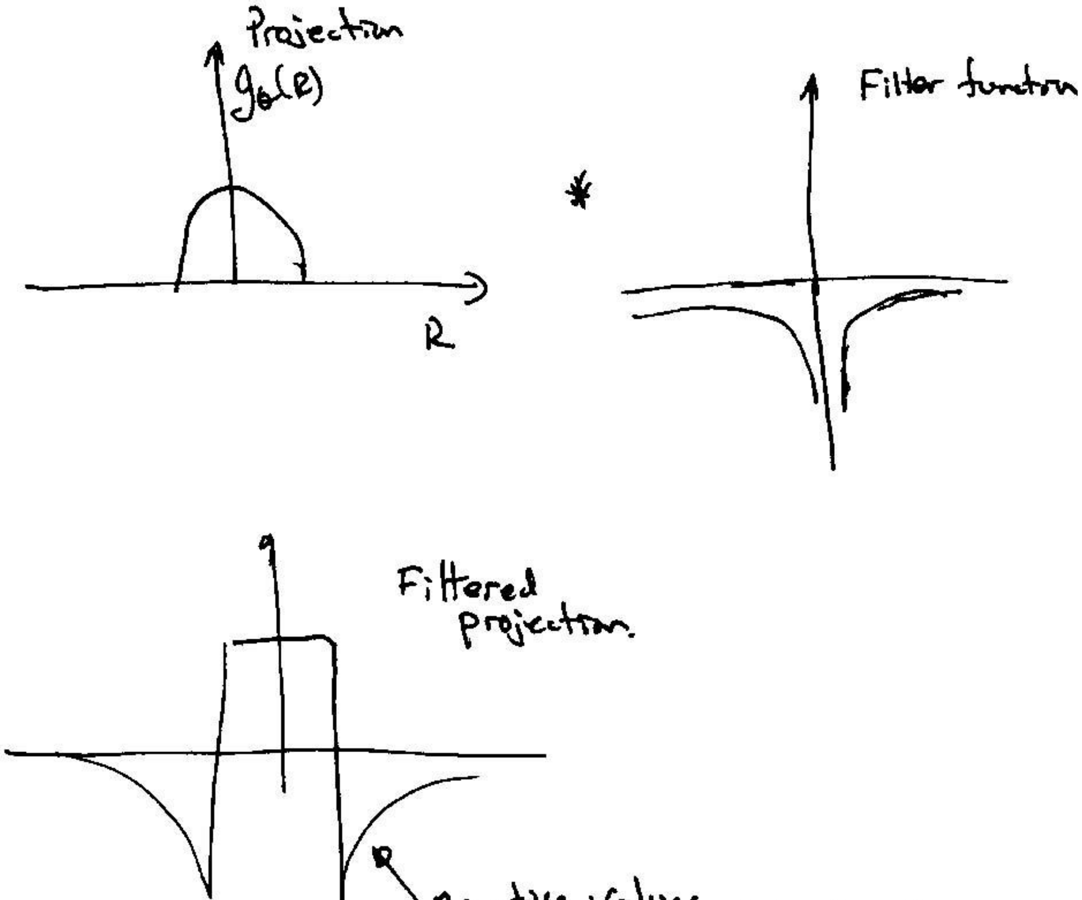
4. Filtered Backprojection Method

- Changing the limits of integration to $(0, 2\pi)$ and $(0, \infty)$,

$$\hat{f}(x, y) = \int_0^{2\pi} \int_0^{\infty} F(\rho, \theta) \exp(i2\pi\rho(x \cos\theta + y \sin\theta)) \rho d\rho d\theta$$

$$= F_{2D}^{-1}\{F(\rho, \theta)\} = f(x, y)$$

4. Filtered Backprojection Method

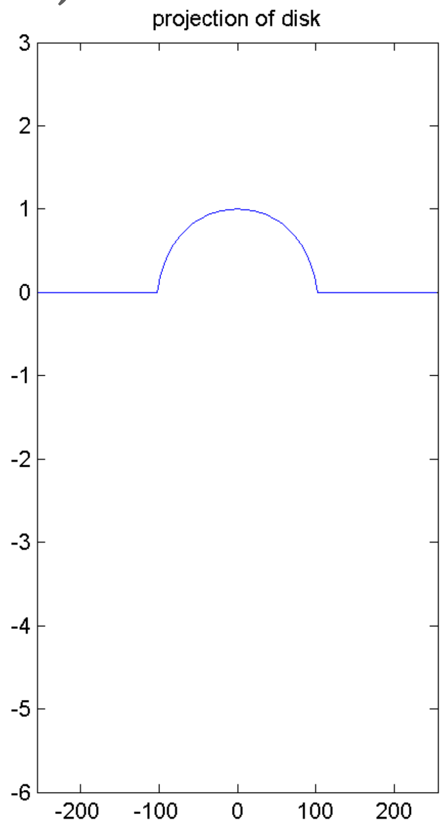


negative values are necessary to correct for projections from other ~~[illegible]~~ angles

