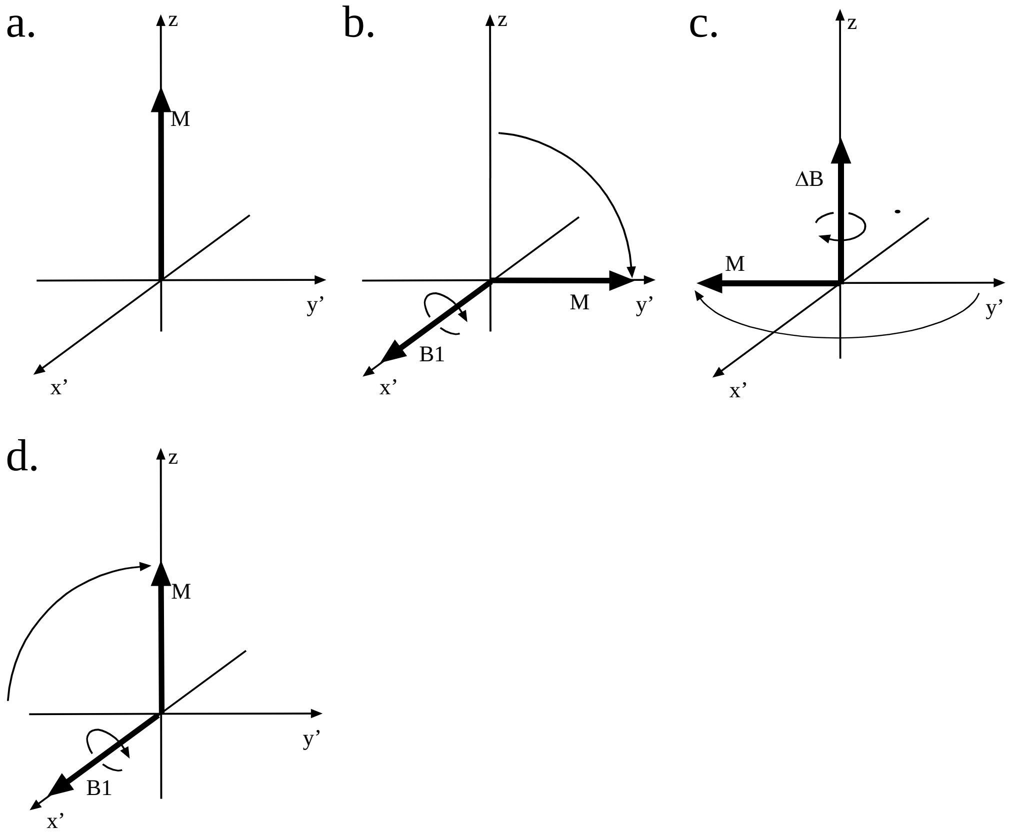
Solutions, Homework #4

1. The magnitude of the cross product is $|\mathbf{M} \times \mathbf{B}_0| = |\mathbf{M}||\mathbf{B}_0|\sin\alpha$ and it points in the direction pointing out of the plane that contains both **M** and $\mathbf{B_0}$. The velocity of the trip of the magnetization (that is, the scalar dM/dt) is γ M B_0 sin α. The x-y component of **M** is M sin α. The path length around a circle traced out by the tip of **M** is given by 2πr where r is the x-y component of **M.** Therefore the path length is 2π M sin α and the velocity along that path is γ M B_0 sin α. The time for one precession is then $\frac{2\pi}{\gamma B_0}$ and the frequency is $\frac{\gamma}{2\pi}B_0$.

   A more rigorous solution has defines M_{xy} in the complex x-y plane and the change of M_{xy} is given by (i($\angle M_{xy}$) γ M B_0 sin α), where i indicates that direction of the derivative is perpendicular to ($\angle M_{xy}$) the x-y direction of **M**. Recognizing that that $\angle M_{xy}$ M sin $\alpha = M_{xy}$, the new scalar differential equation is $dM_{xy}/dt = -i\,\gamma B_0 M_{xy}$, which has solution $M_{xy}(t) = M \sin\alpha \exp(-i\,\gamma B_0\, t)$, which precesses at $\omega_0 = \gamma B_0$.

2. 13C parameters in terms of 1H paramters:
   a. Since $\Delta E = \hbar\gamma B_0$, ΔE scales linearly with γ. Therefore $\Delta E_C = ¼\ \Delta E_H$.
   b. Since $\omega_0 = \gamma B_0$, ω_0 scales linearly with γ. Therefore $\omega_{0,C} = ¼\ \omega_{0,H}$.
   c. Since $N_{diff} \cong \Delta E/kT$, N_{diff} scales linearly with ΔE. Therefore $N_{diff,C} = ¼\ N_{diff,H}$.
   d. Since $|\boldsymbol{\mu}| = \hbar\gamma$, $|\boldsymbol{\mu}|$ scales linearly with γ. Therefore $|\boldsymbol{\mu}|_C = ¼\ |\boldsymbol{\mu}|_H$.
   e. Since $|\mathbf{m}| = N_{diff} \langle\boldsymbol{\mu}\rangle$, $|\mathbf{m}|$ scales linearly with both $|\boldsymbol{\mu}|$ and N_{diff}, and thus scales with the square of γ. Therefore $|\mathbf{m}|_C = \frac{1}{16}\ |\mathbf{m}|_H$.

3. Rotational frequency are for B_1: $\omega_1=\gamma B_1$ and for ΔB: $\Delta\omega=\gamma\Delta B$. Rotation angles are equal to $\phi=\omega\tau$: b. $\pi/2$, c. π, d. $\pi/2$, e.(c.) $\pi/2$, e.(d.) $\pi/2$. Axes of precession are shown below.



4. Relaxation.
   a. To find the maximum of $\Delta s_{xy}(t) = M_{xyA}(t) - M_{xyB}(t) = M_0(e^{-t/T2A}-e^{-t/T2B})$, we must find t, such that $d\{\Delta s_{xy}(t)\}/dt = 0$. The solution
   $$t_{opt} = \frac{T_{2A}T_{2B}}{T_{2A}-T_{2B}}\ln\left(\frac{T_{2A}}{T_{2B}}\right)$$
   This t_{opt} is the "echo time" that will yield the maximum difference between two tissues A and B, that is, will maximize the contrast to noise ratio. A similar solutions works for the T1's.
   b. To find the maximum of $\Delta s_z(t) = M_{zA}(t) - M_{zB}(t) = M_0(1-e^{-t/T1A}-1+e^{-t/T1B})$, we must find t, such that $d\{\Delta s_z(t)\}/dt = 0$. The solution is
   $$t_{opt} = \frac{T_{1A}T_{1B}}{T_{1A}-T_{1B}}\ln\left(\frac{T_{1A}}{T_{1B}}\right)$$
   This t_{opt} is the "repetition time" that will yield the maximum difference between tissues A and B.
   c. $T_{2gray} = 95$ ms, $T_{2white} = 75$ ms → $TE_{opt} = 84.21$ ms
   $T_{1gray} = 1100$ ms, $T_{1white} = 800$ ms → $TR_{opt} = 934.13$ ms