Computed Tomography

BME/EECS 516

CT Lecture #1

Recall X-ray

- Equation governing the image intensity in X-ray projection imaging

$$I_d(x,y) = I_0 \exp\left(-\int \mu(x,y,z)\,dz\right)$$

- Planar x-ray imaging:
  - Inexpensive
  - Widespread usage

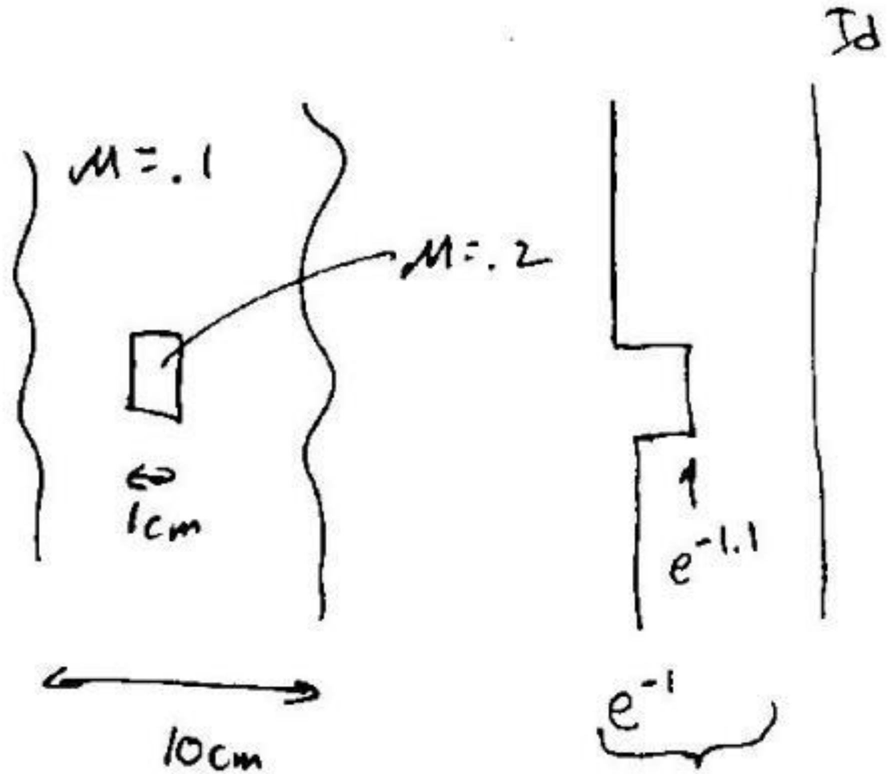
Challenges with Projection X-ray Systems

- No depth (z) information in the images
- Lack of contrast
  - due to integration, large changes in attenuation coefficient may result in very small changes
  - E.g., high attenuation in localized region.

Challenges with Projection X-ray Systems

- Contrast ratio $C = \Delta S / \bar{S}$
- Contrast to noise ratio $CNR = \Delta S / \sigma_s$

Challenges with Projection X-ray Systems



Contrast in μ $\quad \frac{.2 - .1}{.1} = \underline{\underline{1}}$

$$C = \frac{\Delta S}{S} = \frac{e^{-1} - e^{-1.1}}{e^{-1}} = \frac{0.03}{0.3} \approx \underline{\underline{0.1}}$$

Question?

Can we do better?

Yes. Computed Tomography - CT

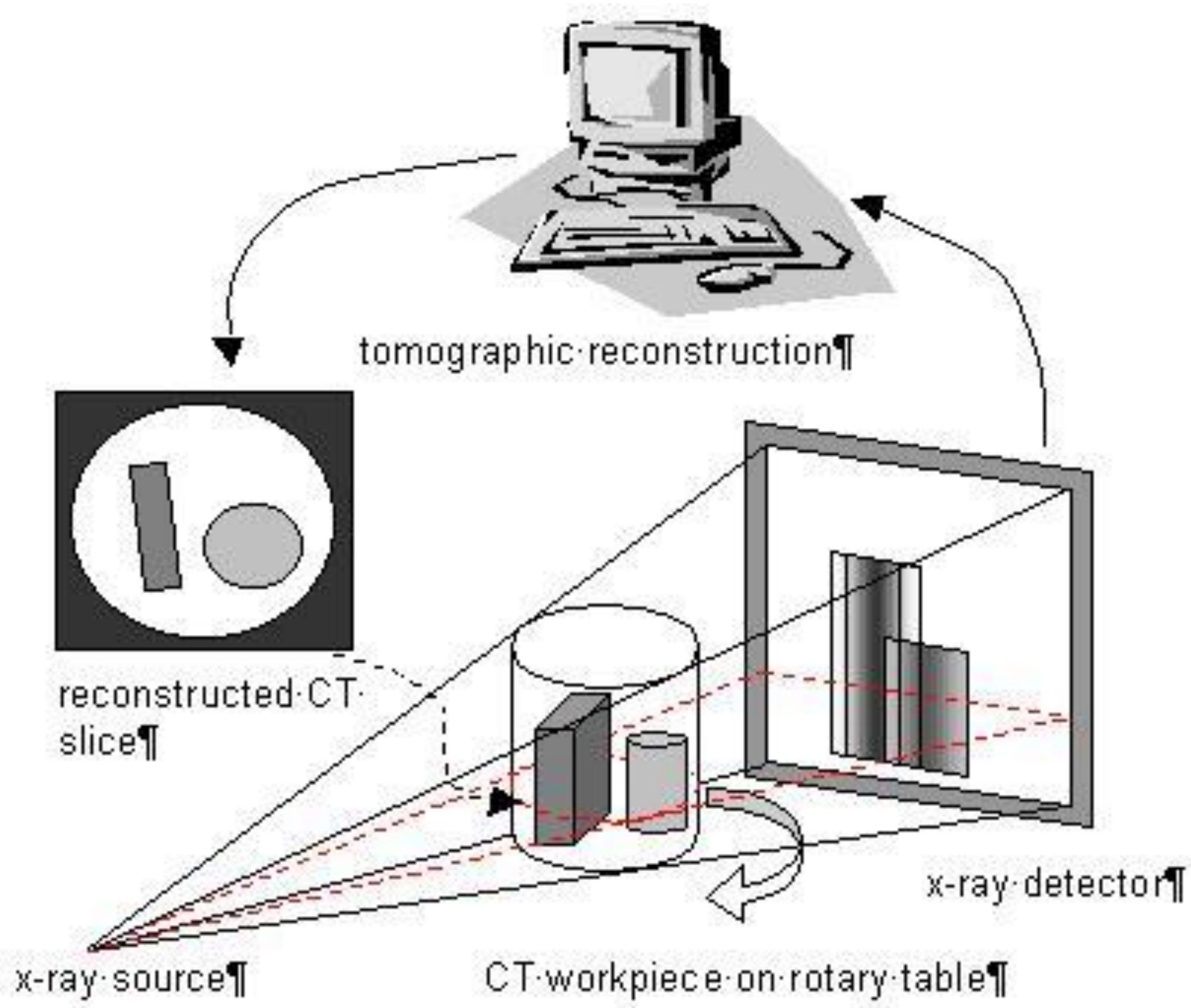
This Lecture

- CT introduction + history + basics
  - Line integral
  - Randon transform
- Central Section Theorem
- Sinograms

Computed Tomography

- Tomography: the generation of cross sectional images of anatomy or structure
- **Computed tomography (CT)**: digital projective geometry processing is used to generate a 3D image of an object from a large series of 2D X-ray images taken around a single axis of rotation.

