Solutions to HW #2

1. Recall that

$$p(t) = a(t)e^{i\omega_0 t}\text{, where } \omega_0 = 2\pi f_0$$

and

$$P(f) = A(f - f_0)\,.$$

With attenuation:

$$a_z(t) = e^{-i\omega_0 t} p_z(t)$$

$$= \int e^{-2z\alpha(f)} A(f - f_0) e^{i2\pi(f-f_0)t} df$$

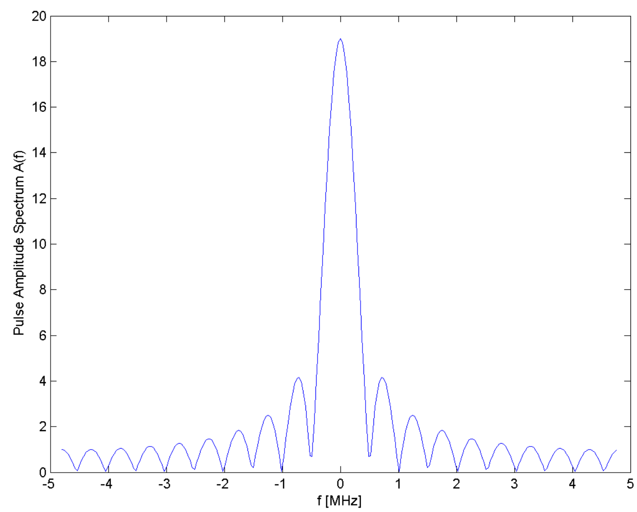
$$= \int e^{-2z\alpha(f+f_0)} A(f) e^{i2\pi ft} df$$

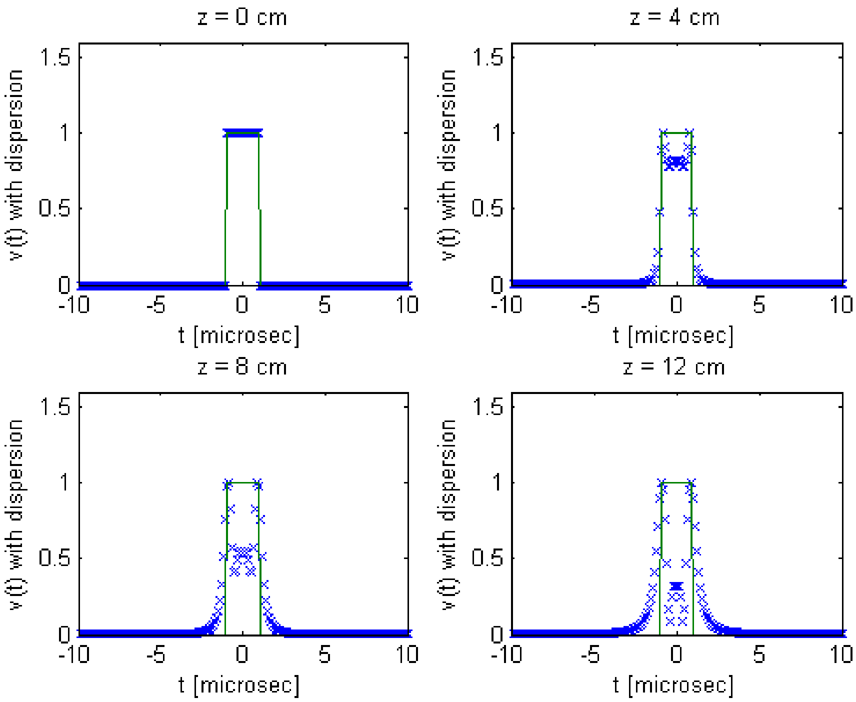
$$= F^{-1}\left\{e^{-2z\alpha(f+f_0)}\right\} * a(t)$$

$$= d_z{}'(t) * a(t)$$

Here, we are given $a(t)$ and we determine $A(f)$. The attenuation function is 1 dB/cm/MHz, which is approximately $\exp(-2z\beta\,|\,f\,|)$, were $\beta = 0.1$ (MHz cm)$^{-1}$. For this problem, we will use $\exp(-2z\beta\,|\,f + f_0\,|)$.

