X-Ray Notes, Part 0

Here is a roadmap for what we are going to cover with regards to x-ray imaging:

1. Physics of x-ray-tissue interactions
2. Geometry of x-ray systems, point spread functions
3. Noise in x-ray systems
4. Tomographic systems (CT) – theory and math
5. Reconstruction algorithms

X-Ray Notes, Part I (Physics)

X-ray Imaging

Images are characterized by the interaction of x-ray photons and tissue.

Physics

Definition: Radiation – a stream of particles or photons.

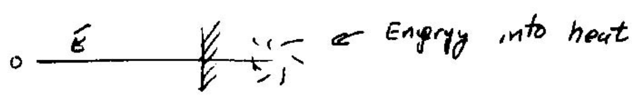
Particles: α ($^{2+}$He), e^- (electrons), β (electrons emitted from nuclei),

β^+ (positrons), p^+ (proton), n^0 (neutrons)

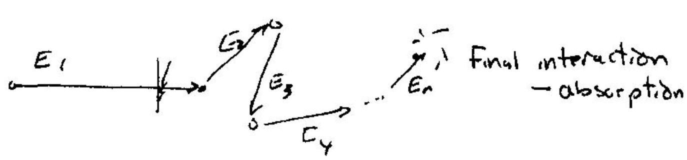
Photons: x-ray, γ, annihilation photons, etc.

Models for interaction of radiation and matter:

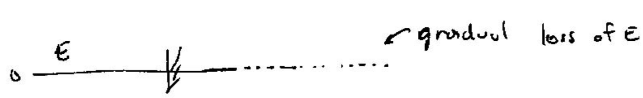
1. Absorption (generally low kinetic energy (KE))



2. Scattering



3. For photons or atomic particles, you don't see this - a gradual loss of energy

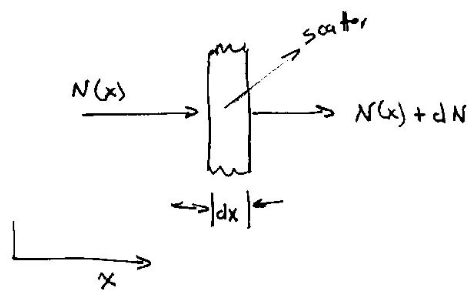


The charged particles above (α, e^-, β, β^+, p^+) interact very strongly with tissue and typically do not pass completely through the human body and thus cannot be used for imaging. Of the above particles photons and neutrons(n^0) pass through the body with an appropriate amount of interaction for imaging (too little is also bad).

Behavior of Radiation Along a Line

Assumptions:

1. Matter consists of discrete particles separated by distances that are large compared to the size of the particles.
2. For a given path length along a line, an x-ray photon either interacts (with prob. *p*) or it doesn't and all interactions are independent.
3. Scattered photons scatter at a different angle and don't contribute to the continuing flux of photons along the line.



The change in the number of photons is:

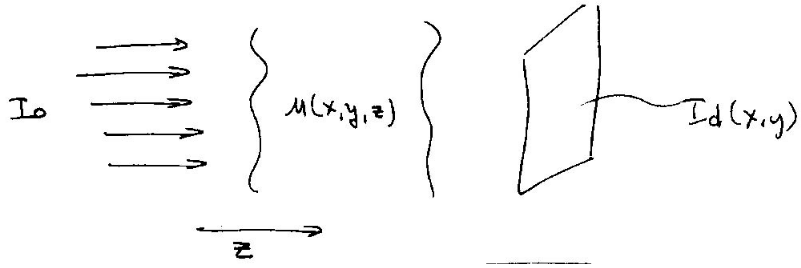
$$dN \propto -N(x)dx$$
$$dN = -\mu N(x)dx$$
$$\frac{dN}{dx} = -\mu N(x)$$
$$N(x) = N(0)\exp\left(-\int_0^x \mu(x')dx'\right)$$

were μ is the "linear attenuation coefficient" and has units (distance)$^{-1}$, e.g. cm^{-1}. For a constant μ:

$$N(x) = N(0)\exp(-\mu x)$$

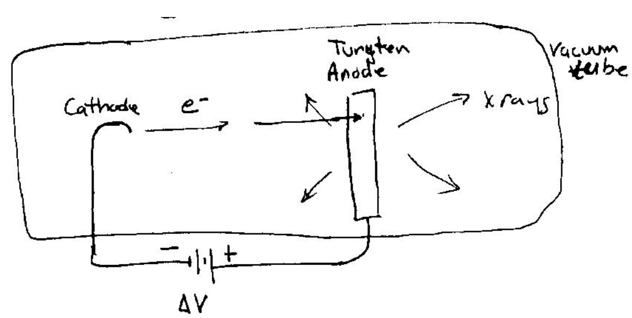
The Basic X-ray Imaging System

Now consider a parallel ray x-ray flux that has intensity I_0 (intensity is photons/unit area/unit time) the passes through a 3D object having a distribution of attenuation coefficients $\mu(x,y,z)$ and projects to an image $I_d(x,y)$:



$$I_d(x,y) = I_0 \exp\left(-\int \mu(x,y,z)dz\right)$$

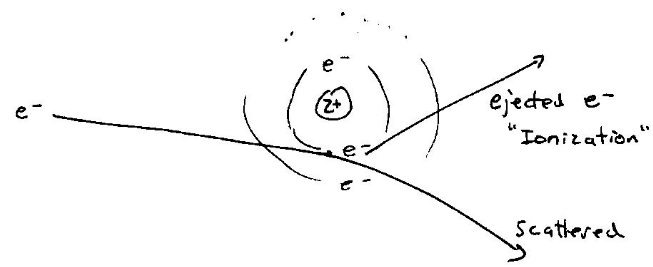
Generation of x-rays



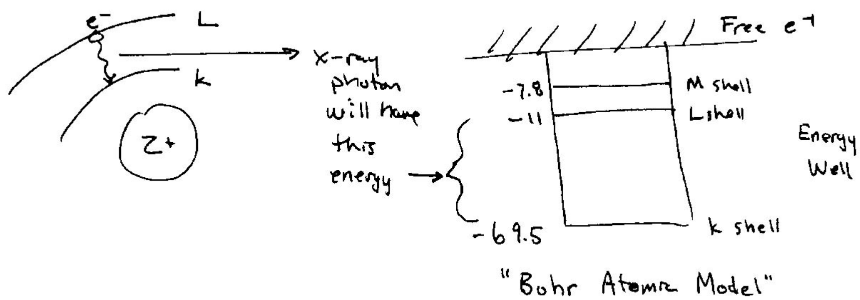
- Target is usually a high-Z, heavy element – typically W, tungsten.
- Electrons are accelerated by the voltage between the cathode and the anode.
- A potential energy of E=qΔv (e.g. e * 150 kV = 150 keV) all gets converted to kinitic energy E = ½ $m_e v^2$ (e.g. also 150 keV).

Kinds of electron interactions:

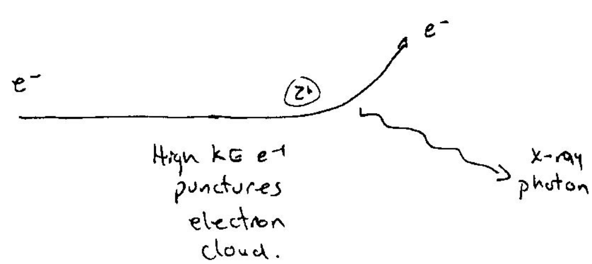
a. Inelastic (energy absorbing) scattering with atomic electrons – the ejection of a bound electron followed by emission of a photons from spontaneous energy state transitions.



