Intersection Theory on Shimura Varieties

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Contents

1 Introduction

In this course we will study *special cycles* on Shimura varieties, mainly of PEL type, their *modularity* properties and the arithmetic properties of their *intersections*.

The known results generally fall into two parallel tracks:

- Classical Intersections:
  - The cycles are Shimura subvarieties.
  - Intersections take place in the complex fiber.
  - There are general results known, especially in the compact case.
  - Conceptually explained by theta lifts, dual reductive pairs, Weil representation.
- Arithmetic Intersections:
  - Cycles are *integral models* of Shimura subvarieties.
  - Intersections take place on integral/local models, often supported in characteristic p.
  - Results known mostly in individual cases, but there have been recent breakthroughs.
  - No satisfactory conceptual explanation - but an “arithmetic theta lift” is conjectured.

2 Historical Overview

In the first part, which is an overview, we will give examples from the literature, concentrating on the simplest and first-known instances of the phenomena. We will briefly mention Kudla-Millson theory as the conceptual framework of the classical picture, but only in general terms. The aim is to provide context for the arithmetic theory, which will be treated in more detail.

2.1 Hirzebruch-Zagier Cycles

Let $F = \mathbb{Q}(\sqrt{p})$, where $p \equiv 1 (\mathrm{mod}\ 4)$ is a prime. Denote its non-trivial automorphism by $x \mapsto x'$. The associated Hilbert modular surface is

$$X = (\mathfrak{h} \times \mathfrak{h}) / \mathrm{SL}_2(\mathcal{O}_F),$$

where the action of $\mathrm{SL}_2(\mathcal{O}_F)$ is $\gamma \cdot (z_1, z_1) = (\gamma \cdot z_1, \gamma' \cdot z_2)$. X is non-compact, and has a compactification $\overline{X}$ obtained by adjoining finitely many cusps $\sigma \in \Sigma$. The number of cusps is $h(F)$, the class number of F. The compactified surface $\overline{X}$ is not smooth, but has a resolution $\widetilde{X} \to \overline{X}$ where the cusps σ are resolved to curves S_σ. $\widetilde{X}$ is smooth and compact.

The image of the natural map $H^2_c(X, \mathbb{C}) \to H^2(\widetilde{X}, \mathbb{C})$ is a direct summand, with complement spanned by the classes $[S_\sigma]$ corresponding to the cusps. Inside $H^2(\widetilde{X}, \mathbb{C})$ one may distinguish the subspace U of classes c with the following properties:

1. c is of type $(1, 1)$ in the Hodge decomposition.
2. $\tau^*(c) = c$, where $\tau(z_1, z_2) = (z_2, z_1)$.
3. c is in the image of $H^2_c(X, \mathbb{C})$.
4. For any ideal $\mathfrak{a} \subset \mathcal{O}_F$, $(\mathfrak{t}^*_{\mathfrak{a}} - \mathfrak{t}^*_{\mathfrak{a}'})c = 0$, where $\mathfrak{t}_{\mathfrak{a}}$ is a Hecke correspondence associated to $\mathfrak{a}$.

F. Hirzebruch in the early 1970s [Hir74] proved that

$$\dim U = \lfloor \frac{p-5}{24} \rfloor + 1. \tag{1}$$

Now let χ_p denote the quadratic character modulo p. Recall $M_2(\Gamma_0(p), \chi_p)$, the space of weight 2 modular forms for $\Gamma_0(p)$ and character χ_p. Consider the subspace

$$M_2^+(\Gamma_0(p), \chi_p) = \left\{ \sum_{n=0}^{\infty} a_n e^{2\pi i n \tau} \ : \ a_n = 0 \text{ if } \chi_p(n) = -1 \right\}.$$

Hecke in 1940 had shown that

$$\dim M_2^+(\Gamma_0(p), \chi_p) = \lfloor \frac{p-5}{24} \rfloor + 1. \tag{2}$$

The coincidence of (1) and (2) was observed by J-P. Serre in a letter to Hirzebruch dated December 8, 1971. In an effort to explain this, Hirzebruch and D. Zagier defined certain curves T_N on the surface X, which eventually became the prototype for special cycles on Shimura varieties.

Definition 2.1 (Hirzebruch-Zagier Cycles)**.** For each $N \geq 1$, let T_N be the image in $X = \mathfrak{H}^2 / \mathrm{SL}_2(\mathcal{O}_F)$ of the points $(z_1, z_2) \in \mathfrak{H}^2$ that satisfy an equation of the form

$$a\sqrt{p} z_1 z_2 + \lambda z_2 - \lambda' z_1 + b\sqrt{p} = 0,$$

for some $a, b \in \mathbb{Z}$, $\lambda \in \mathcal{O}_F$, with $\lambda\lambda' + abp = N$.

Here's a slightly neater formulation. Let

$$\mathrm{Herm}_2(\mathcal{O}_F) = \{A \in M_2(\mathcal{O}_F) : A^* = A\},$$

where $(a_{ij})^* = (a'_{ji})$. Then T_N is the image of the points $(z, A \cdot z) \in \mathfrak{H}^2$ where $z \in \mathfrak{H}$, and $A \in \mathrm{Herm}_2(\mathcal{O}_F)$ has determinant N.

There is also a T_0 defined in an ad-hoc, but justified, manner, which we ignore for now.

We will write $H_2(\widetilde{X})$ for $H_2(\widetilde{X}, \mathbb{C})$ for short. For each N, let T_N^c denote the projection of $T_N \in H_2(\widetilde{X})$ onto the image of $H_2(X, \mathbb{C})$. The cycles T_N^c all belong to the subspace $U \subset H_2(\widetilde{X})$.

Let

$$\Phi(z) = \sum_{N=0}^{\infty} T_N^c q^{Nz} \in H_2(\widetilde{X})[\![q]\!],$$

where $q = e^{2\pi i z}$.

For each $K \in H_2(\widetilde{X})$ put

$$\Phi_K(z) = \sum_{N=0}^{\infty} (T_N^c \cdot K) q^{Nz},$$

where $(A \cdot B)$ denotes the intersection number.

Theorem (2.1.1) (Hirzebruch-Zagier). *The assignment $K \to \Phi_K$ induces a map*

$$H_2(\widetilde{X}) \to M_2^+(\Gamma_0(p), \chi_p).$$

The image is an Eisenstein series if $K = T_0^c$, and a cusp form if $(K \cdot T_0^c) = 0$. Its restriction to the span of $T_N^c \in H_2(\widetilde{X})$, $N = 0, 1, 2, \cdots$, is injective.

Let $U' \subset H_2(\widetilde{X})$ be the span of the cycles T_N^c. It so happens that U' is a non-degenerate subspace of $H_2(\widetilde{X})$, i.e. $U' \cap (U')^\perp = 0$. Thus the map $K \to \Phi_K$ factors through U', and injects U' into $M_2^+(\Gamma_0(p), \chi_p)$. It's not hard to check that $U' \subset U$, where U is the space of cycles studied by Hirzebruch. The final step to explain Serre's observation was taken by Zagier.

Theorem (2.1.2) (Zagier). *U' coincides with U. In particular,*

$$U \to M_2^+(\Gamma_0(p), \chi_p), \quad K \mapsto \Phi_K,$$

is an isomorphism.

On the one hand one has a subspace $U \subset H_2(\widetilde{X})$ of the homology of a Hilbert modular surface, on the other hand a distinguished subspace of modular forms of weight 2 and level $\Gamma_0(p)$. Both subspaces are roughly half the dimension in their ambient spaces. The isomorphism between the two is an example of a 'geometric theta lift'.

2.2 Gross-Keating Invariants

We will first review some classical facts about the modular polynomials, interpreted as results about intersection numbers of graphs of Hecke correspondences. These belong to 'classical intersection theory' as in the introduction. Their arithmetic counterpart appears in the work of Gross-Keating, the first work on 'arithmetic intersection theory' of Shimura varieties. The reference for everything in this section is the paper [GK93].

2.2.1 Classical results on modular polynomials

Let $m \geq 1$ be an integer. The group $\mathrm{SL}_2(\mathbb{Z})$ acts by right-multiplication on 2×2 integer matrices with determinant m. Let S_m be a complete set of representatives for the orbits of this action. The classical modular polynomial $\phi_m(x, y) \in \mathbb{Z}[x, y]$ is characterized by

$$\phi_m(j(\tau_1), j(\tau_2)) = \prod_{T \in S_m} (j(\tau_1) - j(T\tau_2)), \quad \tau_1, \tau_2 \in \mathfrak{H},$$

where $j(\tau)$ is the elliptic j-function. It follows from the fact

$$|S_m| = \sigma_1(m) = \sum_{d|m} d,$$

that for any $\alpha \in \mathbb{C}$, we have

$$\deg_x \phi_m(x, \alpha) = \deg_x \phi_m(\alpha, x) = \sigma_1(m). \tag{3}$$

If m is not a perfect square, a classical result of Kronecker states

$$\deg_x \phi_m(x, x) = \sum_{dd'=m} \max\{d, d'\}. \tag{4}$$

It's possible to interpret these degree identities in terms of intersection numbers of cycles on the self-product of the modular curve, as follows.

Let $X = Y_0(1) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$ be the moduli space of complex elliptic curves. The j-function gives an isomorphism $X \to \mathbb{C}$ of complex analytic spaces, identifying X with the complex points of the affine "j-line" $\operatorname{Spec}\mathbb{C}[j]$. Similarly $S_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[j, j']$ is the moduli space of pairs of elliptic curves (E, E'). We define

$$T_m = \operatorname{Spec}\mathbb{C}[j, j']/(\phi_m(j, j'))$$

as a divisor on $S_{\mathbb{C}}$.

Recall that the classical modular curve $Y_0(m)$ parametrizes triples (f, E, E'), where E, E' are elliptic curves, and $f : E \to E'$ is an isogeny of order m. The cycle T_m is the image of $Y_0(m)$ in $X \times X$ under the mapping $(f, E, E') \mapsto (E, E')$. In other words, its points correspond to pairs of elliptic curves (E, E') that admit an isogeny $f : E \to E'$ of degree m.

For an integer $D > 0$, the (modified) Hurwitz class number $H(D)$ is the number of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of positive-definite integral binary quadratic forms with determinant D, *except* that the forms $x^2 + y^2$ and $x^2 + xy + y^2$ are counted with multiplicity $\frac{1}{2}$ and $\frac{1}{3}$. This is to account for the fact that the points i and $e^{2\pi i/3}$ in $\mathfrak{H}$ have non-trivial stabilizers in $\mathrm{PSL}_2(\mathbb{Z})$. Correspondingly, X is a complex analytic space rather than a complex manifold, and is represented by an algebraic stack rather than an algebraic variety.

For $m \geq 1$ not a perfect square, put

$$G(m) = \sum_{\substack{t \in \mathbb{Z} \\ t^2 \leq 4m}} H(4m - t^2).$$

Proposition (2.2.1) (Hurwitz)**.** *T_{m_1} and T_{m_2} intersect properly on $S_{\mathbb{C}}$ if and only if $m_1 m_2$ is not a perfect square. In that case, the intersection $T_{m_1} \cdot T_{m_2}$ is supported on points (E, E'), where E, E' are CM elliptic curves with CM discriminants $d(E)$, $d(E') \geq -4m_1 m_2$. Furthermore,*

$$(T_{m_1} \cdot T_{m_2})_{S_{\mathbb{C}}} = \sum_{n \mid \gcd(m_1, m_2)} n \cdot G(m/n^2).$$

Suppose that $m \geq 1$ is not a perfect square. Then by Hurwitz's formula, we have

$$(T_1 \cdot T_m) = G(m).$$

On the other hand, T_1 is just the diagonal $X \to X \times X = S_{\mathbb{C}}$. If we compactify X and $S_{\mathbb{C}}$ by setting

$$\widetilde{X} = \mathrm{SL}_2(\mathbb{Z})\backslash(\mathfrak{H} \cup \{\infty\}), \quad \widetilde{S}_{\mathbb{C}} = \widetilde{X} \times \widetilde{X},$$

then $\widetilde{S}_{\mathbb{C}} \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and

$$\widetilde{T}_m = T_m \cup \{(\infty, \infty)\}$$

is the closure of T_m in $\widetilde{S}_{\mathbb{C}}$.

If for $\alpha \in \mathbb{C}$, $U_\alpha \subset S_{\mathbb{C}}$ denotes the cycle defined by $j' = \alpha$, the relation (3) is equivalent to

$$(T_m \cdot U_\alpha) = \sigma_1(m).$$

Now the homology class of the compactification $\widetilde{T}_1$ is the sum of a vertical and a horizontal fiber, both homologous to U_α. It follows that

$$(\widetilde{T}_m \cdot \widetilde{T}_1) = 2\sigma_1(m) = \sum_{dd'=m} (d + d').$$

On the other hand, the contribution of the point (∞, ∞) to the intersection $(\widetilde{T}_m \cdot \widetilde{T}_1)$ may be computed using the Tate curve as

$$(\widetilde{T}_m \cdot \widetilde{T}_1)_{(\infty,\infty)} = \dim_{\mathbb{C}} \mathbb{C}[\![q]\!]/(\prod_{dd'=m} (q^d - q^{d'})) = \sum_{dd'=m} \min\{d, d'\}.$$

Then from

$$(\widetilde{T}_m \cdot \widetilde{T}_1) = (T_m \cdot T_1) + (\widetilde{T}_m \cdot \widetilde{T}_1)_{(\infty,\infty)}$$

we obtain

$$\sum_{dd'=m} (d + d') = G(m) + \sum_{dd'=m} \min\{d, d'\}$$

and hence

$$G(m) = \sum_{dd'=m} \max\{d, d'\}.$$

This is the class number relation of Hurwitz and Kronecker. It is equivalent to Hurwitz's formula for $(T_m \cdot T_1)$, plus Kronecker's degree formula (4).

Note that, as in the work of Hirzebruch-Zagier, one needed to compactify the cycles T_m before intersecting, and then separately account for the contribution at the boundary.

2.2.2 Arithmetic Triple Intersections

Let

$$S = \operatorname{Spec} \mathbb{Z}[j, j']$$

be considered as an integral model of the surface $S_{\mathbb{C}} = X \times X$. Similarly, let

$$T_m = \operatorname{Spec} \mathbb{Z}[j, j']/(\phi_m(j, j'))$$

be the corresponding closed subscheme of S. Over $\operatorname{Spec} \mathbb{Z}$, S has dimension 3, and the T_m have dimension 2. If T_{m_1}, T_{m_2}, T_{m_3} intersect properly on S, the quotient

$$\mathbb{Z}[j, j']/(\phi_{m_1}, \phi_{m_2}, \phi_{m_3}) \tag{5}$$

is a *finite ring.* In that case Gross and Keating compute the *arithmetic intersection number*

$$(T_{m_1} \cdot T_{m_2} \cdot T_{m_3})_S = \log \#\{Z[j, j']/(\phi_{m_1}, \phi_{m_2}, \phi_{m_3})\}$$

in terms of the deformation theory of elliptic curves, using the moduli interpretation of S and T_m,

Compare the first half of Hurwitz's proposition (2.2.1) with the following arithmetic analogue.

Proposition (2.2.2) (Gross-Keating)**.** *The arithmetic divisors T_{m_1}, T_{m_2}, T_{m_3} intersect properly on S if and only if no integral binary quadratic form $Q(x, y) = ax^2 + bxy + cy^2$ represents m_1, m_2, m_3. In that case $T_{m_1} \cdot T_{m_2} \cdot T_{m_3}$ is supported on supersingular pairs of elliptic curves (E, E') in characteristic $p < 4m_1m_2m_3$.*

Proof. Suppose m_1, m_2, m_3 are represented by a positive-definite integral binary quadratic form $Q(x, y)$. Then there exist CM elliptic curves E, E' such that the degree form on $\operatorname{Hom}(E, E')$ is equivalent to Q. In that case (E, E'), and its reductions at almost all primes, correspond to points of $T_{m_1} \cdot T_{m_2} \cdot T_{m_3}$, which consequently must have dimension ≥ 1 over $\mathbb{Z}$, hence the intersection can't be proper.

Now suppose m_1, m_2, m_3 are not represented by any positive-definite integral binary quadratic form. Let $\operatorname{Spec} \bar{k} \to T_{m_1} \cdot T_{m_2} \cdot T_{m_3}$ be a geometric point, corresponding to a pair of elliptic curves (E, E'), along with isogenies f_1, f_2, $f_3 \in \operatorname{Hom}(E, E')$ of degrees m_1, m_2, m_3. Then $\operatorname{Hom}(E, E')$ must have rank > 2, in which case $\operatorname{char}(\bar{k}) = p > 0$, and E, E' are supersingular.

To prove $p < 4m_1m_2m_3$, one may assume p is odd. By an analysis of invariants one shows that p must divide $\frac{1}{2}\det(Q)$. One uses that $\operatorname{Hom}(E, E')$ has rank 4, on which the degree form has square determinant. We omit these details and refer to the paper of Gross-Keating. $\square$

Let us assume T_{m_1}, T_{m_2}, T_{m_3} intersect properly. Note that

$$(T_{m_1} \cdot T_{m_2} \cdot T_{m_3}) = \sum_p n(p) \log p$$

where

$$n(p) = \operatorname{length}_{\mathbb{Z}_p} \mathbb{Z}_p[j, j']/(\phi_{m_1}, \phi_{m_2}, \phi_{m_3}).$$

Letting $W = W(\overline{\mathbb{F}}_p)$, and

$$A = W[j, j']/(\phi_{m_1}, \phi_{m_2}, \phi_{m_3}),$$

we have $n(p) = \mathrm{length}_W(A)$. The ring A is an artinian W-algebra, which factors as

$$A = \prod_{i=1}^{r} A_i,$$

where $(A_i, \mathfrak{m}_i)$ are local Artinian with residue field $\overline{\mathbb{F}}_p$. Then

$$n(p) = \sum_{i=1}^{r} \mathrm{length}_W(A_i).$$

We will interpret each term in the sum as a quantity from deformation theory.

Fix an index i, and let ϕ denote the composition $A \to A_i \to A_i/\mathfrak{m}_i = \overline{\mathbb{F}}_p$. Let j_E, resp. $j_{E'}$ denote the image of j, resp. j', in $\overline{\mathbb{F}}_p$. Choose $\widetilde{j}_E$, $\widetilde{j}_{E'}$ in W lifting j_E, $j_{E'}$, and let $\widetilde{\phi} : W[j, j'] \to W$ map j, j' to $\widetilde{j}_E$, $\widetilde{j}_{E'}$, respectively. We then have a commutative diagram

$$\begin{tikzcd} W[j,j'] \arrow[r] \arrow[d, "\widetilde{\phi}"'] & A \arrow[d, "\phi"] \\ W \arrow[r] & \overline{\mathbb{F}}_p. \end{tikzcd}$$

The map $\phi : A \to \overline{\mathbb{F}}_p$, determines a pair of supersingular elliptic curves (E, E') with j-invariants j_E, $j_{E'}$, along with three isogenies $f_i : E \to E'$, $i = 1, 2, 3$, of degrees m_1, m_2, m_3, respectively. The map $\widetilde{\phi}$ determines lifts $(\widetilde{E}, \widetilde{E}')$ of (E, E') to W, which, being in characteristic 0, do *not* admit isogenies of degrees m_1, m_2, m_3 simultaneously.

The local ring

$$R_0 = W[\![j - \widetilde{j}_E, j' - \widetilde{j}_{E'}]\!]$$

is the completion of $W[j, j']$ at the maximal ideal $(p, j - \widetilde{j}_E, j' - j_{E'})$. Local homomorphisms $R_0 \to T$ correspond to lifts of (E, E') from $\overline{\mathbb{F}}_p$ to T. More precisely, R_0 is the universal local deformation ring of (E, E'), *except* when E, E' have automorphisms others than ± 1. In general the local deformation ring R of is a finite free R_0-algebra of rank $u_E \cdot u_{E'}$ where

$$u_E = \frac{1}{2}\#\mathrm{Aut}(E), \quad u_{E'} = \frac{1}{2}\#\mathrm{Aut}(E).$$

If $p > 3$, it may be explicitly described as

$$R = W[\![t, t']\!], \quad t^{u_E} = j - \widetilde{j}_E, \quad t'^{u_{E'}} = j - \widetilde{j}_{E'}.$$

Now we have a diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\widetilde{\phi}_i"'] & A_i \arrow[d, "\phi_i"] \\ W \arrow[r] \arrow[ur, dotted] & \overline{\mathbb{F}}_p, \end{tikzcd}$$

of complete local W-algebras with residue field $\overline{\mathbb{F}}_p$. Since W is initial in the category of such objects, there is a unique diagonal map making the diagram commute. The composition

$$R \xrightarrow{\widetilde{\phi}_i} W \longrightarrow A_i$$

is surjective, with kernel I. It is the *minimal* ideal of R such that there exist isogenies

$$\widetilde{f}_i : \widetilde{E} \to \widetilde{E}' \pmod{I}, \quad i = 1, 2, 3$$

of degrees m_1, m_2, m_3, lifting f_1, f_2, f_3 from $\overline{\mathbb{F}}_p$.

2.3 Kudla-Millson Theory

Recall that Hirzebruch-Zagier cycles T_N^c are homology classes on a (smooth compact cover of a) Hilbert modular surface $\widetilde{X}$ associated with a real quadratic field $\mathbb{Q}(\sqrt{p})$, for $p \equiv 1 \pmod 4$ prime. The cycles together faciliate a non-trivial map

$$H_2(\widetilde{X}, \mathbb{C}) \to M_2(\Gamma_0(p), \chi_p), \quad K \mapsto \sum_{N \geq 0} (K \cdot T_N^c) e^{2\pi i N \tau}$$

from the middle homology of $\widetilde{X}$ to holomorphic modular forms on SL_2.

In the 1980s, S. Kudla and J. Millson generalized the construction of special cycles to arithmetic quotients of symmetric spaces of arbitrary unitary and orthogonal groups([KM86, KM87, KM88, KM90a, KM90b]). If G is such a group, and M is an arithmetic quotient of its symmetric space, there are analogous maps

$$H_i(M, \mathbb{C}) \to \mathcal{A}(G')$$

from the homology of M to automorphic forms on a different group G'. The Hirzebruch-Zagier cycles then correspond to the case $G = O(2,2)$, $i = 2$, and $G' = \mathrm{SL}_2$. However, Kudla and Millson construct the map above as a special case of an even more general construction, and in the specialization step they assume M is *compact* for simplicity. Thus to obtain proper generalizations of Hirzebruch-Zagier one requires an extension of the work of Kudla-Millson so that it applies to the compactification of M as well as the special cycles. For orthogonal groups this is largely the work of J. Funke and Millson (e.g. [FM02, FM06, FM11, FM13]). Their methods are expected to work for unitary groups, but this appears not to have been published. For Picard modular varieties associated to $U(2,1)$ over a CM field, the compact case was already treated by Kudla in the late 1970s ([Kud78]), and the non-compact case was worked out in Cogdell's thesis ([Cog85]).

We shall give a brief overview of the results of Kudla-Millson. This is meant to serve as a model for the sort of results one might expect in the arithmetic theory, where the picture is less clear.

2.3.1 Geometric Theta Lifts

As with any topic that deals with the Weil representation, there is a clutter of notation and set-up before results can be stated, and everything comes in orthogonal and unitary flavors which take double the space when stated separately, yet become muddled if unified. For this reason, rather than state the results in full generality, we shall restrict to the particular family of examples that will ultimately concern us. The general results have the same form.

Let E be an imaginary quadratic extension of $\mathbb{Q}$, and V a finite dimensional E-vector space equipped with a non-degenerate hermitian pairing $(\ ,\ )$ of signature (p, q). Let $W = E^{2n}$ be equipped with the skew-hermitian form

$$\langle x, y\rangle = {}^t\overline{y} \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix} x.$$

Put

$$G = U(V, (\ ,\ )), \quad G' = U(W, \langle\ ,\ \rangle).$$

Then $G'(\mathbb{R}) = U(n, n)$, and the symmetric space of G' may be identified with the hermitian upper half-space of degree n

$$\mathbb{C}\mathfrak{H}_n = \{z = x + iy \in M_n(\mathbb{C}) : x, y \in \mathrm{Herm}_n(\mathbb{C}), y > 0\}.$$

The symmetric space D of G may be identified with the set of maximal negative-definite subspaces of $V(\mathbb{R})$. In other words,

$$D = \{Z \in \mathrm{Gr}_q(V_{\mathbb{R}}) : (\ ,\ )|_Z < 0\}.$$

As an open subset of the Grassmannian $\mathrm{Gr}_q(V_{\mathbb{R}})$, D acquires a G-invariant complex structure and is a hermitian symmetric domain. If $q = 1$ it may be identified with the open ball in $\mathbb{C}^{p+1}$.

We assume that V contains a self-dual $\mathcal{O}_E$-lattice $L \subset V$ on which $(\ ,\ )$ takes $\mathcal{O}_E$ values. Here self-dual means the hermitian pairing induces an isomorphism $L \simeq L^\vee = \mathrm{Hom}(L, \mathcal{O}_E)$. If $\Gamma(L)$ is the stabilizer of L in $G(\mathbb{Q})$, and $\Gamma \subset \Gamma(L)$ is a torsion-free arithmetic subgroup, then

$$M = \Gamma \backslash D$$

is a quasi-projective algebraic variety. This is our ambient Shimura variety (at a fixed level) which hosts the special cycles.

The group $G'(\mathbb{R})$ has a non-algebraic "meta-unitary" double-cover $\widetilde{G'}(\mathbb{R}) \to G'(\mathbb{R})$ determined by the Cartesian diagram

$$\begin{tikzcd} \widetilde{G'}(\mathbb{R}) \arrow[r] \arrow[d] & \mathbb{C}^\times \arrow[d, "x \mapsto x^2"] \\ G'(\mathbb{R}) \arrow[r, "\det"] & \mathbb{C} \end{tikzcd}.$$

If we had started with a quadratic space V rather than a hermitian one, we would have $G = \mathrm{O}(V)$,

$G' = \mathrm{Sp}_{2n}$, and $\widetilde{G}'(\mathbb{R})$ would be the more familiar metaplectic group Mp_{2n}. In either case there is a *Weil representation*

$$\omega : G(\mathbb{R}) \times G'(\mathbb{R}) \to \mathrm{GL}(\mathscr{S}(V^n)),$$

where $\mathscr{S}(U)$ for an E-vector space U denotes the Schwartz space of rapidly decreasing C^∞ functions $\varphi : U(\mathbb{R}) \to \mathbb{C}$. The action of $g \in G(\mathbb{R})$ on $\varphi \in \mathscr{S}(V^n)$ is given by

$$(\omega(g) \cdot \varphi)(x) = \varphi(g^{-1}x),$$

where G acts diagonally on V^n. The action of $G'(\mathbb{R})$ is more complicated and packages the symmetries of $\mathscr{S}(V^n)$ induced by translations and dilatations of V^n, together with the Fourier transform.

