NOTES ON THE THEORY OF D-MODULES

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This is not a self-contained note. It should be used as a supplement for the book [Kas03]. We fill the detail and prove some propositions whose proof are omitted in the book. We also try to understand some complicated material by considering some examples of Weyl algebra.

Notations

X: complex manifold (or smooth variety)
$\mathcal{O}_X$: sheaf of holomorphic functions on X (or regular functions on X)
$\mathcal{D}_X$: sheaf of ring of differential operators on X
Θ_X: sheaf of vector fields on X

1. Linear PDEs

In the chapter 6 of [Cou95], we learned how to build the associated D-modules for a system of PDEs that has polynomial solutions. We extend this idea to more general PDEs:

$$\sum_{j=1}^{p} P_{ij}(x,\partial)u_j = 0,\ i = 1,\dots q,$$

where $P_{ij}(x,\partial) = \sum a_\alpha(x)\partial^\alpha$ are the linear partial differential operators.

Let D be the ring of linear differential operators and $P : D^{\oplus q} \to D^{\oplus p}$ given by

$$(Q_1,\dots,Q_q) \mapsto (\sum_{i=1}^{q} Q_iP_{i1},\dots,\sum_{i=1}^{q} Q_iP_{ip}).$$

The cokernel $M = D^{\oplus p}/\mathrm{Im}\ P$ is the D-module corresponding to the system of PDEs. In particular, if $D = D(K[x_1,\dots,x_n]) = K[x,\partial] = A_n$ and $P : A_n \to A_n$ by $Q \mapsto \sum QP_i$, then

$$M = A_n/\sum A_nP_i.$$

Let F be the space where we want to find solutions. We regard F as a left D-module. Then

$$\mathcal{H}om_D(D^{\oplus p}, F) = \bigoplus \mathcal{H}om_D(D, F) = F^{\oplus p} = \{(u_1,\dots u_p) : u_1,\dots,u_p \in F\}.$$

Consider the finite presentation of M,

$$D^{\oplus q} \to D^{\oplus p} \to M \to 0.$$

By the left exactness of $\mathcal{H}om_D$,

$$0 \to \mathcal{H}om_D(M, F) \to \mathcal{H}om_D(D^{\oplus p}, F) \xrightarrow{\bar{P}} \mathcal{H}om_D(D^{\oplus q}, F).$$

Thus,

$$\mathcal{H}om_D(M, F) = \mathrm{Ker}\ \bar{P} = \{(u_1,\dots,u_p) \in F^{\oplus p} : \sum_j P_{ij}u_i = 0\}.$$

Remark. The map $\bar{P}$ can be viewed as P by the following correspondences

$$\begin{array}{ccc} \mathcal{H}om_D(D^{\oplus p}, F) & \longrightarrow & \mathcal{H}om_D(D^{\oplus q}, F) \\ \updownarrow & & \updownarrow \\ F^{\oplus p} & \longrightarrow & F^{\oplus q} \end{array}$$

Hence, the solution space of the system of linear PDEs is $\mathcal{H}om_D(M, F)$.

2. The Category of D-modules

This section correspond to chapter 1 of [Kas03]. The goal is to prove the equivalence between the category $\mathrm{Mod}(\mathcal{D}_X)$ of left $\mathcal{D}_X$-modules and the category $\mathrm{Mod}(\mathcal{D}_X^{\mathrm{op}})$ of right $\mathcal{D}_X$-modules.

Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules.

Definition 2.1. A $\mathbb{C}$-linear sheaf homomorphism $f : \mathcal{F} \to \mathcal{G}$ is called a *differential homomorphism* if for each $s \in \mathcal{F}$ there exists finitely many $P_j \in \mathcal{D}_X$ and $v_j \in \mathcal{G}$ such that

$$f(as) = \sum_j P_j(a) v_j$$

for all $a \in \mathcal{O}_X$.