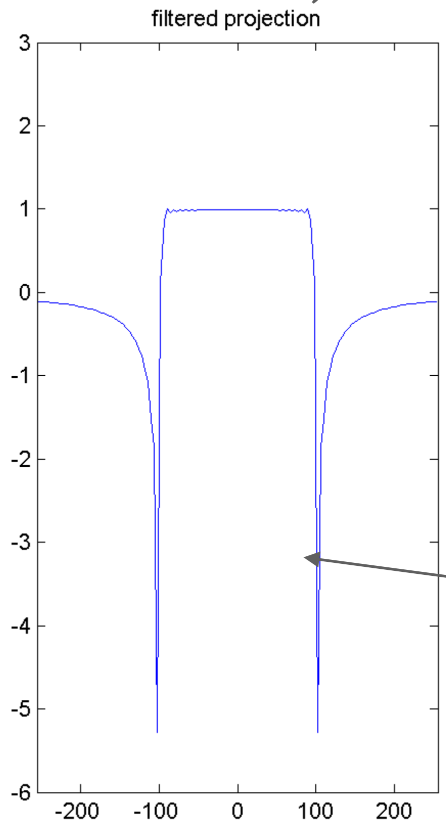
4. Filtered Backprojection Method

One Example – projection of a disk

Projection of disk



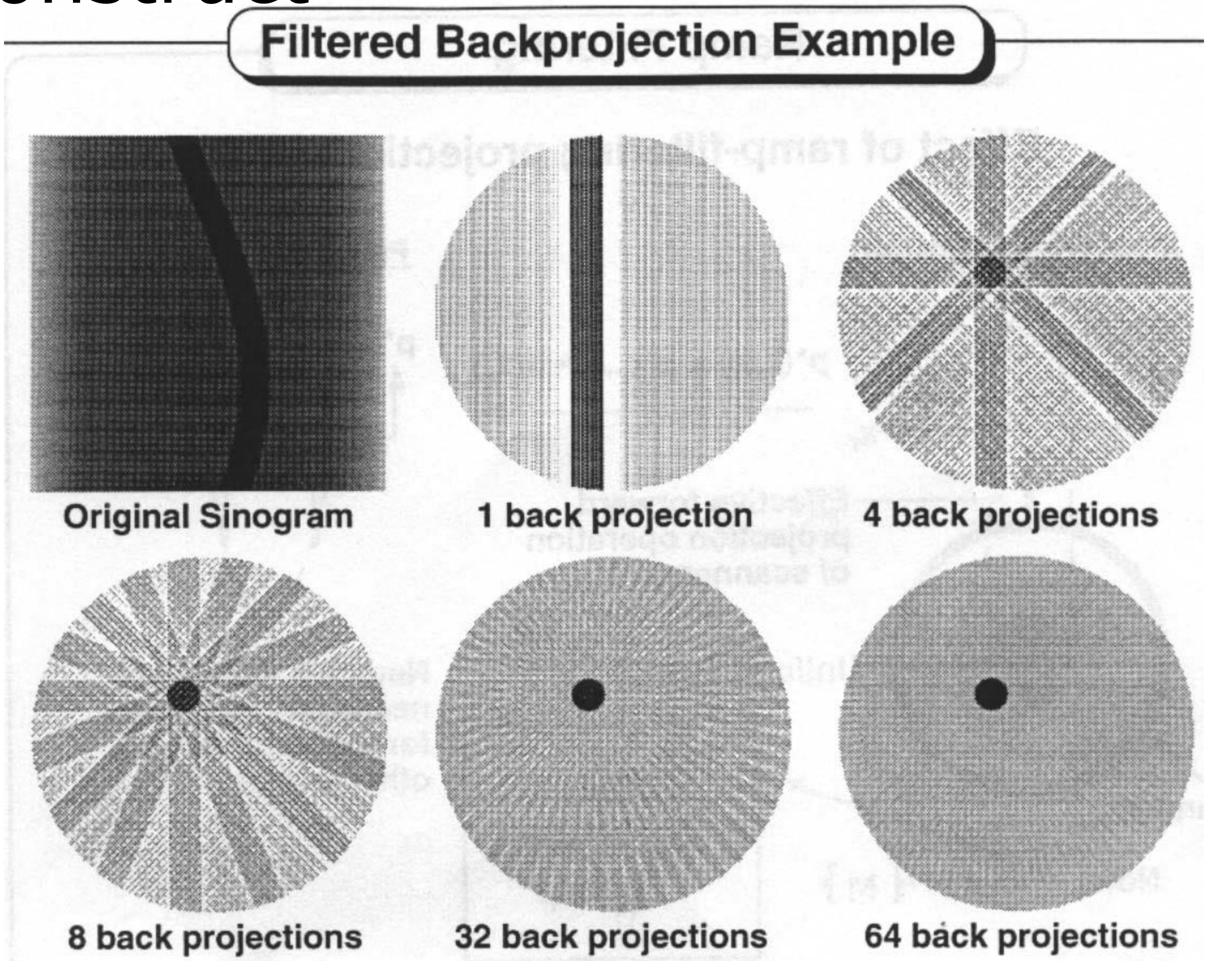
Filtered Projection



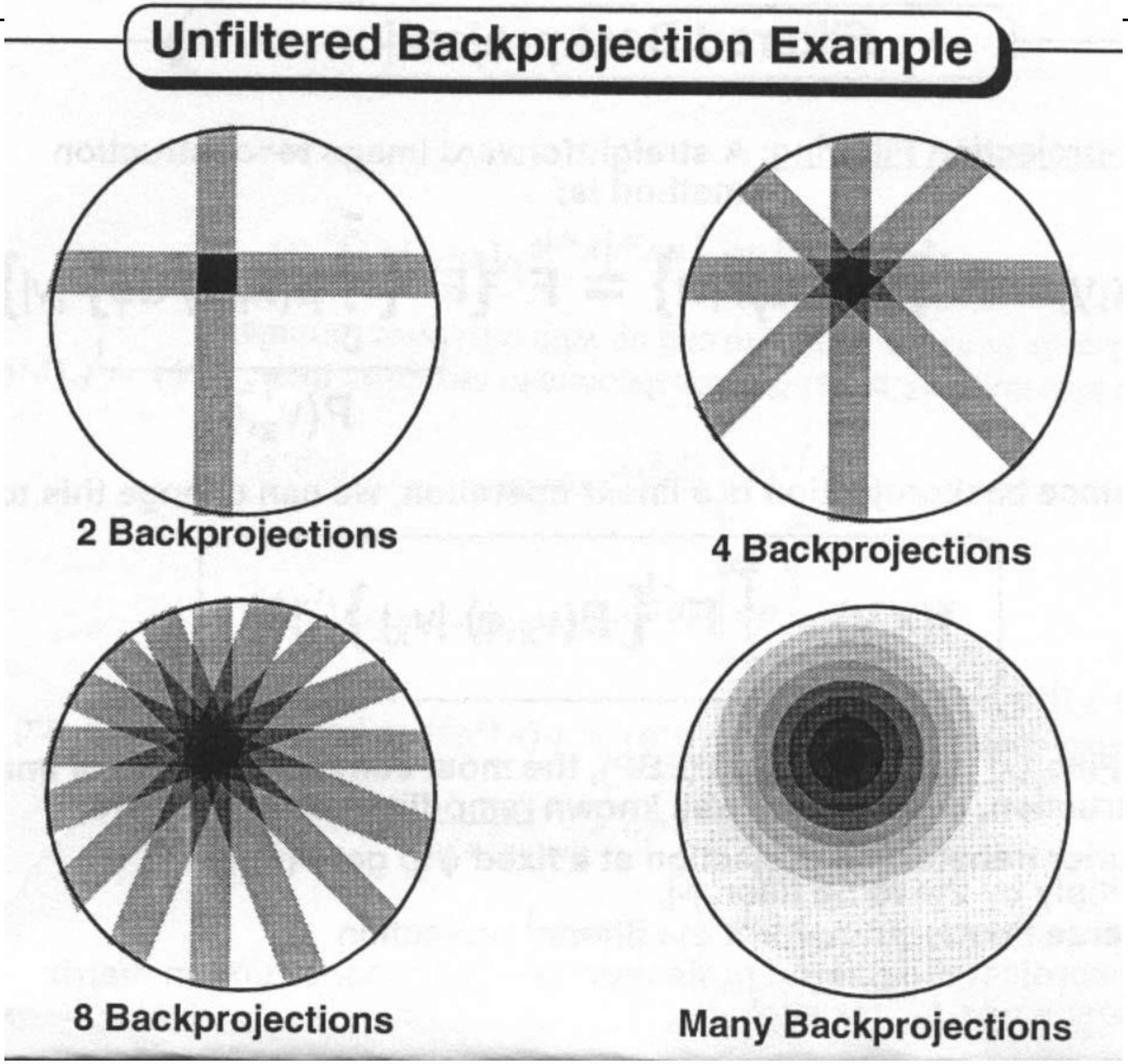
Negative values necessary to correct projections from other angles

4. Filtered Backprojection Method

- Example cont. – backprojecting to reconstruct



Recall Unfiltered Backprojection



4. Filtered Backprojection Method

Steps:

1. Filter the backprojection with a |ρ| filter (ramp filter)
   - Fourier method:

$$g'_\theta(R) = F_{1D}^{-1}\{|\rho| F_{1D}\{g_\theta(R)\}\} = F_{1D}^{-1}\{|\rho| F_{2D}(\rho,\theta)\}$$

   - Convolution method:

$$g'_\theta(R) = g_\theta(R) * c(R), \text{ where } \; c(R) \approx F_{1D}^{-1}\{|\rho|\}$$

2. Backproject for all angles to get $\hat{f}(x, y)$

4. Filtered Backprojection Method

- Filter Methods:
  - Fourier method – needs two 1D transforms (1D FT and 1D inverse FT) -
  - Convolution Method – one convolution – simpler computation, needs filter $c(R)$

4. Filtered Backprojection Method

- Convolution Filtering Method:
- Convolution filter function $-c(R) = F^{-1}\{|\rho|\}$
- Problem: $c(R) = F^{-1}\{|\rho|\} = \int_{-\infty}^{\infty} |\rho| \exp(i2\pi R\rho)\, d\rho$
  does not exist (not integrable), goes to ∞

4. Filtered Backprojection Method

- Find FT of a variety of functions that approach $|\rho|$ in the limit
- Filter 1: $c(R) = \lim_{\varepsilon \to 0} F^{-1}\{|\rho| \exp(-\varepsilon|\rho|)\}$