Kudla and Millson consider a general pairing

$$(\!(\ ,\ )\!) : H_c^{2pq-i}(M, \mathbb{C}) \times H^i_{\text{cts}}(G, \mathscr{S}(V^r)) \to C^\infty(\widetilde{G}'(\mathbb{R})), \tag{6}$$

where $2pq = \dim_{\mathbb{R}} M$, and $H^i_{\text{cts}}(G, \mathscr{S}(V^r))$ is the *continuous cohomology* of G with values in the G-module $\mathscr{S}(V^r)$. This is just group cohomology where the cocycles are required to be continuous. Its role in the pairing is through a map to the complex of closed differential forms $Z^\bullet(M)$

$$\theta : H^\bullet_{\text{cts}}(G, \mathscr{S}(V^n)) \to Z^\bullet(M),$$

defined as follows. First, one considers the map

$$H^\bullet_{\text{cts}}(G, \mathscr{S}(V^n)) \to H^\bullet(\Gamma, \mathbb{C}), \quad \varphi \mapsto \Theta \circ \varphi \circ \iota$$

where $\iota : \Gamma \to G(\mathbb{R})$ is the inclusion map, and

$$\Theta : \mathscr{S}(V^n) \to \mathbb{C}, \quad \Theta(\varphi) = \sum_{x \in L^n} \varphi(x)$$

is the theta distribution determined by the lattice L. Then θ is the composition of the map above with the isomorphism

$$H^\bullet(\Gamma, \mathbb{C}) \cong Z^\bullet(M).$$

Then $f = (\!(\eta, \varphi)\!)$ is defined by

$$f(g') = \int_M \eta \wedge \theta(\omega(g')\varphi), \quad g' \in \widetilde{G}'(\mathbb{R}).$$

We now present a sequence of specializations of the ingredients that appear in this pairing, to arrive at a general analogue of Hirzebruch-Zagier.

(1) Automorphic cocycles

Consider the character of the maximal compact $K' = \mathrm{U}(n) \times \mathrm{U}(n)$ of G' defined by

$$\chi_m : \mathrm{U}(n) \times \mathrm{U}(n) \to \mathbb{C}^\times, \quad \chi_m(k_1, k_2) = \det(k_1)^m \det(k_2)^{-m}.$$

Let $\widetilde{K}'$ denote the pre-image of K' in $\widetilde{G}'$, and consider χ_m as a character of $\widetilde{K}'$ via the projection. To it corresponds a line bundle $\mathscr{L}_m$ on $D' = \widetilde{G}'/\widetilde{K}' = G'/K' = \mathbb{C}\mathfrak{H}_n$.

Let $\varphi \in H^i_{\mathrm{cts}}(G, \mathscr{S}(V^n))^{\widetilde{K}'}_{\chi_m}$ denote a continuous cocycle on which $\widetilde{K}'$ acts via χ_m. It is a fact (well-known according to [KM90b]), that $f = (\!(\eta, \varphi)\!)$ is a "harmonic" section of the quotient bundle $\Gamma'\backslash\mathscr{L}_m$ on $\Gamma'\backslash D'$, for some arithmetic subgroup $\Gamma' \subset G'(\mathbb{Q})$. In other words, it is an *automorphic form of weight m on $\mathbb{C}\mathfrak{H}_n$, for some level* Γ', annihilated by the differential operator $\overline{\partial\partial^*} + \overline{\partial^*\partial}$.

(2) Holomorphic automorphic cocycles

Further conditions on $\varphi \in H^i_{\mathrm{cts}}(G, \mathscr{S}(V^n))^{\widetilde{K}'}_{\chi_m}$ ensure that the harmonic form $(\!(\eta, \varphi)\!)$ on G' is in fact holomorphic, i.e. annihilated by $\overline{\partial}$. To state these we first note that the action of G' on the continuous cocycles may be differentiated to obtain an action of $\mathrm{Lie}(G')$. Let $\mathfrak{g}'_0 = \mathrm{Lie}(G')$, $\mathfrak{k}'_0 = \mathrm{Lie}(K')$, and $\mathfrak{g}'$, $\mathfrak{k}'$ denote their complexifications. The automorphic condition from (1) may also be written as

$$\varphi \in H^i_{\mathrm{cts}}(G, \mathscr{S}(V^n))^{\mathfrak{k}'}_{\chi_m},$$

where by slight abuse we have conflated χ_m with the character of $\mathfrak{k}'$ it induces.

The Killing form on $\mathfrak{g}'_0$ induces an orthogonal decomposition $\mathfrak{g}'_0 = \mathfrak{k}_0 \oplus \mathfrak{p}'_0$, where $\mathfrak{p}'_0$ may be identified with the real tangent space of $D' = G'/K'$ at eK'. Since D' is a complex manifold the complexification $\mathfrak{p}'$ of $\mathfrak{p}'_0$ splits as $\mathfrak{p}'_+ \oplus \mathfrak{p}'_-$, where $\mathfrak{p}'_+, \mathfrak{p}'_-$ are the holomorphic and anti-holomorphic parts of $\mathfrak{p}'$, respectively. We say the cocycle φ is *holomorphic*, and write

$$\varphi \in H^i_{\mathrm{cts}}(G, \mathscr{S}(V^n))^{\mathfrak{p}'_-}$$

if it is annihilated by $\mathfrak{p}'_-$.

Let $\mathfrak{b}' = \mathfrak{k}' \oplus \mathfrak{p}'_-$, and write χ_m again for the character of $\mathfrak{b}'$ induced by the projection $\mathfrak{b}' \to \mathfrak{k}'$. We may combine the two conditions above by writing

$$\varphi \in H^i_{\mathrm{cts}}(G, \mathscr{S}(V^n))^{\mathfrak{b}'}_{\chi_m}.$$

Theorem (2.3.1) (Kudla-Millson)**.** *The pairing*

$$(\!(\ ,\ )\!) : H^{2pq-i}_c(M, \mathbb{C}) \times H^i_{\mathrm{cts}}(G, \mathscr{S}(V^n))^{\mathfrak{b}'}_{\chi_m} \to \Gamma(\mathscr{L}_m)$$

takes values in holomorphic automorphic forms on $\mathbb{C}\mathfrak{H}_n$, of weight m, and some level $\Gamma' \subset G'$. If $f = (\!(\eta, \varphi)\!)$ is such a form, with Fourier expansion

$$f(\tau) = \sum_{T \in \mathrm{Herm}_n(E)} a_T \cdot q^T, \quad q^T = \exp(2\pi i \operatorname{tr} \tau T), \ \tau \in \mathbb{C}\mathfrak{H}_n,$$

then $a_T \neq 0$ only if T is positive-definite or positive semi-definite.

In the orthogonal case, i.e. when $G = \mathrm{O}(V)$, $G' = \mathrm{Sp}_{2n}$, the analogous result says the pairing takes values in holomorphic forms of weight $m/2$ on the Siegel upper half-space $\mathfrak{H}_n$. In particular if m is odd one obtains a form of half-integral weight. There the Fourier coefficients are indexed by symmetric $n \times n$ matrices.

(3) Forms whose Fourier coefficients are periods over special cycles

The choice of a maximal compact subgroup $K \subset G$ is evidently equivalent to choosing a *majorant* of the pairing $(\ , \ )$ on $V(\mathbb{R})$. More explicitly, K is the stabilizer of a unique point z_0 in the symmetric space D of G, which in turn corresponds to a maximal negative-definite subspace Z_0 of $V(\mathbb{R})$. We then have an orthogonal splitting $V(\mathbb{R}) = Z_0 \oplus Z_0^\perp$, where Z_0, $Z_0^\perp$ have signature $(0, q)$, $(p, 0)$, respectively. Let $J_{z_0} \in \mathrm{End}_{\mathbb{C}}(V(\mathbb{R}))$ be the involution that takes the value -1 on Z_0 and $+1$ on $Z_0^\perp$. The hermitian form

$$(u, v)_0 = (J_{z_0} u, v)$$

is then *positive-definite*, in fact a majorant of $(\ , \ )$. This construction is a correspondence between such majorants and maximal compact subgroups of G. For now the only use we make of this is to define a Gaussian $\varphi_0 \in \mathscr{S}(V^n)$, associated to our choice of K, by

$$\varphi_0 : V^n(\mathbb{R}) \to \mathbb{C}, \quad \varphi_0(v_1, \cdots, v_n) = \exp(-\pi \sum_i (v_i, v_i)_0).$$

The "Fock polynomials" $\mathcal{S}(V^n) \subset \mathscr{S}(V^n)$ are the Schwartz functions of the form

$$g(v_1, \cdots, v_n)\varphi_0(v_1, \cdots, v_n),$$

where g is a a polynomial function on V^n. The Fock polynomials are stable under the action of complexified Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$.

There is a canonical "van Est" isomorphism from continuous to $(\mathfrak{g}, K)$-cohomology

$$H^\bullet_{\mathrm{cts}}(G, \mathscr{S}(V^n)) \longrightarrow H^\bullet(\mathfrak{g}, K; \mathscr{S}(V^n)),$$

which on smooth cocycles is given by differentiating the G-action. The right-hand side above admits a natural map $H^\bullet(\mathfrak{g}, K; \mathcal{S}(V^n)) \to H^\bullet(\mathfrak{g}, K; \mathscr{S}(V^n))$. We say $\varphi \in H^\bullet_{\mathrm{cts}}(G, \mathscr{S}(V^n))$ *takes values in* $\mathcal{S}(V)$ if it lands in this subspace under the van Est isomorphism.

Theorem (2.3.2). *Let $\eta \in H_c^{2pq-i}(M, \mathbb{C})$, $\varphi \in H^i_{\mathrm{cts}}(G, \mathscr{S}(V^n))^{\mathfrak{b}'}_{\chi_m}$, and suppose φ takes values in $\mathcal{S}(V^n)$. Then for $f = (\!(\eta, \varphi)\!)$, with Fourier expansion $f(\tau) = \sum_T a_T q^T$, one has*

$$a_T = \int_{C_T} \eta \wedge c(T, \varphi),$$

where C_T is a totally geodesic cycle on M that depends only on $T \in \mathrm{Herm}_n(E)$, and $c(T, \varphi)$ is a differential form on C_T.

The cycles C_T of the theorem are the *special cycles* that generalize those of Hirzebruch-Zagier, which we shall soon define.

(4) The Kudla-Millson Schwartz form φ_{KM}

A central thread through the work of Kudla-Millson is the construction of specific cocycles $\varphi_{nq}^{+} \in H^{nq}_{\mathrm{cts}}(\mathrm{O}(p,q), \mathscr{S}(V^n))$, $\varphi_{nq,nq}^{+} \in H^{nq,nq}_{\mathrm{cts}}(U(p,q), \mathscr{S}(V^n))$ that satisfy all the properties given so far: they are holomorphic, take values in $\mathcal{S}(V^n)$, and $\widetilde{K}'$ acts on them via χ_m. Nowadays these are often denoted φ_{KM} in the literature, when it's clear which one is meant. For such forms,

$$c(T,\varphi) = c_q^{n-t}, \quad t = \mathrm{rank}(T)$$

where c_q is the qth Chern form on D.

In the orthogonal case, a similar "Euler form" e_q plays the role of c_q. It so happens that if q is odd, then $e_q = 0$. In that case, combined with the previous vanishing result, we have $a_T = 0$ unless T is *positive-definite*. In other words $f = (\!(\eta, \varphi_{\mathrm{KM}})\!)$ is a holomorphic *cusp form* on $\mathfrak{H}_n$.

Note that if $\mathrm{rank}(T) = n$, the Fourier coefficient a_T is exactly the period integral of η over C_T. When T is positive semi-definite of rank $< n$, C_T will have dimension larger than the degree of η, and the period integral won't make sense. The map $\eta \mapsto \eta \wedge c_q^{n-t}$ then shifts η to the correct degree for integration over C_T.

(5) Geometric Theta Lifting

Let φ_{KM} be the Kudla-Millson Schwartz function as above. Suppose that $C \in H_i(M,\mathbb{C})$ is the homology class of a *compact* cycle. Then there exists a closed compactly supported differential form η_C whose class in $H_c^i(M,\mathbb{C})$ is the Poincaré dual of C. For positive definite $T \in \mathrm{Herm}_n(E)$, we then have

$$a_T = \int_{C_T} \eta_C = C \cdot C_T,$$

where $C \cdot C_T$ denotes the intersection number of C and C_T. If T is semi-definite, one may replace C by the cap product $C \cap c_q^{n-\mathrm{rank}(T)}$. Then the Fourier expansion of $f = (\!(\eta_C, \phi_{\mathrm{KM}})\!)$ is

$$f(\tau) = \sum_{T \in \mathrm{Herm}_n(E)} C \cdot (C_T \cap c_q^{n-\mathrm{rank}(T)}) q^T, \quad q^T = \exp(2\pi i \,\mathrm{tr}\, T\tau), \quad \tau \in \mathbb{C}\mathfrak{H}_n.$$

Letting

$$\Phi(\tau) = \sum_{T \in \mathrm{Herm}_n(E)} C_T \cap c_q^{n-\mathrm{rank}(T)} q^T,$$

one obtains a formal generating series analogous to $\sum_{N \geq 0} T_N^c e^{2\pi i N\tau}$ of Hirzebruch-Zagier, where $(\!(\eta_C, \varphi_{\mathrm{KM}})\!)$ may be identified with the term-by-term "intersection" Φ_C of C with Φ.

If M is *compact*, the above applies to every homology class $C \in H_i(M,\mathbb{C})$, and one obtains a map

$$H_i(M,\mathbb{C}) \to \mathcal{A}(G'), \quad C \mapsto \Phi_C.$$

A few remarks are now in order:

- To rid of this last compactness assumption on M one might try to imitate Hirzebruch-Zagier and compactify M and C_T. One then has to take into account the contribution of the boundary components to the form $(\!(\eta, \varphi_{\mathrm{KM}})\!)$. This is in general quite complicated and (so far as we know) not yet completely settled. See for example, however, the work of Funke, [Fun02], and that of Funke-Millson mentioned before.

- An appealing property of φ_{KM} that is not apparent in our presentation so far is the fact that it's both explicit and very general. Essentially it allows us to compute the "Poincaré duals" of the cycles C_T by averaging an invariant differential form on M over Γ-orbits of the lattice $L^n \subset V^n$.

- The pairing $(\!(\eta, \phi)\!)$ is a generalization of the *theta correspondence*, which is a relation between automorphic forms on G and G'. The latter are mutual centralizers inside a large symplectic group $\mathrm{Sp}(\mathbb{W})$ that acts on $\mathscr{S}(V^n)$ via the Weil representation ω. One has a *theta distribution*

$$\Theta : \mathscr{S}(V^n) \to \mathbb{C}, \quad \Theta(\varphi) = \sum_{x \in L^n} \varphi(x),$$

and for each $\varphi \in \mathscr{S}(V^n)$ a theta function

$$\Theta_\varphi : \mathrm{Sp}(\mathbb{W}) \to \mathbb{C}, \quad \Theta_\varphi(h) = \Theta(\omega(h)\varphi).$$

If φ satisfies certain invariance conditions, Θ_φ is an automorphic form on $\mathrm{Sp}(\mathbb{W})$. Then given $f \in \mathcal{A}(G)$, integrating it against $\Theta_\varphi|_G$ gives an automorphic form on G' and vice versa. This is the classical theta lift. In the work of Kudla-Millson, one replaces the Schwartz functions with cocycles, and integrates Θ_φ against differential forms on M, which can have geometric origins. Thus one has a *geometric theta lift*.

3 Unitary Shimura Varieties

Let E be an imaginary quadratic extension of $\mathbb{Q}$ embedded in $\mathbb{C}$, V a finite-dimensional E-vector space, and $\langle\ ,\ \rangle$ an E-hermitian form on V of signature (p, q). We use the convention that $\langle x, y\rangle$ is E-linear in y, so that

$$\eta : V \to V^*, \quad \eta(x)(y) = \langle x, y\rangle$$

is an isomorphism of E-vector spaces satisfying $\eta^* = \eta$, where $V^* = \mathrm{Hom}(\overline{V}, E)$.

The unitary similitude group G_V is the algebraic group over $\mathbb{Q}$ such that for all $\mathbb{Q}$-algebras R, its R-valued points are given by

$$G_V(R) = \{g \in \mathrm{GL}_{R\otimes E}(V_R) : \exists \nu(g) \in R^\times,\ \langle gx, gy\rangle_E = \nu(g)\langle x, y\rangle_E,\ \forall x, y \in V_R\}.$$

Here $V_R = V \otimes R$ is a free $E\otimes R$-module, equipped with the skew-hermitian form $\langle x, y\rangle_R = \eta_R(x)(y)$. The scalar $\nu(g)$ is called the *similitude factor*, and induces a homomorphism $\nu : G_V \to \mathbb{G}_m$ of algebraic groups over $\mathbb{Q}$. Note that $\nu(g) \in \mathbb{Q}^\times$ for $g \in G(\mathbb{Q})$.

Let $D = D_V$ denote the set of maximal negative-definite subspaces of $V(\mathbb{R})$:

$$D = \{Z \in \mathrm{Gr}(q, V_\mathbb{R}) : \langle\ ,\ \rangle_\mathbb{R}|_Z < 0\}. \tag{7}$$

As an open subset of the Grassmanian $\mathrm{Gr}(q, V_\mathbb{R})$ of q-dimensional subspaces of $V_\mathbb{R}$, D becomes a hermitian symmetric domain. The group $G_V(\mathbb{R})$ acts transitively on D, and the stabilizer K_0 of any fixed $Z_0 \in D$ is maximal compact, identifying D with the symmetric space $G_V(\mathbb{R})/K_0$ of G_V. The unitary Shimura variety associated to V is the inverse system $\mathrm{Sh}(G, D) = \{\mathrm{Sh}(G, D)_K\}_K$ of complex analytic spaces

$$\mathrm{Sh}(G, D)_K = G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K,$$

indexed by compact open subgroups $K \subset G(\mathbb{A}_f)$. For K small enough $\mathrm{Sh}(G, D)_K$ is a normal quasi-projective variety defined over E.

3.1 Special cycles

3.1.1 Basic cycles associated to subspaces

As before, assume $(V, (\ ,\ ))$ is a non-degenerate hermitian space over E of signature (p, q), $G = \mathrm{U}(V)$, D is the symmetric space of G, $\Gamma \subset G(\mathbb{Q})$ a torsion-free arithmetic subgroup and

$$M = \Gamma\backslash D.$$

Let $X = (x_1, \cdots, x_n) \in V^n$, and suppose $V_X = \mathrm{Span} X_i$ is non-degenerate. To such an X we associate an algebraic cycle on M, by which we mean a locally finite map $C_X \to M$. This will be a *basic cycle*, which are the building blocks of the *special cycles*.

In fact the basic cycles will only depend on V_X, but for the purpose of intersection theory it will turn out to be more convenient to parametrize them by the spanning vectors X. Since V_X is assumed non-degenerate, there's an orthogonal decomposition

$$V = V_X \perp V_X',$$

and associated with it a subgroup

$$G_X = \mathrm{U}(V_X) \times \mathrm{U}(V_X^\perp) \subset G.$$

The symmetric space of G_X may be identified with

$$D_X = \{Z \in D :\ Z = Z \cap V_X(\mathbb{R}) + Z \cap V_X(\mathbb{R})^\perp\} \subset D. \tag{8}$$

Let G_X^0 denote the connected component of identity, and put

$$\Gamma_X = \Gamma \cap G_X^0(\mathbb{Q}).$$

The basic cycle $C_X \to M$ associated to X is

$$\Gamma_X \backslash D_X \to \Gamma \backslash D.$$

In general the cycle C_X may not be well-behaved enough, and one may need to pass from Γ to a normal subgroup $\Gamma^\dagger \subset \Gamma$ of finite index. Then for

$$M^\dagger = \Gamma^\dagger \backslash D, \quad C_X^\dagger = \Gamma_X^\dagger \backslash D_X,$$

we have a commutative diagram

$$\begin{tikzcd} C_X^\dagger \arrow[r] \arrow[d, "\pi^\dagger"'] & C_X \arrow[d, "\pi"] \\ M^\dagger \arrow[r] & M \end{tikzcd}$$

with $C_X^\dagger \to C_X$ a finite cover, and $C_X^\dagger$ having better properties than C_X if $\Gamma^\dagger$ is well-chosen. For instance if C_X is singular, it's possible to choose $\Gamma^\dagger$ so that $C_X^\dagger \to C_X$ resolves its singularities.

In the orthogonal case if D is not a hermitian domain, M and C_X are defined the same way, except they're not guaranteed to be the algebraic varieties. Then to have well-defined intersection numbers one requires C_X to be orientable, which is possible to achieve *sometimes* (e.g. the cases considered by Kudla-Millson) by passing to an appropriate $C_X^\dagger$. In the unitary case, D and D_X are hermitian domains, and have a natural orientation induced by the hermitian metric, so this problem doesn't arise.

3.1.2 Special Cycles

The *special cycles* correspond to closed orbits of $G(\mathbb{R})$ in V, and are made up of basic cycles. Let $\mathcal{O} \subset V(\mathbb{R})^n$ be a $G(\mathbb{R})$-orbit, and $X = (x_1, \cdots, x_n) \in \mathcal{O}$. Denote by (X, X) the $n \times n$ matrix (x_i, x_j), and by $V_X \subset V(\mathbb{R})$ the subspace spanned by $x_1, \cdots, x_n$. Then $\mathcal{O}$ is called:

(i) *non-singular* if $\operatorname{rank}(X, X) = n$,

(ii) *non-degenerate* if $\operatorname{rank}(X, X) = \dim V_X$,

(iii) *degenerate* if $(\ ,\ )|_{V_X}$ is degenerate.

Exercise: Show that the orbit $\mathcal{O} \subset V(\mathbb{R})^n$ is closed if and only if it's non-degenerate.

The significance of closed orbits is via the following theorem, which we quote without proof.

Theorem (3.1.1). *[Bor69, 9.11] Let G be a reductive group over $\mathbb{Q}$, and $\pi : G \to \mathrm{GL}(V)$ a rational representation of G (defined over $\mathbb{Q}$). Suppose that $\mathcal{O} = G(\mathbb{R}) \cdot v \subset V(\mathbb{R})$ is a closed orbit. Then* *for any lattice $L \subset V(\mathbb{Q})$, and any arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$ stabilizing L, $L \cap \mathcal{O}$ is a finite number of Γ-orbits.*

Let $\mathcal{O} \subset V^n(\mathbb{R})$ be a closed orbit of $G(\mathbb{R})$, and assume $L \subset V(\mathbb{Q})$ is an $\mathcal{O}_E$-lattice on which $(\ , \ )$ takes $\mathcal{O}_E$-values. Then by the above theorem there exist $X_1, \dots, X_k \in L^n$ such that $\mathcal{O} \cap L^n$ is a disjoint union of $\Gamma X_1, \cdots, \Gamma X_k$. Since the span each $X_i \in V^n$ is non-degenerate, we have a basic cycle C_{X_i} associated to X_i as in the previous paragraph. The *special cycle* associated with the closed orbit $\mathcal{O}$ is then

$$C_{\mathcal{O}} = \bigsqcup_{i=1}^{k} C_{X_i}.$$

Note that this is short-hand for the collection of locally finite maps $C_{X_i} \to M$.

For $T \in \mathrm{Herm}_n(\mathcal{O}_E)$, let

$$\mathscr{Q}_T = \{X \in V(\mathbb{R})^n : (X, X) = T\}.$$

If $\mathrm{rank}(T) = n$, $\mathscr{Q}_T$ is a single closed $G(\mathbb{R})$-orbit. In general, $\mathscr{Q}_T$ is a union of $G(\mathbb{R})$-orbits, but it contains a a *unique closed orbit*

$$\mathscr{Q}_T^c = \{X \in V(\mathbb{R})^n : (X, X) = T, \quad \dim_{\mathbb{R}} \mathrm{Span} X = \mathrm{rank}(T)\}.$$

Definition 3.1. Let $T \in \mathrm{Herm}_n(\mathcal{O})$. The *special cycle* $C_T \to M$ is the one associated to the closed $G(\mathbb{R})$-orbit $\mathscr{Q}_T^c$. Namely,

$$C_T = C_{X_1} \cup \cdots \cup C_{X_k},$$

where X_i are the distinct Γ-orbits of $\mathscr{Q}_T^c \cap L^n$.

In the orthogonal case, the cycles C_T are parametrized by symmetric matrices $T \in \mathrm{Sym}_n(\mathbb{Q})$ (or more generally $\mathrm{Sym}_n(F)$ for a totally real field F). The Hirzebruch-Zagier cycles then correspond to the case $n = 1$, $F = \mathbb{Q}$, and V a quadratic space over $\mathbb{Q}$ of signature $(2, 2)$.

Next we describe the PEL moduli space corresponding to the Shimura variety M, and interpret the cycles C_T in terms of a moduli space of morphisms. These moduli descriptions then lead to the construction of integral models $\mathcal{C}_T$, over the corresponding integral model $\mathcal{M}$.

3.2 PEL moduli space

Let $E/\mathbb{Q}$ be imaginary quadratic, and $\mathrm{Sch}_{/\mathcal{O}_E}$ the category of locally noetherian scheme over $\mathrm{Spec}\, \mathcal{O}_E$. We write $\underline{A} = (A, \iota, \lambda)$ to denote the data consisting of

- An abelian scheme A over S,
- An action $\iota : \mathcal{O}_E \to \mathrm{End}_S(A)$,
- An $\mathcal{O}_E$-linear prinicipal polarization.

See Definition (11.4) in the appendix for some more detail and references.