Let $\mathcal{D}\mathit{iff}(\mathcal{F}, \mathcal{G})$ be the sheaf of differential homomorphisms from $\mathcal{F}$ to $\mathcal{G}$. Note that $\mathcal{D}_X = \mathcal{D}\mathit{iff}(\mathcal{O}_X, \mathcal{O}_X)$.

Proposition 2.2. *There is an isomorphism*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathcal{D}\mathit{iff}(\mathcal{F}, \mathcal{G}).$$

To prove Corollary 1.4, we first state a well-known theorem.

Lemma 2.3. *(Adjoint Associativity) Let R and S be rings and $A_R, {}_RB_S, C_S$,-bimodules. Then there is an isomorphism of abelian groups*

$$\alpha : \mathcal{H}om_S(A \otimes_R B, C) \xrightarrow{\sim} \mathcal{H}om_R(A, \mathcal{H}om_S(B, C)),$$

defined for each $f : A \otimes_R B \to C$ by

$$((\alpha f)(a))(b) = f(a \otimes b).$$

Proof. See [Hun89]. □

Corollary 2.4 (Corollary 1.4)**.**

$$\mathcal{H}om_{\mathcal{D}_X^{op}}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \simeq \mathcal{D}\mathit{iff}(\mathcal{F}, \mathcal{G}).$$

Proof. It suffices to show that

$$\mathcal{H}om_{\mathcal{D}_X^{op}}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X).$$

Since the right $\mathcal{D}_X$-modules induces left D_X^{op}-modules,

$$\mathcal{H}om_{\mathcal{D}_X^{op}}(\mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \simeq \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X).$$

By the adjoint associativity,

$$\begin{aligned}\mathcal{H}om_{\mathcal{D}_X^{op}}(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{D}_X,\mathcal{G}\otimes_{\mathcal{O}_X}\mathcal{D}_X) &\simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{H}om_{\mathcal{D}_X^{op}}(\mathcal{D}_X,\mathcal{G}\otimes_{\mathcal{O}_X}\mathcal{D}_X))\\ &\simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X,\mathcal{G}\otimes_{\mathcal{O}_X}\mathcal{D}_X))\\ &\simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}\otimes_{\mathcal{O}_X}\mathcal{D}_X)\end{aligned}$$

□

We talk about the invertible modules. The references are [Eis95] §11.3 and [Ati94] Chapter 9.

Definition 2.5. Let R be a ring and I be an R-module. Then I is invertible if I is finitely generated and locally free of rank 1; that is, for all prime ideals $\mathfrak{p}$ of R, $I_{\mathfrak{p}} \cong R_{\mathfrak{p}}$.

Definition 2.6. Let S be the set of non zero-divisors, and $K(R) = S^{-1}R$ be the total quotient ring of R. An R-submodule I of $K(R)$ is a fractional ideal if there exists $r \in R$ such that $rI \subset R$.

Theorem 2.7. *(*[Eis95] *Theorem 11.6) Let R be a noetherian ring.*

(a) *If I is an R-module, then I is invertible iff the natural map $\mu : I^*\otimes I \to R$ is an isomorphism.*
(b) *Every invertible module is isomorphic to a fractional ideal of R. Every invertible fractional ideal contains a nonzerodivisor of R.*
(c) *If $I, J \subset K(R)$ are invertible modules, then the natural maps $I \otimes J \to IJ$, taking $s \otimes t$ to st, and $I^{-1}J \to \mathrm{Hom}_R(I, J)$, taking $t \in I^{-1}J$ to $\varphi_t : I \to J$ defined by $\varphi_t(a) = ta$, are isomorphisms. In particular, $I^{-1} \cong I^*$.*

There are some remarks about Theorem 11.6 in [Eis95]. First, every invertible module is a fractional ideal. Let $I^{-1} = \{s \in K(R) : sI \subset R\}$. Then invertibility is a "dual" property; that is, if I is invertible then so is I^{-1}. To see that $I^{-1} \cong I^* = \mathcal{H}om_R(I, R)$, notice that

$$I^{-1} \cong I^{-1} \otimes R \cong I^{-1}R \cong \mathcal{H}om_R(I, R).$$

As an application in [Kas03] page 9, let $R = \mathcal{O}, I = \mathcal{L}$. Then $\mathcal{L}^{\otimes -1} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) = \mathcal{L}^*$ is invetible. By the theorem above, $\mathcal{L}^{\otimes -1} \otimes_{\mathcal{O}_X} \mathcal{L} \cong \mathcal{O}_X$.