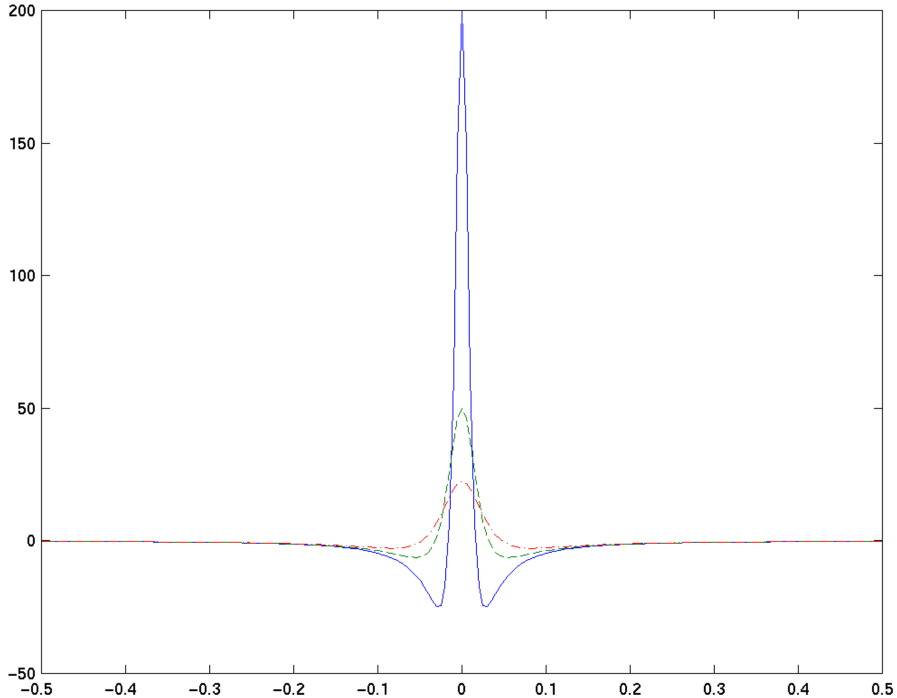
$$= \lim_{\varepsilon \to 0} \frac{2(\varepsilon^2 - 4\pi^2 R^2)}{(\varepsilon^2 + 4\pi^2 R^2)^2}$$

  - for small R, $c(R)$ will approach $2/\varepsilon^2$
  - for large R, $c(R)$ will approach $-1/2\pi^2 R^2$

Macovski

4. Filtered Backprojection Method

$c(R)$



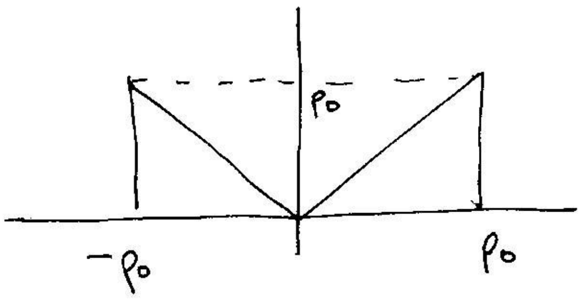
- The function $\exp(-\varepsilon|\rho|)$ cuts off the high frequency parts of $|\rho|$ - cutoff frequency of ρ_0

4. Filtered Backprojection Method

- Filter 2: rect function

$$C(\rho) = |\rho| \operatorname{rect}\left(\frac{\rho}{2\rho_0}\right)$$

$$= \rho_0 \left[\operatorname{rect}\left(\frac{\rho}{2\rho_0}\right) - \operatorname{tri}\left(\frac{\rho}{\rho_0}\right) \right]$$

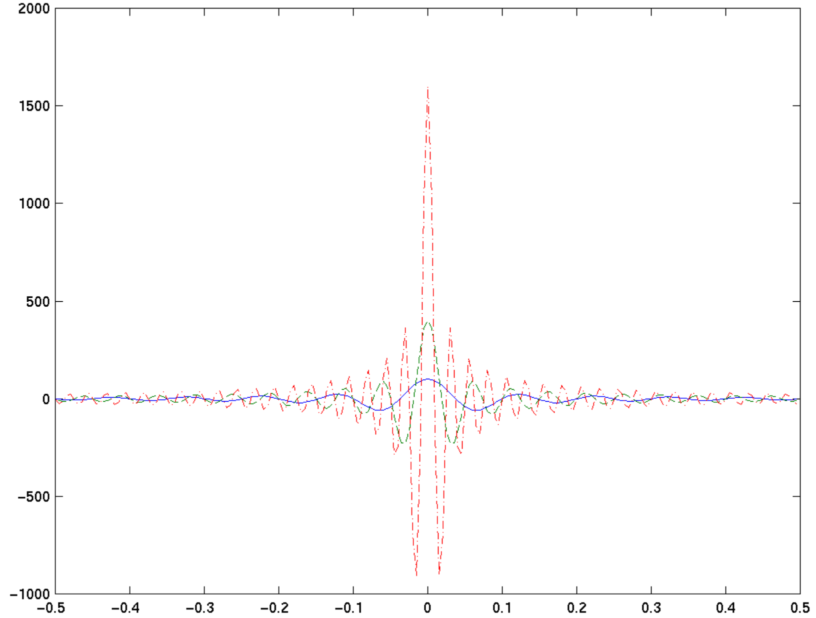


$$c(R) = \rho_0^2 \left(2\operatorname{sinc}(2\rho_0 R) - \operatorname{sinc}^2(\rho_0 R) \right)$$