Computed Tomography



Computed Tomography

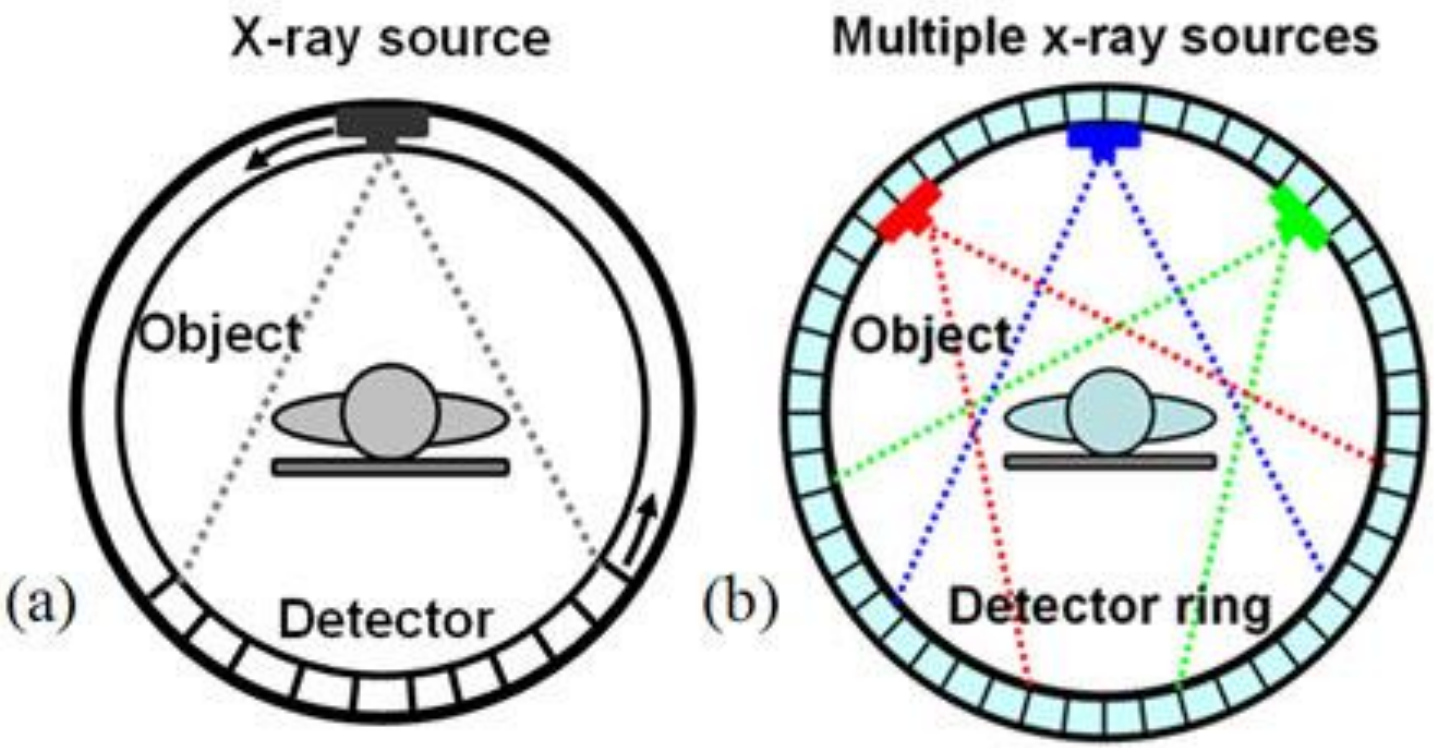
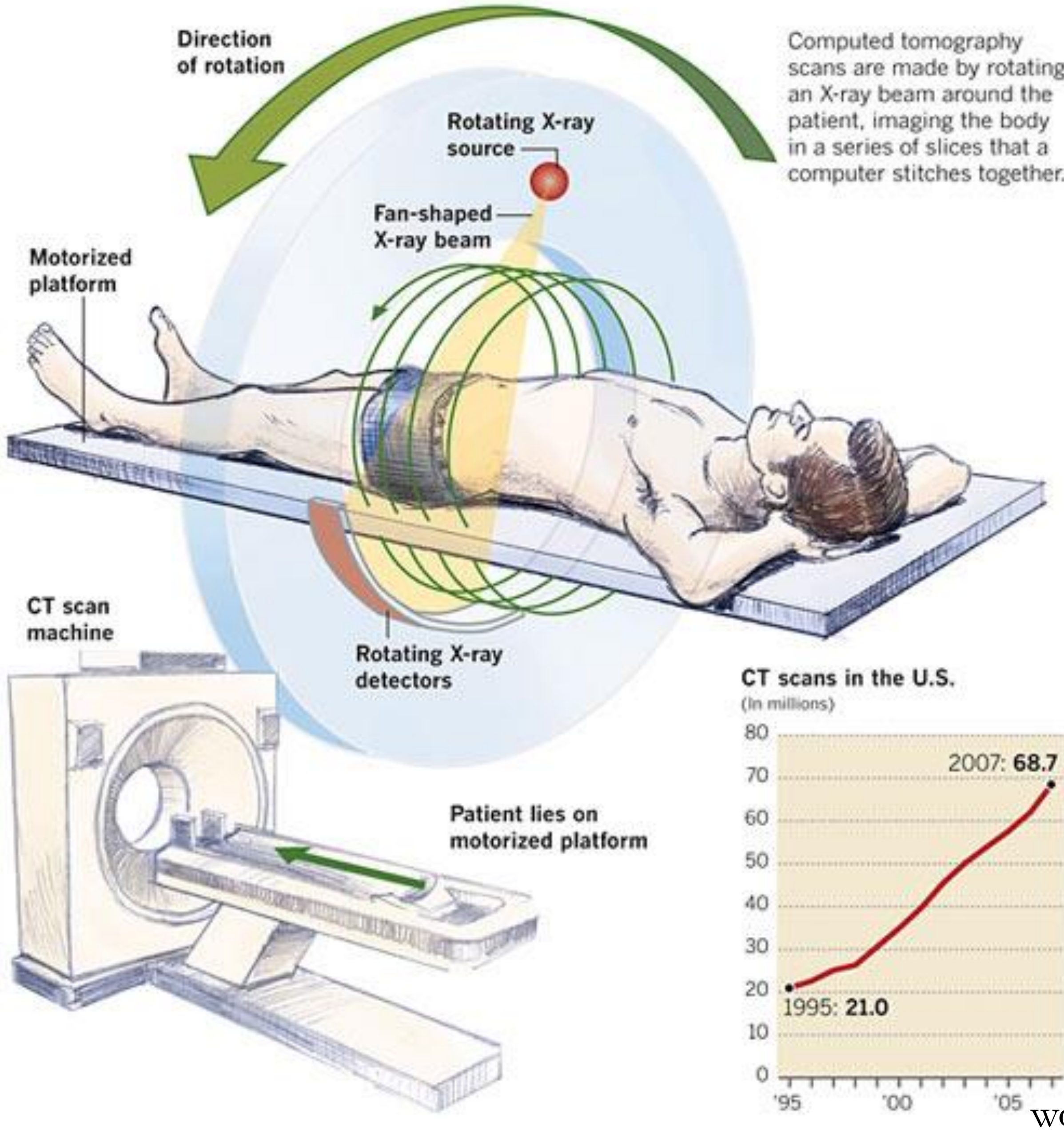


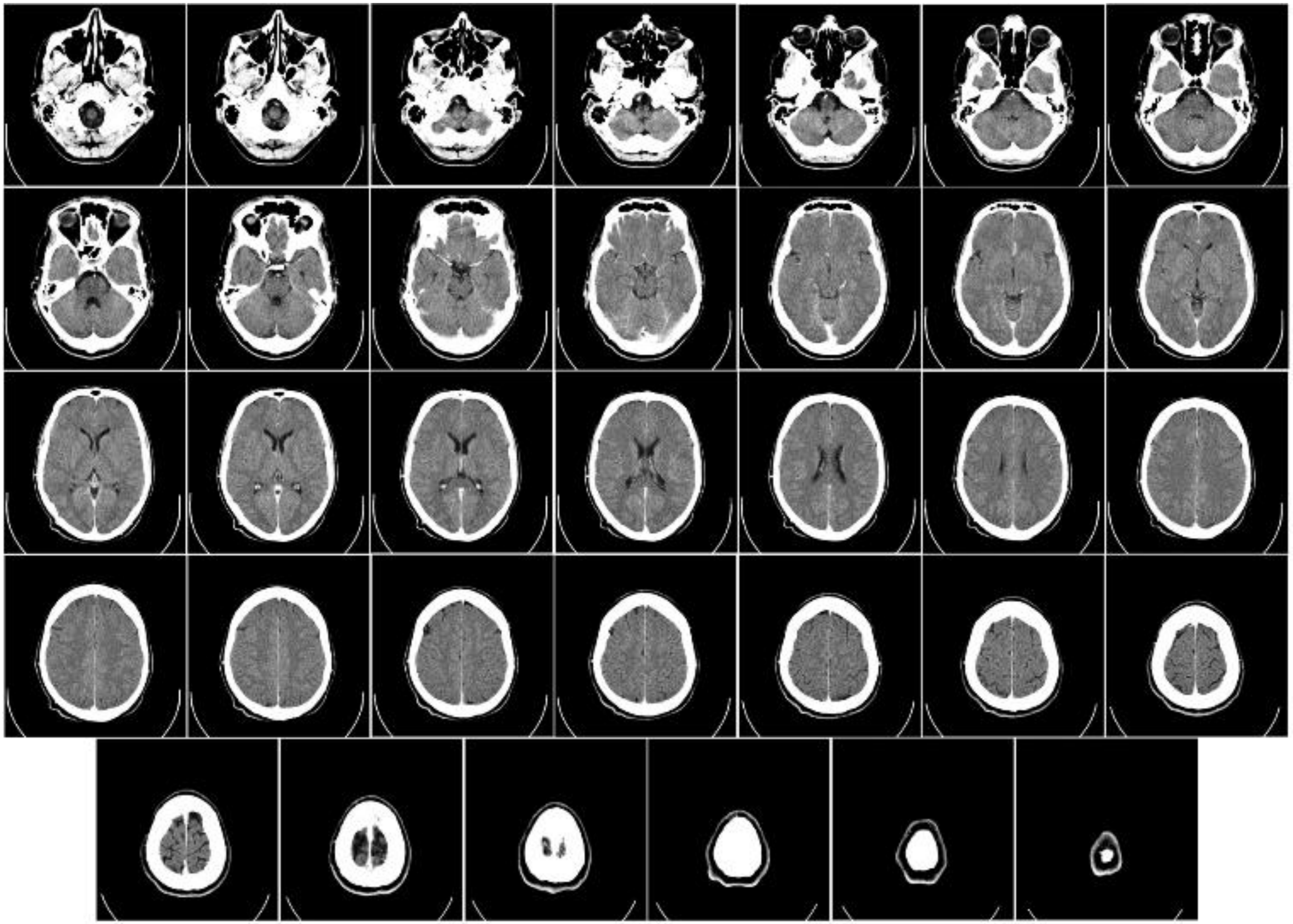
Figure 1: (a) Configuration of the conventional third generation CT scanner. (b) The multiplexing CT scanner enables simultaneous collection of multiple projection images due to the multiplexing principle.