For integers $p, q \geq 0$, not both zero, we define a moduli functor

$$\mathcal{M}_{p,q} : \mathrm{Sch}_{/\mathcal{O}_E} \longrightarrow \text{Groupoids}$$

that associates to each $S \in \mathrm{Sch}_{\mathcal{O}_E}$ the category of data $\underline{A} = (A, \iota, \lambda)$ over S, satisfying the *Kottwitz signature condition*:

$$\mathrm{charpoly}(\iota(a)|_{\mathrm{Lie}_S(A)}, X) = (X - a)^p (X - \overline{a})^q.$$

Definition 3.2. An $\underline{\mathcal{O}_E\text{-hermitian lattice}}$ is a finitely generated projective $\mathcal{O}_E$-module L, equipped with an $\mathcal{O}_E$-hermitian pairing

$$\langle\ ,\ \rangle_L : L \times L \to \mathcal{O}_E.$$

It is called *non-degenerate* if the map $L \to L^\vee,\ x \mapsto \langle x, - \rangle_L$ is an isomorphism. We say L has signature (p, q) if the E-hermitian space $L_\mathbb{Q}$ does.

Let $\mathcal{L}_{p,q}(E)$ denote a complete set of representatives for the isomorphism classes of *non-degenerate* $\mathcal{O}_E$-hermitian lattices of signature (p, q). For $L \in \mathcal{L}_{p,q}(E)$, let $D(L)$ denote the set of maximal negative-definite subspaces of $L_\mathbb{R}$, considered as the symmetric space of $\mathrm{U}(L_\mathbb{R})$, and write $\Gamma(L)$ for the subgroup of $U(L_\mathbb{R})$ stabilizing L.

Let $\underline{A} \in \mathcal{M}_{p,q}(\mathbb{C})$. The polarization λ induces the structure of a non-degenerate $\mathcal{O}_E$-hermitian lattice on $H = H_1(A, \mathbb{Z})$. Since $H_\mathbb{R} = \mathrm{Lie}(A)$, the Kottwitz condition on $\underline{A}$ implies that H has signature (p, q). The complex structure on $\mathrm{Lie}(A)$ then determines a maximal negative-definite subspace $Z_{\underline{A}} \subset H(\mathbb{R})$, i.e. a point of $D(H)$.

Proposition (3.2.1). *For $\underline{A} \in \mathcal{M}_{p,q}(\mathbb{C})$, let $L = L_{\underline{A}}$ be the unique $\mathcal{O}_E$-hermitian lattice for which there exists an isomorphism $\alpha : H_1(A, \mathbb{Z}) \to L$. Then the map $\underline{A} \mapsto \alpha(Z_{\underline{A}}) \in D(L)$ induces an isomorphism of orbifolds*

$$\mathcal{M}_{p,q}(\mathbb{C}) \xrightarrow{\ \sim\ } \coprod_{L \in \mathcal{L}_{p,q}(E)} [\Gamma(L) \backslash D(L)].$$

Proof. The surjectivity follows from Riemann's characterization of abelian varieties in terms of polarizable lattices. If $\underline{A}$ and $\underline{A'}$ are isomorphic objects, the isomorphisms $\alpha : H_1(A, \mathbb{Z}) \to L$ and $\alpha' : H_1(A', \mathbb{Z}) \to L$ differ by an element of $\Gamma(L)$, which the quotient takes into account. See [KR09, Prop. 3.1] for more details. $\square$

Let $L \in \mathcal{L}_{p,q}$, $V = L \otimes \mathbb{Q}$, and $U_V = U(V)$. Let $K(L) \subset U_V(\mathbb{A}_f)$ be the stabilizer of $\widehat{L} = L \otimes \widehat{\mathbb{Z}} \subset V(\mathbb{A}_f)$, so that $\Gamma(L) = U_V(\mathbb{Q}) \cap K(L)$. The connected Shimura variety $\mathrm{Sh}(U_V, D_V)$ at level $K(L)$ is

$$\mathrm{Sh}(U_V, D_V)_{K(L)} = U_V(\mathbb{Q}) \backslash D_V \times U_V(\mathbb{A}_f) / K(L) \simeq \coprod_{L' \in [\![L]\!]} [\Gamma(L') \backslash D_V],$$

where $[\![L]\!] \subset \mathcal{L}_{p,q}$ is the genus of the lattice L.

A *relevant* E-hermitian space V is one that contains a non-degenerate (self-dual) lattice L. Let $\mathcal{R}_{p,q} = \mathcal{R}_{p,q}(E)$ denote a complete set of representatives for the isomorphism classes of relevant hermitian spaces of signature (p,q). Each genus $[\![L]\!] \subset \mathcal{L}_{p,q}$ determines a unique $V \in \mathcal{R}_{p,q}$, isomorphic to $L_{\mathbb{Q}}$. Let $\mathcal{R}^{\#}_{p,q} = \mathcal{R}^{\#}_{p,q}(E)$ denote the set of pairs $(V, [\![L]\!])$, where $V \in \mathcal{R}_{p,q}$ and $[\![L]\!]$ is a genus class represented by some $L \subset V$. Then the orbifold isomorphism of the proposition can be written as

$$\mathcal{M}_{p,q}(\mathbb{C}) \xrightarrow{\sim} \coprod_{(V,[\![L]\!]) \in \mathcal{R}^{\#}_{p,q}} \mathrm{Sh}(U_V, D_V)_{K(L)}. \tag{9}$$

Definition 3.3. Let A be an abelian variety. A principal $\mathbb{Q}$-polarization $\lambda : A \to A^\vee$ is an element of $\mathrm{Hom}(A, A^\vee) \otimes \mathbb{Q}$, such that for some positive rational number r, $r\lambda$ is a principal polarization.

Definition 3.4. Let $\mathcal{M}^V : \mathrm{Sch}_{/\mathcal{O}_E} \longrightarrow \mathrm{Groupoids}$ be the functor that associates to every locally noetherian scheme S over $\mathrm{Spec}\,\mathcal{O}_E$, the category $\mathcal{M}^V(S)$ whose objects are tuples $(A, \iota, \lambda, \eta)$ consisting of:

- An abelian scheme A over S,
- An action $\iota : E \to \mathrm{End}_S(A)^0$,
- An E-linear $\mathbb{Q}$-polarization $A \to A^\vee$,
- An $\mathbb{A}_f$-linear $\eta : V(\mathbb{A}_f) \to V_f(A)$ isomorphism of free $\mathbb{A}_f$-modules mapping the skew-symetric pairing on $V(\mathbb{A}_f)$ to the Weil pairing on $V_f(A)$, up to a constant factor in $\mathbb{A}_f^\times$, and mapping $\widehat{L}$ to $T_f(A)$.

In addition, for each $a \in \mathcal{O}_E$, we require that the Kottwitz condition is satisfied:

$$\mathrm{charpoly}(\iota(a)|_{\mathrm{Lie}(A)}, T) = (T - a)^p (T - \overline{a})^q.$$

The morphisms $(A, \iota, \lambda, \eta) \to (A', \iota', \lambda' \eta')$ in $\mathcal{M}^V_K(S)$ are taken to be E-linear homomorphisms $\phi : A \to A'$ in the isogeny category, such that $\phi^*(\lambda') = r \cdot \lambda$ for some positive rational number r, and $V_f(\phi) \circ \eta = \eta'$.

Note that the condition $\phi^*(\lambda') = r \cdot \lambda$ on ϕ implies that it's an isogeny.

The classical theorem of Riemann implies that the cateogry $\mathcal{M}^V(\mathbb{C})$ has a description in terms of linear algebra data.

Definition 3.5. Let N^V denote the category whose objects are tuples (W, M, J, h) where:

- W, $\langle\ ,\ \rangle_E$ is a non-degenerate E-hermitian form of signature (p,q),
- $M \subset W$ is a lattice on which $\langle\ ,\ \rangle_W$ takes $\mathcal{O}_E$-values,
- J is an E-linear complex structure on $W(\mathbb{R})$, compatible with the skew-symmetric form $\mathcal{S}_W = \mathrm{tr}_{E/\mathbb{Q}}\, \delta_E \langle\ ,\ \rangle$.

- $h : V(\mathbb{A}_f) \to W(\mathbb{A}_f)$ is an $\mathbb{A}_E$-linear isomorphism that preserves the hermitian forms up to a factor in $\mathbb{A}_f^\times$.

The morphisms $(W, M, J, h) \to (W', M', J, h')$ in N^V are E-linear maps $f : W \to W'$ that preserve the forms up to a positive rational factor, restrict to an isogeny $L \to L'$, and map J to J', h to h'.

Theorem (3.2.2) (Riemann)**.** *The homology functor $A \mapsto H^1(A, \mathbb{Q})$ induces an equivalence of categories $\mathcal{M}^V(\mathbb{C}) \to N^V$.*

Proof. This amounts to Riemann's characterization of complex tori which are abelian varieties. We will only describe the functor a bit more precisely.

For $(A, \iota, \lambda, \eta)$ an object in $\mathcal{M}^V(\mathbb{C})$, let $W = H_1(A, \mathbb{Q})$. The E-action on A makes W an E-vector space, and the polarization λ induces a hermitian pairing $\langle\ ,\ \rangle_\lambda$ on W, which takes $\mathcal{O}_E$-values on $M = H_1(A, \mathbb{Z})$. The identification $\mathrm{Lie}(A(\mathbb{C})) \simeq W(\mathbb{R})$ gives a complex structure J on the latter. The rational Tate module $V_f(A)$ is canonically isomorphic to $W(\mathbb{A}_f)$, and its composition with η gives a map $h : W(\mathbb{A}_f) \to V(\mathbb{A}_f)$. ∎

Let $(W, M, J, h) \in N^V$, and suppose $r \in \mathbb{A}_f^\times$ is the scalar factor that $h : V(\mathbb{A}_f) \to W(\mathbb{A}_f)$ maps $\langle\ ,\ \rangle$ to $r \cdot \langle\ ,\ \rangle_W$. Let $r_0 = |r|_{\mathbb{A}_f} \in \mathbb{Q}_+^\times$ and write $r = r_0 u$, $u \in \widehat{\mathbb{Z}}$. Then $h' = u^{-1}h : V(\mathbb{A}_f) \to W(\mathbb{A}_f)$ maps $\langle\ ,\rangle$ to $r_0 \cdot \langle\ ,\rangle_W$. Then these forms are isomorphic locally at every prime p. But they have the same signature (p, q), so they are also isomorphic over $\mathbb{R}$. Then by transport of structure, after replacing (W, M, J, h) with an isomorphic object we can assume $W = V$. In that case $h \in G_V(\mathbb{A}_f)$.

Let (W, M, J, h) be an object in N^V, and $g \in G_V(\mathbb{A})$. Writing $g = (g_0, g_f)$, $g_0 \in G_(\mathbb{R})$ and $g_f \in G(\mathbb{A}_f)$, we

Proposition (3.2.3). *$G_V(\mathbb{A})$ acts transitively on the set of tuples $(A, \iota, \lambda, \eta)$ where $\mathcal{O}_E$-linear principal $\mathbb{Q}$-polarization,*

$\eta : V(\mathbb{A}_f) \to V_f(A)$ is an isomorphism of free $\mathbb{A}_f$-modules preserving the hermitian forms up to a factor in $\mathbb{A}_f^\times$.

Fixing a base point $(A_0, \iota_0, \lambda_0, \eta_0)$ then induces a bijection

$$G_V(\mathbb{Q})\backslash D \times G_V(\mathbb{A}_f) \xrightarrow{\sim} \{\textit{isomorphism classes of tuples } (A, \iota, \lambda, \eta) \text{ over } \mathbb{C}\}.$$

Proof. Let $(A_0, \iota_0, \lambda_0, \eta_0)$ be as in the statement, and put $V_0 = H_1(A, \mathbb{Q})$. The $\mathcal{O}_E$-linear polarization λ_0 induces a hermitian pairing $\langle\ ,\ \rangle_0$ on V_0. Then $V_0(\mathbb{A}_f)$ is canonically identified with $V_f(\mathbb{A}_f)$. Then η is identified with an $\mathbb{A}_f$-linear isomorphism

$$\eta : V(\mathbb{A}_f) \to V_0(\mathbb{A}_f),$$

that maps the form $\langle\ ,\ \rangle$ on V to $r\langle\ ,\ \rangle_0$ on V_0, for some $r \in \mathbb{A}_f^\times$. Let $r_0 = |r|_{\mathbb{A}_f}$ be the adelic norm of r, and write $r = r_0 u$. Then

$$u^{-1}\eta : V(\mathbb{A}_f) \to V_0(\mathbb{A}_f)$$

maps $\langle\ ,\rangle$ to $r_0\langle\ ,\ \rangle_0$. In particular, $(V,\langle\ ,\ \rangle)$ and $(V_0, r_0\langle\ ,\ \rangle)$ are locally isomorphic at every finite prime. In fact they are also isomorphic over $\mathbb{R}$, since the Kottwitz condition implies that $\langle\ ,\ \rangle_0$ has signature (p,q), hence so does $r_0\langle\ ,\rangle_0$ as $r_0 > 0$. This shows that $(V,\langle\ ,\ \rangle)$ and $(V_0, r_0\langle\ ,\ \rangle_0)$ are isomorphic E-hermitian spaces.

If V_0' is V_0 equipped with the hermitian form $\langle\ ,\ \rangle_0' = r\cdot\langle\ ,\ \rangle_0$, then $\eta' = u^{-1}\eta : V(\mathbb{A}_f) \to V_0'(\mathbb{A}_f)$ is an isomorphism of $\mathbb{A}_f$-modules that maps $\langle\ ,\ \rangle$ to $\langle\ ,\ \rangle_0'$. Then V and V_0' are hermitian spaces locally over $\mathbb{Q}_p$ for every prime p. On the other hand, the Kottwitz condition implies that $V_0(\mathbb{R})$ has signature (p,q), hence it is isomorphic to $V(\mathbb{R})$. It follows that V and V_0' are isomorphic hermitian spaces over E, and we may assume η' descends to such an isomorphism $\eta_0 : V \to V_0'$. □

3.3 Moduli description of cycles

As before, let V, $(\ ,\ )$ be a non-degenerate hermitian space over E of signature (p,q), containing a self-dual $\mathcal{O}_E$-lattice L on which $(\ ,\ )$ takes $\mathcal{O}_E$-values. Let $G = \mathrm{U}(V)$, D the symmetric space of G, and $M = \Gamma\backslash D$ for a torsion-free arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$ stabilizing L.

The symmetric space D parametrizes the complex structures on $V(\mathbb{R})$ with respect to which $V(\mathbb{R})/L$ is a complex abelian variety, with multiplication by $\mathcal{O}_E$, and principal polarization given by $(\ ,\ )|_L$ (for more details see the appendix). For $z \in D$, we denote by A_z corresponding abelian variety parametrized in this way.

Assume that $x \in L$ is a vector with $(x,x) > 0$. It may be identified with the $\mathcal{O}_E$-linear map $f : \mathcal{O}_E \to L,\ f(a) = ax$. Identifying $E \otimes \mathbb{R}$ with $\mathbb{C}$, f induces a map of real tori

$$\phi : \mathbb{C}/\mathcal{O}_E \to V(\mathbb{R})/L.$$

Let $A_0 = \mathbb{C}/\mathcal{O}_E$. It is naturally a complex elliptic curve with CM by $\mathcal{O}_E$. The codomain of ϕ is also an abelian variety A, once a point $z \in D$ is chosen, giving a group homomorphism (of *real* tori)

$$\phi_z : A_0 \to A.$$

Note that $X = \mathrm{Span}\{x\}$ is a non-degenerate subspace of V, to which corresponds a hermitian subspace $D_X \subset D$.

Proposition (3.3.1). *For $z \in D$, let A_z denote the corresponding abelian variety over $\mathbb{C}$. Then $\phi_z : A_0 \to A_z$ is a morphism of abelian varieties if and only if $J_z \in D_X$.*

Proof. Let $z \in D$, and suppose $J_z \in \mathrm{End}_R(V(\mathbb{R}))$ is the corresponding complex structure. The map $\phi_z : A_0 \to A_z$ is a morphism of abelian varieties if and only if it is holomorphic.

Assume ϕ is holomorphic. Its differential $d\phi_0 : T_0(A_0) \to T_0(A_z)$ at the identity may be identified with the map $f_\mathbb{R} : \mathbb{C} \to V(\mathbb{R})$, so that J_z stabilizes the image $V_X = \mathrm{Span}_\mathbb{R}\{x\} \subset V(\mathbb{R})$ of $f_\mathbb{R}$. On the other hand, $J_z \in G(\mathbb{R})$ since it is required to preserve the pairing $(\ ,\ )$, so J_z also preserves the orthogonal complement $V_X^\perp$. In other words, $J_z = J_z \cap V_X + J_z \cap V_X^\perp$, and so $J_z \in D_X$.

Conversely, if $J_z \in D_X$, $\varphi_z : A_0 \to A_z$ is holomorphic, hence a morphism of abelian varieties. □

For each $\gamma \in \Gamma$, and $z \in D$, there is an associated isomorphism $\psi_\gamma : A_z \to A_{\gamma z}$ preserving the structures. If $z \in D_X$, and $\gamma \in \Gamma_X$, there is furthermore a commutative diagram

$$\begin{tikzcd} A_0 \arrow[r, "\phi_z"] \arrow[d, equal] & A_z \arrow[d, "\psi_\gamma", "\simeq"'] \\ A_0 \arrow[r, "\phi_{\gamma z}"] & A_{\gamma z}. \end{tikzcd}$$

Then with the usual notion of equivalence of morphisms $A_0 \to A$, one obtains a well-defined map

$$C_X = \Gamma_X \backslash D_X \to \{\phi : A_0 \to A\}_{/\sim}$$

induced by mapping $z \in D_X$ to $\phi_z : A_0 \to A$. To obtain a moduli characterization of C_X, we must understand the image of this map.

The maps $\phi_z : A_0 \to A_z$ in general do *not* preserve the polarizations. Indeed, the canonical principal polarization on A_0 corresponds to the hermitian pairing

$$(a, b)_0 = \overline{a}b,$$

and the map ϕ_z preserves polarizations if and only if $f_{\mathbb{Q}} : E \to V$ is compatible with the hermitian structures. However, we have

$$(f(a), f(b)) = (x, x)(a, b)_0, \quad a, b \in E.$$

Thus the degree to which the morphisms ϕ_z fail to preserve polarizations is exactly measured by the norm (x, x).

3.4 Unitary Shimura data

3.4.1 Tensor and Hom of Hermitian spaces

In this section $V_1 \otimes V_2$ denotes $V_1 \otimes_E V_2$ and $\mathrm{Hom}(V_1, V_2)$ means $\mathrm{Hom}_E(V_1, V_2)$.

Let V_1, $\langle\ ,\ \rangle_1$ and V_2, $\langle\ ,\ \rangle_2$ be non-degenerate hermitian spaces over E of signatures (p_1, q_1) and (p_2, q_2), respectively. Then $V_1 \otimes V_2$ is equipped with a hemitian form $\langle\ ,\ \rangle_{V_1,V_2}$ given by

$$V_1 \otimes V_2 \times V_1 \otimes V_2 \to E, \quad \langle x_1 \otimes x_2, y_1 \otimes y_2\rangle_{V_1,V_2} = \overline{\langle x_1, y_1\rangle_1} \cdot \langle x_2, y_2\rangle_2.$$

Then form above has signature $(p_1q_1 + p_2q_2, p_1q_2 + p_2q_1)$. Now let

$$V = \mathrm{Hom}(V_1, V_2) \cong \mathrm{Hom}(V_1, E) \otimes_E V_2.$$

Using the isomorphism

$$\overline{V_1} \to \mathrm{Hom}(V_1, E), \quad x \mapsto \langle\, -, x\rangle_1,$$

we obtain

$$V \cong \overline{V_1} \otimes V_2,$$

so that V obtains a hermitian structure $\langle\ ,\ \rangle = \langle\ ,\ \rangle_{\overline{V}_1,V_2}$, having the same signature as $V_1 \otimes V_2$.

Another way to construct this form is as follows. Given $f_1, f_2 \in \mathrm{Hom}_E(V_1, V_2)$, let ξ_{f_1,f_2} : $\overline{V}_1 \otimes V_1 \to E$ correspond to the bilinear form

$$\overline{V}_1 \times V_1 \to E, \quad (x, y) \mapsto \langle f_1(x), f_2(y)\rangle_2.$$

Then

$$\langle f_1, f_2\rangle = \xi_{f_1,f_2}(\mathbb{E})$$

where $\mathbb{E} \in \overline{V}_1 \otimes V_1$ is the element mapping to $\mathrm{id} : V_1 \to V_1$ under

$$\overline{V}_1 \otimes V_1 \xrightarrow{\sim} \mathrm{Hom}(V_1, V_1), \quad \alpha \otimes \beta \mapsto (x \mapsto \langle x, \alpha\rangle_1 \beta).$$

For $i = 1, 2$, let $G_i = G_{V_i}$ be the unitary similitude group of V_i, and $\nu_i : G_i \to \mathbb{G}_m$ the similitude factors. If R is a $\mathbb{Q}$-algebra, $g = (g_1, g_2) \in G_1(R) \times G_2(R)$ acts on $f : V_1(R) \to V_2(R)$ by

$$(g \cdot f)(x) = g_2 \cdot f(g_1^{-1} \cdot x).$$

For $f_1, f_2 \in V$, we have

$$\langle g \cdot f_1, g \cdot f_2\rangle = \nu_1(g_1^{-1})\nu_2(g_2)\langle f_1, f_2\rangle.$$

Then we obtain a commutative diagram

$$\begin{tikzcd}
G_1 \times G_2 \arrow[r] \arrow[d, "(\nu_1,\nu_2)"'] & G_V \arrow[d, "\nu_V"] \\
\mathbb{G}_m \times \mathbb{G}_m \arrow[r, "(x,y)\mapsto x^{-1}y"'] & \mathbb{G}_m.
\end{tikzcd}$$

We define G to be the subgroup of $G_1 \times G_2$ on which the similitude factors ν_1 and ν_2 agree:

$$G(R) = \{(g_1, g_2) \in G_1(R) \times G_2(R) : \nu_1(g_1) = \nu_2(g_2)\}, \quad R \in \mathbb{Q}\text{–Alg}.$$

In other words,

$$G = G_1 \times_{\mathbb{G}_m} G_2,$$

where the fiber product is with respect to $\nu_i : G_i \to \mathbb{G}_m$. It's clear from this description that there's a unique map $\nu : G \to \mathbb{G}_m$ agreeing with $G \to G_i$. The map $G_1 \times G_2 \to G_V$ restricts to $G \to \mathrm{SU}(V) = \ker(\nu_V)$.

Let D_1, D_2 be the hermitian symmetric domains of G_1, G_2, realized as maximal negative-definite subspaces of $V_1(\mathbb{R})$, $V_2(\mathbb{R})$. Then $G(\mathbb{R}) \subset G_1(\mathbb{R}) \times G_2(\mathbb{R})$ acts on $D = D_1 \times D_2$, and identifies it with the symmetric space of G.

3.4.2 The ambient Shimura varieties

We now describe the Shimura data whose associated varieties admit special cycles defined by moduli problems.

Let V_1, $\langle\ ,\ \rangle_1$ and V_2, $\langle\ ,\ \rangle_2$ be as in the previous section. We assume further that V_1, V_2 have signatures $(1,0)$ and $(n-1,1)$, respectively. Then the symmetric space D_1 of G_{V_1} is a single point, and $\mathrm{Sh}(G_{V_1}, D_1)$ is the zero-dimensional Shimura variety that parametrizes elliptic curves with CM by $\mathcal{O}_E$. The space $V = \mathrm{Hom}_E(V_1, V_2)$ also has signature $(n-1,1)$, and the group $G = G_1 \times_{\mathbb{G}_m} G_2$ defined previously has symmetric space $D = D_1 \times D_2 \simeq D_2$. However, the Shimura data $h_2 : \mathbb{S} \to G_2$ and $h : \mathbb{S} \to G$ are different, and so are the Shimura varieties $\mathrm{Sh}(G_2, D_2)$.

For example, suppose V_1, V_2 are the corresponding standard hermitian forms on E and E^n. Then $h_1 : \mathbb{S} \to G_{V_1}$ is the identity map $z \mapsto z$ on real points. Then h_2 is the map

$$h_2 : \mathbb{S} \to G_{V_2}, \qquad z \longmapsto \begin{pmatrix} z & & & \\ & \ddots & & \\ & & z & \\ & & & \overline{z} \end{pmatrix}.$$

Now if $V \cong \overline{V}_1 \otimes V_2$ is also put in standard form, the map h is

$$h : \mathbb{S} \to G \subset G_V, \quad z \longmapsto \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & z/\overline{z} \end{pmatrix}.$$

We will see in the next section that $\mathrm{Sh}(G, D)$ can be thought of as the moduli space of pairs (A_0, A) where A_0 is a CM elliptic curve parametrized by $\mathrm{Sh}(G_1, D_1)$, and A an abelian variety determining an object in $\mathrm{Sh}(G_2, D_2)$.

Recall that we interpreted the complex fiber of basic cycles as moduli spaces of certain maps $\phi : A_0 \to A$ from CM elliptic curves A_0 to abelian varieties A. The PEL interpretation allow us to define canonical and integral models for the special cycles, as parameter spaces for maps $\phi : A_0 \to A$ where A_0, A are allowed to be *abelian schemes*. The forgetful map $\phi \mapsto (A_0, A)$ then realizes these moduli spaces of morphisms as cycles over $\mathrm{Sh}(G, D)$.

4 Integral Models

In this part, we take a closer look at the various integral and local models for Shimura varieties of unitary type, with emphasis on the signature $(n-1,1)$ case.

4.1 The naive integral model

There are several closely related variations all based on the following, due to Kottwitz.

Definition 4.1. Let (p, q) be a pair of integers, and $E/\mathbb{Q}$ an imaginary quadratic field. We define a moduli functor $\mathcal{M}_{p,q} = \mathcal{M}_{p,q}^{\text{naive}}$

$$\mathcal{M}_{p,q} : \mathrm{Sch}_{/\mathcal{O}_E} \to \text{Groupoids}$$

whose value on an $\mathcal{O}_E$-scheme S is the groupoid of triples (A, ι, λ) consisting of:

- an abelian scheme A over S,
- a ring homomorphism $\iota : \mathcal{O}_E \to \mathrm{End}_S(A)$,
- an $\mathcal{O}_E$-linear principal polarization $\lambda : A \to A^\vee$,

such that for each $a \in \mathcal{O}_E$, the *Kottwitz signature condition* holds:

$$\mathrm{charpoly}(\iota(a)|_{\mathrm{Lie}(A)}, T) = (T - a)^p (T - \overline{a})^q.$$

The morphisms in $\mathcal{M}_{p,q}(S)$ are taken to be $\mathcal{O}_E$-linear isomorphisms that preserves the polarizations.