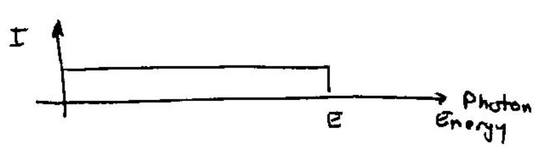
The Bohr model accounts for absorption/generation of discrete valued energies. 58.5 keV is one "characteristic" x-ray for W. Any combination of shell transition energies will also be characteristic energies (e.g. 3.2 and 61.7 keV). Very low energies are hard to observe due to other absorption processes.



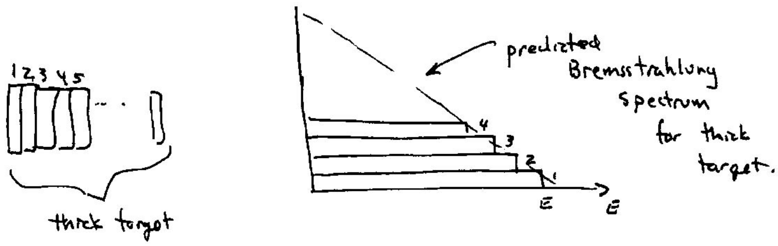
b. Bremsstrahlung "Braking" Radiation – Acceleration (change in direction) of electron by Coulomb attraction to the large, positively charged nucleus leads to the generation of photons (acceleration of any charged particle will do this).



For electrons of a particular energy, E, striking an infinitely thin target, Bremsstrahlung radiation will have a uniform distribution of energy between 0 and E.

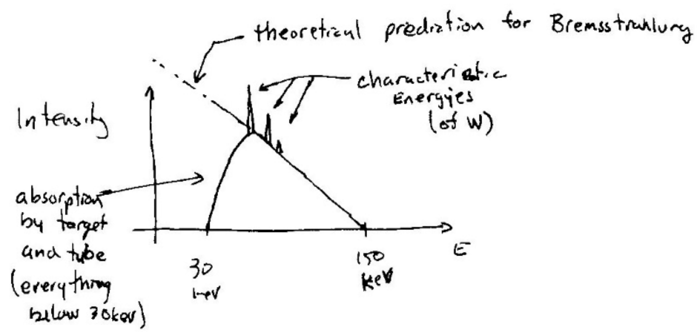


We assume that all electrons interact. For a thick target, it is often modeled as a series of thin targets where the highest energy impinging upon subsequent stages is reduced by the interactions. Each thin target produces a new uniform spectrum, but with a lower peak energy. The resultant spectrum is approximately linear from a peak at 0 keV to 0 at E.



The x-ray Spectrum

- For electrons with energy E, the maximum x-ray photon energy is E.
- $E = h\upsilon = \dfrac{hc}{\lambda}$
- Very low energy photons are absorbed by the target and by the glass in the x-ray tube.
- Spectrum will have a combination of Bohr (discrete) energies and Bremsstrahlung radiation:



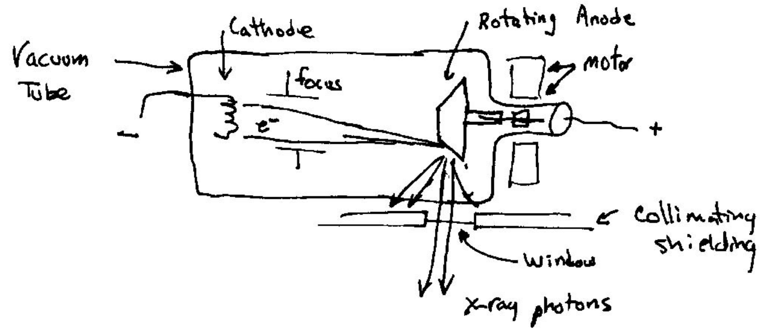
- The x-ray spectrum is function of photon energy: $I_0 = I_0(E)$
- I now represents energy/unit time/unit area or power/unit area.

Practical x-ray tube

Why Tungsten?

- x-ray spectrum in desired range
- High Z (high efficiency in stopping electrons)

- High melting point (3300 deg. C) – typical operation temp is ~2500 deg. C – this is due to the low efficiency of the electron to x-ray conversion (~0.8%). The rest goes into heat.
- Example:



- Rotation of target to reduce peak temp
- Shielding to collimate beam
- Window further filters x-ray spectrum ("hardens beam" – makes it have a higher average E)

The Attenuation Coefficient

We say above that the x-ray spectrum is a function of photon energy E: $I_0 = I_0(E)$. The attenuation function is also a function of E: $\mu = \mu(x,y,z,E)$. The new expression for the intensity at the output will now be:

$$I_d(x,y) = \int_E I_0(E)\exp\left(-\int \mu(x,y,z,E)dz\right)dE$$

Note: I_d tells us nothing about z or E – it only gives us x,y information.

The x-ray attenuation coefficient μ is, of course, also a function of material properties. Two of the most important properties that affect the attenuation coefficient are tissue density, ρ, and the atomic number Z. As most x-ray photon/tissue interactions are photon/electron interactions both ρ and Z will influence μ.

For x-ray photons, there are 4 main types of interactions (listed in order of increasing likelihood with increasing photon energy, *E*):

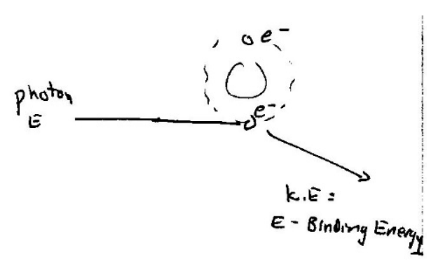
1. Rayleigh-Thompson Scattering
2. Photoelectric Absorption
3. Compton Scattering
4. Pair Production

In general, we can write an expression for the attenuation coefficient as the some of these constituent parts:

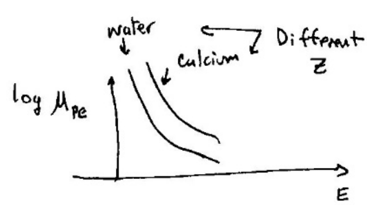
$$\mu(E) = \mu_{rt}(E) + \mu_{pe}(E) + \mu_{cs}(E) + \mu_{pp}(E) + \ldots$$

1. Rayleigh-Thompson Scattering or "coherent" scattering – atomic absorption with spontaneous emission at the same energy *E*. This is the same effect as is seen in x-ray diffraction in crystals. This term is rarely important in the diagnostic energy range (50-200 keV).

2. Photoelectric Absorption – Absorption of photon to ionize and eject an atomic electron. The ejected electron will have an kinetic energy of the photon energy less the binding energy of the electron.



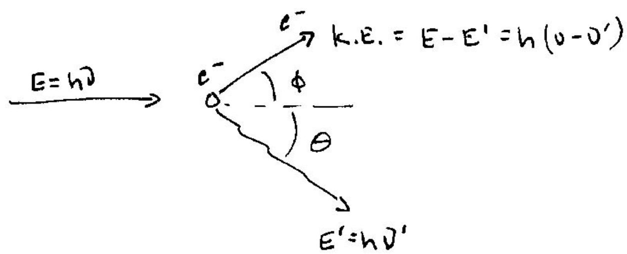
The photoelectric effect increases rapidly with atomic number, Z, and with decreasing energy. The photoelectric effect dominates μ in the lower part of the diagnostic spectrum.



For high Z materials (e.g. Lead, Iodine, Tungsten), the shell energy boundaries are evident in the μ vs. E plots. When the energy gets high enough to make that shell's electrons available to the PE effect (when E exceeds the binding energy), then the probability of a PE interaction increases.