To get $\mathcal{D}iff(\Omega_X, \Omega_X) = \Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}$ in [Kas03] Proposition 1.10, we use Corollary 2.4 and hence

$$\begin{aligned}\mathcal{D}iff(\Omega_X,\Omega_X) &= \mathcal{H}om_{\mathcal{D}_X^{op}}(\Omega_X\otimes_{\mathcal{O}_X}\mathcal{D}_X,\Omega_X\otimes_{\mathcal{O}_X}\mathcal{D}_X)\\ &= \mathcal{H}om_{\mathcal{O}_X}(\Omega_X,\mathcal{H}om_{\mathcal{D}_X^{op}}(\mathcal{D}_X,\Omega_X\otimes_{\mathcal{O}_X}\mathcal{D}_X))\\ &= \mathcal{H}om_{\mathcal{O}_X}(\Omega_X,\Omega_X\otimes_{\mathcal{O}_X}\mathcal{D}_X)\\ &= \Omega_X\otimes_{\mathcal{O}_X}\mathcal{D}_X\otimes_{\mathcal{O}_X}\Omega_X^{\otimes -1}\end{aligned}$$

3. Coherent Modules

Definition 3.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$-module. We say that $\mathcal{M}$ is of finite type if for each $x \in X$ there exists an open neighborhood U such that $\mathcal{M}|_U$ is generated by finitely many sections. i.e. there is a surjective morphism $\mathcal{O}_X^n \to \mathcal{M}|_U$ for some n.

Definition 3.2. We say that $\mathcal{M}$ is a coherent $\mathcal{O}_X$-module if the following conditions hold:

(1) $\mathcal{M}$ is of finite type

(2) for every open $U \subset X$ and every finite collection $s_i \in \mathcal{M}(U),\ i = 1, \ldots, n$, the kernel of the map $\mathcal{O}_X^n \to \mathcal{M}|_U$ is of finite type.

Let $\mathcal{F}$ be a sheaf on X. The support of $s \in \mathcal{F}(U)$ is defined to be $\{P \in U \mid s_P \neq 0\}$, where s_P is the germ of s in the stalk $\mathcal{F}_P$. We can show that $\mathrm{Supp}(s)$ is a closed subset of U. For an element m in a module $\mathcal{M}$, we have $\mathrm{Supp}(m) = V(\mathrm{Ann}\ m)$. (See [Har77])

Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module and Z is a closed analytic subset of a manifold X. Since $\mathrm{Supp}(m) = V(\mathrm{Ann}\ m)$, we can see that $\Gamma_Z(\mathcal{F}) = \{u \in \mathcal{F} : \mathrm{Supp}(u) \subset Z\}$ is a finitely generated submodule and hence is coherent.

4. Cotangent Bundles

Let

$$F_m(\mathcal{D}_X) = \{P \in \mathcal{D}_X : P = \sum_{\substack{|\alpha| \leq m \\ \alpha \in \mathbb{Z}^n_{\geq 0}}} a_\alpha(x) \partial_x^\alpha\}$$

where $|\alpha| = \alpha_1 + \ldots \alpha_n$. Then $\{F_m\}$ is the *order filtration* of $\mathcal{D}_X$. It gives rise to the graded ring

$$\mathrm{Gr}^F(\mathcal{D}_X) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Gr}^F_n(\mathcal{D}_X)$$

where

$$\mathrm{Gr}^F_n(\mathcal{D}_X) = F_n(\mathcal{D}_X)/F_{n-1}(\mathcal{D}_X).$$

Note that $\mathrm{Gr}^F(\mathcal{D}_X) = \bigoplus_\alpha \mathcal{O}_X \partial_x^\alpha$ is a sheaf of commutative algebra over $\mathcal{O}_X$.