Anatomy of a CT scan

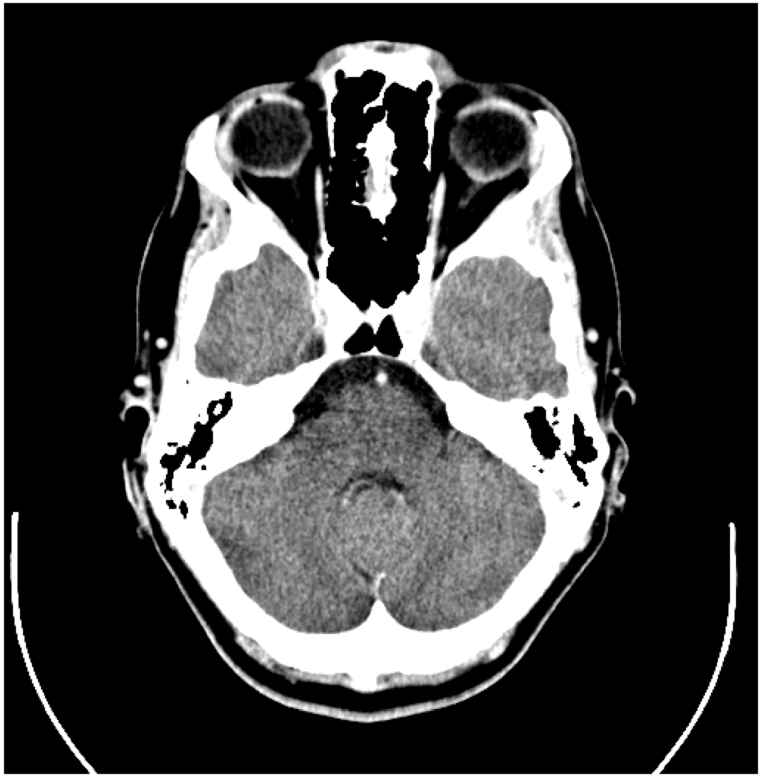
CT scanners give doctors a 3-D view of the body. The images are exquisitely detailed but require a dose of radiation that can be 100 times that of a standard X-ray.



Cranial CT scan



Cranial CT scan



A Little CT History

- Bohemian Mathematician J.H. Radon developed the concept in 1917
- Physicist Allan McLeod Cormack (Tufts University, MA) proved the concept and developed the algorithm in 1957-63.
- English Engineer Godfrey Hounsfield, Engineer (EMI central lab, UK) developed the first CT scanner in 1967-1972

A Little CT History

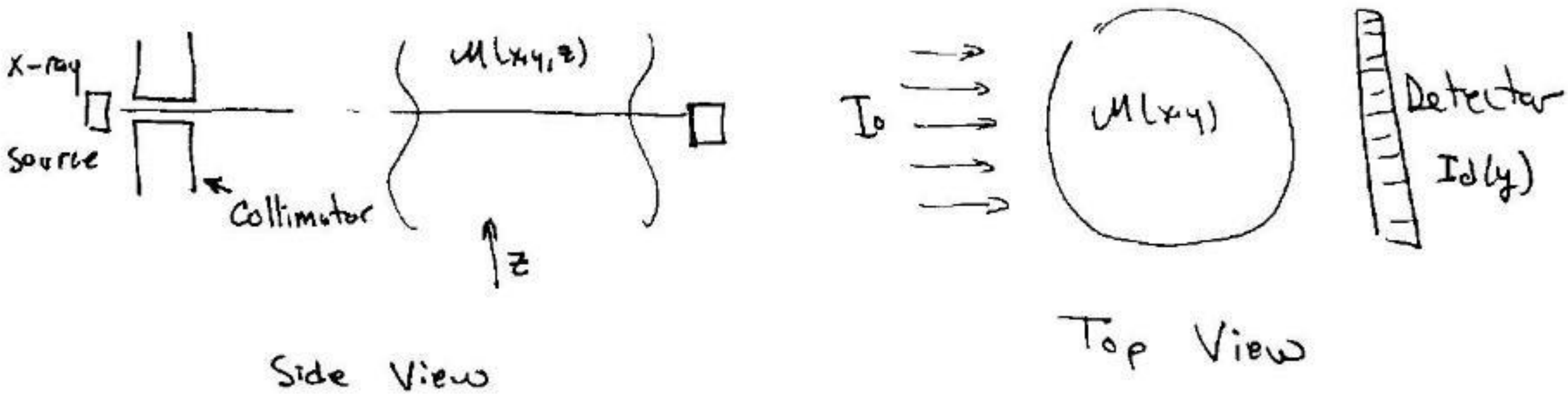
- Hounsfield and Cormack both worked without knowledge of previous study. They shared the Nobel Prize in Medicine in 1979
- Hounsfield is widely recognized as the inventor of CT.
- First commercial CT scanner – EMI scanner
    - Hounsfield worked for EMI

CT Scanner (Toshiba)

http://www.wired.com/gadgets/mis
ous/news/2008/04/Toshiba_CTScar

Computed Tomography

- Collimate a single slice through the object (i.e., $z = z_0$)