Iodine and Barium, common x-ray contrast agents, have k-edges in the diagnostic energy range at 33.2 keV and 37.4 keV, respectively.

3. Compton Scattering – scatting of photons by an elastic collision with a free electron. Elastic collisions preserve E and momentum (p). For loosely bound electrons or very high energy photons, the equations for free electrons hold reasonably well.



Unknowns: ϕ, θ, E', K.E.

Conservation of energy:

$$\text{K.E.} = E - E' = (m - m_0)c^2$$

where $m = \dfrac{m_0}{\sqrt{1 - v^2 / c^2}}$ is the relativistic mass of the electron

Just a check on this equation … for $v^2 << c^2$, then

$$(m - m_0)c^2 = m(1 - \sqrt{1 - v^2 / c^2})c^2$$

$$\approx m(1 - (1 - \frac{1}{2}\frac{v^2}{c^2}))c^2 = \frac{1}{2}mv^2$$

Conservation of momentum in x and y directions:

$$\frac{E}{c} = \frac{E'}{c}\cos\theta + mv\cos\phi$$

$$\frac{E'}{c}\sin\theta = mv\sin\phi$$

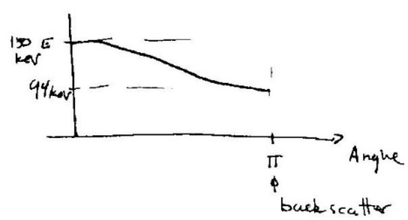
solving these equations we get the energy of the scattered photon:

$$E' = \frac{E}{1 + \frac{E}{E_e}(1-\cos\theta)}$$

where $E_e = m_0c^2 = 511$ keV, the rest energy of an electron.

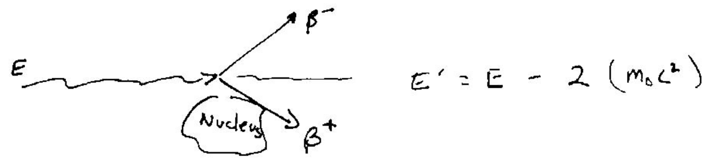
Comments:

- For $E \ll E_e$, there is very little change in energy with angle.
- For higher E:

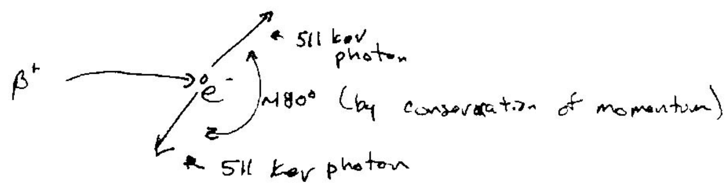


- For low E, scatter is essentially isotropic in angle
- For higher E, scatter is preferentially forward scattered (where there is very little change in photon E).
- It is very hard to discriminate between forward scattered photons and unimpeded photons based on energy.
- μ_{cs} is nearly constant across diagnostic spectrum
- Compton scatter comes mostly from atomic electrons (μ_{cs} is proportional to ρ)
- At higher E, Compton scatter dominates over the PE effect (most important effect in x-ray imaging).

4. Pair Production – the spontaneous creation of an electron/positron pair:



In this interaction, photon energy in transferred to mass energy in the electron and positron. Since the rest energy of each is 511 keV, pair production cannot occur for x-ray photons below 1022 keV (not in the diagnostic spectrum). Positrons will wander around until they bump into an electron, which will result in mutual annihilation and the emission of two 511 keV photons:



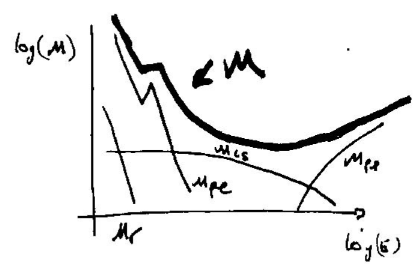
The ejected photons from a positron/electron annihilation is the basis for positron emission tomography [more on this later].

Total Linear attenuation coefficient for photons

Again, the combined coefficient is:

$$\mu(E) = \mu_{rt}(E) + \mu_{pe}(E) + \mu_{cs}(E) + \mu_{pp}(E) + \ldots$$

For example, the combined coefficient for lead is:



Commonly, figures are reported as a "mass attenuation coefficient" which is defined as:

$$\tau = \mu / \rho \text{ (units: cm}^2\text{/gm)}$$

TABLE 2.3. Mass Attenuation Coefficient Versus Photon Energy

X-ray Photon Energy (keV)	Mass Attenuation Coefficient, μ (cm^2/g)			
	Water	Air	Compact Bone	Muscle
10	4.89	4.66	19.0	4.96
15	1.32	1.29	5.89	1.36
20	0.523	0.516	2.51	0.544
30	0.147	0.147	0.743	0.154
40	0.0647	0.640	0.305	0.0677
50	0.0394	0.0384	0.158	0.0409
60	0.0304	0.0292	0.0979	0.0312
80	0.0253	0.0236	0.0520	0.0255
100	0.0252	0.0231	0.0386	0.0252
150	0.0278	0.0251	0.0304	0.0276
200	0.0300	0.0268	0.0302	0.0297
300	0.0320	0.0288	0.0311	0.0317

(taken from: Aston, Richard <u>MEDICAL IMAGING EQUIPMENT THEORY</u>, 2ND EDITION, www.epix.net/~astonr, Wilkes-Barre, Pennsylvania, 2008)

The final attenuation coefficient is calculated by multiplying the density:

TABLE 2.2.
Density of common biological substances

SUBSTANCE	DENSITY (g/cm^3)
Air	0.0013
Water	1.0
Muscle	1.06
Fat	0.91
Bone	1.85

(taken from: Aston, Richard <u>MEDICAL IMAGING EQUIPMENT THEORY</u>, 2ND EDITION, www.epix.net/~astonr, Wilkes-Barre, Pennsylvania, 2008)

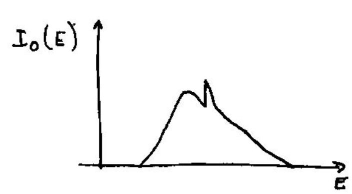
This parameter is convenient when describing the behavior of composite materials with N constituent components:

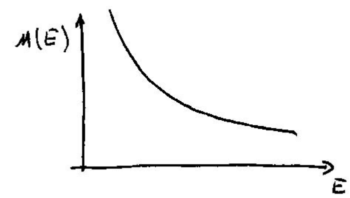
$$\tau = \frac{1}{M}\sum_{i=1}^{N} m_i \tau_i$$

where m_i are the masses of the components and M is the total mass.

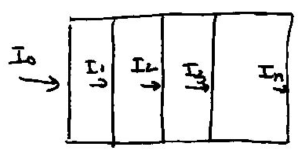
Beam Hardening

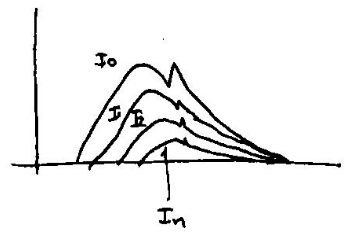
Because the attenuation spectrum is not uniform across the diagnostic energy spectrum, the output spectrum will have a different intensity distribution than the input spectrum, $I_0(E)$.