The polynomial on the right-hand side of the Kottwitz signature condition is considered as a global section of the sheaf $\mathcal{O}_S[T]$ via the morphism $\mathcal{O}_E[T] \to \mathcal{O}_S[T]$ induced by the structure map of S. Note that condition implies A has dimension $m = p + q$.

The functors $\mathcal{M}_{p,q}$ are representable by Deligne-Mumford stacks over $\mathrm{Spec}\, \mathcal{O}_E$, such that $\mathcal{M}_{p,q} \times_{\mathrm{Spec}\, \mathcal{O}_E} \mathrm{Spec}\, \mathcal{O}_E[\Delta^{-1}]$ is smooth over $\mathrm{Spec}\, \mathcal{O}_E[\Delta^{-1}]$, where Δ is the discriminant of $E/\mathbb{Q}$.

Note $\mathcal{M}_{1,0}^E$ is the stack of elliptic curves with CM by $\mathcal{O}_E$.

This moduli space is sometimes denoted $\mathcal{M}_{p,q}^{\text{naive}}$, because at some point it was expected to be flat over $\mathrm{Spec}\, \mathcal{O}_E$. This is not the case, however, since over primes that ramify in E, the signature condition is degenerate, and the moduli space acquires vertical components. It's possible to modify the signature condition to obtain an integral model that is flat or regular. These modifications all coincide over $\mathrm{Spec}\, \mathcal{O}_E[\Delta^{-1}]$, so $\mathcal{M}^{\text{naive}}$ is suitable as long as one avoids the ramified primes. For the purpose of introducing special cycles, they will do.

4.2 Integral Special Cycles

Our ambient moduli space will be

$$\mathcal{M} = \mathcal{M}_{1,0} \times_{\mathrm{Spec}\, \mathcal{O}_E} \mathcal{M}_{m-1,1}.$$

In other words, for each S over $\mathrm{Spec}\, \mathcal{O}_E$, $\mathcal{M}(S)$ will consist of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda)$, where $(A_0, \iota_0, \lambda_0) \in \mathcal{M}_{1,0}(S)$ is an elliptic curve over S with CM by $\mathcal{O}_E$, and $(A, \iota, \lambda) \in \mathcal{M}_{m-1,1}(S)$ is as above. We will often shorten this to $(A_0, A) \in \mathcal{M}(S)$, and forego writing ι_0, λ_0, etc.

Let $(A_0, A) \in \mathcal{M}(S)$, and put

$$L(A_0, A) := \mathrm{Hom}_{\mathcal{O}_E}(A_0, A).$$

It is a finitely generated projective $\mathcal{O}_E$-module, naturally equipped a hermitian pairing as follows. Given $f, g \in L(A_0, A)$, consider the morphism that goes around the following diagram starting at A_0:

$$\begin{tikzcd} A_0 \arrow[r, "f"] & A \arrow[d, "\lambda"] \\ A_0^\vee \arrow[u, "\lambda_0^{-1}"] & A^\vee \arrow[l, "g^\vee"'] \end{tikzcd}$$

Since all the maps are $\mathcal{O}_E$-linear, so is the composition $\lambda_0^{-1} \circ g^\vee \circ \lambda \circ f \in \mathrm{End}_{\mathcal{O}_E}(A_0)$. Now it is a general fact that for $A_0 \in \mathcal{M}_{1,0}(S)$, the map $\iota_0 : \mathcal{O}_E \to \mathrm{End}_{\mathcal{O}_E}(A_0)$ is an isomorphism, meaning that $\mathcal{O}_E$ is its own centralizer in $\mathrm{End}_{\mathcal{O}_E}(A_0)$. Therefore we can define

$$\langle f, g \rangle = \iota_0^{-1}(\lambda_0^{-1} \circ g^\vee \circ \lambda \circ f) \in \mathcal{O}_E.$$

Proposition (4.2.1). *The hermitian pairing $\langle\ ,\ \rangle$ on $L(A_0, A)$ is positive-definite.*

Proof. We must show $\langle f, f \rangle > 0$ if $f : A_0 \to A$ is non-zero. Any such f is an isogeny onto its image. It follows that the pullback $\lambda_1 = f^*(\lambda) = f^\vee \circ \lambda \circ f$ is also a polarization on E_0. Now if $a = \langle f, f \rangle$, we have

$$\lambda_1 = \iota(a) \circ \lambda_0.$$

It follows by a general ampleness criterion of Mumford [Mum08] that any such $\iota(a)$ must be positive in $\mathrm{End}_{\mathcal{O}_E}(A_0)^{\mathrm{sym}} \otimes \mathbb{R} = \mathbb{R}$, where "sym" denotes invariance under the Rosati involution. $\square$

It follows that if $x_1, \cdots, x_n \in L(A_0, A)$, the $n \times n$ matrix $\langle x_i, x_j \rangle \in \mathrm{Herm}_n(\mathcal{O}_E)$ is positive *semi-definite* in general, and positive-definite if non-degenerate.

We are now ready to define the integral version of special cycles. Given any positive-definite $T \in \mathrm{Herm}_n(\mathcal{O}_E)$, we consider the functor

$$\mathcal{Z}_T : \mathrm{Sch}_{/\operatorname{Spec} \mathcal{O}_E} \to \mathrm{Groupoids}$$

that associates to each $\mathcal{O}_E$-scheme S the groupoid of tuples $(A_0, A, x_1, \cdots, x_n)$ where $(A_0, A) \in \mathcal{M}(S)$, and

$$\langle x_i, x_j \rangle = T.$$

Morphisms $(A_0, A, x_i) \to (A_0', A', x_i')$ are taken to be pairs of isomorphisms (f_0, f), i.e. morphisms in $\mathcal{M}(S)$, which preserve the maps x_i, i.e. make the following diagrams commute:

$$\begin{tikzcd} A_0 \arrow[r, "x_i"] \arrow[d, "f_0"'] & A \arrow[d, "f"] \\ A_0' \arrow[r, "x_i'"'] & A'. \end{tikzcd}$$

The moduli functors $\mathcal{Z}_T$ admit natural maps to $\mathcal{M}$ via the forgetful functors

$$\mathcal{Z}_T \to \mathcal{M}, \quad (A_0, A, x_1, \cdots, x_n) \mapsto (A_0, A).$$

They are representable by Deligne-Mumford stacks that are *finite and unramified* over $\mathcal{M}$. Note they are *not* flat in general! In fact we will see that if $m = n$, $\mathcal{Z}_T$ is supported away from the generic fiber, over finitely many primes.

The integral cycles $\mathcal{Z}_T$ are the unitary versions of the integral modular correspondences considered by Gross-Keating. The Shimura variety there, as in the Hirzebruch-Zagier case, can be identified as orthogonal type, attached to $\mathrm{O}(2,2)$. These have generalizations in Shimura varieties attached to $\mathrm{O}(2,n)$. From the point of view of classical theta lifting, the orthogonal Shimura varieties are more convenient than their unitary counterparts: the phenomena are generally the same, but the calculations are tidier, and the theta lifts produce Siegel modular forms on Sp_{2n}, which are more familiar than hermitian modular forms on $U_{n,n}$. This is why the work of Kudla-Millson has an emphasis on the orthogonal groups. However, from point of view of *arithmetic* intersection theory, unitary Shimura varieties are easier to deal with, since their integral models are easier to construct.

4.3 $\mathcal{M}^{\mathrm{Pap}}$ and $\mathcal{M}^{\mathrm{Kra}}$

G. Pappas in [Pap00] observed that the naive integral model $\mathcal{M}^{\mathrm{naive}}_{p,q}$ is not flat over the primes ramified in E, and offered the following fix.

Definition 4.2 (Pappas Model)**.** Fix non-negative integers p, q with $p+q > 0$. The *Pappas integral model* is the functor

$$\mathcal{M}^{\mathrm{Pap}}_{p,q} : \mathrm{Sch}_{\mathcal{O}_E} \to \mathrm{Groupoids}$$

whose value on an $\mathcal{O}_E$-scheme S is the groupoid of triples (A, ι, λ) consisting of:

- an abelian scheme A over S,
- a ring homomorphism $\iota : \mathcal{O}_E \to \mathrm{End}_S(A)$,
- an $\mathcal{O}_E$-linear principal polarization $\lambda : A \to A^\vee$,

such that:

- For each $a \in \mathcal{O}_E$, the following determinant condition holds:

$$\mathrm{charpoly}(\iota(a)|_{\mathrm{Lie}(A)}, T) = (T - a)^p (T - \overline{a})^q.$$

- For each $a \in \mathcal{O}_D$, the *Pappas wedge condition* holds:

$$\bigwedge^{p+1}(\iota(a) - \overline{a})|_{\mathrm{Lie}_S(A)} = 0, \qquad \bigwedge^{q+1}(\iota(a) - a)|_{\mathrm{Lie}_S(A)} = 0.$$

The morphisms in $\mathcal{M}^{\mathrm{Pap}}_{p,q}(S)$ are taken to be $\mathcal{O}_E$-linear isomorphisms that preserves the polarizations.

Thus the objects of $\mathcal{M}^{\mathrm{Pap}}_{p,q}$ are exactly those of $\mathcal{M}^{\mathrm{naive}}_{p,q}$ that also satisfy the Pappas wedge condition. In fact, over the generic fiber the wedge condition and the determinant condition are equivalent, and the two moduli spaces coincide. Unlike $\mathcal{M}^{\mathrm{naive}}_{p,q}$ however, $\mathcal{M}^{\mathrm{Pap}}_{p,q}$ is expected to be flat over $\operatorname{Spec} \mathcal{O}_E$. For $(p, q) = (n-1, 1)$, which is the case of interest to us, this is due to Pappas.

Let

$$\mathrm{Sing}_{n-1,} \subset \mathcal{M}^{\mathrm{Pap}}_{n-1}$$

denote the singular locus, and for $-D = \mathrm{disc}(E)$ let $\delta = \sqrt{-D}$ be the square-root with positive imaginary part.

Theorem (4.3.1) (Pappas[Pap00]). *The stack $\mathcal{M}^{\mathrm{Pap}}_{n-1,1}$ is flat over $\operatorname{Spec} \mathcal{O}_E$, of relative dimension $n-1$, Cohen-Macaulay, and normal. Furthermore,*

(1) The special fibers $\mathcal{M}^{\mathrm{Pap}}_{n-1,1/\mathbb{F}_q}$ are Cohen-Macaulay and geometrically normal, for any $\mathcal{O}_E \twoheadrightarrow \mathbb{F}_q$.

(2) $\mathrm{Sing}_{n-1,1}$ is a finite 0-dimensional stack over $\mathcal{O}_E$, supported over the primes dividing D. It is the reduced substack of the closed subscheme defined by $\delta\mathrm{Lie}(A) = 0$.

(3) The blow-up of $\mathcal{M}^{\mathrm{Pap}}_{n-1,1}$ along $\mathrm{Sing}_{n-1,1}$ is regular.

The regular stack obtained by blowup of $\mathcal{M}^{\mathrm{Pap}}_{n-1,1}$ along $\mathrm{Sing}_{n-1,1}$ also has a moduli description.

Definition 4.3 (Krämer model). The *Krämer integral model* is stack representing the functor

$$\mathcal{M}^{\mathrm{Kra}}_{n-1,1} : \mathrm{Sch}_{\mathcal{O}_E} \to \mathrm{Groupoids}$$

whose value on an $\mathcal{O}_E$-scheme S is the groupoid of tuples $(A, \iota, \lambda, \mathcal{F}_A)$ consisting of:

- an abelian scheme A over S,
- a ring homomorphism $\iota : \mathcal{O}_E \to \mathrm{End}_S(A)$,
- an $\mathcal{O}_E$-linear principal polarization $\lambda : A \to A^\vee$,
- an $\mathcal{O}_E$-stable $\mathcal{O}_S$-submodule $\mathcal{F}_A \subset \mathrm{Lie}_S(A)$ that is locally a direct summand of rank $n-1$, and satisfies the *Krämer condition*: $\mathcal{O}_E$ acts on $\mathcal{F}_A$ via the structure morphism $\mathcal{O}_E \to \mathcal{O}_S$, and on $\mathrm{Lie}_S(A)/\mathcal{F}_A$ via its complex conjugate.

The morphisms in $\mathcal{M}^{\mathrm{Kra}}_{n-1,1}(S)$ are taken to be $\mathcal{O}_E$-linear isomorphisms that preserves the polarizations and the submodules $\mathcal{F}_A$.

There is a map $\mathcal{M}^{\mathrm{Kra}}_{n-1,1} \longrightarrow \mathcal{M}^{\mathrm{Pap}}_{n-1,1}$ that forgets the $\mathcal{O}_S$-submodule $\mathcal{F}_A \subset \mathrm{Lie}_S(A)$. Recall that our ambient Shimura variety hosting the special cycles is

$$\mathcal{M} = \mathcal{M}_{1,0} \times \mathcal{M}^{\mathrm{naive}}_{n-1,1}.$$

The sequence of integral models

$$\mathcal{M}^{\mathrm{Kra}}_{n-1,1} \longrightarrow \mathcal{M}^{\mathrm{Pap}}_{n-1} \longrightarrow \mathcal{M}^{\mathrm{naive}}_{n-1,1}$$

after base change along $\mathcal{M}_{1,0} \times \mathcal{M}^{\mathrm{naive}}_{n-1,1} \to \mathcal{M}^{\mathrm{naive}}_{n-1,1}$, induces a sequence that we denote

$$\mathcal{S}_{\mathrm{Kra}} \longrightarrow \mathcal{S}_{\mathrm{Pap}} \longrightarrow \mathcal{M}.$$

The Exceptional locus $\mathrm{Exc}_{n-1,1}$ of $\mathcal{M}^{\mathrm{Kra}}_{n-1,1}$ is defined by the Cartesian diagram

$$\begin{tikzcd} \mathrm{Exc}_{n-1,1} \arrow[r] \arrow[d] & \mathcal{M}^{\mathrm{Kra}}_{n-1,1} \arrow[d] \\ \mathrm{Sing}_{n-1,1} \arrow[r] & \mathcal{M}^{\mathrm{Pap}}_{n-1,1}. \end{tikzcd}$$

By base change along $\mathcal{S}_{\mathrm{Pap}} \to \mathcal{M}^{\mathrm{Pap}}_{n-1}$ we obtain another Cartesian diagram, that we denote

$$\begin{tikzcd} \mathrm{Exc} \arrow[r] \arrow[d] & \mathcal{S}_{\mathrm{Kra}} \arrow[d] \\ \mathrm{Sing} \arrow[r] & \mathcal{S}_{\mathrm{Pap}}. \end{tikzcd}$$

This diagram displays the relation between the two integral models $\mathcal{S}_{\mathrm{Kra}}$ and $\mathcal{S}_{\mathrm{Pap}}$.

Theorem (4.3.2) (Pappas, Krämer).

- *The stack* $\mathcal{S}_{\mathrm{Kra}}$ *is regular and flat over* $\mathcal{O}_E$*.*
- *The exceptional locus* $\mathrm{Exc} \subset \mathcal{S}_{\mathrm{Kra}}$ *is a disjoint union of smooth Cartier divisors.*
- *The stack* $\mathcal{S}_{\mathrm{Pap}}$ *is flat and Cohen-Macauley over* $\mathcal{O}_E$*, and with geometrically normal and Cohen-Macauley fibers.*
- *The singular locus* $\mathrm{Sing} \subset \mathcal{S}_{\mathrm{Pap}}$ *is a reduced closed stack of dimension* 0*, supported over the ramified primes of* $\mathcal{O}_E$*.*
- *The morphism* $\mathcal{S}_{\mathrm{Kra}} \to \mathcal{S}_{\mathrm{Pap}}$ *is surjective, and outside the exceptional locus restricts to an isomorphism*
$$\mathcal{S}_{\mathrm{Kra}} \backslash \mathrm{Exc} \xrightarrow{\sim} \mathcal{S}_{\mathrm{Pap}} \backslash \mathrm{Sing}.$$

The flatness of the Pappas model $\mathcal{M}^{\mathrm{Pap}}_{p,q}$ may have been resolved relatively recently by one of the experts in the field. On the other hand, the construction of a regular model similar to $\mathcal{M}^{\mathrm{Kra}}_{n-1,1}$ for arbitrary signature (p, q) is an open problem, and apparently quite difficult. It is one of the major obstacles preventing a generalization of the current literature on arithmetic intersection theory of unitary Shimura varieties from the $(n-1, 1)$ case to arbitrary signature.

4.4 Modularity of Special Cycles: Preview

There are canonical toroidal compactifications $\mathcal{S}^*_{\mathrm{Kra}}$ of $\mathcal{S}_{\mathrm{Kra}}$, and $\mathcal{S}^*_{\mathrm{Pap}}$ of $\mathcal{S}_{\mathrm{Pap}}$, each of which hosts compactified special cycles. For the moment suppose we have already constructed these compactifications. For each $m > 0$, let $\mathcal{Z}^*_{\mathrm{Kra}}(m)$ denote the Zariski closure of $\mathcal{Z}_{\mathrm{Kra}}(m) \to \mathcal{S}_{\mathrm{Kra}}$ in $\mathcal{Z}^*_{\mathrm{Kra}}$. To these we must add boundary components. Then the compactification of the cycles $\mathcal{Z}_{\mathrm{Kra}}(m)$ will be

$$\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) = \mathcal{Z}^*_{\mathrm{Kra}}(m) + \mathcal{B}_{\mathrm{Kra}}(m) \in \mathrm{CH}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}}),$$

where $\mathcal{B}_{\mathrm{Kra}}(m)$ is itself a combination of the rational boundary components of $\mathcal{S}^*_{\mathrm{Kra}}$. For $m = 0$, we put

$$\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(0) = \omega^{-1} + \mathrm{Exc} \in \mathrm{CH}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}}),$$

where ω is a line bundle of modular forms on $\mathcal{S}_{\mathrm{Kra}}$, extended to $\mathcal{S}^*_{\mathrm{Kra}}$.

We may then form the generating series

$$\Phi(\tau) = \sum_{m \geq 0} \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \cdot q^m \in \mathrm{CH}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}})[\![q]\!], \quad q = e^{2\pi i \tau}. \tag{10}$$

We will spend the next several sections making sense of the ingredients of this series. We will review the toroidal compactifications $\mathcal{S}^*_{\mathrm{Kra}}$ and $\mathcal{S}^*_{\mathrm{Pap}}$, as well as their rational boundary components. Infinitesimal neighbourhoods of these boundary components have descriptions which play a key role in translating classical analytic formulas into integral arithmetic geometry; these we will also review. For this purpose we will introduce mixed Shimura varieties, which provides a convenient framework for describing toroidal compactifications.

Our motivation is to eventually outline the proof of the following theorem:

Theorem (4.4.1) (Bruinier-Howard-Kudla-Rapoport-Yang [BHK+17])**.** *Assume that the discriminant D of E is odd, and let $\chi_D : (\mathbb{Z}/D\mathbb{Z}) \to \{\pm 1\}$ be the associated Dirichlet character. Then Φ in (10) is a modular form of weight n, level $\Gamma_0(D)$, and character χ_D^n. That is, for every linear functional $\alpha : \mathrm{CH}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}}) \to \mathbb{C}$ the series*

$$\sum_{m \geq 0} \alpha(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)) \cdot q^m \in \mathbb{C}[\![q]\!]$$

is the q-expansion of a classical modular form of the indicated weight, level, and character.

In fact, the authors prove a much stronger modularity result for Arakelov cycles $\widehat{\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}}(m) \in \widehat{\mathrm{CH}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}})$. The construction of this second generating series requires equipping the cycles $\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)$ with certain Green functions. Both for the construction of these Green forms, and the proof of the

Arakelov version of the theorem, one requires explicit descriptions of the boundary components of $\mathcal{S}_{\mathrm{Kra}}*$.

In the next section, we will define mixed Shimura varieties and outline their basic properties. We shall then apply them to describe $\mathcal{S}^*_{\mathrm{Kra}}$, its rational boundary components, and the local structure around each component.

5 Mixed Shimura Varieties

As has been evident in examples given so far, for a good theory of special cycles on Shimura varieties one has to compactify both the Shimura variety and the cycles, and this often involves adding boundary components to both. Furthermore, to have satisfactory knowledge about the intersections of cycles, one needs to understand the structure of the compactifications near these boundary components.

The Baily-Borel compactification can be intrinsically characterized as the minimal normal compactification, and is therefore canonical. It also has the appealing property that its (rational) boundary components are themselves Shimura varieties of smaller dimension. However, beyond the case of the modular curve, the Baily-Borel compactification is usually highly singular, and to resolve its singularities one needs more complicated constructions.

Toroidal compactifications depend on the choice of auxiliary combinatorial data (called cone decompositions), and are therefore usually *not* canonical. On the other hand, they can always be made smooth and proper by choosing the auxiliary data appropriately. In our case this can even be done canonically, as we shall see. However, the boundary components of toroidal compactifications are no longer Shimura varieties, but a more general type of object: they are mixed Shimura varieties.

It turns out that much of the theory of Shimura varieties can be extended to mixed Shimura varieties: they parametrize certain mixed Hodge structures, have a group-theoretic formulation, admit canonical models and their own toroidal compactifications. The boundary components of toroidal compactifications of mixed Shimura varieties are themselves mixed Shimura varieties.

5.1 Rational Mixed Hodge Structures

Let M be a finite-dimensional vector space over $\mathbb{Q}$.

Definition 5.1. A *rational mixed Hodge structure* on M consists of:

(1) An *increasing* “weight filtration” W on M, and

(2) A *decreasing* “Hodge filtration” F on $M_{\mathbb{C}}$,

satisfying the following axiom:

There exists a (unique) bigrading on $\mathrm{Gr}_W(H_{\mathbb{C}})$ by subspaces $H^{p,q}$ such that

(i) $\mathrm{Gr}^n_W(H_{\mathbb{C}}) = \bigoplus_{p+q=n} H^{p,q}$

(ii) The filtration F induces on $\mathrm{Gr}_W(H_{\mathbb{C}})$ the filtration

$$\mathrm{Gr}_W(F)^p = \bigoplus_{p' \geq p} H^{p',q'}$$

(iii) $\overline{H^{p,q}} = H^{q,p}$.

An *integral* mixed Hodge structure is a $\mathbb{Z}$-module of finite rank $M_{\mathbb{Z}}$ and a mixed Hodge structure on $M_{\mathbb{Q}}$ such that the weight filtration is defined over $\mathbb{Z}$.

5.2 1-Motives

Definition 5.2. A semi-abelian scheme over a base S is an S-group scheme G that is an extension of an abelian scheme A by a torus T:

$$0 \to T \to G \to A \to 0.$$

Let us see how semi-abelian varieties arise naturally in compactification problems. Let S be a separated base scheme, and

$$\mathcal{M} : \mathrm{Sch}_{/S} \longrightarrow \mathrm{Groupoids}$$

a functor representable by an algebraic stack that is finite type and quasi-separated over S.

Let R be valuation ring, K the fraction field of A, and $\operatorname{Spec} R \to \mathcal{S}$ a morphism. Then the map of S-schemes $\operatorname{Spec} K \to \operatorname{Spec} R$ induces a functor $\mathcal{M}(R) \to \mathcal{M}(K)$. If K'/K is a finite extension, with $R' \subset K'$ the integral closure of R in K', one also obtains a functor $\mathcal{M}(R') \to \mathcal{M}(K')$. The valuative criterion of properness says $\mathcal{M}$ is proper over S if and only if for each such R, there exists a finite extension K'/K such that $\mathcal{M}(R') \to \mathcal{M}(K')$ is an equivalence of categories.

Now suppose $\mathcal{M}$ is a moduli functor parametrizing PEL-triples (A, ι, λ) satisfying some extra conditions. Then $\mathcal{M}$ is proper if and only if for each $(A, \iota, \lambda) \in \mathcal{M}(K)$, there exists K'/K such that, the triple $(A', \iota', \lambda') \in \mathcal{M}(K')$ obtained the base change lifts $\mathcal{M}(R')$, uniquely so up to isomorphism.

If A is an abelian variety over K, it has a Néron model $\mathscr{A}$ over $\operatorname{Spec} R$. Its connected component of identity $\mathscr{A}^\circ$ is a smooth group scheme over $\operatorname{Spec} R$, but not always an abelian scheme. In particular, if $k = k_v$ is the residue field at the valuation v, the reduction $\mathscr{A}_k^\circ$ may not be an abelian variety, but it is always a smooth connected group scheme. Now recall the following theorem of Chevalley.

Theorem (5.2.1) (Chevalley Decomposition)**.** *Let G be a smooth connected group scheme over a perfect field k. Then there exists a unique (up to isomorphism) short exact sequence of k-group schemes*

$$1 \to G^{\mathrm{aff}} \to G \to A \to 1,$$

where A is an abelian variety, and G^{aff} is affine.

Let A be an abelian variety over K, and $A_k = \mathscr{A}_k^\circ$ the special fiber of $\mathscr{A}^\circ$ over $\operatorname{Spec} k$. By Chevalley's theorem, after extending to an algebraic closure $\overline{k}$, $A_{\overline{k}}$ becomes the extension of an abelian variety by an affine *commutative* group scheme.

Definition 5.3. Let R be a discrete valuation ring, K the field of fractions, and k the residue field. Let $A_k = \mathscr{A}_k^\circ$. An abelian variety A over K has

- *Good reduction* if A_k is an abelian variety.
- *Semistable reduction* if A_k is a semi-abelian variety.

The key result now is the semistable reduction theorem of Grothendieck.

Theorem (5.2.2) (Grothendieck)**.** *Let R be a discrete valuation ring, K the fraction field, and A an abelian variety over K. Then there exists a finite separable extension K'/K such that $A_{K'}$ has semistable reduction over the integral closure of R in K'.*

This suggests that for our moduli stack $\mathcal{M} \to S$ parametrizing (A, ι, λ) to be proper, the extra conditions imposed would have to be strong enough to guarantee the Néron model of (A, ι, λ) over K is an abelian scheme. This is the case for instance for the moduli stacks $\mathcal{M}_{n,0}$, which indeed turn out to be smooth and proper of relative dimension 0 [How12], but not in general. On the other hand, starting with a moduli stack $\mathcal{M} \to S$ that is not proper, we may try to *compactify* it by allowing *new* objects. That is, we would like a moduli functor $\overline{\mathcal{M}} \to S$ that contains all the objects of $\mathcal{M}$, along with enough new objects so as to satisfy the valuative criterion of properness. Grothendieck's semi-stable reduction theorem implies that the class of semi-abelian varieties is a good candidate for this enlargment.