4. Filtered Backprojection Method

- Has substantial ringing effect

$c(R)$

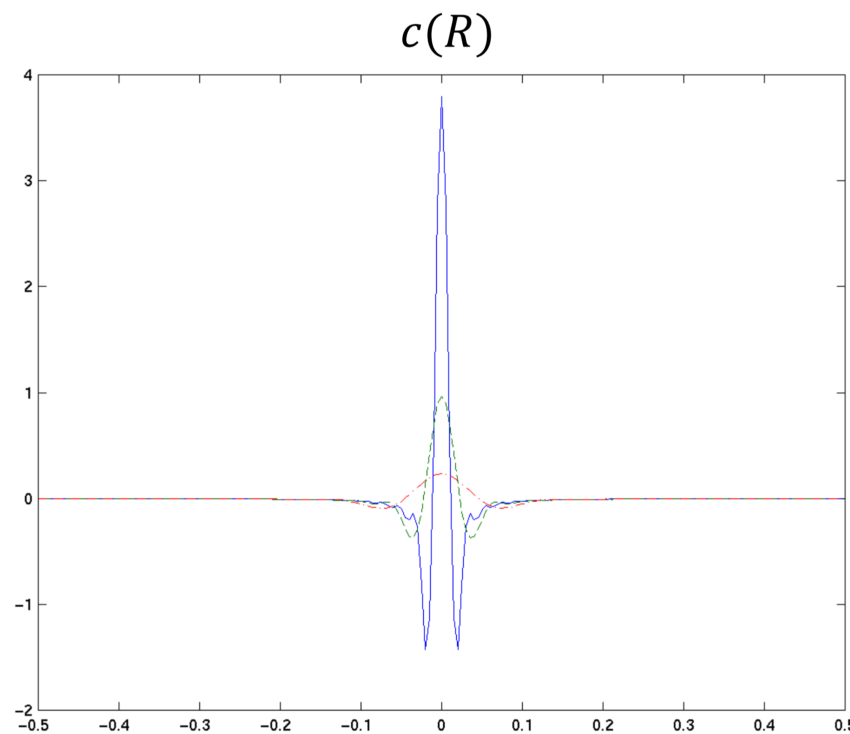


4. Filtered Backprojection Method

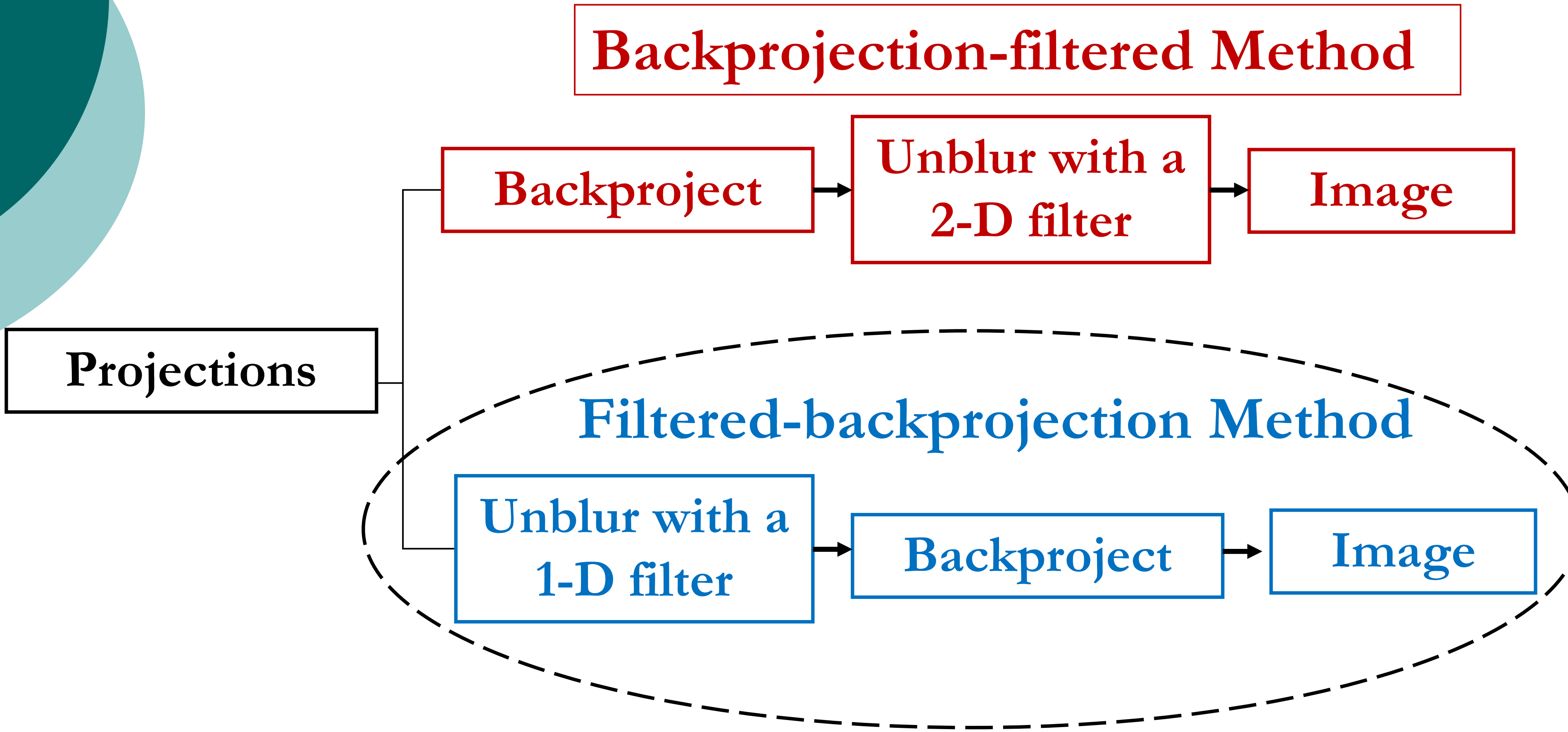
- Filter 3: Gaussian or Hanning filter (suppress ringing)

$$C(r)=|r|\exp\left(-p\left(\frac{r}{r_0}\right)^2\right)$$

$$C(r)=|r|\,\mathrm{rect}\left(\frac{r}{2r_0}\right)\left(\tfrac{1}{2}+\tfrac{1}{2}\cos\left(p\frac{r}{r_0}\right)\right)$$



Comparison between Two Backprojection Methods



Only need 1D transform
Less computation
Widely used

Questions?

Filtered-backprojection Method

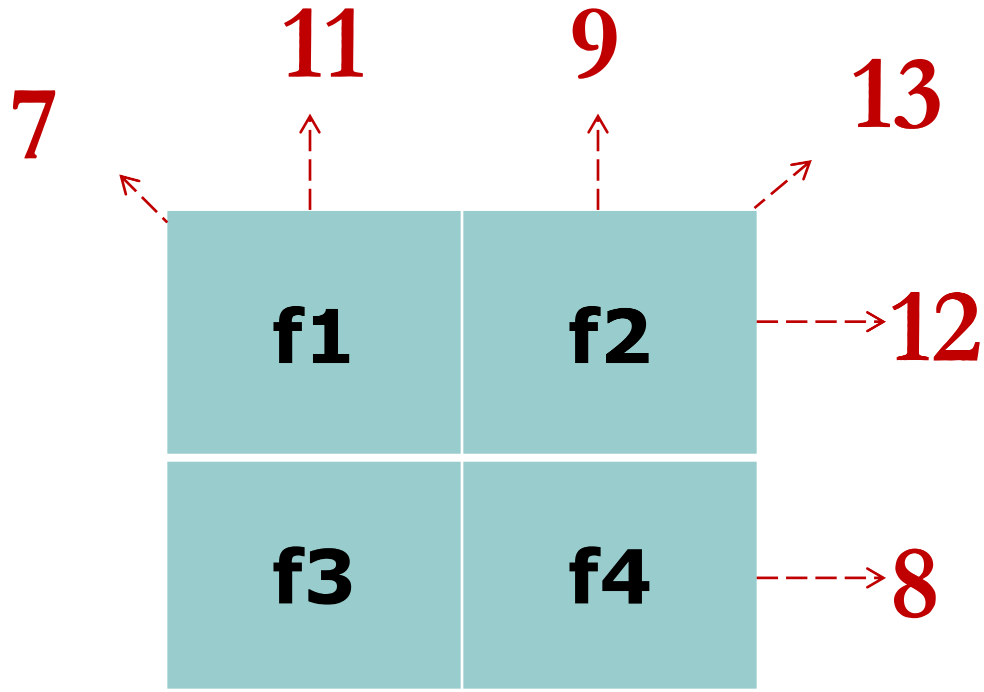
5. Algebraic Reconstruction Technique (ART)

- Like regular backprojection, but uses iterative corrections
- Straightforward, although computationally inefficient solution involves linear algebra
- Used initially, relatively old and now seldomly used now

5. Algebraic Reconstruction Technique (ART)

- Example

Object and projection



5. Algebraic Reconstruction Technique (ART)

- g_i be the measured projections and f_{ij}^{q} be the image at iteration q

Image @ q

Image @ q+1

$$f_{ij}^{q+1} = f_{ij}^{q} + \beta \underbrace{\frac{g_i - \sum_j f_{ij}^{q}}{N}}$$

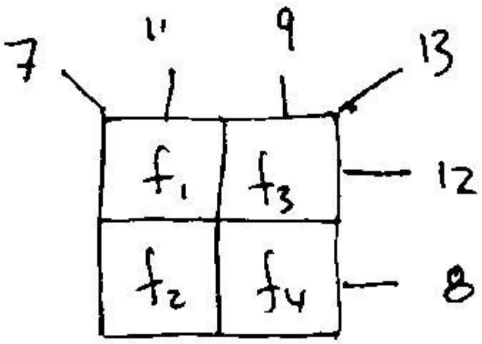
Backproject the difference

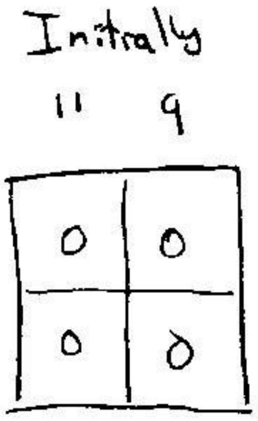
Regularization parameter controls rate of convergence (1 for example)

This represents the calculated projection *j* from current iteration (*q*) of the object.