If we split an object into smaller parts, and look at then energy spectrum at for each part:





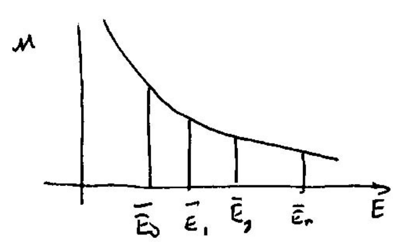
we will find that the mean energy:

$$\overline{E} = \frac{\int EI(E)dE}{\int I(E)dE}$$

will increase (get harder) as we move through the object:

$$\overline{E}_0 < \overline{E}_1 < \overline{E}_2 < \ldots < \overline{E}_n .$$

For medical imaging, this has the unfortunate consequence that a particular tissue type will have a μ that changes as a function of position along the path.



In particular, as we move deeper into the object, we will find that there is less attenuation than expected, given the initial spectrum, $I_0(E)$.

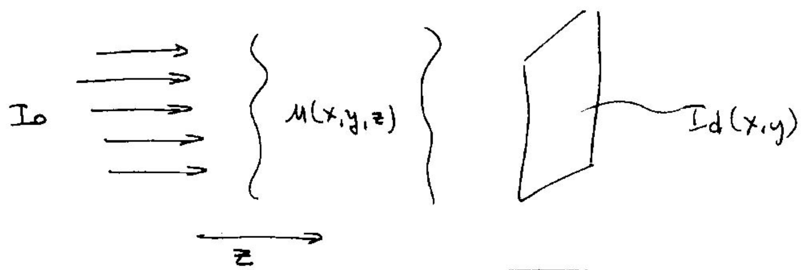
One solution is to make the beam "hard" to begin with. This is often accomplished by filtering out the low E photons with a thin metal plate (often use aluminum).

X-Ray Notes, Part II (Systems)

Source Issues

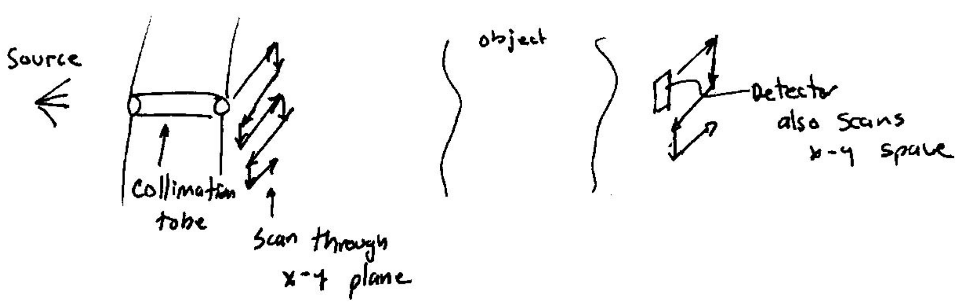
The Parallel X-ray Imaging System

Earlier, we considered a parallel ray system with an incident intensity I_0 that passes through a 3D object having a distribution of attenuation coefficients $\mu(x,y,z)$ and projects to an image $I_d(x,y)$:



$$I_d(x,y) = I_0 \exp\left(-\int \mu(x,y,z)dz\right)$$

There are essentially no practical medical project x-ray systems where the source has parallel rays. There are some scanning systems that might be appropriate for industrial inspection operations, for example:

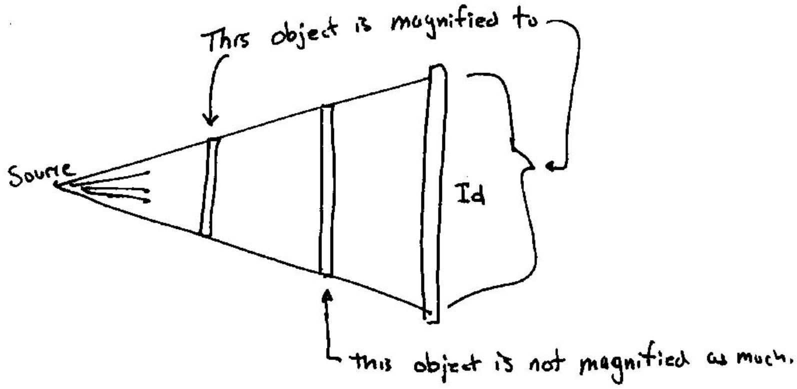


but these kinds of systems are too slow for medical applications.

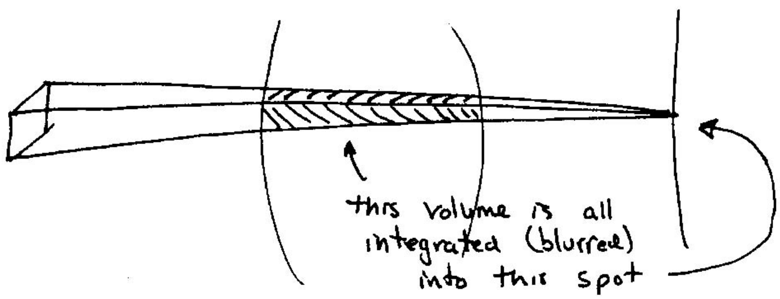
Practical X-ray Sources

There are two main issues associated with practical x-ray sources:

1. Geometric distortions due to point geometry – "depth dependent magnification."

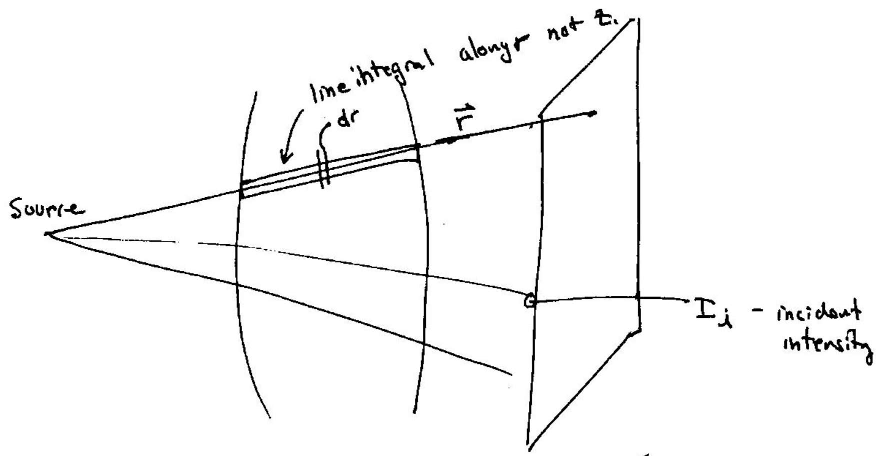


2. Resolution loss (blurring) due to finite (large) source sizes



Point Source Geometry

First, we will find expressions for the image intensity, $I_d(x_d, y_d)$, for a point source geometry:



$$I_d(x_d, y_d) = I_i(x_d, y_d)\exp\left(-\int \mu(x, y, z)dr\right)$$

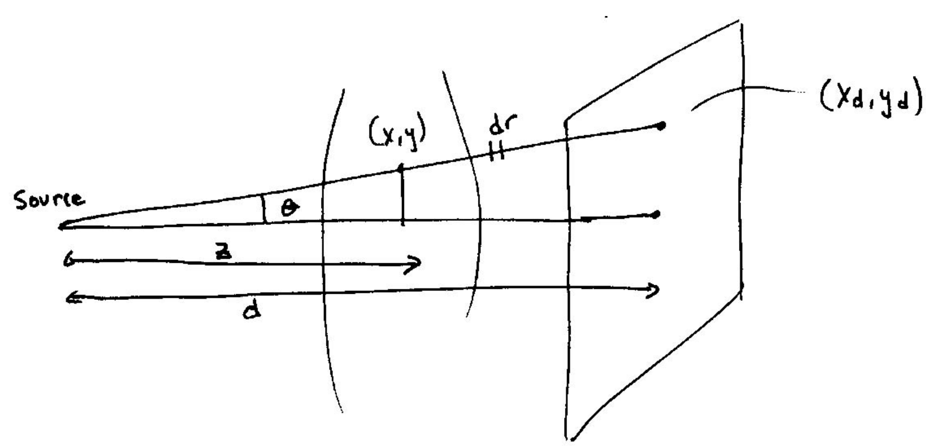
Comments:

1. (x_d, y_d) is the coordinate system in the output detector plane.
2. (x, y, z) is the coordinate system of the object.

