Homework #6
Solution

1. A region of the body has a 10cm thickness of muscle with 2cm of bone imbedded as shown in Figure 1. The densities (ρ) of muscle and bone are 1.0 and 1.75 g/cm^3, respectively. Use parallel x-ray beams for this question. (30 points, 10 each for parts a, b, and c).

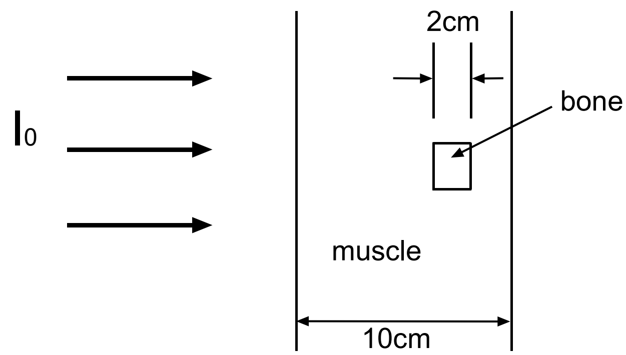

Figure 1. Object to be imaged

Table 1. Mass attenuation coefficient of muscle and bone

Photon Energy	Tissue Type	μ/ρ (cm^2/g)
30 keV	Muscle	0.29
	Bone	0.70
100 keV	Muscle	0.17
	Bone	0.19

a. Calculate the x-ray transmission along paths through muscle alone and through muscle and bone for photon energies of 30 and 100 keV using Table 1. Assume mono-energetic beams of photons.

First, determine μ's based on Table 1.

Photon Energy	Tissue Type	μ/ρ (cm^2/g)	ρ (g/cm^3)	μ (cm^{-1})
30 keV	Muscle	0.29	1.0	0.29
	Bone	0.70	1.75	1.225
100 keV	Muscle	0.17	1.0	0.17
	Bone	0.19	1.75	0.3325

30keV Muscle only: $I_d = I_0 e^{-\int \mu dz} = I_0 e^{-0.29*10} = I_0 e^{-2.9} = 0.055 I_0$

30keV Muscle+Bone: $I_d = I_0 e^{-\int \mu dz} = I_0 e^{-(0.29*8+1.225*2)} = I_0 e^{-4.77} = 0.0085 I_0$

100keV Muscle only: $I_d = I_0 e^{-\int \mu dz} = I_0 e^{-0.17*10} = I_0 e^{-1.7} = 0.183 I_0$

100keV Muscle+Bone: $I_d = I_0 e^{-\int \mu dz} = I_0 e^{-(0.17*8+0.3325*2)} = I_0 e^{-2.025} = 0.132 I_0$

b. Determine the contrast between the background (muscle only) and muscle + bone. Which energy has the best contrast? ($C = \Delta N / \overline{N}_{background}$)

Bone-muscle contrast: $C = \Delta N / \overline{N}_{background} = \dfrac{I_m - I_{m+B}}{I_m}$

30keV $$C = \frac{0.055I_0 - 0.0085I_0}{0.055I_0} = 0.846$$

100keV $$C = \frac{0.183I_0 - 0.132I_0}{0.183I_0} = 0.277$$

c. Assuming that number of photons in I_0 is inversely proportional to the photon energy and that I_0 is fixed (e.g. the total energy transmitted is the same), which energy has the best contrast to noise ratio? ($C = \Delta N / \overline{N}_{background}$)

$$N_0 \propto \frac{1}{E} \quad SNR = \sqrt{N} \propto \sqrt{\frac{\exp(-\mu l)}{E}} \qquad CNR = C \cdot SNR \propto C \frac{\exp\left(-\frac{\mu l}{2}\right)}{\sqrt{E}}$$

CNR @ 30 keV $\propto \frac{0.846}{\sqrt{30}} e^{-1.45} = 0.036$

CNR @ 100 keV $\propto \frac{0.277}{\sqrt{100}} e^{-0.85} = 0.012$

Therefore, CNR is better at 30keV.

2. Projection properties: (30 points, 15 each for a and b)
   a. If the area of a function is $A = \iint f(x,y)dxdy$, find an expression for A in terms of $g_\theta(R)$.
   b. Suppose the project function is separable, e.g. $g_\theta(R) = h_R(R)h_\theta(\theta)$, show that this is not a valid project unless $h_\theta(\theta)$ is a constant.

   a. Let $A = \iint f(x,y)dxdy = \iint f(x,y)e^{-i2\pi(ux+vy)}dxdy \,|_{u=0,v=0} = F(u,v)\,|_{u=0,v=0}$
   F(0,0) is the value of the central point in k-space
   The 1D FT of $g_\theta(R)$ is: $G_\theta(\rho) = F_{1D}\{ g_\theta(R)\} = F(\rho,\theta)$
   $F(\rho,\theta)|_{\rho=0} = F(u,v)|_{u=0,v=0}$
   Therefore, $A = F_{1D}\{g_\theta(R)\}\,|_{\rho=0} = \int g_\theta(R)e^{-i2\pi R\rho}\,|_{\rho=0}\, dR = \int g_\theta(R)dR$
   b. Let $A = \int g_\theta(R)dR = \int h_R(R)h_\theta(\theta)dR = \int h_R(R)dR \cdot h_\theta(\theta) = k \cdot h_\theta(\theta) \quad \forall\theta$
   Where $k = \int h_R(R)dR \quad \forall\theta$
   Therefore, $h_\theta(\theta) = A/k \quad \forall\theta$ is constant
   [The area is the same, irrespective of projection angle.]

3. Suppose we have a CT system with a parallel ray source. Let's assume that the detector system contains a scintillator that disperses light to the photodetectors in roughly a sinc pattern. That is, we can assume that the projection detected by the computer is the ideal project convolved with a sinc in the form $g_{\theta(R)} * h(R)$, where $h(R) = \mathrm{sinc}\left(\frac{R}{W}\right)$.

   Our acquired projection in terms of the true projection is:
   $g'_\theta(R) = g_{\theta(R)} * h(R), h(R) = \mathrm{sinc}\left(\frac{R}{W}\right)$

   In the 1D F.T. domain:
   $G'_\theta(\rho) = G_\theta(\rho) \cdot W\,\mathrm{rect}(W\rho)$
   and by the central section theorem
   $\hat{F}(\rho,\theta) = G'_\theta(\rho) = G_\theta(\rho) \cdot W\,\mathrm{rect}(W\rho) = F(\rho,\theta) \cdot W\,\mathrm{rect}(W\rho)$

   Estimated FT of object true FT of object Filtering function
   Take inverse **2D** FT:
   $\hat{f}(x,y) = f(x,y) ** \frac{1}{W}\mathrm{jinc}\left(\frac{R}{W}\right)$ using 2D IFT for a circ symmetric function from FT notes where $R = \sqrt{x^2+y^2}$.
   Blurring function