$$I_d(y) = I_0 \exp\left(-\int \mu(x,y)\,dx\right)$$

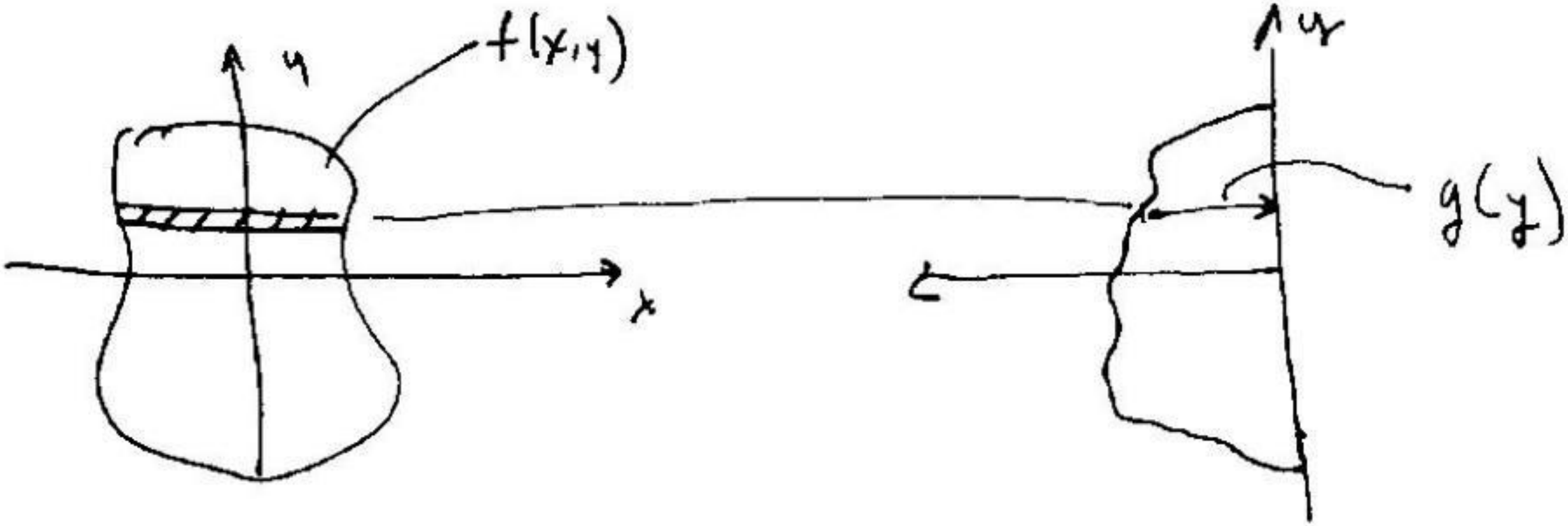
Computed Tomography

- Define the "line integral" through *f* in the *x* direction

$$g(y) = \ln \frac{I_0}{I_d(y)} = \int \mu(x,y)dx = \int f(x,y)dx$$

- Goal: solve f(x,y) from g(y) contain spatial attenuation coefficient info

Computed Tomography

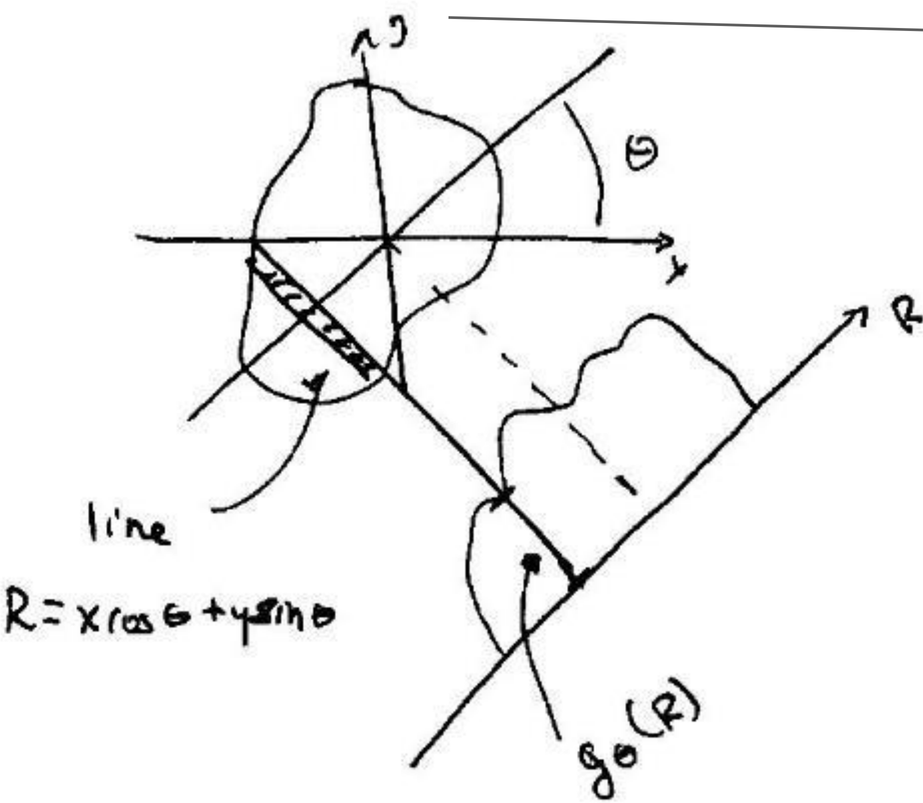


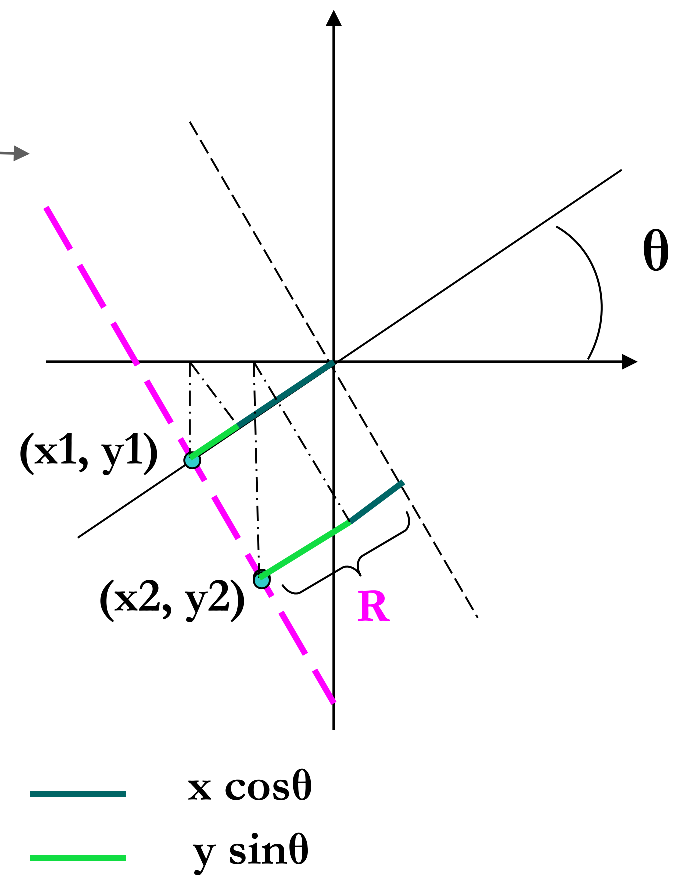
- Sampling at y = R

$$g(R) = \iint f(x,y)\delta(y-R)\,dx\,dy = \int f(x,R)\,dx$$

Computed Tomography

- **Describe line integral at an arbitrary angle, θ**





Computed Tomography

- Line integral:

$$g(R) = \iint f(x,y)\delta(y-R)dxdy = \int f(x,R)dx$$

- When $R = x\cos\theta + y\sin\theta$:

$$g_\theta(R) = \iint f(x,y)\delta(x\cos\theta + y\sin\theta - R)dxdy$$

- $g_\theta(R)$ is known as the *Radon transform* of $f(x,y)$.

True or False

- A CT scanner directly collects the Radon transform (i.e., $g_\theta(R)$)of the attenuation coefficient (i.e., $f(x, y) = \mu(x, y)$) at a series of θ and R values. Ignore the oblique path issues from a point source and blurring issues from a finite source.

True or False

- A CT scanner directly collects the Radon transform (i.e., $g_\theta(R)$)of the attenuation coefficient (i.e., $f(x,y) = \mu(x,y)$) at a series of θ and R values. Ignore the oblique path issues from a point source and blurring issues from a finite source.

- False. The x-ray detector in the CT scanner collects $I_d(R,\theta)$, which then can be used to calculate:

$$g_\theta(R) = \ln \frac{I_0}{I_d(R,\theta)} = \int \mu(x,y)\, dx\, dy$$

Questions?

CT introduction

Key to CT

Central Section Theorem

Central Section Theorem

The central section theorem:

- The 1D FT of a projection $g_\theta(R)$ is the 2D FT of $f(x,y)$ evaluated at angle θ.