Indeed compactifications of moduli spaces of abelian varieties are often moduli spaces of semi-abelian varieties. To obtain proper *integral models*, the semi-abelian varieties have to be augmented with an extra lattice-type structure.

Definition 5.4. A *1-motive* over S is a morphism of group schemes $u : L \to G$, where G is a semi-abelian scheme over S, and X is étale locally a constant $\mathbb{Z}$-module of finite rank.

A 1-motive can be represented as a diagram of group schemes:

$$\begin{tikzcd} & & X \arrow[d] & & \\ 0 \arrow[r] & T \arrow[r] & G \arrow[r] & A \arrow[r] & 0 \end{tikzcd} \qquad .$$

Let (X, A, T, G, u) be a 1-motive over $\mathbb{C}$. Then there is a map $\exp : \mathrm{Lie}(G) \to G$, whose kernel is $H_1(G)$. Let $M_{\mathbb{Z}}$ be the fiber product of $H_1(G)$ and X over G:

$$\begin{tikzcd} 0 \arrow[r] & H_1(G) \arrow[r] \arrow[d, equal] & M_{\mathbb{Z}} \arrow[r, "\beta"] \arrow[d, "\alpha"'] & X \arrow[r] \arrow[d, "u"] & 0 \\ 0 \arrow[r] & H_1(G) \arrow[r] & \mathrm{Lie}(G) \arrow[r, "\exp"] & G \arrow[r] & 0. \end{tikzcd}$$

Define a weight filtration on $M_{\mathbb{Z}}$ by setting

$$\begin{aligned} W_{-1}(M_{\mathbb{Z}}) &= H_1(G) = \ker(\beta) \\ W_{-2}(M_{\mathbb{Z}}) &= H_1(T) = \ker(H_1(G) \to H_1(A)), \end{aligned}$$

and $W_0(M_{\mathbb{Z}}) = M_{\mathbb{Z}}$, $W_{-3}(M_{\mathbb{Z}}) = 0$.

The map $\alpha : M_{\mathbb{Z}} \to \mathrm{Lie}(G)$ extends to $\alpha_{\mathbb{C}} : M_{\mathbb{C}} = M_{\mathbb{Z}} \otimes \mathbb{C} \to \mathrm{Lie}(G)$. Define a two-step Hodge filtration

$$M_{\mathbb{C}} = F_{-1} \supset F_0 \supset F_1 = 0$$

on $M_{\mathbb{C}}$ by setting

$$F_0(M_{\mathbb{C}}) = \ker(\alpha_{\mathbb{C}}).$$

Theorem (5.2.3) (Deligne)**.** *The map $(X, A, T, G, u) \mapsto (M_{\mathbb{Z}}, F, W)$ is an equivalence of categories from 1-motives over $\mathbb{C}$ to torsion-free integral mixed Hodge structures of type*

$$\{(0,0), (0,-1), (-1,0), (-1,-1)\}$$

such that $\mathrm{gr}^W_{-1} M_{\mathbb{Z}}$ is polarizable.

Proof. Essentially by construction. □

Let P be a connected linear algebraic group over $\mathbb{Q}$, $W = \mathrm{R}_u(P) \subset P$ the unipotent radical of P, and $G = P/W$ the reductive quotient. Denote by $\pi : P \to G$ the quotient map.

Proposition (5.2.4). *Suppose that $h : \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$ is a homomorphism of complex algebraic groups such that*

(i) The composition $\pi \circ h : \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}} \twoheadrightarrow G_{\mathbb{C}}$ is defined over $\mathbb{R}$.

(ii) The weight homomorphism $\pi \circ w_h : \mathbb{G}_{m,\mathbb{R}} \to G_{\mathbb{R}}$ (which makes sense by (i)) defines a cocharacter of the <u>center</u> of G, which is furthermore defined over $\mathbb{Q}$.

(iii) In the weight filtration $\{W_n \mathrm{Lie}(P)\}_{n \in \mathbb{Z}}$ defined by $\mathrm{Ad}_P \circ h$, we have $W_{-1}\mathrm{Lie}(P) = \mathrm{Lie}(W)$.

<u>Then</u>: For any object $\rho : P \to \mathrm{GL}(M)$ in $\mathrm{Rep}_{\mathbb{Q}}(P)$,

(b) The (weight) filtration $\{W_n M_{\mathbb{R}}\}$ on $M_{\mathbb{R}}$ induced by $\mathrm{Ad}_P \circ w_h : \mathbb{G}_{m,\mathbb{R}} \to \mathrm{GL}(M_{\mathbb{R}})$ is defined over $\mathbb{Q}$, and stable under P.

(a) The (Hodge) filtration $\{F^p M_{\mathbb{C}}\}$ on $M_{\mathbb{C}}$ induced by $\rho_{\mathbb{C}} \circ h : \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(M_{\mathbb{C}})$ induces a pure Hodge structure of weight n on $W_n M / W_{n-1} M$.

In particular, ρ defines a rational mixed Hodge structure on M. Furthermore, if $p \in P(\mathbb{R}) \cdot W(\mathbb{C})$, the same is true for $\mathrm{ad}(p) \circ h$.

Proof. See [Pin90, Prop. 1.4] □

5.3 Mixed Shimura Data

Recall that in a Shimura datum (G, X), X is a $G(\mathbb{R})$-conjugacy class of morphisms $h : \mathbb{S} \to G_{\mathbb{R}}$ such that, among other properties, the associated Hodge structure on $\mathrm{Lie}(G)$ is of type

$$\{(-1,1),(0,0),(1,-1)\}.$$

A mixed Shimura datum will likewise correspond to a family of mixed Hodge structures of a particular type on the Lie algebra of a connected linear algebraic group P defined over $\mathbb{Q}$. More precisely, the weight filtration will have the shape

$$0 \subseteq \mathrm{Lie}(U) \subseteq \mathrm{Lie}(W) \subseteq \mathrm{Lie}(P) \tag{11}$$

where W is the unipotent radical of P, and $U \subset W$ is a subgroup that's normal in P. Here $\mathrm{Lie}(U)$, $\mathrm{Lie}(W)$, and $\mathrm{Lie}(P)$ will correspond to weights -2, -1, and 0, respectively. The weight filtration will be *constant* for the entire family, and the Hodge filtration will then induce Hodge structures of weight -2, -1, and 0 on the graded pieces

$$\mathrm{gr}^W_{-2} = \mathrm{Lie}(U), \quad gr^W_{-1} = \mathrm{Lie}(W)/\mathrm{Lie}(U), \quad \mathrm{gr}^W_0 = \mathrm{Lie}(P)/\mathrm{Lie}(W).$$

These Hodge structures will be required to have types

$$\{(-1,-1)\}, \quad \{(-1,0),(0,-1)\}, \quad \{(-1,1),(0,0),(1,-1)\},$$

respectively.

Note that the group $G = P/W$ is reductive, and $\mathrm{Lie}(G) = \mathrm{Lie}(P)/\mathrm{Lie}(W) = \mathrm{gr}^W_0$, so that a mixed Hodge structure as above induces on it the same type of Hodge structure as appears in a pure Shimura datum.

Definition 5.5. A mixed Shimura datum consists of a tuple $(P, U, \mathfrak{X}, h)$, where:

- P is a connected linear algebraic group over $\mathbb{Q}$,
- U is a subgroup of the unipotent radical $W = \mathrm{R}_u(P)$, such that U is normal in P.
- $\mathfrak{X}$ is a left-homogeneous space for the subgroup $P(\mathbb{R}) \cdot U(\mathbb{C})$ of $P(\mathbb{C})$,
- $h : \mathfrak{X} \to \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}}), x \mapsto h_x$ is a quasi-finite $P(\mathbb{R}) \cdot U(\mathbb{C})$-equivariant map,

such that for some (all) $x \in \mathfrak{X}$:

(i) If $G = P/W$ and $\pi_G : P \to G$ is the projection, $\pi_G \circ h_x : \mathbb{S}_{\mathbb{C}} \to G_{\mathbb{C}}$ is defined over $\mathbb{R}$.

(ii) Z_G acts on U and P/U through a $\mathbb{Q}$-torus isogenous to $T_s \times T_c$, where T_s is $\mathbb{Q}$-split and $T_c(\mathbb{R})$ is compact.

(iii) $\mathrm{Ad}_P \circ h_x$ induces on $\mathrm{Lie}(P)$ a rational mixed Hodge structure of type

$$\{(-1,-1)\} \cup \{(-1,0),(0,-1)\} \cup \{(-1,1),(0,0),(1,-1)\},$$

such that the weight filtration W_n has $W_{-1} = \mathrm{Lie}(W)$ and $W_{-2} = \mathrm{Lie}(U)$.

(iv) $\mathrm{ad} \circ \pi_G \circ h_x(\sqrt{-1})$ induces a Cartan involution on $G^{\mathrm{ad}}_{\mathbb{R}}$.

(v) G^{ad} has no non-trivial $\mathbb{Q}$-factor H with $H(\mathbb{R})$ compact.

Remarks:

- Axiom *(iii)* implies $W_n = \mathrm{Lie}(P)$ for $n \geq 0$, and $W_n = 0$ for $n < -2$. Then the pure Hodge structure on $W_0/W_{-1} = \mathrm{Lie}(P)/\mathrm{Lie}(W) = \mathrm{Lie}(G)$ has weight zero, so that $\mathrm{Ad}_G \circ \pi_G \circ w_{h_x}$ is trivial on $\mathrm{Lie}(G)$. In other words, $\pi_G \circ w_{h_x}$ has image in Z_G. In the literature this fact is often added as redundant axiom.

- The group U is determined by the weight filtration, and hence by any h_x. A mixed Shimura datum is then in fact a triple $(P, \mathfrak{X}, h)$.

- The map $h : \mathfrak{X} \to \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, G_{\mathbb{C}})$ is a local diffeomorphism onto a complex analytic space, through which $\mathfrak{X}$ obtains a complex structure.

- If P is reductive, then $W = U = 1$, and $(P, h(\mathfrak{X}))$ is a (pure) Shimura datum.

Definition 5.6. Let $(P, \mathfrak{X}, h)$ be a mixed Shimura datum, and $K \subset P(\mathbb{A}_f)$ a compact open subgroup. The *mixed Shimura variety* of level K corresponding to $(P, \mathfrak{X})$ is

$$\mathrm{Sh}_K(P, \mathfrak{X}) = P(\mathbb{Q}) \backslash \mathfrak{X} \times P(\mathbb{A}_f)/K.$$

5.4 Rational Boundary Components

Let $(P, \mathfrak{X}, h)$ be a mixed Shimura datum, $W = \mathrm{R}_u(P)$, and $G = P/W$. The $\mathbb{Q}$-parabolic subgroups of P are all inverse images of parabolic subgroups of G^{ad}. Let $G^{\mathrm{ad}} = G_1 \times \cdots \times G_r$ be the decomposition into simple $\mathbb{Q}$-factors.

Definition 5.7. An *admissible* $\mathbb{Q}$*-parabolic subgroup* of P is the inverse image Q in P of $P_1 \times \cdots \times P_r < G^{\mathrm{ad}} = G_1 \times \cdots \times G_r$, where for each i either P_i is a maximal proper $\mathbb{Q}$-parabolic subgroup of G_i, or $P_i = G_i$.

To every admissible $\mathbb{Q}$-parabolic subgroup Q of P one associates a mixed Shimura datum $(P_1, \mathfrak{X}_1)$, where P_1 is a canonical subgroup of Q. The corresponding mixed Shimura variety then occurs in the description of the boundary components of the toroidal compactification of $\mathrm{Sh}_K(P, \mathfrak{X})$. It is called a rational boundary component by loose analogy with the Baily-Borel compactification, but in fact it describes a formal neighbourhood of the boundary components, after some modification.

For a detailed description of mixed Shimura varieties and their toroidal compactifications we refer to Pink's Thesis [Pin90], available from his websites.

Let (G, X) denote a Shimura datum. We will give a rough outline of how to enumerate the boundary components of a toroidal compactification of $\mathrm{Sh}(G, X)_K$. We will complement this with a more detailed example from unitary Shimura varieties.

Denote by $\mathscr{P}$ the set of $\mathbb{Q}$-parabolic subgroups of G.

Definition 5.8. A *cusp label* $\Phi = (P, g)$ is an element of $\mathscr{P} \times G(\mathbb{A}_f)$. It is called *proper* if P is a proper parabolic subgroup, and *improper* if $P = G$.

To every cusp label $\Phi = (P, g)$ is associated a distinguished normal subgroup $Q_\Phi \lhd P$, which in the improper case is P itself. For every compact open $K \subset G(\mathbb{A}_f)$, one defines a K-*equivalence* relation

$$(P, g) \sim_K (P', g') \iff P' = \gamma P \gamma^{-1}, \quad g' = \gamma h g k \text{ for some } \gamma \in G(\mathbb{Q}), h \in Q_\Phi(\mathbb{A}_f), k \in K.$$

Let $(P, \mathfrak{X})$ be a mixed Shimura datum, and $K_f \subset P(\mathbb{A}_f)$ an open compact subgroup. A toroidal compactification of $\mathrm{Sh}(P, \mathfrak{X})_{K_f}$ depends on the choice of a K_f-admissible cone decomposition Σ for $(P, \mathfrak{X})$. We will not define a cone decomposition here, as we only plan to enumerate the boundary components rather than construct them, but the details can be found in [Pin90].

For every rational boundary component $(P_1, \mathfrak{X}_1)$ of $(P, \mathfrak{X})$, and every $p \in P(\mathbb{A}_f)$, put

$$K_1^p = P_1(\mathbb{A}_f) \cap p K_f p^{-1}.$$

Then one has an open immersion

$$\mathrm{Sh}_{K_1^p}(P_1, \mathfrak{X}^+) \subset \mathrm{Sh}_{K_1^p}(P_1, \mathfrak{X}_1).$$

The toroidal compactification of $\mathrm{Sh}(P, \mathfrak{X})$ is a quotient, by an equivalence relation, of the disjoint union

$$\overline{\mathcal{S}} = \bigsqcup \overline{\mathrm{Sh}}_{K_1^p}(P_1, \mathfrak{X}^+),$$

for all rational boundary components $(P_1, \mathfrak{X}_1)$ of $(P, \mathfrak{X})$, and $p \in P(\mathbb{A}_f)$. This is a huge disjoint union, but only finitely many components remain after taking the quotient. Here $\overline{\mathrm{Sh}}_{K_1^p}(P_1, \mathfrak{X}_1)$ is the interior of the closure of $\mathrm{Sh}_{K_1^p}(P_1, \mathfrak{X}_1)$ in a larger space that depends on the cone decomposition Σ. Of course, this step is where the entire construction lies, but we have at least described a parameter space for the boundary components. In the next section we will give more details for the example we care about.

5.5 The toroidal compactification of $\mathcal{S}_{\mathrm{Kra}}$

We will now describe the toroidal compactification $\mathcal{S}^*_{\mathrm{Kra}}$ of the integral model $\mathcal{S}_{\mathrm{Kra}}$, as constructed in [BHK+17]. The proofs of these statements are mostly in the papers of B. Howard [How12], [How].

We have simplified some of the constructions, and so take credit for the mistakes introduced.

Let $\Phi = (P, g) \in \mathscr{P} \times G(\mathbb{A}_f)$ be a cusp label representative for $(G, \mathcal{D})$, where $G \subset G_{V_0} \times G_{V_1}$ is the group whose Shimura variety is the moduli space of pairs $(\underline{E}, \underline{A})$. The distinguished normal subgroup $Q_\Phi \lhd P$ has the following description. If Φ is improper, $Q_\Phi = P = G$. If Φ is proper, P is the the stabilizer of a rational istropic E-line $J \subset V_1$. P acts on the filtration $\mathcal{F} : 0 \subset J \subset J^\perp \subset V_1$, and hence also on its associated graded pieces. We thus have a morphism

$$P \to G(\mathcal{F}) := \mathrm{GL}(J) \times \mathrm{GU}(J^\perp/J) \times \mathrm{GL}(V_1/J^\perp).$$

It also acts on G_{V_0} since G does, so we get a map $P \to G_{V_0} \times G(\mathcal{F})$. The group Q_Φ is defined by the fiber product

$$\begin{array}{ccc} Q_\Phi & \longrightarrow & \mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m) \\ \downarrow & & \downarrow \\ P & \longrightarrow & G_{V_0} \times G(\mathcal{F}) \end{array}$$

where $\mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m) \to G_{V_0}$ is the identity, and $\mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m) \to G_J$ is the map $(a \mapsto (\mathrm{Nm}(a), a, \mathrm{id})$.

For each compact open $K_f \subset G(\mathbb{A}_f)$, we have

$$K_\Phi = gKg^{-1} \cap Q_\Phi(\mathbb{A}_f), \quad K'_\Phi = gKg^{-1} \cap P(\mathbb{A}_f),$$

and a finite group

$$\Delta_\Phi = (P(\mathbb{Q}) \cap Q_\Phi(\mathbb{A}_f)\widetilde{K}_\Phi)/Q_\Phi(\mathbb{Q}).$$

The subspace $J^\perp$ orthogonal to J has codimension 1 in V_1. Let $D_{V_1}(\Phi)$ denote the space of all complementary $E \otimes \mathbb{R} \simeq \mathbb{C}$-lines $J'_\mathbb{R}$ in $V_1(\mathbb{R})$:

$$D_{V_1}(\Phi) = \{J'_\mathbb{R} \subset V_1(\mathbb{R}) : J'_\mathbb{R} \oplus J^\perp(\mathbb{R}) = V_1(\mathbb{R})\},$$

and put

$$D_\Phi = D_{V_0}(\Phi) \times D_{V_1}(\Phi),$$

where $D_{V_0}(\Phi) = D_{V_0}$ is the single point corresponding to the symmetric space of G_{V_0}. Then (Q_Φ, D_Φ) is a mixed Shimura datum.

The toroidal compactification $\mathcal{S}^*_{\mathrm{Kra}}$ of $\mathcal{S}_{\mathrm{Kra}}$ is a disjoint union

$$\mathcal{S}^*_{\mathrm{Kra}} = \bigsqcup_\Phi \mathcal{S}^*_{\mathrm{Kra}}(\Phi),$$

where Φ ranges over K-equivalence classes of cusp labels. Each component $\mathcal{S}^*_{\mathrm{Kra}}(\Phi)$ is a locally-closed substack. There is a unique improper K-equivalence class Φ, for which $\mathcal{S}^*_{\mathrm{Kra}}(\Phi) = \mathcal{S}_{\mathrm{Kra}}$ is an open dense substack of $\mathcal{S}^*_{\mathrm{Kra}}$. For each proper class Φ, there is a tower of $\mathcal{O}_E$-stacks

$$\mathcal{C}_\Phi \to \mathcal{B}_\Phi \to \mathcal{A}_\Phi,$$

with

$$\mathcal{S}^*_{\mathrm{Kra}}(\Phi) = \Delta_\Phi \backslash \mathcal{B}_\Phi$$

and

$$\mathcal{C}_\Phi \otimes_{\mathcal{O}_E} E \simeq \mathrm{Sh}(Q_\Phi, D_\Phi).$$

There is an isomorphism of stacks $\mathcal{A}_\Phi \to \mathcal{M}_{1,0}$ that depends on the cusp label Φ. In particular, $\mathcal{A}_\Phi$ is smooth and proper of relative dimension 0 over $\operatorname{Spec} \mathcal{O}_E$.

We now describe these components in a little more detail. Let $L \subset V_1$ be a self-dual lattice, that we assume exists. Given a cusp label (P, g), the action of g gives another self-dual lattice

$$L^g := (g\widehat{L}) \cap V(\mathbb{Q}).$$

Intersecting the filtration $\mathcal{F}$ with L^g gives a filtration of lattices

$$0 \subset (L^g \cap J) \subset (L^g \cap J^\perp) \subset L^g$$

with associated graded pieces denoted

$$\mathfrak{n} = L^g \cap J, \quad \mathfrak{h} = (L^g \cap J^\perp)/(L^g \cap J), \quad \mathfrak{m} = L^g/(L^g \cap J^\perp).$$

The middle graded piece $\mathfrak{h}$ inherits a self-dual hermitian form from L^g. The symplectic form of the same hermitian form on L^g induces a perfect bilinear pairing

$$\mathfrak{m} \times \mathfrak{n} \to \mathbb{Z}.$$

Now, let $L_0 \subset V_0$ be the other fixed self-dual lattice (of rank-1), and L_0^g the corresponding translate of it by g. Put

$$\Lambda = \mathrm{Hom}_{\mathcal{O}_E}(L_0^g, \mathfrak{h}).$$

Then Λ obtains a self-dual hermitian structure from L_0^g and $\mathfrak{h}$. Two cusp labels Φ and Φ' are K-equivalent if and only if the pairs $(\Lambda_\Phi, \mathfrak{n}_\Phi)$ and $(\Lambda_{\Phi'}, \mathfrak{n}_{\Phi'})$ are isomorphic. That means $\Lambda_\Phi \simeq \Lambda_{\Phi'}$ as $\mathcal{O}_E$-hermitian lattices, and $\mathfrak{n}_\Phi \simeq \mathfrak{n}_{\Phi'}$ as rank-one projective $\mathcal{O}_E$-modules.

Let Φ be a fixed cusp label, and $(\Lambda_\Phi, \mathfrak{n}_\Phi)$ the corresponding pair. The stack $\mathcal{A}_\Phi$ represents the functor

$$\underline{\mathcal{A}_\Phi} : \mathrm{Sch}_{\mathcal{O}_E} \longrightarrow \mathrm{Groupoids}$$

that to each $S \in \mathrm{Sch}_{/\mathcal{O}_E}$ associates the category of triples (A_0, B, ρ), where

$$(A_0, B) \in \mathcal{M}_{1,0} \times_{\mathcal{O}_E} \mathcal{M}_{n-2,0}(S),$$

and

$$\rho : \Lambda_\Phi \otimes_{\mathcal{O}_E} A_0 \to B.$$

is an isomorphism of objects in $\mathcal{M}_{n-2,0}$. Note that $\mathcal{A}_\Phi$ only depends on Λ_Φ. In fact, it is essentially the zero-dimensional special cycle $\mathcal{Z}(\Lambda_\Phi)$ on the zero-dimensional moduli stack $\mathcal{M}_{1,0} \times \mathcal{M}_{n-2,0}$.

The stack $\mathcal{B}_\Phi$ represents tuples (A_0, B, ρ, σ), where (A_0, B, ρ) is an object of $\mathcal{A}_\Phi$, and $\sigma : \mathfrak{n}_\Phi \to B$ is a morpshim of $\mathcal{O}_E$-group schemes. We leave out the definition of $\mathcal{C}_\Phi$, and refer to [BHK$^+$17].

6 Arithmetic Cycles

First we define the usual Chow groups and their intersection pairing, then we introduce Green currents and arithmetic Chow groups. The reference for everything here is [Sou92].

6.1 Chow groups

Let X be a separated noetherian scheme of dimension d. Let $X^{(p)}$ denote the set of codimension-p generic points of X, so that $x \mapsto \overline{\{x\}}$ is a bijection between $X^{(p)}$ and closed irreducible codimension-p subschemes of X.

Definition 6.1. The group $Z^p(X)$ of *p-cycles* on X is the free abelian group generated by $X^{(p)}$.

Equivalently, elements of $Z^p(X)$ may be thought of as finite integral linear combinations of closed irreducible subschemes of X.

For each $y \in X^{(p-1)}$, with closure $Y = \overline{\{y\}}$, and $f \in k(y)^*$, we put

$$\operatorname{div}(f) = \sum_{x \in X^{(p)} \cap Y} \operatorname{ord}_{\mathcal{O}_{Y,x}}(f) \cdot \overline{\{x\}}.$$

Let $R^p(X)$ denote the subgroup of $Z^p(X)$ generated by $\operatorname{div}(f)$ for all $y \in X^{(p-1)}$ and $f \in k(y)^*$. Two cycles Z_1, $Z_1 \in Z^p(X)$ are said to be *rationally equivalent* if $Z_1 - Z_2 \in R^p(X)$.

Definition 6.2. The Chow group $\mathrm{CH}^p(X)$ of X is the quotient group $Z^p(X)/R^p(X)$ of p-cycles on X up to rational equivalence.

The image of $Z \in Z^p(X)$ in $\mathrm{CH}^p(X)$ is denoted $[Z]$.

Two cycles $Y \in Z^p(X)$ and $Z \in Z^q(X)$ are said to *intersect properly* if either $Y \cap Z = \emptyset$ or

$$\operatorname{codim}_X(Y \cap Z) = \operatorname{codim}_X Y + \operatorname{codim}_X Z.$$

Suppose that Y and Z intersect properly, and $x \in Y \cap Z \cap X^{(p+q)}$. The *intersection multiplicity* $\chi^x(Y, Z)$ of Y and Z at the (generic) point x is given by Serre's Tor formula:

$$\chi^x(Y, Z) = \sum_{i \geq 0} (-1)^i \operatorname{length}_{\mathcal{O}_{X,x}}(\operatorname{Tor}_i^{\mathcal{O}_{X,x}}(\mathcal{O}_{Y,x}, \mathcal{O}_{Z,x})).$$

Theorem (6.1.1). *Let X be a regular scheme. There exists a unique pairing*

$$\mathrm{CH}^p(X)_{\mathbb{Q}} \otimes \mathrm{CH}^q(X)_{\mathbb{Q}} \longrightarrow \mathrm{CH}^{p+q}(X)_{\mathbb{Q}}, \quad [Y] \otimes [Z] \longmapsto [Y] \cdot [Z],$$

such that if $Y \in Z^p(X)$ and $Z \in Z^q(X)$ intersect properly,

$$[Y] \cdot [Z] = \left[\sum_{x \in Y \cap Z \cap X^{(p+q)}} \chi^x(Y, Z) \cdot \overline{\{x\}} \right].$$

The Arithmetic Chow groups $\widehat{\mathrm{CH}}^p(X)$ are defined by augmenting the p-cycles $Y \in Z^p(X)$ with a certain distribution on the space of $(p-1, p-1)$ forms on $X(\mathbb{C})$. In the next few sections we give the precise definitions and the basic properties.