3. Notice that $I_i(x_d,y_d)$ a spatially variant incident intensity replaces I_0. There are two components to this – fall off in intensity as distance increases and adjustment for the fact that photons strike the detector at an angle. For the detector a distance d from the source: $I_i(x_d, y_d) = I_0 \cos^3\theta = I_0\left(\frac{d}{\sqrt{d^2 + r_d^2}}\right)^3 = I_0 \frac{1}{\left(1+\left(\frac{r_d}{d}\right)^2\right)^{3/2}}$ where $r_d^2 = x_d^2 + y_d^2$.
4. Notice that the integration is along some path r with variable of integration dr rather than dz. (dr is at an angle and is a slightly longer path than dz).

Magnification

If we look at some point in the object (x,y) at depth z, we see that it will strike the detector at a position $\left(x_d, y_d\right) = \left(x\frac{d}{z}, y\frac{d}{z}\right)$:



where $M(z) = \frac{d}{z}$ is the magnification factor for an object at depth z. We can now write the attenuation coefficient at location (x,y) in terms of the output coordinate system:

$$\mu(x,y,z)=\mu\left(\frac{x_d}{M(z)},\frac{y_d}{M(z)},z\right)$$

Thin Objects and Finite (Large) Sources

To gain an understanding of this issue, we will first consider a "thin" object. Specifically, we will let the attenuation coefficient be:

$$\mu(x,y,z)=\tau(x,y)\delta(z-z_0)$$

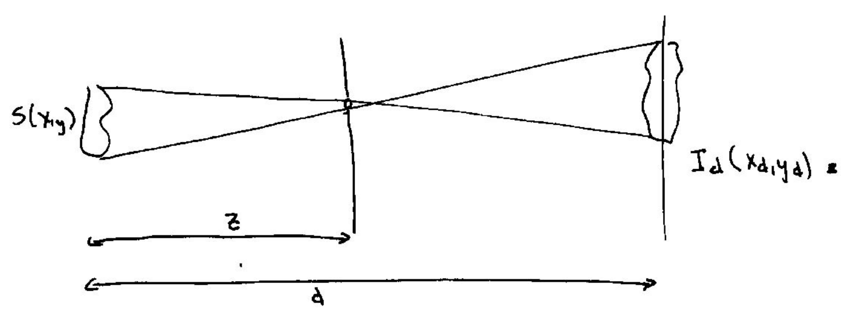
and then, after the ignoring the pathlength obliquity term (replace *dr* with *dz*) and neglecting fall of the intensity (replace I_i with I_0):

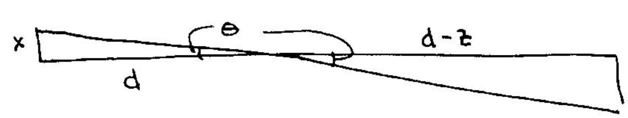
$$I_d(x_d,y_d)=I_0\exp\left(-\int\tau\left(\frac{x_d}{M(z)},\frac{y_d}{M(z)}\right)\delta(z-z_0)dz\right)$$
$$=I_0\exp\left(-\tau\left(\frac{x_d}{M(z_0)},\frac{y_d}{M(z_0)}\right)\right)$$

We let $M=M(z_0)=d/z_0$ the *object magnification factor*, and term to get:

$$I_d(x_d,y_d)=I_0\exp\left(-\tau\left(\frac{x_d}{M},\frac{y_d}{M}\right)\right)=I_0t\left(\frac{x_d}{M},\frac{y_d}{M}\right)$$

where $t=\exp(-\tau)$ is the transmission function. Now we consider a finite source function $s(x,y)$ and a very small pinhole transmission function:





$$\frac{-x_d}{d-z}=\frac{x}{z}$$
$$x_d=-\left(\frac{d-z}{z}\right)\cdot x$$
$$=m(z)\cdot x$$

The image will now be an image of the source with the *source magnification factor*, $m = m(z) = -\frac{d-z}{z}$:

$$I_d(x_d, y_d) = ks\left(\frac{x_d}{m}, \frac{y_d}{m}\right)$$

where k is a scaling factor that is proportional to the area of the pinhole, $1/d^2$, etc. If we want the above I_d to represent the impulse response of the system, we need to make the pinhole equal to $\delta(x,y)$ and account for all of the scaling terms [$t(x,y) = \delta(x,y)$ is not a realizable transmission function since t can never exceed 1, nevertheless, we will allow it for mathematical convenience.]

The area of the pinhole is $\iint \delta(x,y)dxdy = 1$. The capture efficiency of the pinhole is the fraction of all photons emitted from the source that pass through the pinhole. This will be equal to:

$$\eta = \frac{\text{pinhole area}}{4\pi z^2} = \frac{1}{4\pi z^2}$$

Letting the total number of photon emitted be:

$$N = \iint s(x,y)dxdy$$

and the total number of photons to get through the pinhole will be:

$$N\eta = \frac{N}{4\pi z^2}.$$

This must be the same number at the detector:

$$\iint ks\left(\frac{x_d}{m}, \frac{y_d}{m}\right)dx_d dy_d = kNm^2 = \frac{N}{4\pi z^2}$$

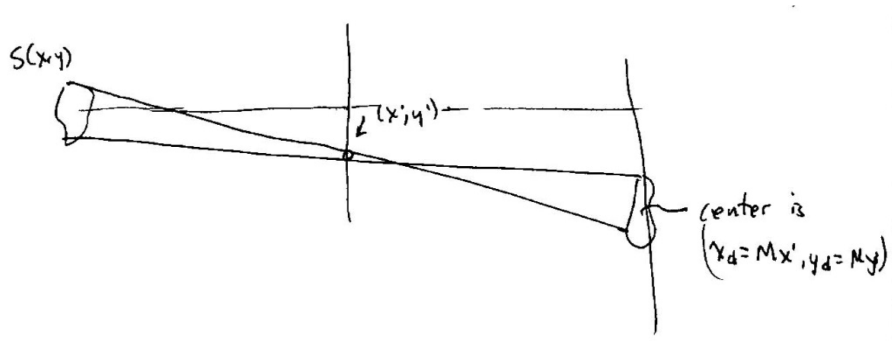
The scaling coefficient will therefore be:

$$k = \frac{1}{4\pi z^2 m^2}$$

so:

$$I_d(x_d, y_d) = \frac{1}{4\pi z^2 m^2} s\left(\frac{x_d}{m}, \frac{y_d}{m}\right)$$

Now we let the pinhole be at position (x',y'), that is, $t(x,y) = \delta(x-x', y-y')$:



The image of the source is not located at (x_d=Mx', y_d=My') where M is the object magnification factor. Thus, the impulse response function is:

$$h(x_d, y_d; x', y') = I_d(x_d, y_d) = \frac{1}{4\pi z^2 m^2} s\left(\frac{x_d - Mx'}{m}, \frac{y_d - My'}{m}\right)$$

Now we can calculate the image for an arbitrary transmission function using the superposition integral:

$$I_d(x_d, y_d) = \iint t(x' y') h(x_d, y_d; x', y') dx' dy'$$

$$= \frac{1}{4\pi z^2 m^2} \iint t(x' y') s\left(\frac{x_d - Mx'}{m}, \frac{y_d - My'}{m}\right) dx' dy' \text{ and sub } Mx' = x$$

$$= \frac{1}{4\pi z^2 m^2 M^2} \iint t\left(\frac{x}{M}, \frac{y}{M}\right) s\left(\frac{x_d - x}{m}, \frac{y_d - y}{m}\right) dx dy$$

$$= \frac{1}{4\pi d^2 m^2} s\left(\frac{x_d}{m}, \frac{y_d}{m}\right) ** t\left(\frac{x_d}{M}, \frac{y_d}{M}\right)$$

<u>Thus, the final image is equal to the convolution of the magnified source and the magnified object. The object is blurred by the source function.</u>

The frequency domain equivalent is:

$$F_{2D}\{I_d(x_d, y_d)\} = \frac{1}{4\pi z^2} S(mu, mv) T(Mu, Mv)$$

Consider $z_0 = d/2$ which yields M=2 and $|m|$=1. The object is magnified by a factor of 2 and is blurred by the unmagnified source.