5. Algebraic Reconstruction Technique (ART)

- q = 1, Looking at the projections from top to bottom



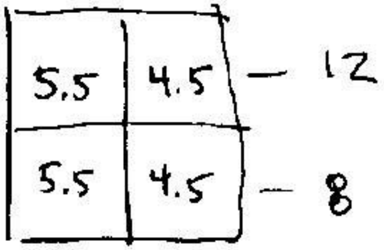


$$f_1^1 = 0 + \frac{11-0}{2} = 5.5; \qquad f_3^1 = 0 + \frac{9-0}{2} = 4.5$$

$$f_2^1 = 0 + \frac{11-0}{2} = 5.5; \qquad f_4^1 = 0 + \frac{9-0}{2} = 4.5$$

5. Algebraic Reconstruction Technique (ART)

- q = 2, looking at the left-right projections:

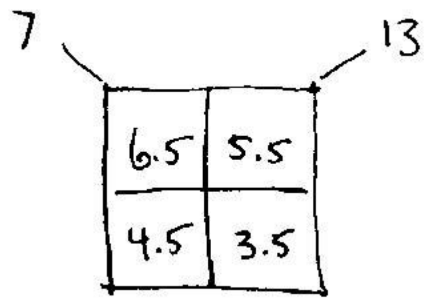


$$f_1^2 = 5.5 + \frac{12-(5.5+4.5)}{2} = 6.5; \quad f_3^2 = 4.5 + \frac{12-(5.5+4.5)}{2} = 5.5$$

$$f_2^2 = 5.5 + \frac{8-(5.5+4.5)}{2} = 4.5; \quad f_4^2 = 4.5 + \frac{8-(5.5+4.5)}{2} = 3.5$$

5. Algebraic Reconstruction Technique (ART)

- q = 3, looking at the diagonal projections



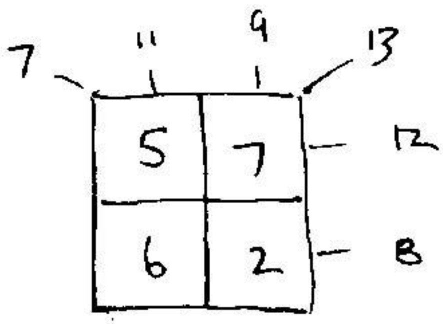
$$f_1^3 = 6.5 + \frac{7-(6.5+3.5)}{2} = 5; \qquad f_3^3 = 5.5 + \frac{13-(5.5+4.5)}{2} = 7$$

$$f_2^3 = 4.5 + \frac{13-(5.5+4.5)}{2} = 6; \qquad f_4^3 = 3.5 + \frac{7-(6.5+3.5)}{2} = 2$$

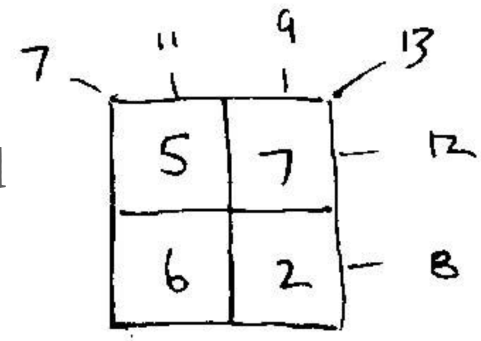
5. Algebraic Reconstruction Technique (ART)

- Final Solution

Final Sol.



Object and projection



5. Algebraic Reconstruction Technique (ART)

- Drawbacks

- In the presence of noise (that is, the projections aren’t exactly consistent), this method converges very slowly, and in some cases, not at all.

- Usually, one goes through the entire set of projections multiple times.

Questions?

Algebraic Reconstruction Technique (ART)

Reconstruction Methods

1. **Direct Fourier Interpolation Method – CT project**
2. **Backprojection-Filtering Method – Introduce backprojection concept**
3. **Direct Fourier Superposition and Filtering Method**
4. **Filtered Backprojection Method – CT project**
5. **Algebraic Reconstruction Technique (ART) – conventional method**

Image Recon

- Matlab Demos

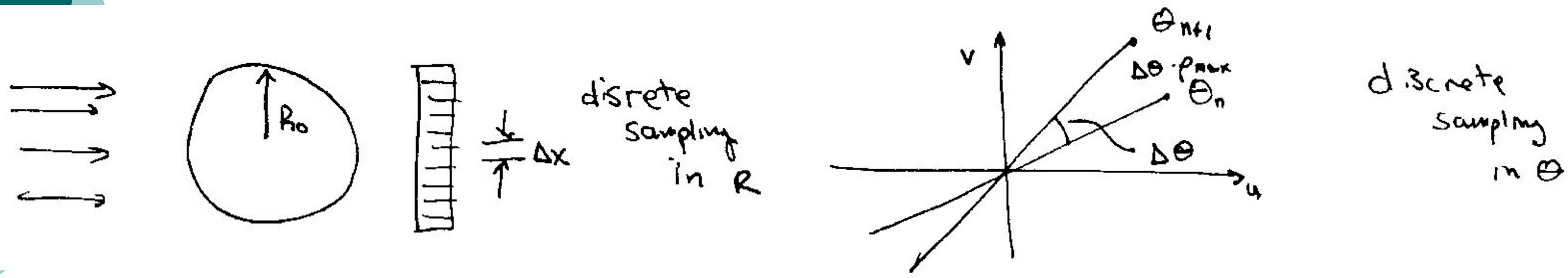
Practical considerations

Sampling

Fan-beam geometric

Circular convolution

Sampling in R and θ



- Usually we think about sampling θ finely enough for the information content in each projection

Sampling in R and θ

- Recall the minimum required sampling distance is given by FOV:

$$\Delta k_{\max} \le \frac{1}{FOV} = \frac{1}{2R_0}$$

- And the maximum extent in k-space is given by the spatial resolution:

$$\rho_{\max} = \frac{1}{2\Delta x}$$

Sampling in R and θ

- Recognizing the arc length in k-space between radial lines is approx:

$$\Delta k_{\max} = \Delta\theta \cdot \rho_{\max}$$

- Then we can set the critical angular sampling as:

$$\Delta\theta \leq \frac{\Delta x}{R_0}$$

Number of angles

- Number of projection angles:

$$N_{proj} = \frac{\theta_{\max} - \theta_{\min}}{\Delta\theta}$$

- Which for 512 samples on each projection over π radians is:

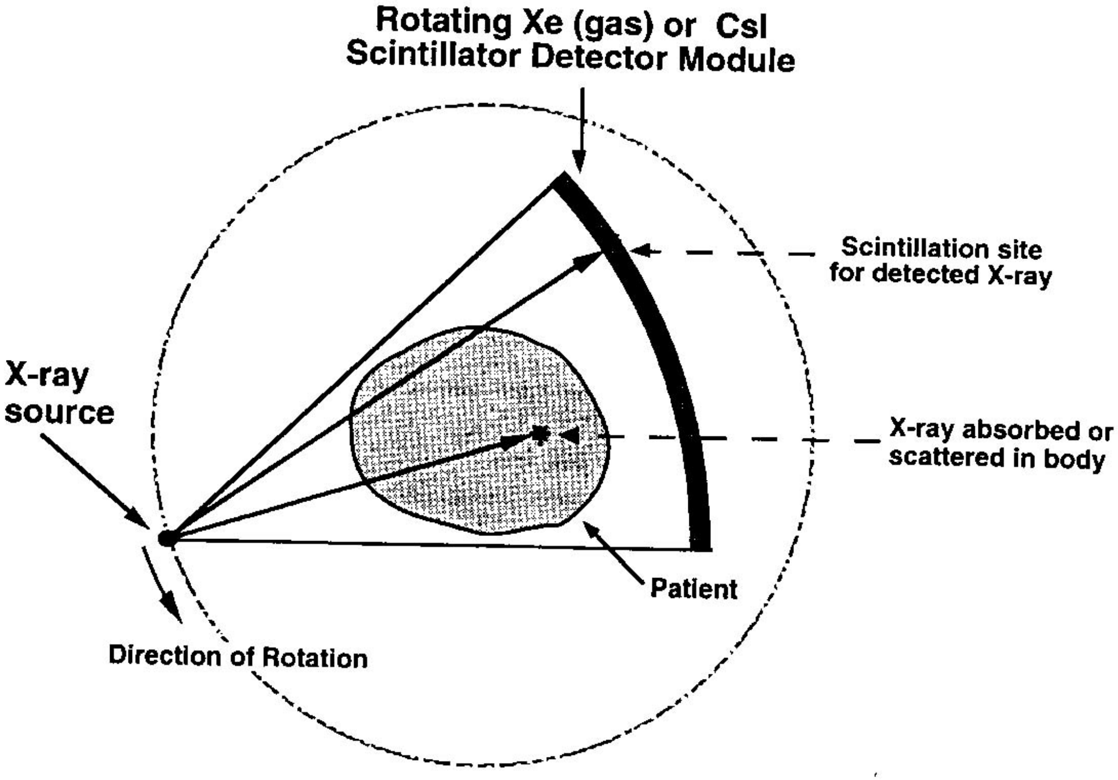
$$\frac{2R_0}{\Delta x} = 512 \quad \Delta\theta \le \frac{1}{256}$$

and then

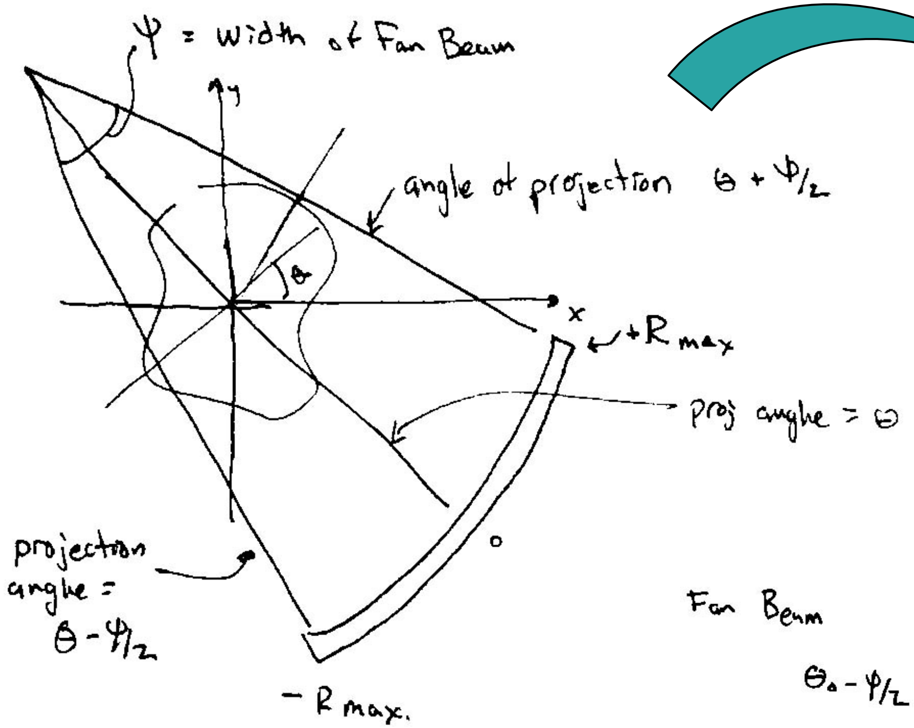
$$N_{proj} = \frac{\theta_{\max} - \theta_{\min}}{\Delta\theta} = 256\pi \approx 804$$

Fan Beam Geometry

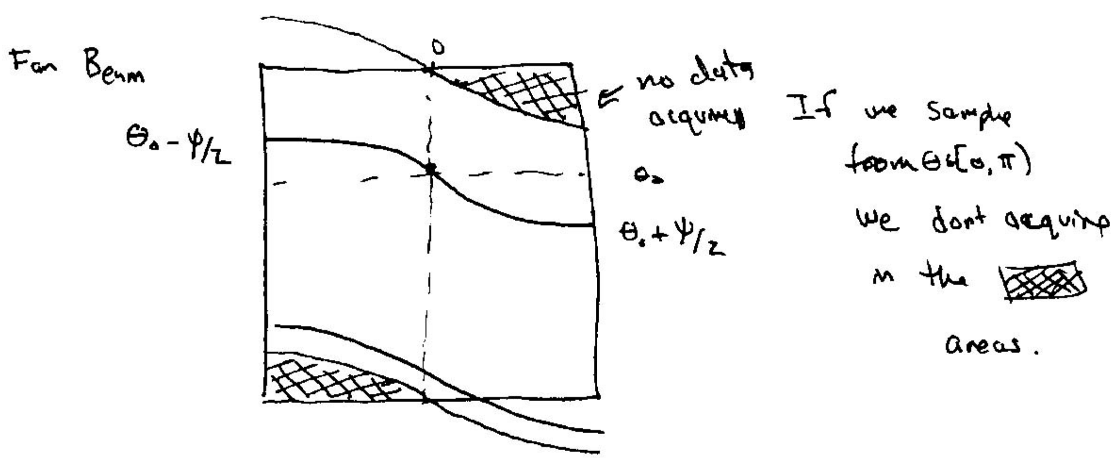
- Beams are no longer parallel



Fan Beam Geometry

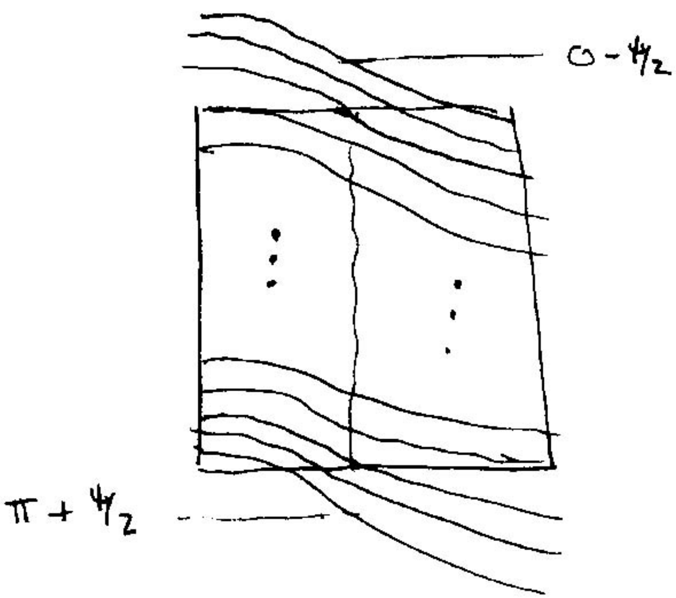


Sinogram



Fan Beam Geometry

- Need to acquire slightly more than π radians to fill sinogram space
- Need $\theta \in \left[-\frac{\psi}{2}, \pi + \frac{\psi}{2}\right]$ or $\pi + \psi$ radians