6.2 Differentials

Let X be a smooth projective variety over $\mathbb{C}$ with connected components of complex dimension d. By $A^{p,q}(X)$ we denote the vector space of complex-valued differential forms of type (p, q) on $X(\mathbb{C})$, and we put

$$A^n(X) = \bigoplus_{p+q=n} A^{p,q}(X). \tag{12}$$

We have the usual differential operators

$$\partial : A^{p,q}(X) \to A^{p+1,q}(X),$$
$$\overline{\partial} : A^{p,q}(X) \to A^{p,q+1}(X)$$

and

$$d : A^n(X) \to A^{n+1}(X), \quad d = \partial + \overline{\partial}.$$

If $z = (z_1, \cdots, z_d)$ are local coordinates on X, each $\omega \in A^n(X)$ is locally of the form

$$\omega = \sum_{I,J} f_{I,J}(z, \overline{z}) dz_I \wedge d\overline{z}_J$$

where $I = (i_1, \cdots, i_p)$, $J = (j_1, \cdots, j_q)$ are multi-indices satisfying

$$0 \le i_1 < \cdots < i_p \le d, \quad 0 \le j_1 < \cdots < j_q \le d,$$

and $f_{I,J}(z, \overline{z})$ are C^∞-functions, with

$$dz_I = dz_{i_1} \wedge dz_{i_2} \wedge \cdots dz_{i_p}, \quad d\overline{z}_J = d\overline{z}_{j_1} \wedge d\overline{z}_{j_2} \wedge \cdots d\overline{z}_{j_q}.$$

If $\{\omega_k\} \subset A^n(X)$ is a sequence of differential forms supported on a compact (analytic) subspace

$K \subset X$, we say ω_k *goes to zero*, and write $\omega_k \to 0$ if, in local coordinates

$$\omega_k = \sum_{I,J} f_{k,I,J}(z, \overline{z}) dz_I \wedge d\overline{z}_J,$$

on K, the functions $f_{k,I,J}$ along with any finite collection of their derivatives go to 0 uniformly on K as $k \to \infty$.

The dual $D_n(X)$ of $A^n(X)$ will then denote the space of linear functionals $T : A^n(X) \to \mathbb{C}$ that are *continuous* in the following sense: $T(\omega_k) \to 0$ if $\omega_k \to 0$. The decomposition (12) then induces

$$D_n(X) = \bigoplus_{p+q=n} D_{p,q}(X).$$

If we put

$$D^{p,q}(X) := D_{d-p,d-q}(X), \quad D^n(X) := D_{d-n}(X)$$

we obtain embeddings

$$A^{p,q}(X) \hookrightarrow D^{p,q}(X), \quad \omega \mapsto [\omega]$$

defined by

$$[\omega](\alpha) := \int_X \omega \wedge \alpha, \quad \alpha \in A^{d-p,d-q}(X).$$

The differentials ∂, $\overline{\partial}$, and d induce adjoints

$$\begin{aligned}
\partial' &: D^{p,q}(X) \to D^{p+1,q}(X), \\
\overline{\partial}' &: D^{p,q}(X) \to D^{p,q+1}(X) \\
d' &: D^n(X) \to D^{n+1}(X)
\end{aligned}$$

where for instance $(\partial' T)(\alpha) = T(\partial\alpha)$ for $T \in D^{p,q}(X)$ and $\alpha \in A^{d-(p+1),d-q}(X)$. These must be modified by appropriate signs in order for the embeddings $A^{p,q}(X) \to D^{p,q}(X)$ to be compatible with differentials. Indeed, for $\omega \in A^{p,q}(X)$ and $\alpha \in A^{d-p,d-q}(X)$, we have

$$\begin{aligned}
[d\omega](\alpha) &= \int_X d\omega \wedge \alpha \\
&= \int_X d(\omega \wedge \alpha) - \int_X (-1)^n \omega \wedge d\alpha \quad \text{(by Stokes')} \\
&= (-1)^{n+1} \int_X \omega \wedge d\alpha \\
&= (-1)^{n+1} (d'[\omega])(\alpha).
\end{aligned}$$

Thus if we put

$$\begin{aligned} \partial &= (-1)^{n+1}\partial' \\ \overline{\partial} &= (-1)^{n+1}\overline{\partial}' \\ \overline{d} &= (-1)^{n+1}d' \end{aligned} \tag{13}$$

we obtain commutative diagrams

$$\begin{tikzcd} A^{p,q}(X) \arrow[r] \arrow[d, "\partial"'] & D^{p,q}(X) \arrow[d, "\partial"] \\ A^{p+1,q}(X) \arrow[r] & D^{p+1,q}(X). \end{tikzcd}$$

We also put

$$d^c = \frac{1}{4\pi i}(\partial - \overline{\partial}) \tag{14}$$

so that in particular

$$dd^c = -\frac{1}{2\pi i}\partial\overline{\partial}.$$

Now, suppose that $i : Y \hookrightarrow X$ is a subvariety of codimension p. It defines an element

$$\delta_Y \in D^{p,p}(X)$$

by

$$\delta_Y(\alpha) = \int_{Y^{\mathrm{ns}}} \iota^*(\alpha),$$

where Y^{ns} is the non-singular locus. It may be extended linearly to arbitrary codimension-p analytic subvarieties of X

6.3 Green currents

Definition 6.3. An *arithmetic variety* X is a flat regular projective scheme X over $\operatorname{Spec}\mathbb{Z}$.

Let X be an arithmetic variety. Denote by

$$F_\infty : X(\mathbb{C}) \to X(\mathbb{C}),$$

the continuous map induced by complex conjugation.

Put

$$A^{p,p}(X)_{=}\{\omega \in A^{p,p}(X(\mathbb{C})) : \omega \text{ real}, \;\; F_\infty^*\omega = (-1)^p\omega\}$$

and

$$Z^{p,p}(X) = \ker(d : A^{p,p}(X) \to A^{2p+1}(X(\mathbb{C})) \subset A^{p,p}(X).$$

If we similarly put

$$D^{p,p}(X) = \{T \in D^{p,p}(X(\mathbb{C})) : T \text{ real}, \ \ F_\infty^* T = (-1)^p T\},$$

then we again have an embedding

$$A^{p,p}(X) \hookrightarrow D^{p,p}(X), \quad \omega \to [\omega].$$

The map $Z \mapsto \delta_Z$ on subvarieties $Y \subset X(\mathbb{C})$ of codimension p extends linearly to p-cycles

$$Z^p(X) \to D^{p,p}(X), \quad Z \mapsto \delta_Z.$$

Definition 6.4. A *Green current* $\mathfrak{g}$ for $Z \in Z^p(X)$ is an element of $D^{p-1,p-1}(X)$ such that

$$dd^c \mathfrak{g} + \delta_Z = [\omega],$$

for some $\omega \in D^{p,p}(X)$.

6.4 Arithmetic Chow Groups

Let X be an arithmetic variety.

Definition 6.5. An *arithmetic p-cycle* on X is a pair (Z, g), where $Z \in Z^p(X)$, and g_Z is a Green current for Z. The arithmetic p-cycles form a group $\widehat{Z}^p(X)$ under componentwise addition.

Let $y \in X^{(p-1)}$ be a codimension $p-1$ generic point, and $f \in k(y)^*$ a rational function. Then $(\mathrm{div} f, -\log \|f\|^2)$ is an element of $\widehat{Z}^p(X)$. Let $\widehat{R}^p(X)$ be the subgroup generated by all such elements, plus elements of the form $(0, \partial u + \overline{\partial} v)$, where u and v are currents of type $(p-2, p-1)$ and $(p-1, p-2)$ respectively.

Definition 6.6. The arithmetic Chow groups are defined as

$$\widehat{\mathrm{CH}}^p(X) = \widehat{Z}^p(X) / \widehat{R}^p(X).$$

There is a map

$$\zeta : \widehat{\mathrm{CH}}^p(X) \to \mathrm{CH}^p(X),$$

that sends a class $[(Z, g_Z)]$ to $[Z]$. There is also a map

$$\omega : \widehat{\mathrm{CH}}^p(X) \to Z^{p,p}(X),$$

that sends $[(Z, g_Z)]$ to $[\omega_Z]$, where

$$dd^c g_Z + \delta_Z = \omega_Z \tag{15}$$

is the differential equation satisfied by the current g_Z.

Theorem (6.4.1) (Intersection Pairing)**.** *There exists a pairing*

$$\widehat{\mathrm{CH}}^p(X) \otimes \widehat{\mathrm{CH}}^q(X) \to \widehat{\mathrm{CH}}^{p+q}(X)_{\mathbb{Q}}, \quad \alpha \otimes \beta \mapsto \alpha \cdot \beta$$

such that

(a) $\oplus_{p \geq 0} \widehat{\mathrm{CH}}^p(X)_{\mathbb{Q}}$ *is a graded commutative algebra over* $\mathbb{Q}$ *with unity.*

(b) The map

$$(\zeta, \omega) : \widehat{\mathrm{CH}}^p(X) \to \mathrm{CH}^p(X) \oplus Z^{p,p}(X),$$

is a homomorphism of $\mathbb{Q}$*-algebras.*

Proof. See [Sou92, p. 60, Theorem 2]. □

The intersection pairing of two arithmetic cycles $[(Z, g_Z)]$ and $[(Y, g_Y)]$ are defined first for irreducible Z and Y, then extended by linearity. For irreducible Y and Z one first assumes the intersection of $Y_{\mathbb{Q}}$ and $Z_{\mathbb{Q}}$ is proper, and defines

$$[(Z, g_Z)] \cdot [(Y, g_Y)] = [([Z] \cdot [Y], g_Z * g_Y)],$$

where $g_Z * g_Y$ is a certain *-product of Green currents. Its definition is rather elaborate and uses Hironaka's resolution of singularities, so we refer to [Sou92, II]. Then for improper intersections, one uses a Moving Lemma to reduce to the proper case. This is the reason why the codomain of the intersection pairing is tensored with $\mathbb{Q}$; the Moving Lemma requires it.

Theorem (6.4.2). *Let* $f : Y \to X$ *be a morphism of arithmetic varieties.*

(i) There is a pullback morphism $f^* : \widehat{\mathrm{CH}}^p(X) \to \widehat{\mathrm{CH}}^p(Y)$ *satisfying*

$$f^*(\alpha \cdot \beta) = f^*(\alpha) \cdot f^*(\beta).$$

(ii) If f *is proper,* $f_{\mathbb{Q}} : Y_{\mathbb{Q}} \to X_{\mathbb{Q}}$ *is smooth, and* X, Y *are equidimensional, there is a pushforward map* $f_* : \widehat{\mathrm{CH}}^p(Y) \to \widehat{\mathrm{CH}}^{p-\delta}(X)$*, where* $\delta = \dim Y - \dim X$.

(iii) The projection formula holds:

$$f_*(f^*(\alpha) \cdot \beta) = \alpha \cdot f_*(\beta) \in \widehat{\mathrm{CH}}^{p+q-\delta}(X)_{\mathbb{Q}}.$$

(iv) If $g : Z \to Y$ *is another map, then*

$$(fg)^* = g^* f^*, \quad (fg)_* = f_* g_*$$

whenever these are defined.

Proof. See [Sou92, p. 64, Theorem 3]. $\square$

Let us describe how the pushforward map in the theorem is defined. Suppose that $f : Y \to X$ satisfies the conditions in (ii), and $(Z, g_Z) \in \widehat{Z}^p(Y)$, where $Z = \overline{\{z\}}$ is irreducible, with $z \in Y^{(p)}$. The pushforward of Z along f is defined by

$$f_*(Z) = \begin{cases} [k(z) : k(f(z))] \cdot \overline{\{f(z)\}} & \text{if } \dim f(z) = \dim z, \\ 0 & \text{if } \dim f(z) < \dim z. \end{cases}$$

Then

$$f_*([Z, g_Z]) = [(f_*(Z), f_*g_Z)],$$

where f_*g_Z is the pushforward of the current g_Z. If ω_Z is the differential form satisfying the current equation (15) for g_Z, then f_*g_Z satisfies

$$dd^c(f_*g_Z) + \delta_{f_*(Z)} = [f_*\omega_Z],$$

where the differential $f_*\omega_Z$ is defined by integrating ω_Z along the fibers of f. This is possible because $f_{\mathbb{Q}}$ is smooth.

6.5 Arithmetic Degree

Let X be an arithmetic variety.

Definition 6.7. A *hermitian line bundle* on X is a pair $(L, \| \ \|)$, where L is a complex line bundle on $X(\mathbb{C})$, and $\| \ \|$ is a smooth hermitian metric on L that is invariant under F_∞.

An isomorphism of hermitian line bundles $(L_1, \| \ |_1) \xrightarrow{\sim} (L_2, \| \ \|_2)$ is an isomorphism of line bundles that preserves the hermitian metric. The product of $(L_1, \| \ \|_1)$ and $(L_2, \| \ \|_2)$ is $L_1 \otimes_{\mathcal{O}_X} L_2$ equipped with the product form.

Definition 6.8. $\widehat{\mathrm{Pic}}(X)$ is the group of isomorphism classes of hermitian line bundles on X.

Let $(L, \| \ \|)$ be a hermitian line bundle on X, and s any rational section of L. Then $(\mathrm{div}(s), -\log \|s\|^2)$ is an element of $\widehat{\mathrm{CH}}^1(X)$ whose class does not depend on the choice of s. The induced map

$$\widehat{c}_1 : \widehat{\mathrm{Pic}}(X) \to \widehat{\mathrm{CH}}^1(X), \quad (L, \| \ \|) \mapsto (\mathrm{div}(s), -\log \|s\|^2) \tag{16}$$

is then a group isomorphism. It is the arithmetic Chern class map.

Now let K be a number field, and $X = \operatorname{Spec} \mathcal{O}_K$. A line bundle on X is equivalent to a rank-one projective $\mathcal{O}_K$-module L, and a rational section s is any non-zero element $s \in L$. Let Σ denote the complex embeddings of K. There is a *degree* map

$$\widehat{\deg} : \widehat{\mathrm{Pic}}(X) \to \mathbb{R}$$

defined by

$$\widehat{\deg}(L, \| \ \|) = \log([L : \mathcal{O}_K s]) - \sum_{\sigma \in \Sigma} \log \|s\|_\sigma,$$

where s is any rational section of L. Then one also obtains a map

$$\widehat{\deg} : \widehat{\mathrm{CH}}^1(X) \to \mathbb{R}$$

via the isomorphism (16).

7 Borcherds Lifts and Products

To construct arithmetic (Arakelov) special cycles on compactified moduli spaces one needs to equip the special cycles on the integral model $\mathcal{S}^*_{\mathrm{Kra}}$ with suitable Green functions (as currents). A principal technique for the construction of Green functions is by systematic use of Borcherds lifts and their product expansions. In this section we outline the main ideas.

7.1 Orthogonal Shimura Varieties

Let (V, Q) be a non-degenerate quadratic space of signature $(2, q)$ over $\mathbb{Q}$, $L \subset V$ a lattice on which Q takes integer values, and $L' \supset L$ its dual in V with respect to Q. We denote by $(\ ,\ )$ the inner product on V corresonding to Q. The symmetric space of $G = O(V)$ can be described as the set of 2-dimensional positive-definite planes in $V(\mathbb{R})$:

$$D = \{Z \subset V(\mathbb{R}) : \dim_{\mathbb{R}} Z = 2,\ (\ ,\ )|_Z > 0\}.$$

There are two ways to choose a compatible system of orientations on $Z \in D$, and we fix one such choice. Let $\mathcal{F} \to D$ be the vector bundle whose fiber $\mathcal{F}_Z$ for $Z \in D$ is the set of oriented orthogonal bases (X, Y) for Z. The group $\mathbb{R}^\times \times \mathrm{O}(2)$ acts on orthogonal bases for Z, and its subgroup $\mathbb{R}^\times_+ \times \mathrm{SO}(2)^+ \simeq \mathbb{C}^\times$ acts simply transitively on each fiber $\mathcal{F}_Z$. Now, there is a $\mathbb{C}^\times$-equivariant embedding

$$\mathcal{F} \to V(\mathbb{C}), \quad (X, Y) \mapsto X + iY$$

whose image lies on the quadric

$$V(\mathbb{C})_0 = \{W \in V(\mathbb{C}) : Q(W) = 0\} \subset V(\mathbb{C}).$$

Since $\mathcal{F}/\mathbb{C}^\times \simeq D$, we obtain a morphism of $\mathbb{C}^\times$ bundles

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & V(\mathbb{C})_0 \backslash \{0\} \arrow[d] \\ D \arrow[r] & \mathbb{P}(V(\mathbb{C})). \end{tikzcd}$$

The bottom map is an embedding of D into complex projective space, through which it obtains the structure of a hermitian symmetric domain.

The quotient map $V(\mathbb{C})\backslash\{0\} \to \mathbb{P}(V(\mathbb{C}))$ is called the *tautological bundle* over $\mathbb{P}(V(\mathbb{C}))$. Let $P \to D$ denote its pullback to D. Denote by $O(V)^+$ the index-2 subgroup of elements of $O(V)$ that are orientation preserving. Let $\Gamma(L) \subset O(V)$ be the arithmetic subgroup that preserves $L \subset V$, and put $\Gamma(L)^+ = \Gamma(L) \cap O(V)^+$.

Definition 7.1 (Borcherds)**.** Let $\Gamma \subset \Gamma(L)^+$ be a finite index subgroup, and $\chi : \Gamma \to \mathbb{C}^\times$ a character. An automorphic form on D of weight $-k$, level Γ, and character χ is a function Ψ on P which is homogeneous of degree $-k$, and satisfies

$$\Psi(\gamma \cdot x) = \chi(\gamma)\Psi(x), \text{ for all } \gamma \in \Gamma, \text{ and } x \in P.$$

Borcherds' lifts provide a way to produce automorphic forms on orthogonal groups, in the above sense, that have remarkable properties. For instance they have explicit product expansions in a neighbourhood of the rational "cusps" of D. We first explain how to construct these neighbourhoods.

7.2 Neighbourhoods of Cusps

Let $Z \subset V$ be a rational isotropic line. Then $Z \cap L = \mathbb{Z} \cdot z$ for some $z \in L$ such that $Q(z) = 0$. The subspace $Z^\perp$ of V has dimension $q+1$ and contains Z. The restriction of Q to $Z^\perp$ induces a non-degenerate quadratic form Q_Z on the quotient $V_Z = Z^\perp/Z$, of signature $(1, q-1)$. The lattice $L \subset V$ induces a lattice $L_Z = (L \cap Z^\perp)/(L \cap Z)$ in V_Z.

Now choose $z' \in L'$ such that $(z, z') = 1$. Then $H = \mathrm{Span}_{\mathbb{R}}\{z, z'\}$ is a hyperbolic plane, and we can write $V = H \perp V_{z,z'}$. Since $V_{z,z'} \subset Z^\perp$, the quotient $Z^\perp \to V_Z$ restricts to a map $V_{z,z'} \to V_Z$, which is in fact an isometry of quadratic spaces.

Given $x \in V_{z,z'}(\mathbb{C})$, there is a unique $x' \in H(\mathbb{C})$ such that $Q(x + x') = 0$, and $(x', z) = 1$. It is given by

$$x' = z' - \frac{1}{2}(Q(x) + Q(z'))z.$$

Then we have a map

$$\beta = \beta_{z,z'} : V_{z,z'}(\mathbb{C}) \to V(\mathbb{C})_0, \quad \beta(x) = x + x'.$$

Since

$$x = \beta(x) - z' - (\beta(x), z') \cdot z,$$

the map β is injective. Composing it with the isomorphism $V_Z \to V_{z,z'}$ we obtain an injection $V_Z(\mathbb{C}) \to V(\mathbb{C})_0$ that we also denote β. Now recall we also had an inclusion $P \to V(\mathbb{C})_0$.

Proposition (7.2.1). *There is a connected "positive cone" $C \subset V_Z(\mathbb{R})$ such that there's a (unique)* *induced map $V_Z(\mathbb{R}) + iC \to P$ forming a commutative digram*

$$\begin{tikzcd} V_Z(\mathbb{R}) + iC \arrow[r] \arrow[d, dotted] & V_Z(\mathbb{C}) \arrow[d, "\beta"] \\ P \arrow[r] & V(\mathbb{C})_0. \end{tikzcd}$$

Furthermore, the composition $V_Z(\mathbb{R}) + iC \to P \to D$ is an isomorphism. The cone C is one of two connected components of the positive-norm vectors in $V_Z(\mathbb{R})$, and is characterized as follows: If $\widetilde{y} \in Z^{\perp}(\mathbb{R})$ maps to $y \in V_Z(\mathbb{R})$, and $(\widetilde{x}, \widetilde{y})$ is an oriented basis for some $Z_0 \in D$, then $y \in C$ if and only if $(\widetilde{x}, z) > 0$.

Proof. This is essentially a reformulation of [Bor99, p.542]. □

The consequence is that given an automorphic form Ψ on P, and rational isotropic vectors z, z' spanning a hyperbolic plane in V, one can "restrict" Ψ to a new automorphic form Ψ_z on $V_Z(\mathbb{R}) + iC \simeq D$. The function Ψ is uniquely determined by this restriction.

When $\Psi = \psi(f)$ is a Borcherds' lift from a modular form f, the restriction Ψ_z has an explicit infinite-product expansion in terms of the q-expansion of f and the geometry of the lattices L and L_Z. We explain this in the next section. A key step in the proof of the modularity of special cycles on unitary Shimura varieties is an algebraic interpretation of this product expansion.

7.3 Borcherds Lifts

The Borcherds lift is a map $f \mapsto \psi(f)$ from a certain space of meromorphic vector-valued modular forms on $\mathfrak{H}$, to automorphic forms on $G = SO(V)$, where V is a quadratic space over $\mathbb{Q}$ of signature $(2, q)$ as in the previous section.

Let us describe the domain of this lift more precisely. The metaplectic double-cover $\mathrm{Mp}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$ is the group of pairs

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm\sqrt{c\tau + d} \right)$$

where the first member belongs to $\mathrm{SL}_2(\mathbb{R})$, and the second is a holomorphic function of a variable $\tau \in \mathfrak{H}$ whose square is $c\tau + d$. The subgroup $\mathrm{Mp}_2(\mathbb{Z})$ has a similar description, and acts on $\mathfrak{H}$ via Möbius transformations by its projection onto $\mathrm{SL}_2(\mathbb{Z})$.

Definition 7.2. Let $\rho : \mathrm{Mp}_2(\mathbb{Z}) \to U$ be a finite-dimensional complex representation, and $\chi : \mathrm{Mp}_2(\mathbb{Z}) \to \mathbb{C}^{\times}$ a unitary character. A function $f : \mathfrak{H} \to U$ is called a *weakly holomorphic vector-valued modular form* of weight k and representation ρ if:

- f is holomorphic on $\mathfrak{H}$, and meromorphic at the cusps,
- $f(\gamma\tau) = (c\tau + d)^k \rho(\gamma, \sqrt{c\tau + d}) f(\tau)$, for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Note that this definition does not involve a level subgroup Γ. That's because forms that are invariant only under a finite index subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ can be expressed as vector-valued forms for $\mathrm{SL}_2(\mathbb{Z})$ at the cost of inflating the dimension of their codomain. We will see an instance of this in the next section.

A weakly holomorphic form $f : \mathfrak{H} \to U$ has a q-expansion

$$f(\tau) = \sum_{m \in \mathbb{Q}} c(m)\mathbf{e}(m\tau), \quad \mathbf{e}(x) = e^{2\pi i x} \tag{17}$$

where $c(m) \in U$ and $c(m) = 0$ for $m \ll 0$.

The input f to Borcherd's lift $\psi(f)$ is a weakly holomorphic modular forms for a particular representation $\rho = \rho_L$ that we now define. Fixing an additive character of $\mathbb{A}/\mathbb{Q}$, the group $\mathrm{Mp}_2(\mathbb{A})$ acts on the space of Schwartz-Bruhat functions $\mathscr{S}(V(\mathbb{A}))$ via the associated Weil representation ω. Let $\mathscr{S}_L$ denote the space of Schwartz-Bruhat functions supported on $\widehat{L}' = L' \otimes \mathbb{A}_f \subset V(\mathbb{A}_f) \subset V(\mathbb{A})$ that are invariant under translation by $\widehat{L}$. The restriction of ω to $\mathrm{Mp}_2(\mathbb{Z})$ then acts on $\mathscr{S}_L$. The representation ρ_L is the *conjugate* of this representation:

$$\rho_L : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(\mathscr{S}_L), \quad \rho_L(\gamma)\varphi = \overline{\omega(\gamma)(\overline{\varphi})}.$$

The space $\mathscr{S}_L$ has a more concrete description via an isomorphism

$$\mathbb{C}[L'/L] \xrightarrow{\sim} \mathscr{S}_L$$

that sends $\lambda \in L'/L$ to the characteristic function $\varphi_\lambda : V(\mathbb{A}) \to \mathbb{C}$ of $\lambda + \widehat{L} \subset V(\mathbb{A}_f)$. We will identify $\mathscr{S}_L$ and $\mathbb{C}[L'/L]$ through this isomorphism.

If f is weakly holomorphic of weight k for ρ_L, with q-expansion (17), we write $c_\lambda(m)$ for the coefficient of $\lambda \in L'/L$ in $c(m)$.

We consider functions $f : \mathfrak{H} \to \mathbb{C}[L/L']$ whose projections $f_\lambda : \mathfrak{H} \to \mathbb{C}$ onto each coordinate $\lambda \in L/L'$ is holomorphic.

Theorem (7.3.1) (Borcherds)**.** *Let f be a vector-valued weakly holomorphic modular form of weight $1 - \frac{q}{2}$ and representation ρ_L such that $c_\lambda(m) \in \mathbb{Z}$ for all $\lambda \in L'/L$ and $m \leq 0$. Then there is a meromorphic function $\psi(f)$ on P such that*

(1) $\psi(f)$ is an automorphic form of weight $c_0(0)/2$ for a subgroup $\Gamma \subset \Gamma(L)$ and some unitary character $\chi : \Gamma \to \mathbb{C}^\times$.