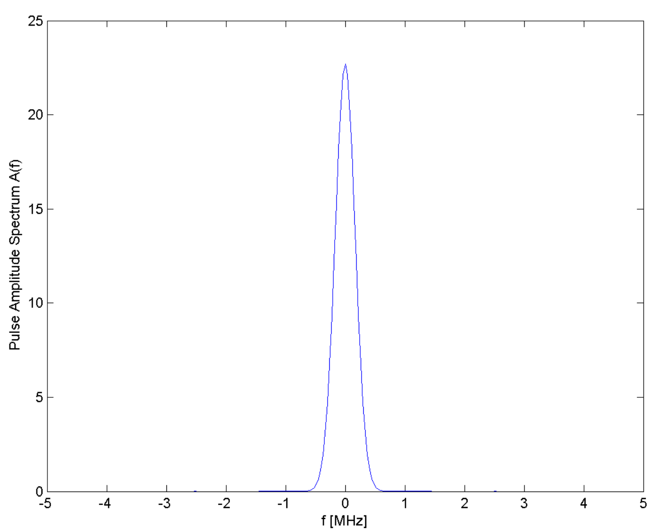
(a)

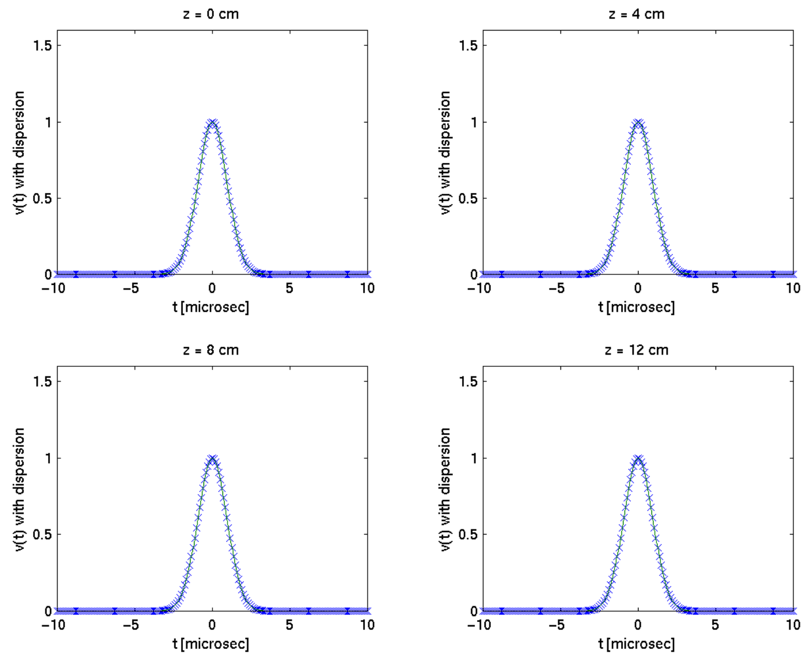




depth = 0 mm, FWHM = 1.979167.1 us
depth = 4 mm, FWHM = 1.979167.1 us
depth = 8 mm, FWHM = 2.395833.1 us
depth = 12 mm, FWHM = 2.604167.1 us

(b)





depth = 0 mm, FWHM = 1.979167.1 us
depth = 4 mm, FWHM = 1.979167.1 us
depth = 8 mm, FWHM = 1.979167.1 us
depth = 12 mm, FWHM = 1.979167.1 us

(c) Clearly, the Gaussian profile is better:
- No depth dependent resolution changes due to dispersion
- Envelop function is not bimodal

While not obvious here, there is a small shift in f_0 due to dispersion. This occurs with both envelope functions.

Comment: In principle, it is possible to do this problem by modulating a(t) with ω_0 and then applying the dispersion and then demodulating back to baseband. For that, however, you may not be sampled adequately to do properly represent p(t) at the sampling rate provided. It is better (more accurate) to do it as described above.

2. The intensity (power/unit area) of the reflected and transmitted waves are $I_r = I_i R^2$ and $I_t = I_i(1-R^2)$, respectively.

(a) $R = \dfrac{Z_B - Z_A}{Z_B + Z_A} = \dfrac{5Z_A - Z_A}{5Z_A + Z_A} = \dfrac{2}{3}$, so are $I_r = I_i \dfrac{4}{9}$ and $I_t = I_i \dfrac{5}{9}$ (56% transmitted).