Take a coordinate system $(x_1, \ldots, x_n)$. We introduce the symbols

$$\xi_i = \frac{\partial}{\partial x_i} \mod F_0(\mathcal{D}_X) = \mathcal{O}_X.$$

i.e. $\xi_i - \frac{\partial}{\partial x_i}$ is a holomorphic function. Denote by Θ_X the sheaf of vector fields on X. With local coordinates, $\Theta_X = \bigoplus_{i=1}^n \mathcal{O}_X \frac{\partial}{\partial x_i}$. Then, there is an isomorphism

$$\mathrm{Sym}_{\mathcal{O}_X}(\Theta_X) \xrightarrow{\sim} \mathcal{O}_X \otimes \mathbb{C}[\xi_1, \ldots, \xi_n] = \mathcal{O}_X[\xi_1, \ldots, \xi_n]$$

by $\frac{\partial}{\partial x_i} \to \xi_i$. Similarly, we can identify $\mathrm{Gr}^F(\mathcal{D}_X)$ with $\mathcal{O}_X[\xi_1, \ldots, \xi_n]$. We have thus obtained the following theorem.

Theorem 4.1. *There is an isomorphism of graded rings*

$$\mathrm{Sym}_{\mathcal{O}_X}(\Theta_X) \xrightarrow{\sim} \mathrm{Gr}^F(\mathcal{D}_X).$$

Consider the the ring homomorphisms

$$\sigma_m : F_m(\mathcal{D}_X) \to \mathrm{Gr}^F_m(\mathcal{D}_X) \subset \mathrm{Gr}^F(\mathcal{D}_X) \simeq \mathcal{O}_X[\xi_1, \ldots, \xi_n]$$

given by

$$\sigma_m(P) = \sum_{|\alpha| = m} a_\alpha \xi^\alpha \in \mathrm{Sym}^m_{\mathcal{O}_X}(\Theta_X).$$

The image $\sigma_m(P)$ is call the *principal symbol* of P.

Let $\pi : T^*X \to X$ be the cotangent bundle. It induces a map $\mathcal{O}_{T^*X} \to \pi_*\mathcal{O}_{T^*X}$. We may regard $\xi_1, \dots, \xi_n$ as a coordinate system of T^*_xX.(see [THT07]) Indeed, if $\theta_x = \sum_j \xi_j dx^j \big|_x \in T^*_xX$, then

$$\xi_i(\theta_x) = \theta_x\left(\frac{\partial}{\partial x_i}\right) = \sum_j \xi_j dx^j \bigg|_x \left(\frac{\partial}{\partial x_i}\right) = \xi_i.$$

These $\xi_1, \dots, \xi_n$ gives the *natural chart* $(\pi^{-1}(U), \{\xi_i\})$ of T^*X. Hence, we have a canonical identification

$$\pi_*\mathcal{O}_{T^*X}(V) = \mathcal{O}_{T^*X}(\pi^{-1}(V)) \simeq \mathcal{O}_X[\xi_1, \dots, \xi_n]$$

as $\mathcal{O}_X$-algebras. Thus, we have the proved the following

Corollary 4.2. *There exists isomorphisms of $\mathcal{O}_X$-algebras*

$$\mathrm{Gr}^F(\mathcal{D}_X) \simeq \pi_*\mathcal{O}_{T^*X} \simeq \mathrm{Sym}_{\mathcal{O}_X}(\Theta_X).$$

Thus, for all $P \in F_m(\mathcal{D}_X)$ we can associate to it a regular function $\sigma_m(P)$ defined on the cotangent bundle T^*X.