(2) The divisor of $\psi(f)$ is supported on rational quadratic divisors $\lambda^\perp$ for $\lambda \in L$ and $Q(\lambda) < 0$, where it has order

$$\sum_{\lambda' \in L' \cap \mathbb{R}^+\lambda} c_{\lambda'}(\frac{Q(\lambda')}{2}).$$

(3) For each rational isotropic vector $z \in L$, and $z' \in L'$ with $(z, z') = 1$, the restriction Ψ_z of $\Psi = \psi(f)$ has an infinite product expansion converging on a neighbourhood of the cusp z

consisting of $Z = X + iY$ where Y is restricted to a "Weyl chamber" W in C, and $X \in V_Z(\mathbb{R})$. The product expansion takes the form

$$C_0 \cdot \mathbf{e}((z,\rho)) \prod_{\substack{\lambda \in L_z' \\ (\lambda, W) > 0}} \prod_{\substack{\delta \in L'/L \\ \delta|_L = \lambda}} (1 - \mathbf{e}((\lambda, Z) + (\delta, z')))^{c_\delta(\frac{Q(\lambda)}{2})},$$

where ρ is an explicit "Weyl vector" depending on L_z, K and f, and C_0 is a constant with

$$|C_0| = \prod_{\substack{\delta \in \frac{1}{N}\mathbb{Z}/\mathbb{Z} \\ \delta \neq 0}} (1 - \mathbf{e}(\delta))^{\frac{c_{\delta z}(0)}{2}}.$$

Proof. Theorem 13.3 in [Bor99]. □

7.4 Unitary Borcherds Products

Let $\chi_E : (\mathbb{Z}/D\mathbb{Z})^\times \to \{\pm 1\}$ be the Dirichlet character for $E/\mathbb{Q}$, and put $\chi = \chi_E^{n-2}$. Fix a weakly holomorphic form

$$f(\tau) = \sum_{m \gg -\infty} c(m) q^m \in M_{2-n}^{!,\infty}(D, \chi)$$

with $c(m) \in \mathbb{Z}$ for all $m \leq 0$.

The restriction of ρ_L to $\Gamma_0(D)$ acts on $\mathbb{C} \cdot \varphi_0$ by the character χ.

Then

$$\widetilde{f}\tau = \sum_{\gamma \in \Gamma_0(D) \backslash \mathrm{SL}_2(\mathbb{Z})} (f|_{2-n}\gamma)(\tau) \cdot \rho_L(\gamma)^{-1} \varphi_0$$

is $\mathscr{S}_L$-valued, of weight $2-n$ and representation ρ_L. One puts $\psi(f) = \psi(\widetilde{f})$.

The Borcherds lift $\psi(f)$, being a holomorphic modular form, is *a priori* an analytic section of a line bundle on an orthogonal Shimura variety. Now, given a E-vector space V with a hermitian form $H : V \times V \to E$ of signature $(n-1, 1)$, we can define a symmetric $\mathbb{Q}$-bilinear form by

$$(x, y) = \mathrm{Tr}_{E/\mathbb{Q}}\, \delta_E H(x, y),$$

where δ_E is any purely imaginary element of E. It's often convenient to take it to be a generator of the inverse different $\mathfrak{d}_E^{-1}$ whose imaginary part is positive. Thus setting $Q(x) = (x, x)$, (V, Q) becomes a quadratic space over $\mathbb{Q}$ of signature $(2n-2, 2)$, and the natural map $U(V, H) \to O(V, Q)$ induces an closed immersion of hermitian symmetric domains $D_H \hookrightarrow D_Q$. This induces maps on the connected components of the correpsonding Shimura varieties, and so via pullback we obtain a unitary Borcherds lift, again denoted $\psi(f)$, on the Shimura variety for $\mathrm{GU}(V)$.

7.5 Borcherds' Modularity Criterion

Let $k \geq 2$ be an integer, D an odd square-free positive integer, and χ a quadratic character modulo D. Let $M_k^{\infty}(D, \chi)$ denote the space of holomorphic modular forms of weight k and character χ for $\Gamma_0(D)$, that vanish at all cusps except possibly at ∞. In [BHK+17], a criterion is proved for determining when a formal power series

$$f = \sum_{m \geq 0} a_m q^m \in \mathbb{C}[\![q]\!] \tag{18}$$

defines an element of $M_k^{\infty}(D, \chi)$, in terms of infinitely many *linear* relations between the coefficients a_m. It is a variant of a similar result of Borcherds [Bor99], and ultimately follows from Serre duality.

The main construction is as follows. Let $M_{2-k}^{!,\infty}(D, \chi)$ denote the space of weakly holomorphic modular forms of weight $2-k$ and character χ for $\Gamma_0(D)$ that are holomorphic at all cusps except possibly at ∞. Each element of $M_{2-k}^{!,\infty}(D, \chi)$ is uniquely determined by its principal part at ∞, so there is a $\mathbb{C}$-linear embedding

$$M_{2-k}^{!,\infty}(D, \chi) \longrightarrow \mathbb{C}[q^{-1}], \quad \sum_m c_m q^m \longmapsto \sum_{m \leq 0} c_m q^m.$$

On the other hand, taking the q-expansion of a modular form also gives $\mathbb{C}$-linear embedding

$$M_k^{\infty}(D, \chi) \to \mathbb{C}[\![q]\!].$$

Now there is a bilinear pairing

$$\mathbb{C}[q^{-1}] \times \mathbb{C}[\![q]\!] \to \mathbb{C}$$

that sends (f, g) to the constant term of $fg \in \mathbb{C}[\![q]\!][q^{-1}]$. The key fact is that under this pairing the images of $M_{2-k}^{!,\infty}(D, \chi)$ in $\mathbb{C}[q^{-1}]$ and $M_k^{\infty}(D, \chi)$ in $\mathbb{C}[\![q]\!]$ are exactly mutual annihilators. The following is then an immediate consequence.

Theorem (7.5.1). *[BHK+17] A formal q-series $\sum_m a_m q^m$ defines an element of $M_k^{\infty}(D, \chi)$ if and only if*

$$\sum_{m \geq 0} a_m \cdot c_{-m} = 0$$

for all $\sum_m c_m q^m \in M_{2-k}^{!,\infty}(D, \chi)$.

8 Modularity of Special Cycles

The complex fiber $\mathcal{M}(\mathbb{C})$ of our PEL-moduli space $\mathcal{M} = \mathcal{M}_{1,0} \times \mathcal{M}_{n-1,1}$ is a disjoint union of finitely many unitary Shimura varieties. Thus one obtains a unitary Borcherds lift to $\mathcal{M}(\mathbb{C})$, which we again denote $\psi(f)$. It is a holomorphic section of a line bundle $(\omega^{\mathrm{an}})^k$ on $\mathcal{M}(\mathbb{C})$.

Now the line bundle ω^{an} is the analytification of an algebraic line bundle ω on the DM stack

$\mathcal{M}$ over $\operatorname{Spec}\mathcal{O}_E[D^{-1}]$. It extends canonically to the integral model $\mathcal{S}_{\mathrm{Kra}}$ of $\mathcal{M}$, and indeed to the toroidal compactification $\mathcal{S}^*_{\mathrm{Kra}}$. It then makes sense to ask whether $\psi(f)$ can be extended to a rational section of ω^k on the entire integral model $\mathcal{S}^*_{\mathrm{Kra}}$. This is indeed the case, and takes serious work to prove in [BHK$^+$17]. The main idea is to interpret the Borcherds products for $\psi(f)$, *a priori* an analytic expression in a neighbourhood of a cusp, in algebraic terms. This is done via an algebraic interpretation of Fourier-Jacobi series, as formal expansions of sections of line bundles in a formal neighbourhood of a rational boundary component of $\mathcal{S}^*_{\mathrm{Kra}}$.

Once $\operatorname{div}(f)$ is interpreted as a rational section on $\mathcal{S}^*_{\mathrm{Kra}}$, its divisor can be expressed in terms of special cycles, and the singular locus Sing of $\mathcal{S}_{\mathrm{Pap}}$. To state it, we recall that Sing is a reduced 0-dimensional stack supported over the ramified primes of $\mathcal{O}_E$. Each connected component $s \in \pi_0(\mathrm{Sing})$ has an étale cover by some $\operatorname{Spec}\mathbb{F}$, where $\mathbb{F}$ is a field of characteristic p. A geometric point $\operatorname{Spec}\overline{\mathbb{F}} \to \mathcal{S}_{\mathrm{Pap}}$ then defines a pair (E_s, A_s), corresponding to which there's a positive-definite $\mathcal{O}_E$-hermitian lattice

$$L_s = \operatorname{Hom}_{\mathcal{O}_E}(E_s, A_s),$$

whose isomorphism class depends only on s. For $m > 0$ let

$$N_s(m) = \#\{x \in L_s : \langle x, x\rangle = m\}.$$

Theorem (8.0.1). *[BHK$^+$17, Theorem 5.3.3] As a rational section $\psi(f)$ of ω^k on $\mathcal{S}^*_{\mathrm{Kra}}$,*

$$\operatorname{div}(\psi(f)) = \sum_{m>0} c(-m)\cdot \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) + k\cdot\left(\frac{\mathrm{Exc}}{2} - \operatorname{div}(\delta)\right) + \sum_{r|D} \gamma_r c_r(0)\cdot\mathcal{V}_r - \sum_{m>0}\frac{c(-m)}{2}\sum_{s\in\pi_0(\mathrm{Sing})} N_s(m)\cdot \mathrm{Exc}_s,$$

where:

- $\mathcal{V}_r = \prod_{p|r} \mathcal{S}_{\mathrm{Kra}/\mathbb{F}_\mathfrak{p}}$*, with* $\mathfrak{p}$ *the unique (ramified) prime of* $\mathcal{O}_E$ *over* $p|D$*,*
- $\gamma_r = \prod_{p|r}\gamma_p$*, with* $\gamma_p(L)$ *the Weil index of the quadratic space* $V_p/\mathbb{Q}_p$*, which has an explicit expression in terms of discrete invariants including Legendre symbols and signs of Gauss sums.*
- Exc_s *is the fiber over* s *of the exceptional divisor* $\mathrm{Exc} \subset \mathcal{S}_{\mathrm{Kra}}$*, which covers* $\mathrm{Sing} \subset \mathcal{S}_{\mathrm{Pap}}$*.*

Note that the complicated terms are supported on special fibers. Over the generic fiber one has

$$\operatorname{div}(\psi(f))_{/E} = \sum_{m>0} c(-m)\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)_{/E}.$$

The space $M^{!,\infty}_{2-n}(D,\chi)$ is spanned by those forms $f(\tau) = \sum_m c_m q^m$ having $c_m \in \mathbb{Z}$ for $m \geq 0$. On the other hand such forms can be the input for Borcherds' lift. By Theorem (8.0.1) For each such f, after possible replacement with a positive integer multiple, its unitary Borcherds' lift to $\mathcal{S}^*_{\mathrm{Kra}}$ has an explicit divisor of the form

$$\operatorname{div}(\psi(f)) = J + \sum_{m>0} c_{-m}\cdot \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m), \tag{19}$$

where J is a linear combination of some exceptional divisors and vertical ones supported over the ramified primes $p|D$.

Now recall the generating series of cycles

$$\phi(\tau) = \sum_{m \geq 0} \mathcal{Z}^{\text{tot}}_{\text{Kra}}(m) \cdot q^m \in \text{CH}^1(\mathcal{S}^*_{\text{Kra}})[\![q]\!].$$

By definition, such a generating series is modular if for any linear functional

$$\alpha : \text{CH}^1(\mathcal{S}^*_{\text{Kra}})_{\mathbb{R}} \to \mathbb{C},$$

the resulting formal q-series

$$\sum_{m \geq 0} \alpha(\mathcal{Z}^{\text{tot}}_{\text{Kra}}(m)) q^m \tag{20}$$

is an actual holomorphic modular form. From (19) we obtain

$$\sum_{m > 0} \alpha(\mathcal{Z}^{\text{tot}}_{\text{Kra}}(m)) \cdot c_{-m} + \alpha(J) = 0.$$

This is similar to the modularity criterion of Theorem (7.5.1) except that the term $m = 0$ is missing, and there is an extra $\alpha(J)$. In fact the modularity criterion can not apply directly. If it did, it would prove (20) is an element of $M_k^\infty(D, \chi)$, and that it vanishes at all cusps other than ∞; but that is not necessarily the case. It turns out that one can still apply the criterion to (20), by first modifying it by certain Eisenstein series to get it to vanish at the cusps $\neq \infty$. This is rather technical and requires explicit descriptions of the various exceptional and vertical divisors, so we will simply refer to [BHK+17] for the details.

To prove the modularity of the generating series of Arakelov cycles in $\widehat{\text{CH}}^1(\mathcal{S}^*_{\text{Kra}})$ the same way, one has to show an analogue of the relation (19) that holds in the arithmetic Chow groups. In particular, one has to construct Green currents on $\mathcal{S}^*_{\text{Kra}}$ for the cycles $\mathcal{Z}^{\text{tot}}_{\text{Kra}}(m)$.

The key result here is an identity due to Borcherds [Bor99]

$$\Theta^{\text{reg}}(f) = -\log \|\psi(f)\|^2,$$

where

$$\Theta^{\text{reg}}(z, g, f) = \int^{\text{reg}}_{\text{SL}_2(\mathbb{Z})\backslash\mathfrak{H}} (\widetilde{f}(\tau), \theta(\tau, z, g)) \frac{du dv}{v^2}$$

is a *regularized theta lift* of f. Note that $\Theta^{\text{reg}}(f)$ has logarithmic singularities on the divisor of $\psi(f)$.

9 Relation with Eisenstein Series

Let $\mathcal{M} = \mathcal{M}_{1,0} \times_{\text{Spec}\,\mathcal{O}_E} \mathcal{M}_{n-1,1}$, where $\mathcal{M}_{p,q} = \mathcal{M}^{\text{naive}}_{p,q}$.

Definition 9.1. Let N be a positive-definite $\mathcal{O}_E$-hermitian module of rank n. The cycle $\mathcal{Z}_N \to \mathcal{M}$ is called *non-degenerate* if it is pure of dimension $m - n$. A positive-definite $T \in \mathrm{Herm}_n(\mathcal{O}_E)$ is called *non-degenerate* if $\mathcal{Z}(T)$ is.

Now let $m = n$, so that $\mathcal{M} = \mathcal{M}_{1,0} \times \mathcal{M}_{n-1,1}$ and N rank-n $\mathcal{O}_E$-hermitian module. Then $\mathcal{Z}_N$ is non-degenerate if and only if it is of pure dimension 0. In that case

10 Appendix A: Shimura Varieties

Here we collect some standard definitions and facts about Shimura varieties. A good introductory text on Shimura varieties is Milne's notes [Mil05]. The canonical references are P. Deligne's Bourbaki seminar notes [Del71] and Corvallis paper [Del79].

10.1 Families of Hodge Structures

(10.1.1) The *Deligne torus* is the real algebraic group

$$\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}).$$

Its scalar extension to $\mathbb{C}$ is then $\mathbb{S}_\mathbb{C} \simeq \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$. The action of complex multiplication on $\mathbb{S}_\mathbb{C}(\mathbb{C})$ is given by

$$(z, w) \mapsto (\overline{w}, \overline{z}), \tag{21}$$

which identifies $\mathbb{S}(\mathbb{R}) \subset \mathbb{S}(\mathbb{C})$ with $(z, \overline{z}) \in \mathbb{C}^\times$.

(10.1.2) Suppose that $V_\mathbb{C}$ is a finite-dimensional complex vector space, and

$$h_\mathbb{C} : \mathbb{S}_\mathbb{C} \to \mathrm{GL}(V_\mathbb{C})$$

is an algebraic representation of $\mathbb{S}_\mathbb{C}$. Then $h_\mathbb{C}$ induces a $\mathbb{Z}$-bigrading

$$V_\mathbb{C} = \bigoplus_{p,q\in\mathbb{Z}} V^{p,q}, \tag{22}$$

that is characterized by

$$h_\mathbb{C}(z, w) \cdot v = z^{-p} w^{-q} v, \qquad \text{for } v \in V^{p,q}. \tag{23}$$

Conversely, given a $\mathbb{Z}$-bigrading as in (22), one may define an algebraic representation of $h_\mathbb{C}$ by (23).

Definition 10.1. A *Hodge decomposition* on V is a $\mathbb{Z}$-bigrading $V_\mathbb{C} = \oplus_{p,q} V^{p,q}$, such that

$$\overline{V^{p,q}} = V^{q,p}, \qquad \text{for } p, q \in \mathbb{Z}.$$

A *Hodge structure* is a pair $(V, \{V^{p,q}\})$ consisting of a real vector space V, equipped with a Hodge decomposition.

Lemma (10.1.3). *The bigrading corresponding to an algebraic homorphism $h_\mathbb{C} : \mathbb{S}_\mathbb{C} \to \mathrm{GL}(V_\mathbb{C})$ defines a Hodge decomposition on V if and only if it is defined over $\mathbb{R}$.*

Proof. Since $\mathbb{S}_\mathbb{C}$ and $V_\mathbb{C}$ are both defined over $\mathbb{R}$, $h_\mathbb{C}$ is defined over $\mathbb{R}$ if and only if it commutes with complex conjugation. The lemma is then the result of (21) and (22). □

In fancy language, $\mathbb{S}$ is the fundamental group of the rigid Tannakian category Hod of Hodge structures $(V, \{V_{p,q}\})$, with respect to the fiber functor $\mathrm{Hod} \to \mathrm{Vect}_{\mathbb{R}}$ that forgets the Hodge structure $\{V_{p,q}\}$.

(10.1.4) Let $G_{\mathbb{R}}$ be a connected reductive algebraic group over $\mathbb{R}$. The reductive property ensures there exists a real vector space V such that $G_{\mathbb{R}}$ is the subgroup of $\mathrm{GL}(V)$ fixing a finite collection of tensors $T_\alpha \in V^{\otimes n_\alpha} \otimes (V^*)^{\otimes m_\alpha}$. If such an embedding $G_{\mathbb{R}} \subset \mathrm{GL}(V)$ is fixed, a map $h : \mathbb{S} \to G_{\mathbb{R}}$ corresponds to a Hodge structure on V which is compatible with the tensors T_α. Then the $G(\mathbb{R})$-conjugacy class of h is a family of such Hodge structures parametrized by $X = G(\mathbb{R})/K$, where K is the stabilizer of h.

(10.1.5) A homomorphism $h : \mathbb{S} \to G$ corresponds to a Hodge structure on $\mathfrak{g}_0 = \mathrm{Lie}(G)$, hence a decomposition

$$\mathfrak{g} = \bigoplus_{p,q} \mathfrak{g}_{p,q}, \tag{24}$$

where $\mathfrak{g}$ is the complexification of $\mathfrak{g}_0$. The subspace $\mathfrak{g}_{0,0} \subset \mathfrak{g}$ is stable under complex conjugation, hence is the complexification of some $\mathfrak{g}_{00} \subset \mathfrak{g}_0$.

Let $K \subset G(\mathbb{R})$ be the stabilizer of h under conjugation. Then $\mathfrak{g}_{00} = \mathrm{Lie}(K)$, and $\mathfrak{p}_0 = \mathfrak{g}_0/\mathfrak{g}_{00}$ may be identified with the tangent space of the homogeneous space $X = G(\mathbb{R})/K$ at $p = eK$. Letting $\mathfrak{p} = \mathbb{C} \otimes \mathfrak{p}_0$, we obtain a decomposition

$$\mathfrak{p} = \bigoplus_{(p,q)\neq(0,0)} \mathfrak{g}_{p,q} = \mathfrak{p}^+ \oplus \mathfrak{p}^-,$$

where

$$\mathfrak{p}^+ = \bigoplus_{p>q} \mathfrak{g}_{p,q}, \quad \mathfrak{p}^- = \bigoplus_{p<q} \mathfrak{g}_{p,q}.$$

The subspaces $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are isomorphic as real vector spaces via complex conjugation, and in particular have the same dimension. If we put $\mathfrak{b} = \mathfrak{g}_{0,0} \oplus \mathfrak{p}^-$, then

$$\mathfrak{p}_0 = \mathfrak{g}/\mathfrak{b} \to \mathfrak{p}^+$$

is an isomorphism of real vector spaces. The complex structure on the right hand side can be transferred to $\mathfrak{p}_0$, endowing $G(\mathbb{R})/K$ with an invariant almost-complex structure.

(10.1.6) The most interesting case of the construction above is when the stabilizer K of h is maximal compact. In that case $G(\mathbb{R})/K$ is identified with the symmetric space of G, and is endowed with a(n almost) complex structure. Note that if the Hodge structure on $\mathfrak{g}_0$ is replaced with $\mathfrak{g} = \mathfrak{p}^+ \oplus \mathfrak{g}_{0,0} \oplus \mathfrak{p}^-$, the resulting isomorphism $\mathfrak{p}_0 \to \mathfrak{p}^+$ is the same. Thus for the purpose of endowing $\mathfrak{p}_0$ with complex structures, it's no loss to restrict to Hodge structures of type $\{(-1,1),(0,0),(1,-1))\}$.

(10.1.7) Let μ be the complex cocharacter defined by

$$\mu : \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}, \quad \mu(z) = h_{\mathbb{C}}(z,1).$$

Then we have

$$h_{\mathbb{C}}(z_1, z_2) = \mu(z_1)\overline{\mu(\overline{z_2})}.$$

In particular, h is completely determined by μ, plus the action of complex conjugation on $G_{\mathbb{C}}$. Often μ is more convenient to work with than h. For instance if h corresponds to a Hodge structure of type $\{(-1,1),(0,0),(1,-1)\}$, the complex structure on $\mathfrak{p}^+$ is the one given by $\mathrm{ad}\circ\mu(i)$.

(10.1.8) Let T be a maximal torus in $G_{\mathbb{C}}$ containing the image of $h_{\mathbb{C}}$, and $\mathfrak{t} = \mathrm{Lie}(T)$. Let $\Phi = \Phi(\mathfrak{g}, \mathfrak{t})$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Then we have a root space decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha.$$

The root decomposition of $\mathfrak{g}$ is strictly finer than the Hodge decomposition afforded by μ, since $\mu(\mathbb{G}_{m,\mathbb{C}}) \subset T$. Thus each $\mathfrak{g}_{p,q}$ is a direct sum of root spaces $\mathfrak{g}_\alpha$. Let $\langle -, - \rangle$ denote the canonical pairing

$$X_*(T) \times X^*(T) \to \mathbb{Z},$$

and put $\overline{\mu}(z) = \overline{\mu(\overline{z})}$.

Proposition (10.1.9). *We have $\mathfrak{g}_\alpha \subset \mathfrak{g}_{p,q}$ if and only if $\langle \mu, \alpha\rangle = -p$, $\langle \mu, \overline{\alpha}\rangle = -q$.*

Proof. Let $d\mu : \mathbb{C} \to \mathfrak{t}$ denote the derivative of $\mu : \mathbb{G}_{m,\mathbb{C}} \to T$ at $z = 1$. Then for $X \in \mathfrak{g}_\alpha$, we have

$$d(\mathrm{Ad}\circ\mu)(z)|_{z=1}X = [d\mu(z), X] = \alpha(d\mu(z))X.$$

On the other hand if $X \in \mathfrak{g}_{p,q}$,

$$(\mathrm{Ad}\circ\mu)(z)|X = z^{-p}X \Longrightarrow [d\mu(z), X] = -pX,$$

so that $\langle \mu, \alpha\rangle = -p$. The same calculation for $\overline{\mu}$ also shows $\langle \overline{\mu}, \alpha\rangle = -q$, which is the same as $\langle \mu, \overline{\alpha}\rangle$. ☐

(10.1.10) Suppose that $h : \mathbb{S} \to G$ corresponds to a Hodge structure of type $\{(-1,1),(0,0),(1,-1)\}$ on $\mathfrak{g}_0$. As before, let K be the stabilizer of h, and μ the corresponding complex cocharacter. Let $T \subset G_{\mathbb{C}}$ be a maximal torus containing the image of $h_{\mathbb{C}}$. Then $\mathfrak{t} = \mathrm{Lie}(T)$ is contained in $\mathfrak{k}$.

10.2 Shimura data

Definition 10.2. A *Shimura datum* is a pair (G, D) consisting of a reductive algebraic group G defined over $\mathbb{Q}$, and a $G(\mathbb{R})$-conjugacy class D of homomorphisms of real algebraic groups $h : \mathbb{S} \to G_{\mathbb{R}}$ such that:

(S1) The adjoint representation

$$\mathrm{ad}\circ h : \mathbb{S} \to \mathrm{End}(\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{R}}))$$

corresponds to a Hodge structure of type $\{(-1,1),(0,0),(1,-1)\}$.

(S2) $\mathrm{Ad} \circ h(i)$ is a Cartan involution.

(S3) G^{ad} has no factor defined over $\mathbb{Q}$ on which the projection of h is trivial.

Let us briefly recall what these mean. The conjugacy class D is isomorphic to the symmetric space of G. More precisely, the conjugation action of $G(\mathbb{R})$ on D factors through $G^{\mathrm{ad}}(\mathbb{R})$, and the stabilizer K_0 of a fixed $h_0 \in D$ is maximal compact, so that D may be identified with $G^{\mathrm{ad}}(\mathbb{R})/K_0$. There is an isogeny $G^{\mathrm{ad}} \to \prod_i G_i$, where G_i are simple algebraic groups over $\mathbb{Q}$, which induces $D \simeq \prod_i D_i$, where D_i are the symmetric spaces of G_i. Axiom (S3) implies that if we compose h with each projection $G \to G_i$, the axioms (S1) and (S2) still hold for each h_i. Now assume G is simple.

Let $\mathfrak{g}_0 = \mathrm{Lie}(G^{\mathrm{ad}})$, $\mathfrak{k}_0 = \mathrm{Lie}(K_0)$, and $\mathfrak{p}_0 = \mathfrak{g}_0/\mathfrak{k}_0$. Then $\mathfrak{p}_0$ may be identified with the tangent space $T_p(D)$ at the point $p = eK_0 \in D$. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ denote the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, and $\mathfrak{p}_0$. Axiom $(S1)$ implies that h determines a Hodge decomposition

$$\mathfrak{g} = \mathfrak{g}_{1,-1} \oplus \mathfrak{p}_{0,0} \oplus \mathfrak{g}_{-1,1}$$

Axiom (S2) implies we may choose the maximal compact K_0 in such a way that $\mathfrak{g}_{0,0} = \mathfrak{k}$. Then

$$\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

where $\mathfrak{p}^+ = \mathfrak{g}_{1,-1}$, $\mathfrak{p}^- = \mathfrak{g}_{-1,1}$. We obtain an isomorphism of real vector spaces

$$T_p(D) = \mathfrak{g}_0/\mathfrak{k}_0 \to \mathfrak{g}/\mathfrak{b} = \mathfrak{p}^+.$$

Then the complex structure on $\mathfrak{p}^+$ given by $\mathrm{ad} \circ h_{\mathbb{C}}(i,1)$ transports to a complex structure on $T_p(D)$, and then an almost complex structure on D by the $G^{\mathrm{ad}}(\mathbb{R})$ action, which turns out to be integrable. This makes D a complex manifold, and in fact a hermitian symmetric domain. Conversely, if the symmetric space of G may be equipped with the structure of a hermitian symmetric domain, that structure arises from the choice of a Shimura datum.