(b) $R_{AC} = \dfrac{Z_C - Z_A}{Z_C + Z_A} = \dfrac{1}{3}$, $R_{CD} = \dfrac{Z_D - Z_C}{Z_D + Z_C} = \dfrac{1}{5}$, $R_{DB} = \dfrac{Z_B - Z_D}{Z_B + Z_D} = \dfrac{1}{4}$.

and $I_{t,AC} = I_i(1 - R_{AC}{}^2)$, $I_{t,CD} = I_{t,AC}(1 - R_{CD}{}^2)$, $I_{t,DB} = I_{t,CD}(1 - R_{DB}{}^2)$, so

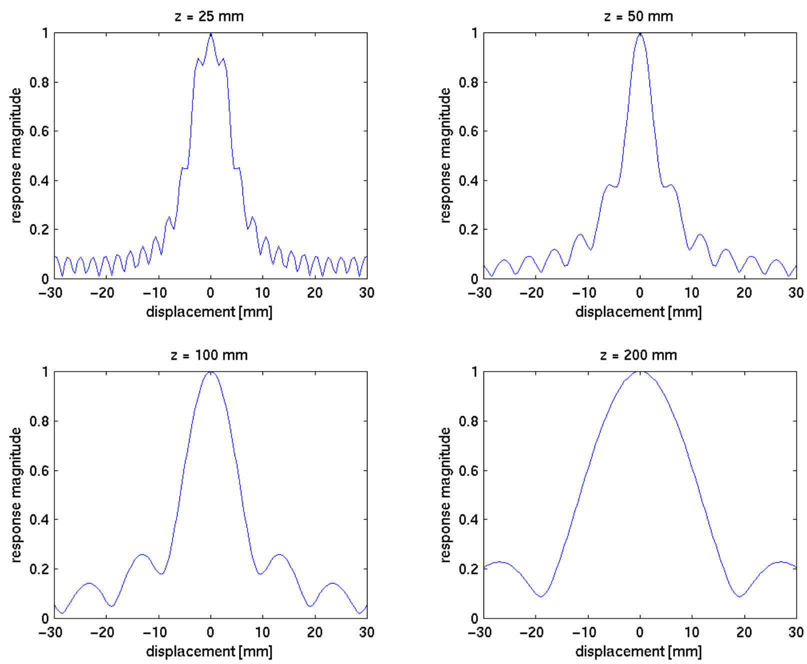
so $I_t = I_i(1 - R_{AC}{}^2)(1 - R_{CD}{}^2)(1 - R_{DB}{}^2) = \dfrac{4}{5} I_i$ (80% transmitted).

3. First, the US system with no focusing.

(a) $\lambda = c / f_0 = 1$ mm, $k = 2\pi/\lambda = 2\pi\,\text{mm}^{-1}$, Fraunhoffer zone $z > (2a)^2/\lambda = 100$ mm.

(b) Using the convolution form of the Fresnel approximation $p(z, x_z) \propto \left[c(x_z) * e^{ik\frac{x_z^2}{2z}} \right]$,

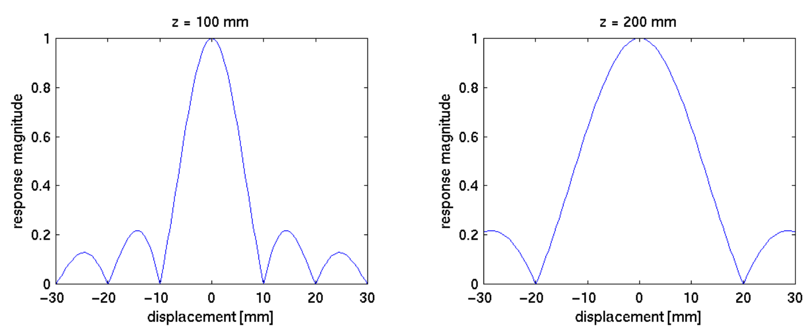
where $c(x_z) = rect(x/2a)$, we get:



(c) Calculated FWHM:

z	FWHM (mm)
25	8.5
50	6.5
100	11.5
200	23.5

(d) The analytical form is $|p(z, x_z)| = |\text{sinc}(2ax/z_0)|$. Plotting:

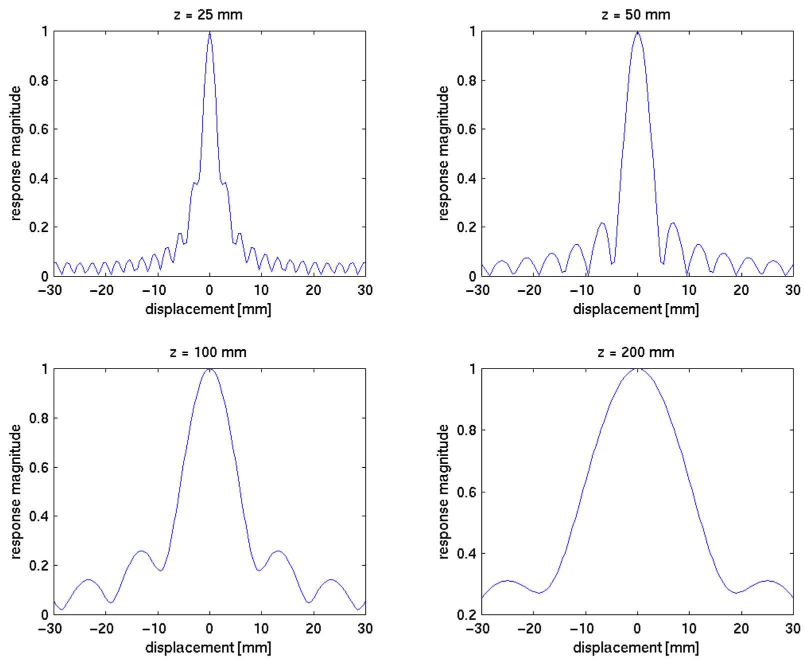


The Fraunhoffer approximation is good, particularly for z = 200mm, less so for z = 100mm.

4. Now, the focused system.

(a) $c(x) = rect(x/10)\exp(-ikx^2/100)$ where scalars (10, 100) and x are in mm.

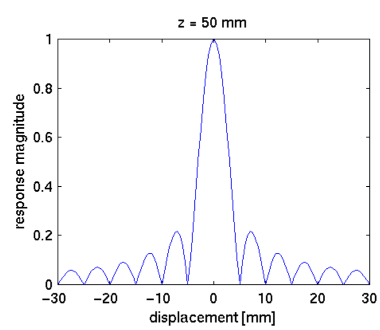
(b) Using the convolution form of the Fresnel approximation:



(c) Calculated FWHM:

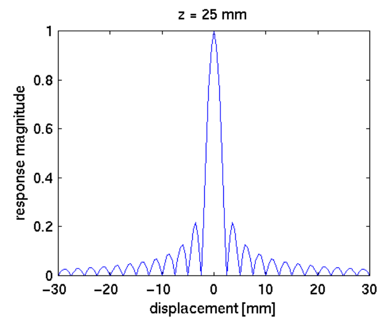
z	FWHM (mm)
25	3.5
50	5.5
100	11.5
200	24.5

(d) At the focal plane (z = 50 mm), the Fourier relationship between the aperture and PSF holds exactly as shown here:



Remaining differences results from numerical errors in convolution form (limited extent of convolution kernels).

Comment 1: Even though z=25 mm is not at the focal plane (in 4.), it is still more narrow than z = 50 mm. Nevertheless, the PSF at z = 25 mm is not as sharp as it would be if the focal plane were located at z=25 as shown here:



Comment 2: Note that the PSF is the same for z = 100 mm for questions 3. and 4. This is because z = 100 mm is, in some sense, equidistant from the focal planes in these two cases ($z_0 = \infty$ and 50 mm, respectively). More specifically, the function

$$s_{eff}(x_0) = |s(x_0)| e^{-ik\frac{x_0^2}{2}\left(\frac{1}{z}-\frac{1}{z_0}\right)}$$

is the same (except for a sign in the exponent) when z = 100 for $z_0 = \infty$ and 50 mm, where $\left(\frac{1}{z}-\frac{1}{z_0}\right)$ represents the "distance" from the focal plane.