Comments:

1. The least source blurring come when $|m|$ is made small. Thus, it is desirable to make the depth plane as far from the source as possible: $z_0 \rightarrow d$. Then $|m| = (d-z)/z \rightarrow 0$ and $M \rightarrow 1$. As we noted above, making $z_0 \rightarrow d$ also reduces geometric distortions. The common practice for x-ray imaging, then, is to position the subject immediately next to or on top of the detector.
2. If the thickness of the body is a limiting factor, then let $d, z \rightarrow \infty$. This will make the system close to a parallel ray geometry with $|m| = \rightarrow 0$ and $M \rightarrow 1$. The main problem with this approach is $I_0 \propto 1/d^2 \rightarrow 0$ and SNR $\propto \sqrt{I_0} \rightarrow 0$.
3. We would also like the make $s(x,y)$ as small as possible to reduce blurring, but $I_0 \propto \iint s(x,y)dxdy$ and making it small might reduce the number of photons created and thus reduce SNR.
4. For a complex object, we can make $\mu(x,y,z) = \sum \tau_i(x,y)\delta(z-z_i)$ and each plane will have its own magnification factors. This is not particularly useful, but it can give you some idea of how blurring and magnification might affect different parts of a real object differently.

Overall System Response

Now we can add the detector response to the other system elements:

$$I_d(x_d, y_d) = \frac{1}{4\pi d^2 m^2} s\left(\frac{x_d}{m}, \frac{y_d}{m}\right) ** t\left(\frac{x_d}{M}, \frac{y_d}{M}\right) ** h(r_d)$$

The impulse response function will then be:

$$h(x_d, y_d) = \frac{1}{4\pi d^2 m^2} s\left(\frac{x_d}{m}, \frac{y_d}{m}\right) ** h(r_d)$$

or for a circularly symmetric source function:

$$h(x_d, y_d) = \frac{1}{4\pi d^2 m^2} s\left(\frac{r_d}{m}\right) ** h(r_d)$$

Object Blurring

One issue is how much does the detector response blur the object. It is important to realize that the detector blurs the magnified object. Our intuition would be to make the object as large as possible by making $M = d/z$ very large. This would dictate moving the object as close to the source as possible, which is exactly opposite as what we would like to do to minimize source blurring.

Consider also, that the magnified source also blurs the magnified object (source and object have different magnification factors). One way to look at this is to examine the response in the coordinate system of the object (x,y) rather than the detector (x_d,y_d):

$$I(x,y) = ks\left(\frac{Mx}{m},\frac{My}{m}\right)**t(x,y)**h(Mr_d)$$

the effective magnification of the source is:

$$\left|\frac{m}{M}\right| = \frac{d-z}{d}$$

and the effective magnification of the detector response is:

$$\frac{1}{M} = \frac{z}{d}$$

These are in competition:

- to make the source blurring small, make $z \rightarrow d$
- to make the detector response small, make $z \rightarrow 0$

Comments:

1. For most film systems, the detector response is very small and the source is almost always bigger. Therefore, we would like to make $z \rightarrow d$.
2. For other kinds of systems, e.g. digital fluoroscopy systems, the detector resolution is much larger (e.g. 0.5 mm) and for these systems an intermediate z may be appropriate.

X-Ray Notes, Part III (Noise)

Noise in X-ray Systems

In an x-ray system, images typically are created from intensity values that are related to the number of photons that strike a detector element in a finite period of time. The photons are generated by electrons randomly striking a source and thus the photons at the detector are also random in nature. We typically describe this kind of random process as one having a rate parameter, λ (units: events/time), and an observation time, T. Let X be the random variable (R.V.) that describes the number of events (photons striking the detector element) in time T.

X will be a Poisson distributed random variable with parameter λT. E.g.

$$X \sim \text{Poisson}(\lambda T)$$

The mean and variance are:

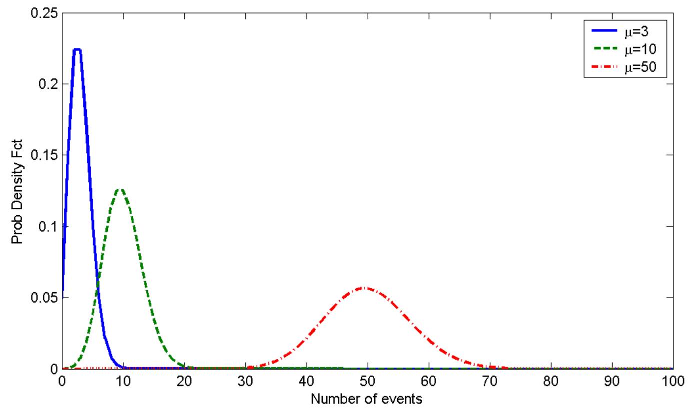
$$\overline{X} = \lambda T$$
$$\sigma_X^2 = \lambda T$$

SNR of a Poisson Measurement

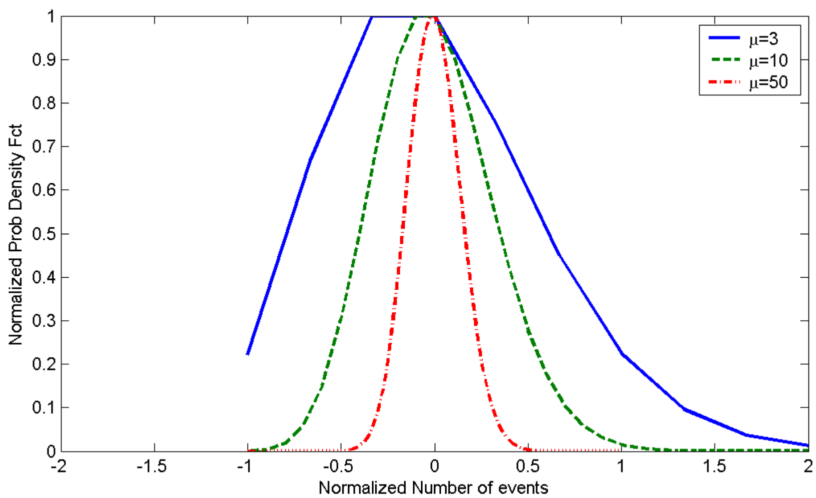
In general, the pixel values in an x-ray image are distributed according to a Poisson R.V. If the mean value of the photon counts for a pixel is μ, then the signal to noise ratio of for that pixel will be:

$$SNR = \frac{\overline{X}}{\sigma_X} = \frac{\mu}{\sqrt{\mu}} = \sqrt{\mu}$$

The SNR increases as the square root of the number of photons. *Thus, the SNR increases as the square root of the dose to the patient.* Finally, by averaging together two neighboring pixels, we can roughly double the photon counts and improve the SNR by $\sqrt{2}$.