Thus the essential content of the axioms is as follows: the *possibility* of the existence of a Shimura datum (G, D) is equivalent to the possibility of equipping the symmetric space of G with the structure of a hermitian symmetric domain. The choice of a Shimura datum then fixes such a hermitian structure.

10.3 Shimura varieties

Let (G, D) be a Shimura datum. For each compact open subgroup K of $G(\mathbb{A}_f)$, the double coset space

$$\mathrm{Sh}(G, D)_K = G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K$$

is canonically a *quasi-projective algebraic variety* over $\mathbb{C}$, at least when K is small enough (and in general a finite quotient of such). The *Shimura variety* attached to the pair (G, D) is the inverse system of quasi-projective varieties

$$\mathrm{Sh}(G,D) = \{\mathrm{Sh}(G,D)_K\}_{K\subset G(\mathbb{A}_f)}$$

as K varies over compact open subgroups of $G(\mathbb{A}_f)$. For each $g \in G(\mathbb{A}_f)$, right-multiplication by g induces a map

$$\mathcal{T}_g : \mathrm{Sh}(G,D)_K \to \mathrm{Sh}(G,D)_{g^{-1}Kg}.$$

These are the *Hecke operators* associated to $g \in G(\mathbb{A}_f)$.

The first fundamental fact about the Shimura variety $\mathrm{Sh}(G,D)$ is that there is a number field $E = E(G,D)$, the *reflex field* of (G,D) such that each $\mathrm{Sh}(G,D)_K$ may be defined over E, with each $\mathcal{T}_g$ a map of algebraic varieties defined over E. Furthermore, this model of $\mathrm{Sh}(G,D)_K$ is uniquely determined by a precise recipe for the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ on a dense set of points in each $\mathrm{Sh}(G,D)_K$.

11 Appendix B: Abelian Varieties

The standard reference for abelian varieties is the book of Mumford [Mum08]. For complex abelian varieties, a comprehensive reference is [BL04].

11.1 Complex abelian varieties

(11.1.1) Let A be an abelian variety over $\mathbb{C}$. Then $A(\mathbb{C})$ is a complex torus, i.e. isomorphic to V/L where V is a complex vector space and $L \subset V$ a lattice. In fact, there is a canonical way to realize $A(\mathbb{C})$ as such a quotient, namely

$$A(\mathbb{C}) \simeq H_1(A,\mathbb{Z})\backslash H^0(A,\Omega^1)^*.$$

Here $H^0(A,\Omega^1)$ denotes the vector space of closed holomorphic differential 1-forms on $A(\mathbb{C})$, and $H^0(A,\Omega^1)^*$ its dual vector space. Given a class $[\gamma] \in H_1(A,\mathbb{Z})$ represented by a closed loop γ, one has a linear map

$$H^0(A,\Omega^1) \to \mathbb{C}, \quad I_{[\gamma]}(\omega) = \int_\gamma \omega.$$

This gives an embedding $H_1(A,\mathbb{Z}) \to H^0(A,\Omega^1)^*$. If $g = \dim A$, then $H_1(A,\mathbb{Z}) \simeq \mathbb{Z}^{2g}$ and $H^0(A,\Omega^1) \simeq \mathbb{C}^g$. The quotient

$$H_1(A,\mathbb{Z})\backslash H^0(A,\Omega^1)^*$$

is then a complex torus.

Let $O \in A(\mathbb{C})$ denote the identity. For each $z \in A(\mathbb{C})$, choose a path $\gamma_z : [0,1] \to A(\mathbb{C})$ such that $\gamma_z(0) = O$, $\gamma_z(1) = z$. Let

$$\Phi : A(\mathbb{C}) \to H^0(A,\Omega^1)^* \quad \Phi(z)\omega = \int_{\gamma_z} \omega.$$

Then the composition of Φ with the quotient map

$$H_0(A,\Omega^1)^* \to H_1(A,\mathbb{Z})\backslash H^0(A,\Omega^1)^*$$

is independent of the choice of the paths γ_z. This gives the canonical uniformization

$$A(\mathbb{C}) \cong H_1(A,\mathbb{Z})\backslash H^0(A,\Omega^1)^*$$

of $A(\mathbb{C})$ as a complex torus. It identifies $\mathrm{Lie}(A)$ with $H^0(A,\Omega^1)^* \cong H_1(A,\mathbb{R})$.

11.1.1 Riemann Forms

Let $X = V/L$ be a complex torus, where V is a complex vector space, and $L \subset V$ a lattice. Then X is an abelian variety if and only if as a complex manifold it is *projective*. Equivalently, X admits an *ample line bundle*. Line bundles on complex tori are classified by Appell-Humbert data.

Definition 11.1. An *Appell-Humbert* datum on a complex torus V/L is a pair (H,α) consisting of:

(1) A hermitian form $H : V \times V \to \mathbb{C}$ such that $E = \mathrm{Im}(iH)|_{L\times L}$ takes values in $\mathbb{Z}$.

(2) $\alpha : L \to \mathbb{C}^1$ is a map satisfying

$$\alpha(x+y) = e^{\pi i E(x,y)}\alpha(x)\alpha(y).$$

It's a fact that given H as in (1), there is always some α satisfying (2). By a theorem of Lefschetz, the line bundle $\mathscr{L}(H,\alpha)$ associated to a pair (H,α) as above is ample if and only if H is positive-definite. It follows that a complex torus V/L is an abelian variety if and only if there exists a positive definite H satisfying (1).

Definition 11.2. A *Riemann form* E on a complex torus V/L is a non-degenerate skew-symmetric bilinear form

$$E : L \times L \to \mathbb{Z}$$

with linear extension $E_{\mathbb{R}} : V \times V \to \mathbb{R}$, such that $E_{\mathbb{R}}(ix,y)$ is positive-definite.

Thus a complex torus V/L is an abelian variety if and only if it admits a Riemann form.

11.2 Moduli of abelian varieties

Let (A,λ) be a complex abelian variety A of dimension g, and $\lambda : A \to A^\vee$ a principal polarization. Up to isomorphism the data (A,λ) is determined by (L,E,J) where:

- L is a free $\mathbb{Z}$-module of rank $2g$,
- $E : L \times L \to \mathbb{Z}$ is a non-degenerate skew-symmetric form,

- $J \in \mathrm{End}_{\mathbb{R}}(L_{\mathbb{R}})$ is a complex structure for which E a Riemann form.

In fact, up to isomorphism, the only variable part of (L, E, J) is the complex structure J. Indeed, by the elementary divisor theorem there exists a $\mathbb{Z}$-basis for L with respect to which the form E is the standard symplectic pairing

$$\mathbb{Z}^{2g} \times \mathbb{Z}^{2g} \to \mathbb{Z}, \quad E_g(x, y) = {}^t y \mathbb{J}_g x, \quad \mathbb{J}_g := \begin{pmatrix} & 1_g \\ -1_g & \end{pmatrix}.$$

By transport of structure, (E, L, J) is isomorphic to $(\mathbb{Z}^{2g}, E_g, J')$ for some complex structure J' acting on $\mathbb{R}^{2g}$. Thus the isomorphism class of (A, λ) is uniquely determined by J'.

Let (L, E, J) be as above, and put $V = L_{\mathbb{R}}$. The possible $J \in \mathrm{End}_{\mathbb{R}}(V)$ are those satisfying

$$J^{-1} = J^* = -J, \quad E_{\mathbb{R}}(Jx, x) > 0, \quad 0 \neq x \in V,$$

where J^* is the adjoint of J with respect to the symplectic pairing. Let $G = \mathrm{Sp}_{2g}$ be identified with the symplectic isometries of V. If $g \in G(\mathbb{R})$, and J satisfies the above properties, then so does gJg^{-1}. This action is *transitive* on the set of all such J. Fixing a particular J, one obtains a surjective map

$$\mathrm{Sp}_{2g}(\mathbb{R}) \longrightarrow \{(A, \lambda) \text{ of dimension } g\}_{/\simeq}, \quad g \mapsto (L, E, gJg^{-1}).$$

The map clearly factors through the quotient by the subgroup K of elements that commute with J. That subgroup is maximal compact, and the quotient $\mathrm{Sp}_{2n}(\mathbb{R})/K$ is isomorphic to the Siegel upper half-space of degree n

$$\mathfrak{H}_n = \{X + iY : X, Y \in \mathrm{Sym}_n(\mathbb{C}), \ Y > 0\}.$$

Then by fixing a complex structure J_0 as a base point, the map above factors through

$$\mathfrak{H}_n \longrightarrow \{(A, \lambda) \text{ of dimension } g\}_{/\simeq}.$$

Accounting for isomorphic pairs (A, λ) then induces a bijection

$$\mathrm{Sp}_{2n}(\mathbb{Z})\backslash\mathfrak{H}_n = \mathrm{Sp}_{2n}(\mathbb{Z})\backslash \mathrm{Sp}_{2n}(\mathbb{R})/K \longleftrightarrow \{(A, \lambda) \text{ of dimension } g\}_{/\simeq}.$$

These are the Siegel modular varieties $\mathscr{A}_g$. Many general properties of Shimura varieties are defined by analogy to properties of $\mathscr{A}_g$, and the proofs often reduce to that case.

11.3 Abelian varieties with multiplication

Let V, $\langle \ , \ \rangle$ be a hermitian space over an imaginary quadratic field E, and G_V the unitary group of G. Its symmetric space D_V is the set of maximal negative definite lines in $V_{\mathbb{R}}$. We will now describe the points of the Shimura variety $\mathrm{Sh}(G_V, D_V)$ in terms of a moduli space of abelian varieties.

Let $L \subset V$ be a full-rank $\mathcal{O}_E$-lattice on which $\langle\ ,\ \rangle$ takes values in $\mathcal{O}_E$. Let δ_E denote the purely imaginary generator of the inverse different $\mathfrak{d}_E^{-1}$, such that $\mathrm{im}(\delta_E) > 0$. Then

$$\mathcal{E} : L \times L \to \mathbb{Z} \quad \mathcal{E}(x, y) = \mathrm{tr}_{E/\mathbb{Q}}(\delta_E \langle x, y \rangle) \tag{25}$$

is a $\mathbb{Z}$-valued skew-symmetric form on L. This construction is a one-to-one correspondence between $\mathfrak{d}_E^{-1}$-valued hermitian forms $\langle\ ,\ \rangle$ on L, and skew-symmetric $\mathbb{Z}$-valued forms $\mathcal{E}$ satisfying

$$\mathcal{E}(x, \overline{a}y) = \mathcal{E}(ax, y).$$

In particular $\langle\ ,\ \rangle$ can be recovered from $\mathcal{E}$.

Let $J \in \mathrm{GL}_\mathbb{R}(V_\mathbb{R})$ be an E-linear complex structure on $V_\mathbb{R}$ with respect to which $\mathcal{E}$ is a Riemann form on L. That means $J^2 = -1$, and:

$$J \in \mathrm{GL}_{E\otimes\mathbb{R}}(V_\mathbb{R}), \tag{26}$$

$$\mathcal{E}(Jx, Jy) = \mathcal{E}(x, y) \text{ for all } x, y \in V_\mathbb{R}, \tag{27}$$

$$\mathcal{E}(Jx, x) > 0 \text{ for non-zero } x. \tag{28}$$

Then

$$A = V(\mathbb{R})/L$$

is a polarized complex torus, hence an abelian variety. Since J commutes with the $\mathcal{O}_E$-action and L is $\mathcal{O}_E$-stable, A obtains an action $\iota : \mathcal{O} \to \mathrm{End}(A)$, and since $\mathcal{E}(ax, y) = \mathcal{E}(x, \overline{a}y)$, the polarization $\lambda : A \to A^\vee$ corresponding to $\mathcal{E}$ is $\mathcal{O}_E$-linear. Thus (A, ι, λ) is a *PEL-triple*.

Properties (26) and (27) are equivalent to $J \in G_V(\mathbb{R})$, with $\nu(J) = 1$. The fixed inclusion $E \subset \mathbb{C}$ induces an isomorphism $E \otimes \mathbb{R} \cong \mathbb{C}$, through which $V_\mathbb{R}$ becomes a $\mathbb{C}$-vector space. As such, $J \in G_V(\mathbb{R})$ is a $\mathbb{C}$-linear map with $J^2 = -1$, so there exists an orthogonal decomposition $V_\mathbb{R} = V^+ \perp V^-$ such that V^+, V^- are the $+i$, $-i$ eigenspaces of J. Let $S = iJ \in \mathrm{End}(V_\mathbb{R})$. If $x \in V^\pm$, then

$$\mathcal{E}(Jx, x) = \mathrm{tr}_{E/\mathbb{Q}}\, \delta_E \mathcal{H}(\pm ix, x) = \pm 2|\delta_E|\mathcal{H}(x, x).$$

The condition (28) is then equivalent to V^- being a maximal negative-definite subspace of $V_\mathbb{R}$, i.e. an element of D. Conversely, any $z \in D$ arises uniquely from a complex structure J_z satisfying (26), (27), (28).

The triples (A, ι, λ) arising from some $z \in D$, are characterized up to isomorphism by the fact that $H_1(A, \mathbb{Z})$, equipped with the skew-symmetric structure induced by λ, is isomorphic to L. Equivalently, there is an isomorphism

$$H_1(A, \mathbb{Q}) \xrightarrow{\sim} V$$

of E-hermitian spaces mapping $H_1(A,\mathbb{Z})$ onto L. Now the Tate-module

$$T_f(A) = \varprojlim_n A[n](\mathbb{C})$$

is canonically isomorphic to $H_1(A,\mathbb{A}_f)$. Then there exists an $\mathbb{A}_f$-linear isomorphism

$$\eta : V_f(A) \xrightarrow{\sim} V(\mathbb{A}_f), \quad V_f(A) = T_f(A) \otimes \mathbb{Q},$$

mapping $T_f(A)$ onto $\widehat{L} = L \otimes \widehat{Z}$ and the form $\langle\ ,\ \rangle_{\mathbb{A}_f}$ on $V(\mathbb{A}_f)$ to the (hermitian) Weil pairing on $T_f(A)$. A *full level-structure* on (A,ι,λ) is such a map η, except that the preservation of forms is allowed to only hold up to a factor $r \in \mathbb{A}_f^\times$.

The group $G(\mathbb{A})$ acts on tuples (A,ι,λ,η) defined over $\mathbb{C}$ as follows. The real points $g_0 \in G(\mathbb{R})$ act on the points $z \in D$ parametrizing (A,ι,λ) (or equivalently by conjugating the complex structure $J_z \mapsto gJ_zg^{-1} \in \mathrm{End}(V_\mathbb{R})$). The finite adelic points $g_f \in G(\mathbb{A}_f)$ act on η by pre-composition, and the lattice L by

$$g \cdot L = g\widehat{L} \cap V(\mathbb{Q}), \quad \widehat{L} = L \otimes \widehat{\mathbb{Z}} \subset V(\mathbb{A}_f).$$

Note that g_f preserves the hermitian form on $V(\mathbb{A}_f)$ only up to scalars, but that $\eta \circ g_f$ is again a full-level structure.

11.4 Abelian Schemes

A first introduction to the topic is the book of Mumford [Mum08].

Let S be a locally noetherian scheme.

Definition 11.3. An abelian scheme over S is a smooth proper group scheme A over S with geometrically connected fibers. A morphism of abelian schemes $A \to B$ over S is an S-group homomorphism.

An abelian variety is the same as an abelian scheme over $S = \mathrm{Spec}(k)$ where k is a field.

Every abelian scheme $A \to S$ has a *dual abelian scheme* $A^\vee \to S$, defined up to isomorphism, that represents the Picard functor $\mathrm{Pic}(A/S)^0$ [FC90]. The functor $A \rightsquigarrow A^\vee$ is a contravariant duality on the category of abelian schemes over S.

The additive group of homomorphisms $\mathrm{Hom}_S(A,B)$ is a free $\mathbb{Z}$-module of finite rank. It is standard notation to write

$$\mathrm{Hom}_S^0(A,B) = \mathrm{Hom}_S(A,B) \otimes \mathbb{Q}.$$

The isogeny category of abelian schemes over S is the one obtained by replacing the hom groups $\mathrm{Hom}_S(A,B)$ with $\mathrm{Hom}_S^0(A,B)$. A morphism $A \to B$ of abelian schemes is an isogeny if and only if its image in the isogeny category is an isomorphism.

A *polarization* on A is a symmetric isogeny $\lambda : A \to A^\vee$ that corresponds to an ample line bundle on A. A polarization is called *principal* if it's an isomorphism. See notes of Brian Conrad titled "polarizations" for a formulation in terms of correspondences. If $\lambda : A \to A^\vee$ is a polarization, let λ^{-1} denote its inverse in the isogeny category. The map $\mathrm{End}^0_S(A) \to \mathrm{End}^0_S(A)$, $r \mapsto r^\dagger$ given by

$$r^\dagger = \lambda^{-1} \circ r^\vee \circ \lambda$$

is called the Rosati involution on $\mathrm{End}^0_S(A)$ induced by λ. If λ is principal, the Rosati involution is well-defined on $\mathrm{End}_S(A)$.

Let R be a (possibly non-commutative) ring, free of finite rank over $\mathbb{Z}$. An R-action on A is a ring homomorphism $\iota : R \to \mathrm{End}_S(A)$. If (A, ι), $(B, \jmath)$ are abelian schemes with R-action, a morphism $\phi : A \to B$ is called R-linear if

$$\phi \circ \iota(a) = \jmath(a) \circ \phi, \quad \text{for all } a \in R.$$

Suppose that R is equipped with an involution $a \mapsto a^*$. If (A, ι) is an abelian scheme with R-action, the dual abelian scheme $A^\vee$ has the dual R-action

$$\iota^\vee(a) = \iota(a^*)^\vee.$$

Let $E/\mathbb{Q}$ be quadratic imaginary, and S a locally noetherian scheme over $\mathrm{Spec}\,\mathcal{O}_E$. Consider $\mathcal{O}_E$ as a ring with involution $a \mapsto a^*$ given by complex conjugation. The following definition is convenient, but not standard.

Definition 11.4. A PEL-triple over S is the data $\underline{A} = (A, \iota, \lambda)$ consisting of:

- An abelian scheme A over S,
- A ring homomorphism $\iota : \mathcal{O}_E \to \mathrm{End}_S(A)$,
- An $\mathcal{O}_E$-linear polarization $\lambda : A \to A^\vee$.

A PEL-triple (A, ι, λ) will be called principal if λ is a principal polarization.

Let $\underline{A}_1 = (A_1, \iota_1, \lambda_1)$, $\underline{A}_2 = (A_2, \iota_2, \lambda_2)$ be PEL-triples over S. A morphism of PEL-triples $\underline{A}_1 \to \underline{A}_2$ is an $\mathcal{O}_E$*-linear* morphism of abelian schemes $\phi : A_1 \to A_2$ that is *compatible with polarizations*, meaning $\phi^*(\lambda_2) = \lambda_1$, where

$$\phi^*(\lambda_2) := \phi^\vee \circ \lambda_2 \circ \phi : A_1 \to A_1^\vee. \tag{29}$$

12 Appendix C: Deformation Theory

Here we collect some standard results on deformation theory of abelian schemes and p-divisible groups.

12.1 Ridigity

An essential property of abelian schemes is the rigidity of deformations of their homomorphisms. We will treat this in some detail, following the standard reference, which is [MFK94, Ch. 6].

A basic example of rigidity is as follows. Let E be an elliptic curve with CM by the ring of integers $\mathcal{O}_K$ of a quadratic imaginary field K. Then E and its full ring of endomorphisms are defined over $S = \operatorname{Spec} \mathcal{O}_H[N^{-1}]$, where H is the Hilbert class field of K, and N is a product of finitely many (bad reduction) primes. For any prime ideal $\mathfrak{p}$ of H outside N, with $k(\mathfrak{p}) = \mathcal{O}_H/\mathfrak{p}$, we obtain an elliptic curve $E_{k(\mathfrak{p})}$ over $\operatorname{Spec} k(\mathfrak{p})$ via base change. Rigidity here says the reduction map

$$\mathcal{O}_K = \operatorname{End}_S(E) \to \operatorname{End}_{k(\mathfrak{p})}(E_{k(\mathfrak{p})}), \quad \phi \mapsto \phi \times_S \operatorname{Spec} k(\mathfrak{p})$$

is *injective.* In particular, for all good reduction primes $\mathfrak{p}$, the endomorphism ring $\operatorname{End}_k(E_k)$ *contains* $\mathcal{O}_K$ (and is strictly larger half the time). A similar injectivity under base change holds for homomorphism groups of arbitrary abelian schemes over a connected locally noetherian base. It is a consequence of the following ([MFK94, Prop 6.1]).

Proposition (12.1.1) (Rigidity Lemma)**.** *Let S be a connected locally noetherian scheme, and*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

a diagram of S-schemes, where p is flat, q is separated, and $p_*(\mathscr{O}_X) = \mathscr{O}_S$. Furthermore, suppose that one of the following holds:

(1) X/S admits a section $\epsilon : S \to X$, and S is an artinian scheme.

(2) X/S admits a section $\epsilon : S \to X$, and p is a closed map.

(3) p is proper.

If there exists $s \in S$, such that $f(X_s)$ is a single point, then Y/S admits a section $\eta : S \to Y$ such that $f = \eta \circ p$.

A number of fundamental facts about abelian schemes are a consequence of the rigidity lemma. See Mumford's book for several of them in a row. For the purpose of deformation theory, the most directly relevant is the following.

Corollary (12.1.2). *Let A, B denote abelian schemes over a base S, and $f, g : A \to B$ two homomorphisms of abelian schemes. Let $T \hookrightarrow S$ be a closed immersion. If $f_T = g_T$, then $f = g$.*

Proof. We can assume T is a closed point of S by pre-composing $T \hookrightarrow S$ with $t \hookrightarrow T$ for a closed point $t \in T$. Then we can apply the rigidity lemma to $h = f - g$. □

In other words, with notation as in the corollary, there is *at most one extension* of $f_T : A_T \to B_T$ to a homomorphism $f : A \to B$.

12.2 Properties of Moduli Functors

The functor of points approach is especially suited to the study of moduli problems.

Definition 12.1. Let S_0 be a base scheme. A functor

$$F : (\mathrm{Sch}_{/S_0})^{\mathrm{op}}_{fppf} \longrightarrow \mathrm{Groupoids}$$

is *locally of finite presentation* if for any inverse system of affine schemes $\{S_\alpha\}$ in $\mathrm{Sch}_{/S_0}$, the natural functor

$$\varinjlim F(S_\alpha) \to F(\varprojlim S_\alpha)$$

is an equivalence of categories.

This definition of 'locally of finite presentation' agrees with all the other ones for schemes, algebraic spaces, stacks, functors, etc. Suppose that S is finite type over a field or an excellent Dedekind domain. If

$$\mathcal{M} : \mathrm{Sch}^{\mathrm{op}}_{/S} \to \mathrm{Groupoids}$$

is representable by an algebraic stack locally of finite type over S, then it is certainly locally of finite presentation. In that case $\mathcal{M} \to S$ is smooth, unramified, or étale, if and only if it is formally smooth, unramified, or étale, respectively. The verification of these formal properties is often more convenient from the functorial point of view, and reduces to meaningful facts about the objects being parametrized.

12.3 Deformation Functors

12.4 Serre-Tate Theory

Definition 12.2. A **nilpotent thickening** is a closed immersion $S_0 \hookrightarrow S$ given by a locally nilpotent sheaf of ideals $I \subset \mathcal{O}_S$.

Let $S_0 \hookrightarrow S$ be a closed immersion induced by a coherent and locally nilpotent sheaf of ideals $I \subset \mathcal{O}_S$.

Let Nilp_{S_0} be the category whose objects are nilpotent thickenings $S_0 \hookrightarrow S$. If (S_0, S_1), (S_0, S_2) are two objects in Nilp_{S_0}, corresponding to closed immersions $\iota_1 : S_0 \hookrightarrow S_1$, $\iota_2 : S_0 \hookrightarrow S_2$, the morphisms $f : (S_0, S_1) \to (S_0, S_2)$ are taken to be closed immersions $f : S_1 \hookrightarrow S_2$ such that $f \circ \iota_1 = \iota_2$.

Let AV_S denote the category of abelian varieties over S, and BT_S the category of p-divisible groups over S, so that there's a functor

$$\mathrm{AV}_S \to \mathrm{BT}_S, \quad A \mapsto A[p^\infty].$$

(this section to be completed)

13 Appendix D: Select Annotated References

- F. Hirzebruch and D. Zagier. Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus. *Invent. Math.*, 36:57–113, 1976

  This is the first instance of the modularity of special cycles in the literature. Here the ambient Shimura variety is a Hilbert modular surface, which may have internal singularities. Much of the technical difficulty is dealing with compactifications and resolutions of singularities.

- Benedict H. Gross and Kevin Keating. On the intersection of modular correspondences. *Invent. Math.*, 112(2):225–245, 1993

  This is the first work where arithmetic intersection numbers of special cycles occur. It is the simplest case of what is now part of the Kudla Program. In fact several of the key results in the field can be reduced to the calculations of Gross-Keating.

- Stephen S. Kudla. Central derivatives of Eisenstein series and height pairings. *Ann. of Math. (2)*, 146(3):545–646, 1997

  This is the essential work at the beginning of the Kudla Program, relating central derivatives of Eisenstein Series with quantities from arithmetic intersection theory.

- Stephen S. Kudla and Michael Rapoport. Height pairings on Shimura curves and p-adic uniformization. *Invent. Math.*, 142(1):153–223, 2000

  In this work, Kudla and Rapoport establish a relation between intersection numbers of formal arithmetic cycles on the Drinfeld upper half-space and derivatives of representation densities of quadratic forms. This is an essential local ingredient for [KRY99], which deals with the global aspect.

- Stephen S. Kudla, Michael Rapoport, and Tonghai Yang. On the derivative of an Eisenstein series of weight one. *Internat. Math. Res. Notices*, (7):347–385, 1999

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