Central Section Theorem

- Taking the 1D FT of the projection, we get:

$$G_\theta(\rho) = F_{1D(R)}\{g_\theta(R)\}$$

$$= \iiint f(x,y)\delta(x\cos\theta + y\sin\theta - R)\exp(-i2\pi\rho R)\,dx\,dy\,dR$$

$$= \iint f(x,y)\exp(-i2\pi\rho(x\cos\theta + y\sin\theta))\,dx\,dy$$

$$= \iint f(x,y)\exp(-i2\pi(\rho\cos\theta x + \rho\sin\theta y))\,dx\,dy$$

Let $\boldsymbol{u = \rho cos\theta,\ v = \rho sin\theta}$

Central Section Theorem

- 2D FT of $f(x,y)$

$$F(u,v) = \iint f(x,y)\exp(-i2\pi(ux+vy))\,dx\,dy$$

- 1D FT of $g_\theta(R)$

$$G_\theta(\rho) = F_{1D(R)}\{g_\theta(R)\}$$

$$= \iint f(x,y)\exp(-i2\pi(\rho\cos\theta x + \rho\sin\theta y))\,dx\,dy$$

- Compare the two - same when $u = \rho\cos\theta$, $v = \rho\sin\theta$ (recall polar coordinates)

Central Section Theorem

- $u = \rho cos\theta$, $v = \rho sin\theta$ – 2D FT in Polar Coordinates.
- Therefore:

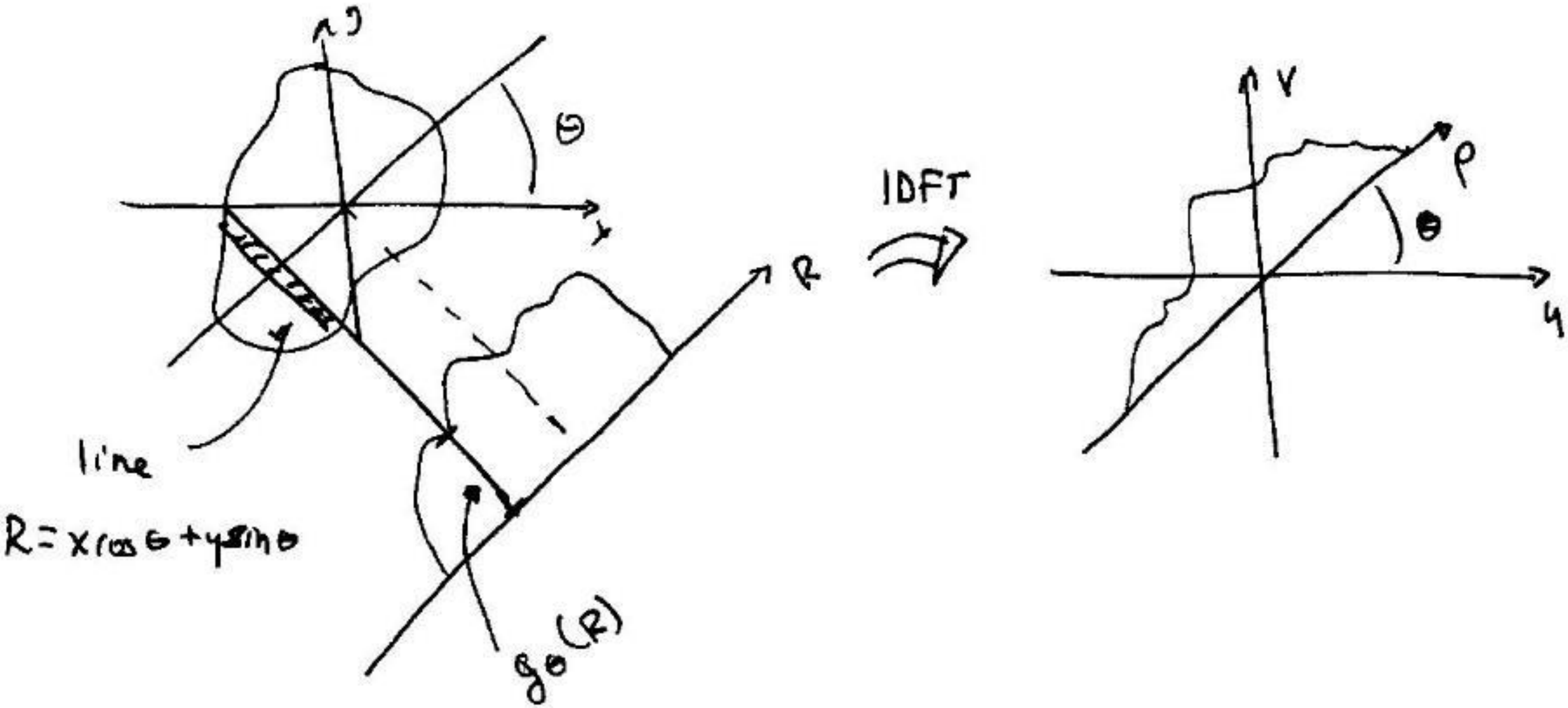
$$G_\theta(\rho) = F(u,v)\Big|_{u=\rho\cos\theta, v=\rho\sin\theta} = F(\rho,\theta)$$

Central Section Theorem

- Make an image using Central Section Theorem
- Step 1: Obtain $g_\theta(R)$ and calculate 1D FT
  - Acquired projections $g_\theta(R)$ at many different angles over $(0,\pi)$ to fill in the $F(u,v)$ space
- Step 2: inverse 2D FT to get the input image $f(x,y)$:

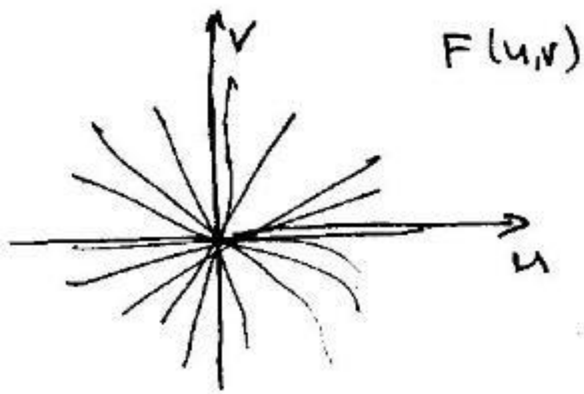
Central Section Theorem

- Step 1: 1D FT of $g_\theta(R)$



Central Section Theorem

- Step 2: inverse 2D FT to get the input



$$f(x,y) = \iint F(u,v)\exp(i2\pi(ux+vy))dudv$$

$$= \int_0^{2\pi}\int_0^{\infty} G_\theta(\rho)\exp(i2\pi(\rho\cos\theta x + \rho\sin\theta y))\rho d\rho d\theta$$

Central Section Theorem
Example 1

- Projection → Object

- $g_\theta(R)$ and 1D FT of $g_\theta(R)$ are:

$$g_\theta(R) = 2\,\mathrm{sinc}\,(2R)$$

$$F(\rho,\theta) = F_{1D}\{2\,\mathrm{sinc}\,(2R)\} = \mathrm{rect}(\rho/2)$$

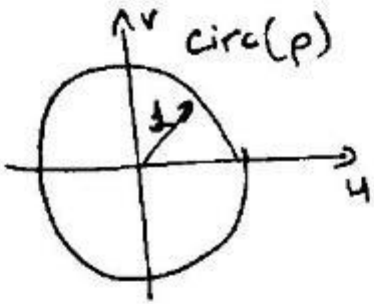
- So 2D FT of f(x,y) is

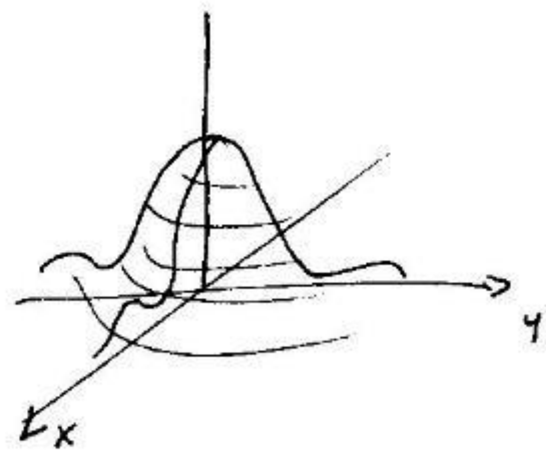
$$F(\rho) = F(\rho,\theta) = \mathrm{circ}(\rho)$$

Example 1

- Inverse 2D FT of $F(\rho)$ is

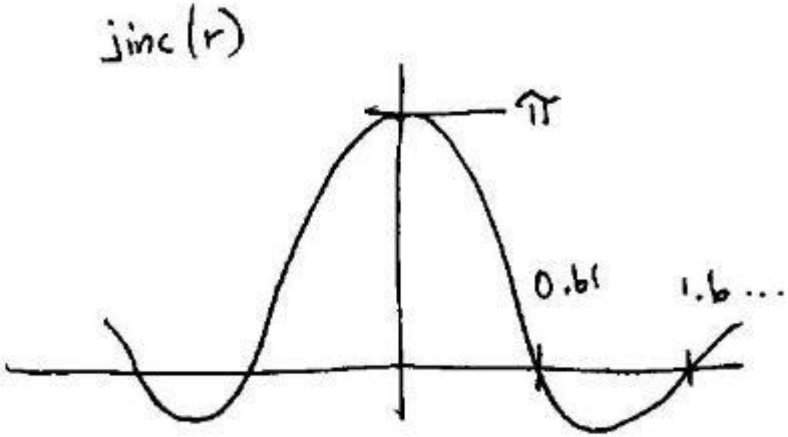
$$f(r) = \text{jinc}(r) = \frac{J_1(2\pi r)}{r}$$