Let $P \in F_{m_1}(\mathcal{D}_X), Q \in F_{m_2}(\mathcal{D}_X)$. Then, $[P, Q] \in F_{m_1+m_2-1}(\mathcal{D}_X)$. This induces a map

$$\mathrm{Gr}^F_{m_1}(\mathcal{D}_X) \times \mathrm{Gr}^F_{m_2}(\mathcal{D}_X) \to \mathrm{Gr}^F_{m_1+m_2-1}(\mathcal{D}_X).$$

Given functions $f(x, \xi)$ and $g(x, \xi)$, set

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial x_i} - \frac{\partial g}{\partial \xi_i}\frac{\partial f}{\partial x_i}\right).$$

Then,

$$\sigma_{m_1+m_2-1}([P, Q]) = \{\sigma_{m_1}(P), \sigma_{m_2}(Q)\}.$$

Let $p \in T^*X$ be a point. be Let ω_X be a 1-form (i.e. a section of $T^*(T^*X)$) on T^*X defined by $\omega_X(p) = \pi_X^*\omega_p$, where ω_p is a 1-form (i.e. a section of T^*X) at the point $\pi(p) \in X$. In local coordinates,

$$\omega_X = \sum_i \xi_i dx_i.$$

At every point f, the 2-form $\theta_X = d\omega_X$ gives an anti-symmetric bilinear form on $T_p(T^*X)$. This is nondegenerate. Let $H : T_p(T^*X) \xrightarrow{\sim} T_p(T^*X)$ be given by the paring

$$\langle \theta_X, v \wedge H(\eta) \rangle = \langle \eta, v \rangle$$

for $v \in T_p(T^*X), \eta \in T^*_p(T^*X)$. In local coordinates,

$$d\xi_i \xrightarrow{H} \frac{\partial}{\partial x_i}$$
$$dx_i \xrightarrow{H} -\frac{\partial}{\partial \xi_i}$$

In particular, $H_f = H(df)$ is a vector field on T^*X, which is called the *Hamiltonian* of f.

Definition 4.3. For functions f, g on T^*X, $\{f, g\} = H_f(g)$ is called the *Poisson bracket* of f and g.

Definition 4.4. Let X be a manifold, and θ be a closed 2-form on X that gives a nondegenerate anti-symmetric bilinear form on T_pX for every p. A tuple (X, θ) is called a *symplectic manifold*.

For example, (T^*X, θ_X) is a symplectic manifold.

If X is the affine space $\mathbb{C}^n$, then $T^*X = \mathbb{C}^{2n}$. Let

$$\Omega = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$

Given $u, v \in \mathbb{C}^{2n}$, define

$$\omega(u,v) = u\Omega v^t.$$

This is called the standard symplectic structure.

5. Characteristic Varieties

Definition 5.1. Let M be an R-module. The support of M is

$$\begin{aligned} \mathrm{Supp}(M) &= \{\mathfrak{p} \in \mathrm{Spec}(R) : M_{\mathfrak{p}} \neq 0\} \\ &= \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \supset \mathrm{Ann}(M)\} \end{aligned}$$

Lemma 5.2. *If M is a finitely generated R-module, then*

$$\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$$

Let $\{F_m(\mathcal{M})\}$ be a coherent filtration of a coherent $\mathcal{D}_X$-module $\mathcal{M}$.

Set

$$\mathrm{Gr}^F(\mathcal{M}) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Gr}^F_n(\mathcal{M})$$

where

$$\mathrm{Gr}^F_n(\mathcal{D}_X) = F_n(\mathcal{M})/F_{n-1}(\mathcal{M}).$$

Then, $\mathrm{Gr}^F(\mathcal{M})$ is $\mathrm{Gr}^F(\mathcal{D}_X)$-module.