The above figure shows Poisson distributions as the mean increases from 3 to 50. We can see that the distribution becomes more symmetric and Gaussian.



The above figure takes Poisson distributions and normalizes them by their mean, that is, we subtract the mean and divide the x-axis by the mean. This plot show demonstrates that the width of the distribution as a fraction of the mean. As the mean gets larger, the distribution gets proportionately narrower – the std. dev. vs. mean ratio is smaller (SNR is higher).

Noise in Detectors

Consider an output to an x-ray system that looks like this:

We define a number of quantities (slightly different definitions that used by Macovski):

Contrast: $C = \Delta S / \overline{S}$

Signal to Noise Ratio: $SNR = \overline{S} / \sigma_s$

Contrast to Noise Ratio: $CNR = \Delta S / \sigma_s = C \cdot SNR$

Previously, we described the SNR for a system having pixels distributed according to a Poisson R.V. If the mean value of the photon counts for a pixel is $\mu = N$, then the signal to noise ratio of for that pixel will be:

$$SNR = \frac{\overline{S}}{\sigma_S} = \frac{N}{\sqrt{N}} = \sqrt{N}$$

The probability density function for the Poisson R.V. is:

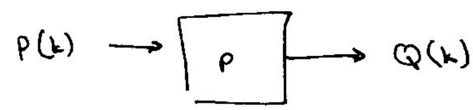
$$p(k) = \frac{e^{-\mu}(\mu)^k}{k!}$$

and

$$E[k] = N$$

$$E[k^2] = N^2 + N$$

We will now describe the probability distribution of detected photons. Suppose the incident x-ray photons arriving at the detector are Poisson(N) and that the detector has efficiency η, as describe previously. We can view the detector as a binary random system in which the photon is detected with probability $p = \eta$:



where

$$Q(k) = \Pr\{k \text{ photons are detected}\}$$

$$= \sum_{n=0}^{\infty} \Pr\{k \text{ photons are detected}/n+k \text{ photons are incident}\} \cdot \Pr\{n+k \text{ photons are incident}\}$$

$$= \sum_{n=0}^{\infty} \text{Binomial}(n+k)_k \cdot \text{Poisson}(N)_{n+k}$$

$$= \sum_{n=0}^{\infty} \frac{(n+k)!}{k!n!} p^k (1-p)^n \cdot \frac{e^{-N} N^{n+k}}{(n+k)!}$$

$$= \frac{e^{-N} p^k N^k}{k!} \sum_{n=0}^{\infty} \frac{N^n (1-p)^n}{n!}$$

$$= \frac{e^{-N} p^k N^k}{k!} e^{N(1-p)}$$

$$= \frac{e^{-pN} (pN)^k}{k!}$$

$$= \text{Poisson}(pN)$$

Thus, the detected photons are also Poisson distributed, but will have probability ηN and the SNR of the detected photons is now:

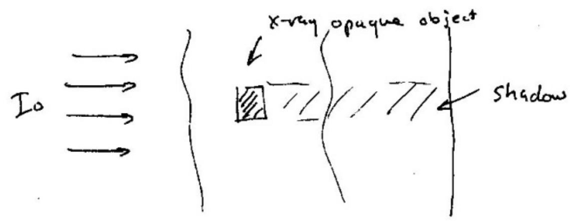
$$SNR_{\text{det}} = \frac{\eta N}{\sqrt{\eta N}} = \sqrt{\eta N}$$

Comments:

1. It is also easy to show that the number of photons that are not detected is also a Poisson process with parameter probability $(1-\eta)N$.
2. The sum of Poisson processes is also Poisson.
3. Finally, if the incident photons are Poisson, then the number of photons that reach the detector will also be Poisson. Attenuation processes that independently affect photons work exactly as above.

Compton Scattered X-rays

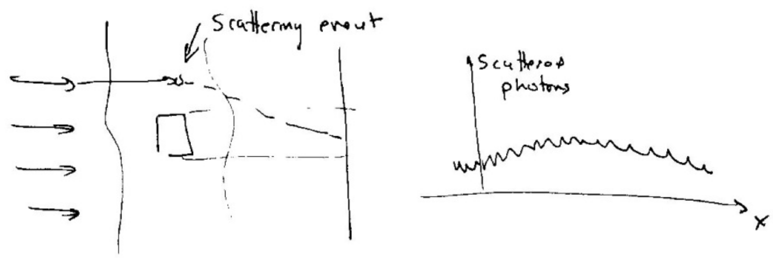
Consider the following object with an x-ray opaque core:



the output image might look like this:

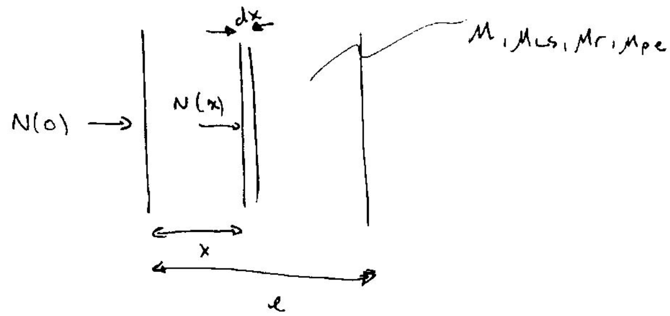
on which we can define a contrast $C = \Delta S / \bar{S}$ and a contrast to noise ratio $CNR = \Delta S / \sigma_s$. Now, consider that the scattered photons – here some fraction of the scattered photons will scatter forward and will generate additional photons in the final image.

The distribution of the scattered photons will look something like the object convolved with the forward scattering distribution. The final image will be the sum of the transmitted photons and the scattered photons.



By increasing $\bar{S}$ *and* σ_s *the scattered photons will reduce both the contrast and the contrast to noise ratio.*

How many photons are scattered? (Derived from Macovski, Problem 3.4) Let's look at an object of length *l* having an attenuation coefficient $\mu = \mu_{rt} + \mu_{pe} + \mu_{cs}$. Let $N(0)$ be the number photons incident upon the object and that the number of photons that have not interacted at depth x is $N(x)$.



The number of scattered photons in an interval *dx* will be:

$$N_{cs}(x) = \mu_{cs} N(x) dx$$

and the total number of scattered photons will be:

$$N_{cs} = \int_0^l N_{cs}(x) dx$$

$$= \int_0^l \mu_{cs} N(x) dx$$

$$= \int_0^l \mu_{cs} N(0) \exp(-\mu x) dx$$

$$= \frac{\mu_{cs}}{\mu} N(0)(1 - \exp(-\mu l))$$

Note that $N(0)(1\text{-}\exp(-\mu l))$ is the total number of photons that interact with the object.