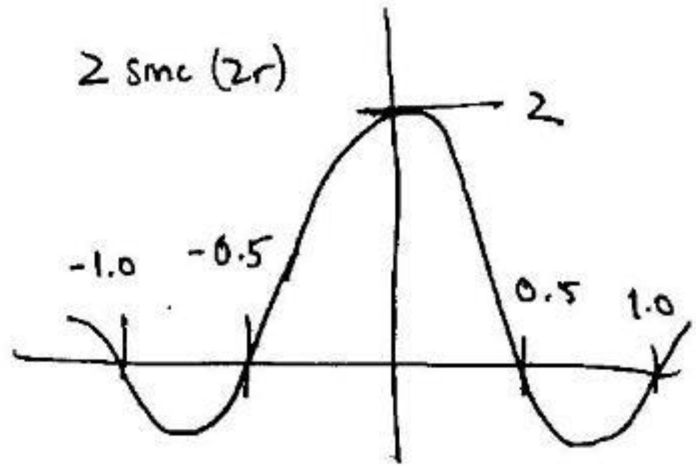




Example 1

- Projection of Jinc(r) is 2*Sinc(2r)





Example 2

- Object → Projection

- $f(x,y)$ = rect(x)rect(y) at and angle of $\theta = \pi/4$.

- 2D FT of f(x,y)

$$F(u,v) = \mathrm{sinc}(u)\mathrm{sinc}(v)$$

Example 2

- 1D FT of $g_\theta(R)$

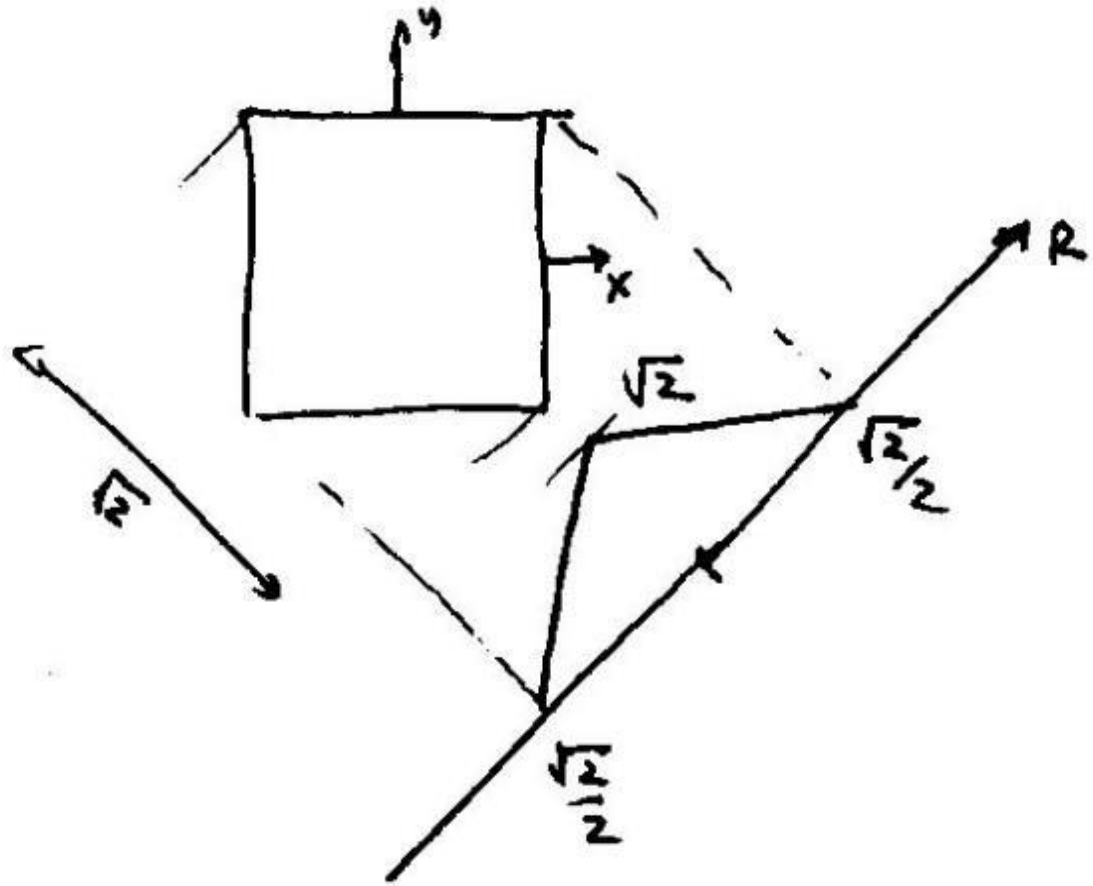
$$G_\theta(\rho) = \operatorname{sinc}(\rho\cos\theta)\operatorname{sinc}(\rho\sin\theta)$$

$$= \operatorname{sinc}(\rho/\sqrt{2})\operatorname{sinc}(\rho/\sqrt{2}) = \operatorname{sinc}^2(\rho/\sqrt{2})$$

- Inverse 1D FT → $g_\theta(R)$

$$g_\theta(r) = F_{1D}^{-1}\left\{\operatorname{sinc}^2(\rho/\sqrt{2})\right\} = \sqrt{2}\ \operatorname{tri}(\sqrt{2}r)$$

Example 2



Example 2

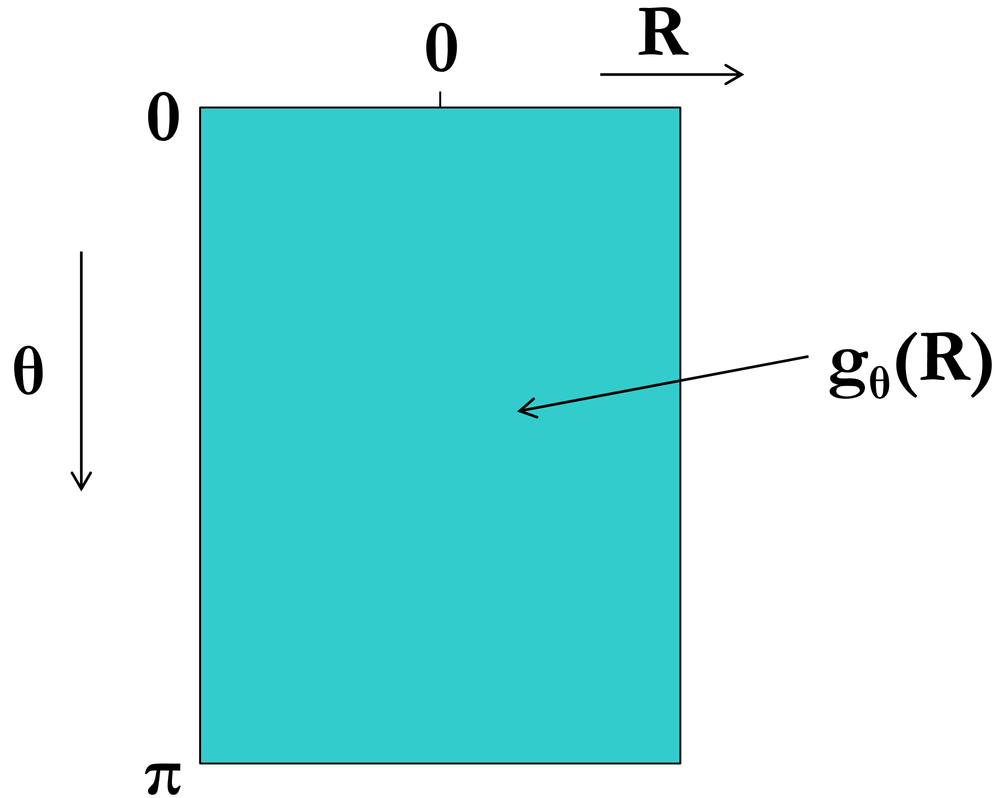
- Matlab simulation

Question?