Denote

$$\widetilde{\mathrm{Gr}^F\mathcal{M}} = \mathcal{O}_{T^*X} \otimes_{\pi^{-1}\mathrm{Gr}^F\mathcal{D}_X} \pi^{-1}\mathrm{Gr}^F\mathcal{M}.$$

This is an $\mathcal{O}_{T^*X}$-module. i.e. $\sim$ is an exact functor from $\mathrm{Mod}(\mathrm{Gr}^F(\mathcal{D}_X))$ to $\mathrm{Mod}(\mathcal{O}_{T^*X})$. Hence, we can define a variety in T^*X.

Definition 5.3. The characteristic variety of a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is

$$\mathrm{Ch}(\mathcal{M}) = \mathrm{Supp}(\widetilde{\mathrm{Gr}^F\mathcal{M}}) = \mathrm{Supp}(\mathcal{O}_{T^*X} \otimes_{\pi^{-1}\mathrm{Gr}^F\mathcal{D}_X} \pi^{-1}\mathrm{Gr}^F\mathcal{M}).$$

Theorem 2.6 on [Kas03] implies this definition is valid. Denote

$$J_{\mathcal{M}} = \sqrt{\mathrm{Ann}_{\mathcal{O}_X}(\widetilde{\mathrm{Gr}^F\mathcal{M}})}.$$

By the lemma above,

$$\mathrm{Supp}(\widetilde{\mathrm{Gr}^F\mathcal{M}}) = V(\mathrm{Ann}(\widetilde{\mathrm{Gr}^F\mathcal{M}})) = V(J_{\mathcal{M}}) \subset T^*X.$$

If U is an open set of X, then

$$\begin{aligned} \mathrm{Ch}(\mathcal{M}) \cap T^*U &= \mathrm{Supp}(\mathcal{O}_{T^*U} \otimes_{\pi^{-1}\mathrm{Gr}^F\mathcal{D}_U} \pi^{-1}\mathrm{Gr}^F\mathcal{M}|_U) \\ &= V(J_{\mathcal{M}(U)}) \\ &= \{p \in T^*U : f(p) = 0 \text{ for all } f \in J_{\mathcal{M}(U)}\}. \end{aligned}$$

Clearly, $\mathrm{Ch}(\mathcal{M})$ is a closed subset.

If X is affine space $\mathbb{A}^n$, then

$$J_{\mathcal{M}} = \sqrt{\mathrm{Ann}(\mathcal{O}_{T^*X} \otimes_{\mathcal{O}_{T^*X}} \pi^{-1}\mathrm{Gr}^F(\mathcal{M}))} = \sqrt{\mathrm{Ann}_{\mathrm{Gr}^F\mathcal{D}_X}(\mathrm{Gr}^F(\mathcal{M}))} \subset T^*X = \mathbb{A}^{2n}.$$

Note that $\mathrm{Gr}^F\mathcal{D}_X = \mathrm{Gr}^F A_n \simeq k[y_1, \dots, y_{2n}]$. (See [Cou95])

We can further write

$$\mathrm{Ch}(\mathcal{M}) \cap T^*U = \bigcup_{\mathfrak{p} \in \mathrm{Supp}(\mathcal{M}(U))} \{p \in T^*U : f(p) = 0 \text{ for all } f \in \mathfrak{p}\}.$$

Then,

$$\pi(\mathrm{Ch}(\mathcal{M})) = \bigcup_{\mathfrak{p} \in \mathrm{Supp}(\mathcal{M}(U))} \{\pi(p) \in T^*U : f(p) = 0 \text{ for all } f \in \mathfrak{p}\}.$$

Thus, we obtain the following properties of $\mathrm{Ch}(\mathcal{M})$.

Proposition 5.4. *(Proposition 2.8)*

$$\mathrm{Supp}(\mathcal{M}) = \pi(\mathrm{Ch}(\mathcal{M})).$$

Lemma 5.5. *If J is a homogeneous ideal of a graded ring R, then $\sqrt{J}$ is also homogeneous.*

Proof. Let $a \in \sqrt{J}$, then $a^n \in J$ for some n. Write $a = a_0 + \cdots a_s$. Then, $a^n = (a_0 + \cdots a_s)^n \in J$. Clearly, $a_s^n \in J$ i.e. $a_s \in \sqrt{J}$. Thus, $a' = a - a_s \in \sqrt{J}$. Apply the same process to a'. By induction, $a_i \in \sqrt{J}$ for all i. □

By the lemma, $J_{\mathcal{M}}$ is an homogeneous ideal. Thus, $\mathrm{Ch}(\mathcal{M})$ is a homogeneous closed set.