Additive Noise in X-Ray Imaging

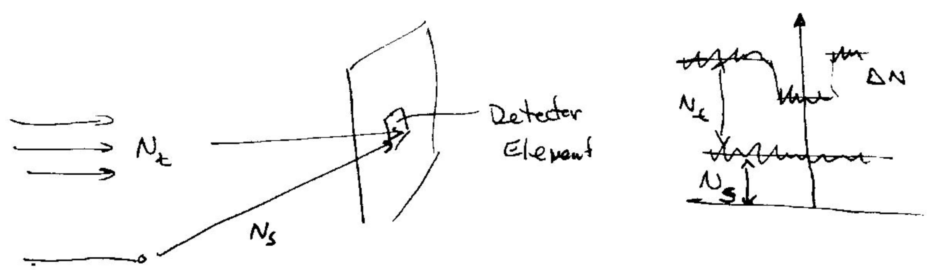
There are two kinds of additive noise that we will consider. The first is zero-mean additive noise, for example, electronic noise in digitized images. In this case, the mean value doesn't change, but the variance does. In most cases, the additive noise will be independent of the Poisson variation in the received photons and thus, the variances will add:

$$\sigma^2 = \sigma_s^2 + \sigma_a^2$$

where σ_a^2 is the variance of the additive noise and σ_s^2 is the Poisson variance. Thus, the SNR is:

$$SNR = \frac{\bar{S}}{\sigma} = \frac{\eta N}{\sqrt{\eta N + \sigma_a^2}} = \sqrt{\eta N}\,\frac{1}{\sqrt{1 + \sigma_a^2 / \eta N}}$$

The other kind of additive noise that we will consider is scatter. Since scatter is also Poisson distributed, it isn't zero mean, and as we've discussed before, it does affect the contrast.



Consider the case where we have $N_t = N$ transmitted photons and N_s scattered photons and a signal difference of ΔN. The original contrast was:

$$C = \frac{\Delta N}{N}$$

and our reduced contrast is:

$$C_r = \frac{\Delta N}{N + N_s}$$

We also know that the variance of the background signal will be the sum of the variances of the to constituent Poisson processes:

$$\sigma = \sqrt{\eta N + \eta N_s}$$

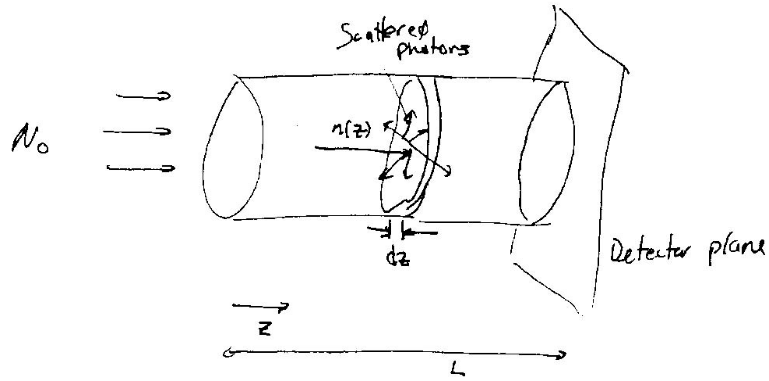
and thus, the reduced contrast to noise ratio will be:

$$CNR_r = \frac{\eta \Delta N}{\sqrt{\eta N + \eta N_s}} = C\,\frac{\eta N}{\sqrt{\eta N + \eta N_s}} = C\sqrt{\eta N}\,\frac{1}{\sqrt{1 + N_s/N}}$$

The scatter CNR reduction factor is $\sqrt{1 + N_s/N}$ or $\sqrt{1+\Psi}$, where $\Psi = \frac{N_s}{N}$ the ratio of scatter photons to transmitted photons.

SNR Reduction

To get an idea of how many photons are scattered and strike the detector, we can look at an example with an isotropic object with attenuation coefficient μ and an scattering component μ_s:



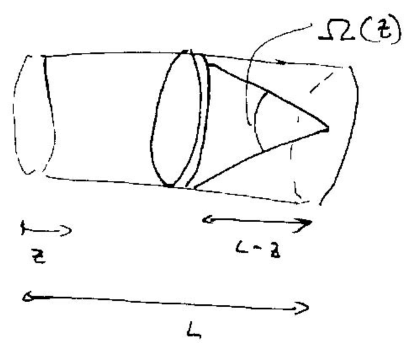
First, the number of scattered photons generated in each incremental thickness is:

$$n_s(z) = n(z)\mu_s dz = N_0 e^{-\mu z} \mu_s dz$$

We make a variety of assumptions:

1. Ignore obliquities
2. Assume a parallel ray geometry for incident intensity
3. Assume μ is energy independent
4. Neglect multiple scatters
5. Assume isotropic scattering

Thus, for the number of scattered photons, some fraction, $F(z)$, will be captured:



$$F(z) = \frac{\Omega(z)}{4\pi} e^{-\mu(L-z)}$$

The number of scattered photons at the detector will then be:

$$N_s = \int_0^L F(z)n_s(z)dz = \int_0^L \frac{\Omega(z)}{4\pi} e^{-\mu(L-z)} N_0 e^{-\mu z} \mu_s dz = e^{-\mu L} N_0 \mu_s \int_0^L \frac{\Omega(z)}{4\pi} dz = N_t \mu_s G$$

where G is a geometric, object dependent factor (which has units of length). For thin and wide object, $G \rightarrow 0.5L$ (this results from $\Omega(z) = 2\pi$) and for long, slim objects $G \rightarrow 0$. Therefore:

$$\Psi = \frac{N_s}{N} = \mu_s G$$

If we take typical values for attenuation coefficients for water at 100 keV, $\mu \approx \mu_s \approx 0.16\,\text{cm}^{-1}$, and L = 20 cm and we will let $G = 0.4L$, then:

$$\Psi = 1.28$$

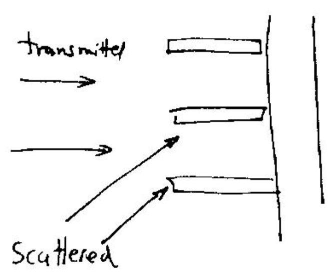
resulting in a reduction of SNR of:

$$\sqrt{1+\Psi} = 1.5$$

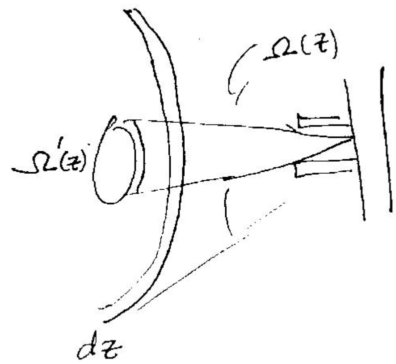
a 50% reduction in SNR.

Scatter Reduction Grids

The most common way of reducing scatter is through the use of a scatter reduction grid:



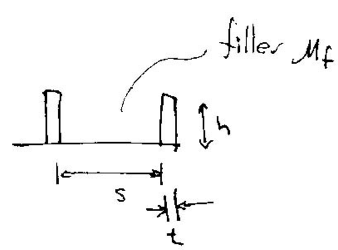
where the gird is made out of some high μ material (like Pb, W) that will block any photons that strike it. The grid works principally by cutting down on the acceptance angle for scattered photons, Ω:



We can define a scatter reduction factor:

$$R_s = \frac{\int \Omega'(z)dz}{\int \Omega(z)dz}$$

where Ω'(*z*) is the acceptance angle of the scatter reduction grid. In addition to reducing the scatter, this also results in a reduction of transmitted photons. We can define an efficiency of the grid by considering transmitted photons blocked by grid and attenuated by the filler material:



in equations, this will be:

$$\eta_t = \frac{s-t}{s}\exp(-\mu_f h)$$

The CNR will now be:

$$CNR = C\frac{\eta\eta_t N}{\sqrt{\eta\eta_t N + \eta R_s N_s}} = C\sqrt{\eta N}\frac{\sqrt{\eta_t}}{\sqrt{1+\frac{R_s\Psi}{\eta_t}}}$$

where $\sqrt{\frac{1}{\eta_t}}\sqrt{1+\frac{R_s\Psi}{\eta_t}}$ is the new SNR reduction factor. You want η_t to be close to 1 and $R_s \ll 1$.