Central Section Theorem

Sinograms

- Sinpgrams - $g_\theta(R)$ data matrix for many different angles θ



Sinograms

- Example – point projection at (x_0, y_0)

$$g_\theta(R) = \iint \delta(x - x_0)\delta(y - y_0)\delta(x\cos\theta + y\sin\theta - R)\,dx\,dy$$

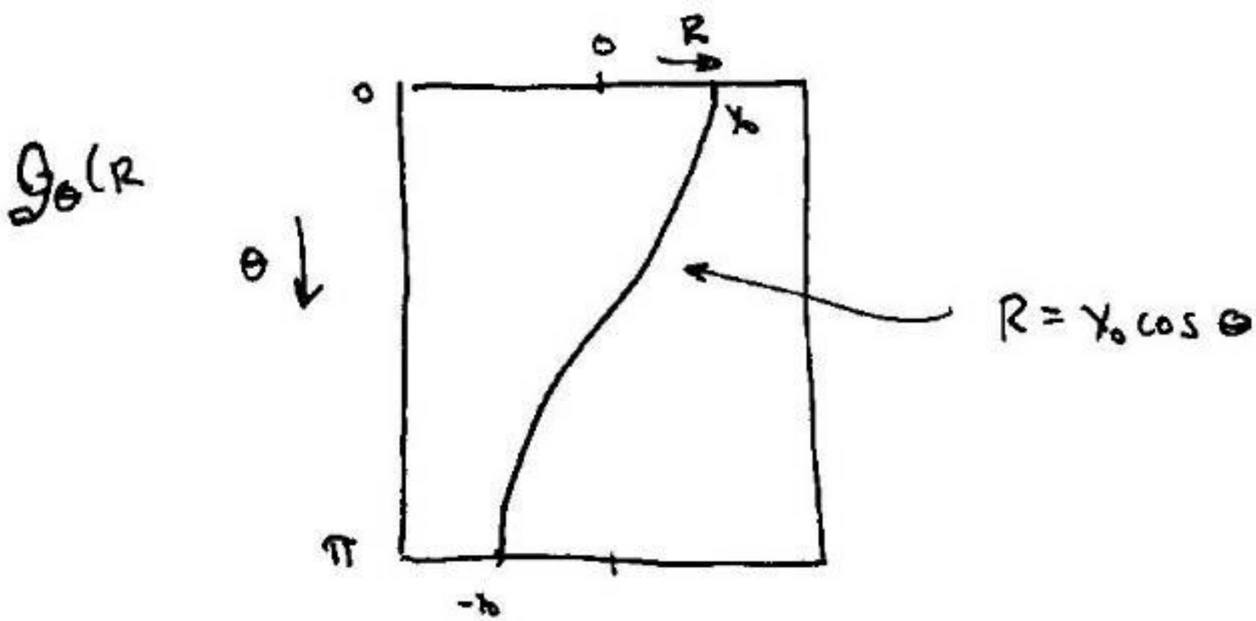
$$= \delta(x_0\cos\theta + y_0\sin\theta - R)$$

- Letting $y_0 = 0$, then:

$$g_\theta(R) = \delta(x_0\cos\theta - R)$$

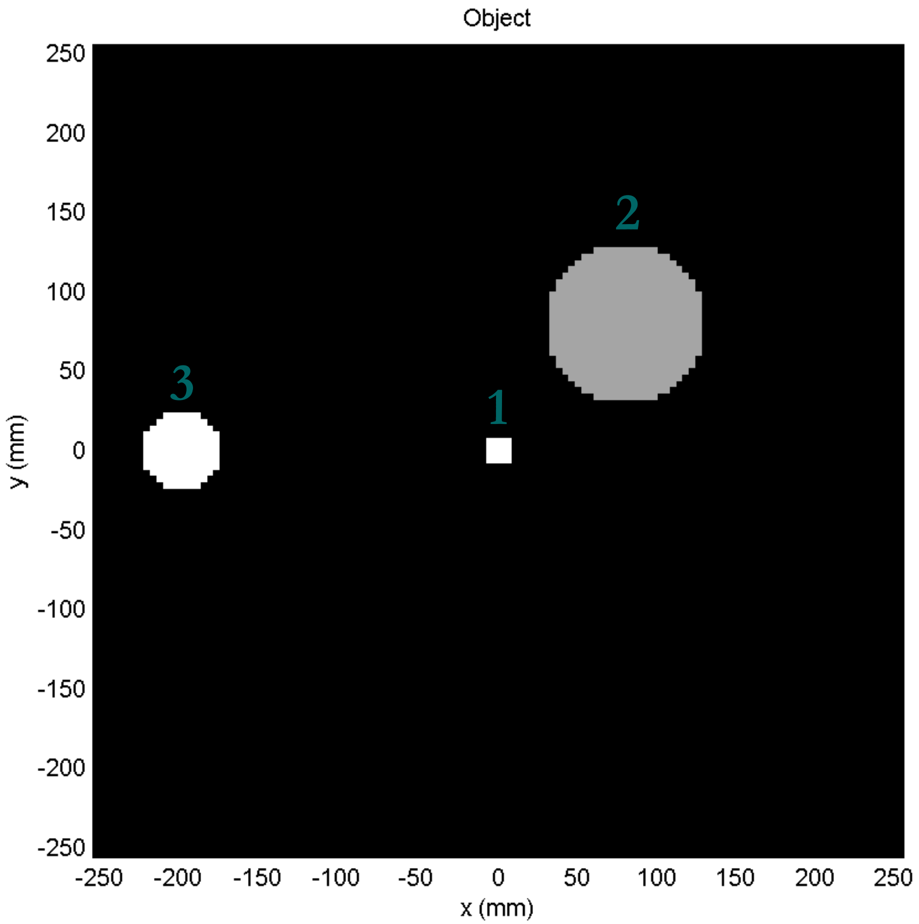
Sinograms

- A point traces out a sinusoid in the *R*-*θ* space --thus the name sinogram.