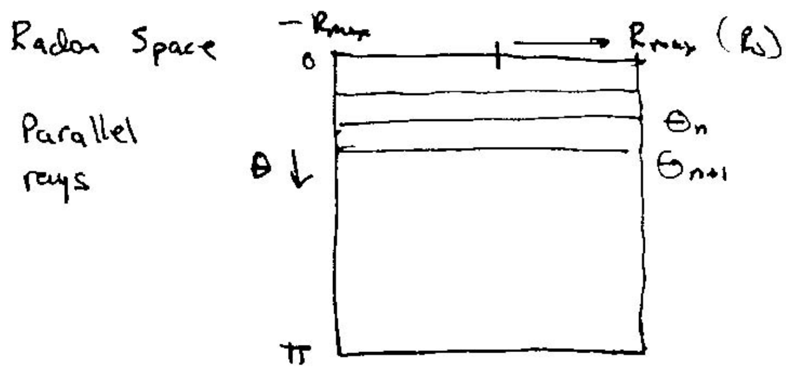
For Full Data

we need

$\theta \in [-\psi/2, \pi + \psi/2]$

Reconstruction of Fan Beam Data

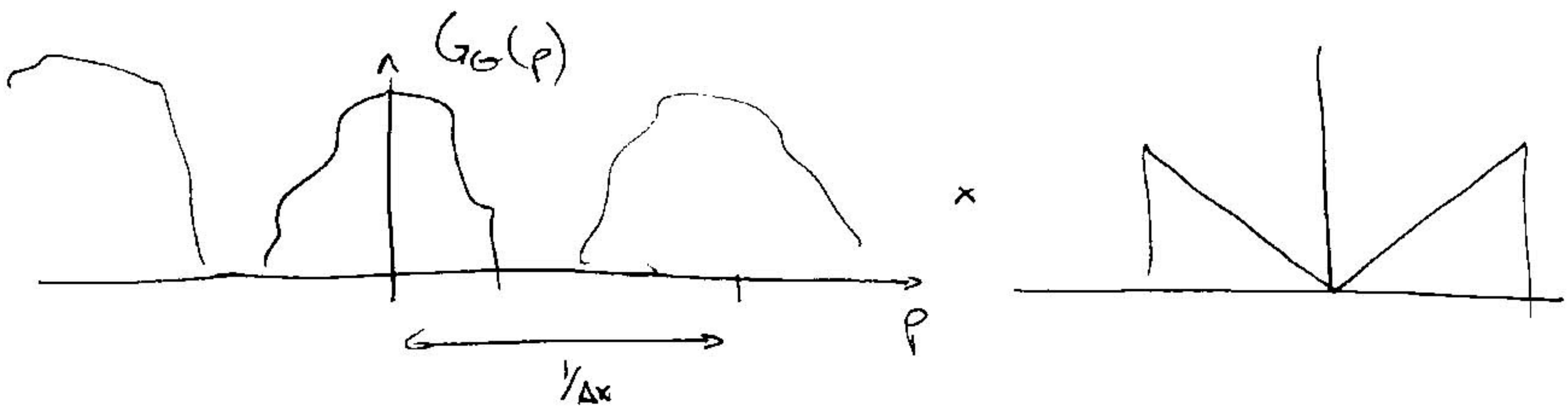
- Interpolate to parallel ray sinogram and then use any method including Fourier interpolation



- Other methods
  - Fan beam filtered backprojection
  - Fan beam ART

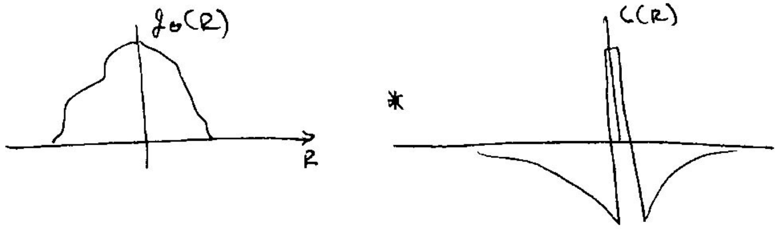
Circular convolution

- When filtering in the Fourier domain for filtered backprojection, one needs to worry about circular convolution

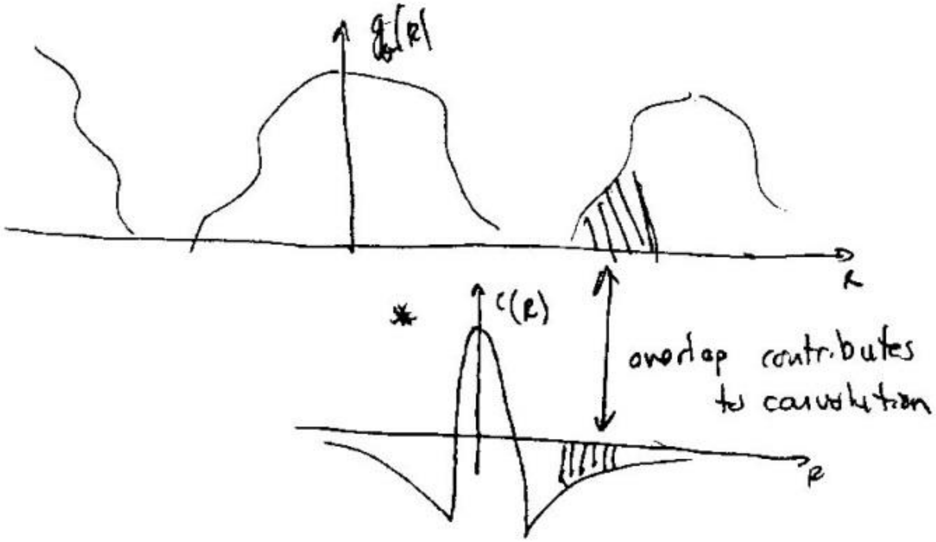


Circular convolution

- In the image domain instead of this:

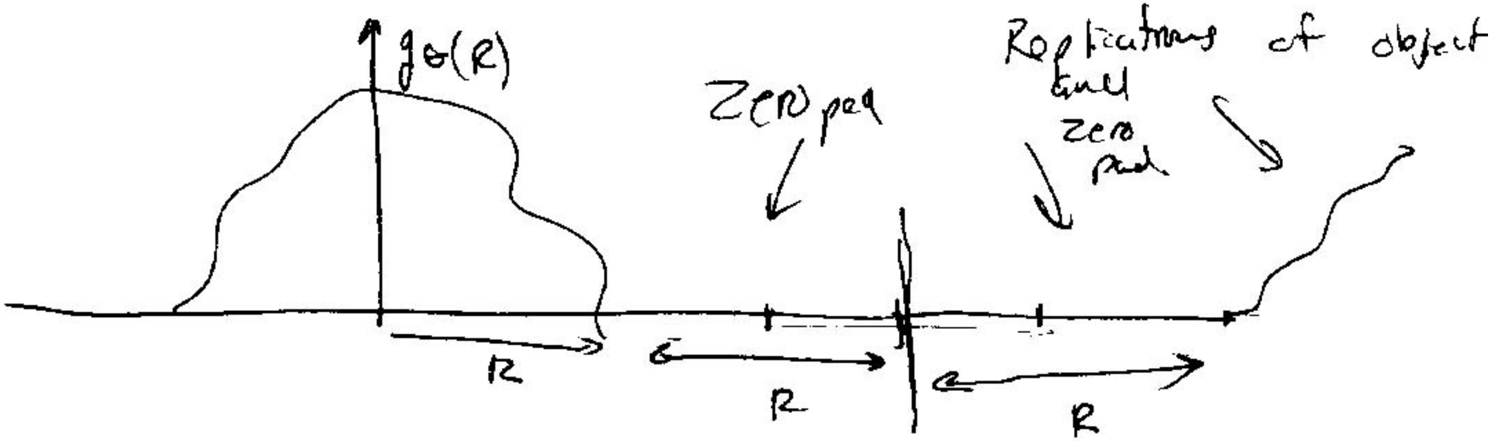


- We get this:



Circular convolution

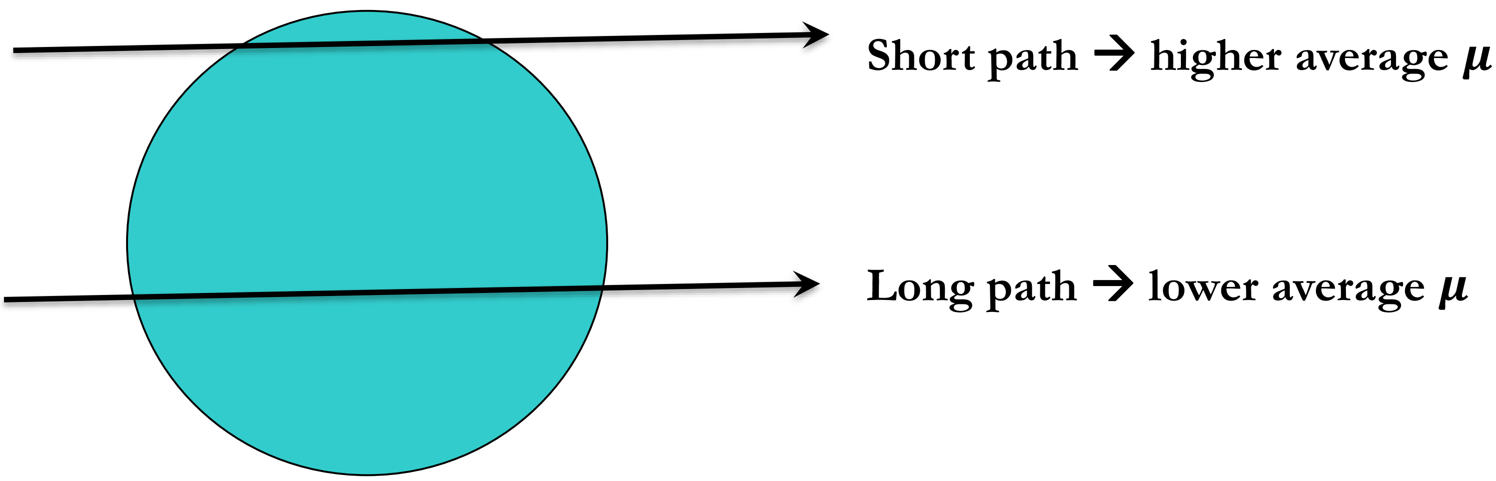
- Solution: zero pad before taking 1D FT:



which pushes the replicates out farther, minimizing these effects

Beam Hardening

- We talked about beam hardening making deeper structure appear to have a lower μ than expected.

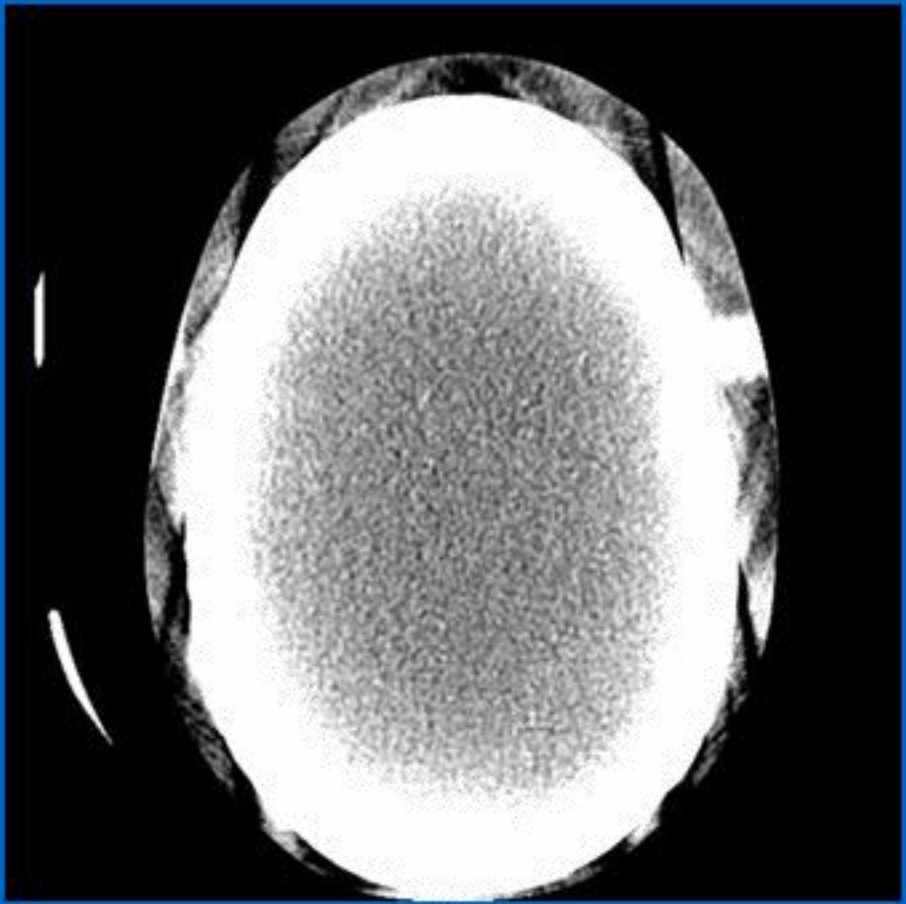


Beam Hardening

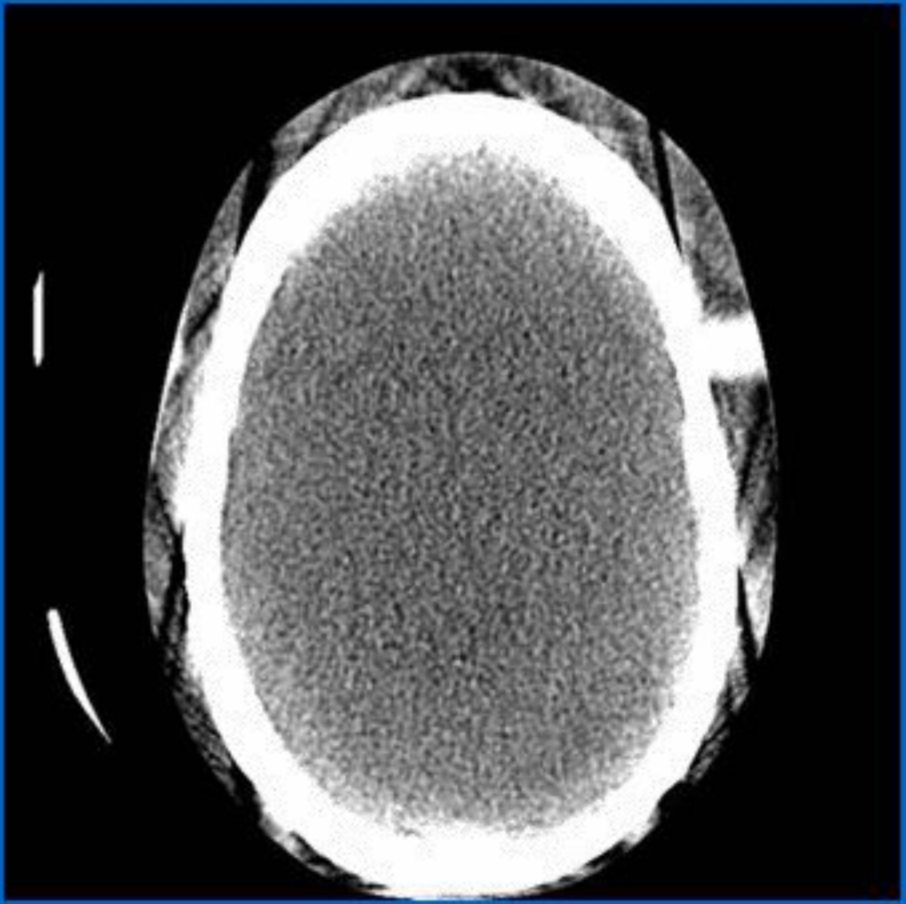
- Results in apparent lower μ in center of objects (cupping)

- Also shadows around implants

Beam hardening correction software



Raw image



After beam hardening correction

http://www.impactscan.org/slides/impactcourse/artefacts/img10.gif

Beam Hardening Artifact