6. Involutivity

Suppose that a $\mathcal{D}_X$-module $\mathcal{M}$ is generated by a single element u. Let $\mathcal{I} = \mathrm{Ann}(u) = \{P \in \mathcal{D}_X : Pu = 0\}$. Then $\mathcal{M} \simeq \mathcal{D}_X/\mathcal{I}$.

Introduce a filtration by

$$F_m(\mathcal{M}) = F_m(\mathcal{D}_X)u$$
$$F_m(\mathcal{I}) = \mathcal{I} \cap F_m(\mathcal{D}_X).$$

Then $\mathrm{Gr}^F\mathcal{M} = \mathrm{Gr}^F\mathcal{D}_X/\mathrm{Gr}^F\mathcal{I}$. Since $\sigma_m(P) \in \mathrm{Ann}(\mathrm{Gr}^F\mathcal{M})$ for $P \in F_m(\mathcal{I})$, we have

$$\mathrm{Ch}(\mathcal{M}) = \{p \in T^*X : \sigma_m(P)(p) = 0 \text{ for all } m \text{ and for all } P \in F_m(\mathcal{I})\}.$$

If $\mathcal{I} = \sum \mathcal{D}_X P_j$ with $P_j \in F_{m_j}(\mathcal{D}_X)$, then

$$\mathrm{Ch}(\mathcal{M}) \subset \bigcap_j \sigma_{m_j}(P_j)^{-1}(0).$$

The equality does not hold in general.

Definition 6.1. The generators $\{P_j\}$ is an involutive system of generators of $\mathcal{I}$ if

$$\mathrm{Gr}^F\mathcal{I} = \sum (\mathrm{Gr}^F\mathcal{D}_X)\sigma_{m_j}(P_j).$$

Lemma 2.1 of [Kas03] gives a characterization of this definition.

In general, we have the following defintiton.

Definition 6.2. A closed subset V of T^*X is involutive if its defining ideal

$$I_V = \{a \in \mathcal{O}_{T^*X} : a|_V = 0\}$$

satisfies $\{I_V, I_V\} \subset I_V$.

If $\mathcal{M} = \mathcal{D}_X/\mathcal{I}$, then $\mathrm{Ch}(\mathcal{M}) = \{p \in T^*X : a(p) = 0 \text{ for all } a \in \mathrm{Gr}^F\mathcal{I}\}$. Conversely, suppose that $\mathrm{Gr}^F\mathcal{I} = \{a \in \mathrm{Gr}^F\mathcal{D}_X : a|_{\mathrm{Ch}(\mathcal{M})} = 0\}$. Let $a, b \in \mathrm{Gr}^F\mathcal{I}$ and $A, B \in \mathcal{I}$ be such that $\sigma(A) = a, \sigma(B) = b$. Here σ apply to $\mathcal{I}$ by apply σ_m to the graded pieces. Since $[A, B] \in \mathcal{I}$, we have

$$\{a, b\} = \sigma([A, B])$$

vanishing on $\mathrm{Ch}(\mathcal{M})$. Hence, $\mathrm{Ch}(\mathcal{M})$ is an involutive variety.

Generally, we have the following theorem.

Theorem 6.3. *If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, then* $\mathrm{Ch}(\mathcal{M})$ *is involutive.*

In chapter 11 of the book [Cou95], the involutive subspaces are the ones that contain their skew-complement. Then the proposition 2.4 gives an equivalent condition as above definition. i.e. an affine variety V in $\mathbb{C}^{2n}$ is involutive if and only if its ideal I_V is closed for the Poisson bracket.

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