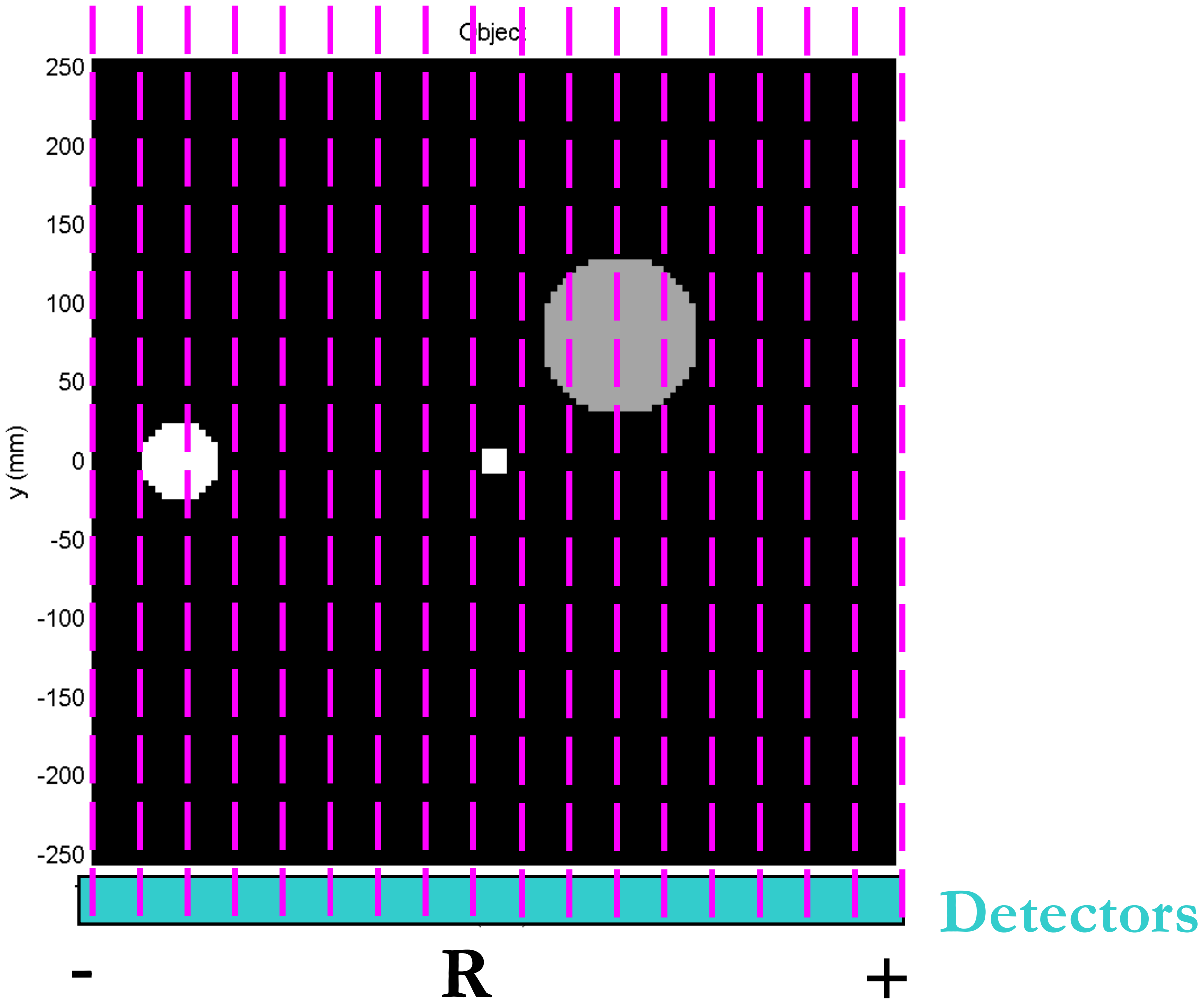
Sinograms

- Objects



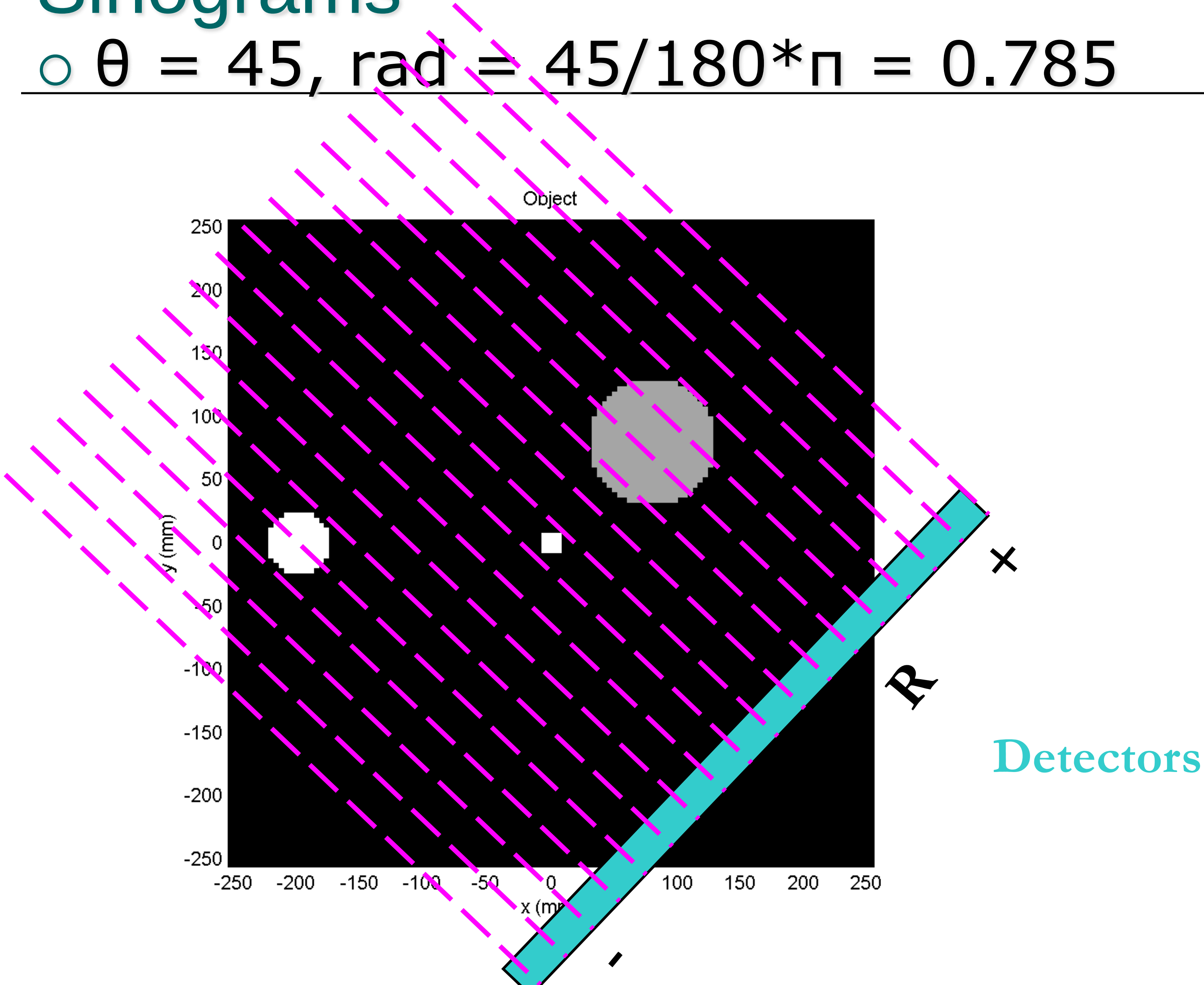
Sinograms

- θ = 0, get project

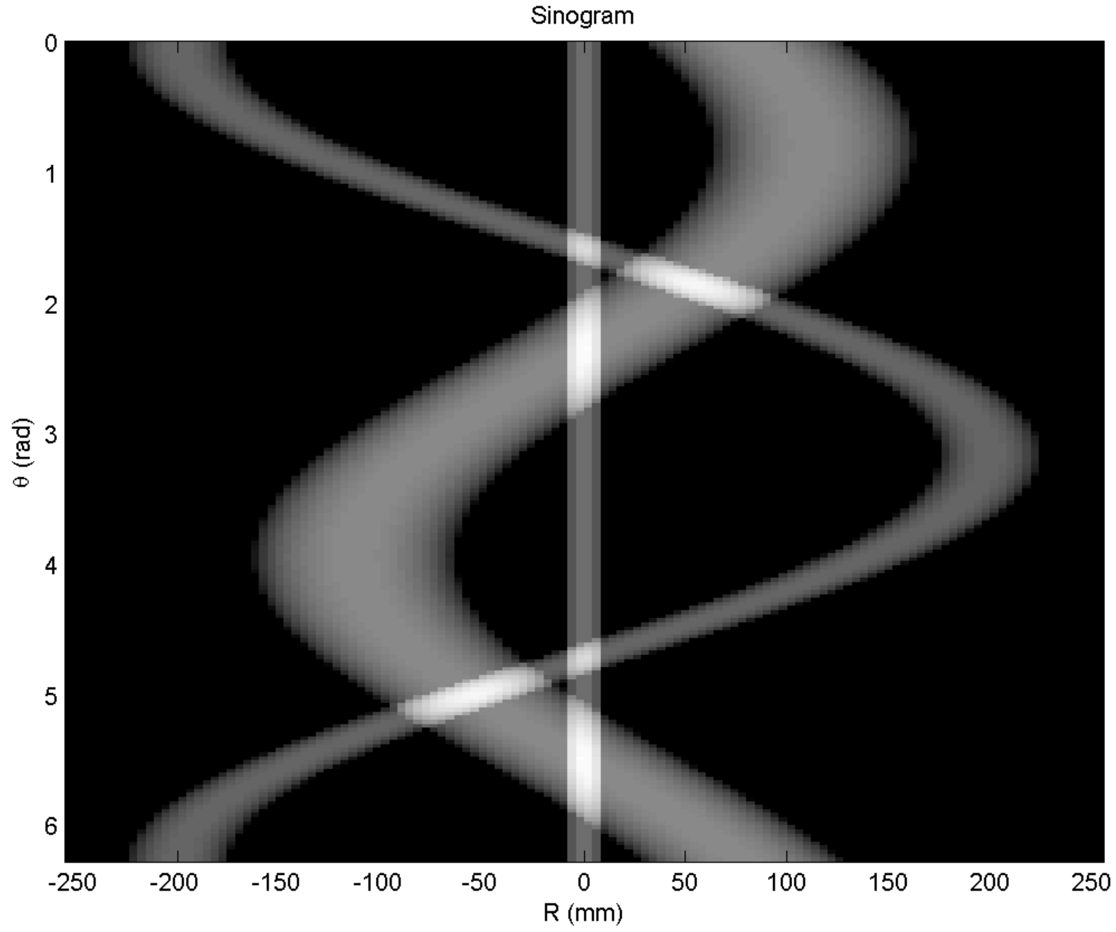


Sinograms

- θ = 45, rad = 45/180*π = 0.785



Sinograms



Sinogram
0
1
2
3
4
5
6
θ (rad)
-250
-200
-150
-100
-50
0
50
100
150
200
250
R (mm)

Sinograms

- The maximum deviation describes an object's distance from the origin
- The point of peak deviation describes the angular location of object.
- Three objects above, from smallest to largest are located at (r,θ) = (0,0), (113, π /4), and (-200,0) or (200, π).
- Symmetry: $\rho(r,\theta) = \rho(-r,\theta+\pi)$

Question?

Sinograms