Deterministic Classical Information

A physical device X has some finite, non-empty set of states.

$\{H, T\}$ $\quad \Sigma = \{0, 1\}$ $\quad \{a, b, c, d\}$

How does the state of the system change?

$f : \{0,1\} \to \{0,1\}$

x	f_1	f_2	f_3	f_4
0	0	0	1	1
1	0	1	0	1

$\Sigma^2 = \{00, 01, 10, 11\}$ **What about multiple such devices?**

$|\Sigma^n| = 2^n$

$\underbrace{\underbrace{2}_{} \times \underbrace{2}_{} \times \dots \times \underbrace{2}_{}}_{n}$

2^n possibilities $\leftarrow$

x_1	x_2	$\dots$	x_n	f
0	0	$\dots$	0	0
0	0	$\dots$	1	1
		$\vdots$		$\vdots$
1	1	$\dots$	1	0

$f : \{0,1\}^n \longrightarrow \{0,1\}$

$N = 2^n$

$2^N = 2^{2^n}$ # of functions

What if we have Incomplete Information?

What if we have Incomplete Information?

Probability

Probability (State of $X=0$) $= p$

Probability (State of $X=1$) $= 1-p$

$$\begin{array}{cc} & \hat{v} \\ \hat{v} = & \begin{pmatrix} p \\ 1-p \end{pmatrix} \end{array} \begin{array}{c} \\ \to v_0 \\ \to v_1 \end{array}$$

Note: When we look at X we do not see $\hat{v}, p$

but rather some state $\sigma \in \Sigma$ with probability v_σ

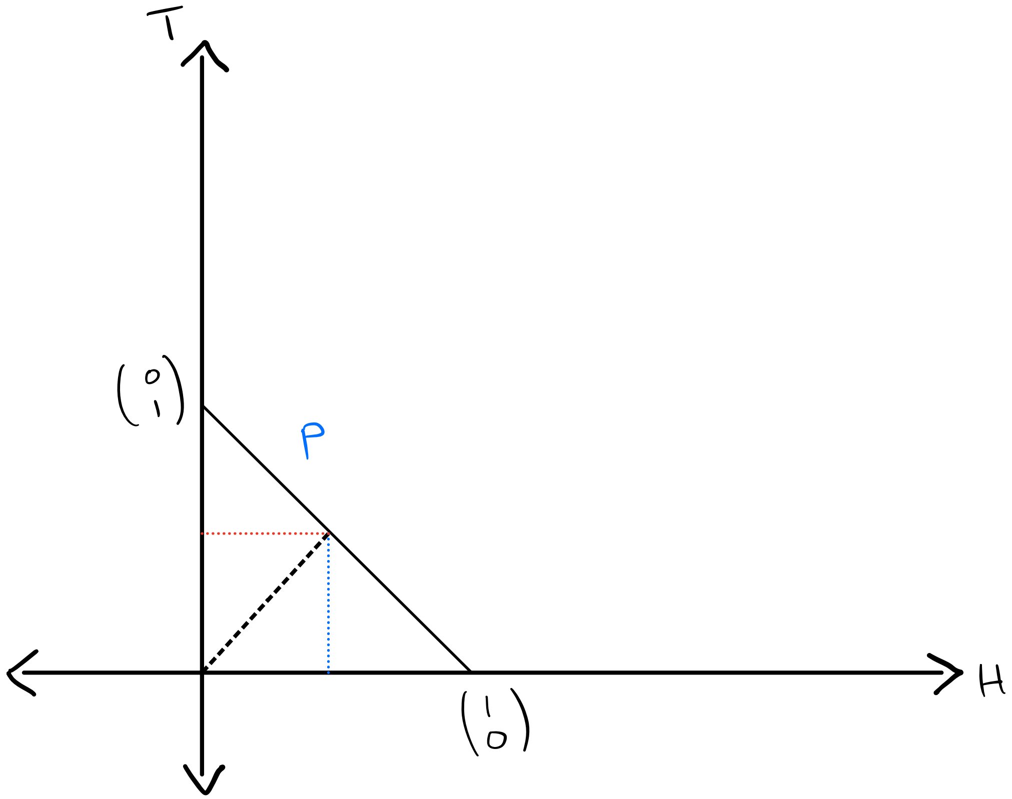
What if we have Incomplete Information?

$$\hat{v} = \begin{pmatrix} p \\ 1-p \end{pmatrix}$$

$$= \underbrace{p\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (1-p)\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{convex combination}}$$

$$p + (1-p) = 1$$

$$p \geq 0$$



What if we have Incomplete Information?

$p(H|T)$

$p(H|H)$

$$\begin{array}{cc} & \begin{array}{cc} H & T \end{array} \\ \begin{array}{c} H \\ T \end{array} & \begin{pmatrix} \boxed{a} & \boxed{b} \\ \boxed{c} & d \end{pmatrix} \end{array} \begin{pmatrix} p \\ 1-p \end{pmatrix} \begin{array}{c} H \\ T \end{array} = \underbrace{\begin{pmatrix} ap + b(1-p) \\ cp + d(1-p) \end{pmatrix}}_{\text{This should be a probability vetor.}}$$

$p(T|H)$

$p(T|T)$

A classical transformation is <u>stochastic</u>

1) all entries ≥ 0

2) Sum of columns $= 1$

What if we have Incomplete Information?

Deterministic Ops as Matrices

$$A_0 = \begin{array}{c c} & \begin{array}{cc} H & T \end{array} \\ \begin{array}{c} H \\ T \end{array} & \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \end{array}, \quad A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

always 'H' identity NOT always 'T'

e.g., $A_3 \begin{pmatrix} p \\ 1-p \end{pmatrix} = \begin{pmatrix} 1-p \\ p \end{pmatrix}$

How does the State of the System change?

Probabilist op.

$$B = \sum_{i=0}^{3} p_i A_i \quad , \quad p_i \geq 0, \ \sum p_i = 1$$

e.g. $F = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} p \\ 1-p \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

How does the State of the System change?

Qs. We flip a coin $\begin{array}{c c} T & \\ H & \end{array}\!\!\begin{pmatrix} p \\ q \end{pmatrix}$. $\begin{cases} \text{if we get a head, we flip again.} \\ \text{if we get a tail, we turn the coin over, i.e., we make it a head.} \end{cases}$

$$\begin{array}{c c} & \begin{array}{c c} T & H \end{array} \\ \begin{array}{c} T \\ H \end{array} & \begin{pmatrix} & p \\ 1 & q \end{pmatrix} \end{array} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} pq \\ p+q^2 \end{pmatrix}$$

Multiple Devices with Incomplete Information

Coin X_1 $\begin{pmatrix} p \\ q \end{pmatrix} \begin{matrix} H \\ T \end{matrix}$ Coin X_2 $\begin{pmatrix} r \\ s \end{pmatrix} \begin{matrix} H \\ T \end{matrix}$

Tensor Product

$$\begin{pmatrix} p \\ q \end{pmatrix} \otimes \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} pr \\ ps \\ qr \\ qs \end{pmatrix} \begin{matrix} HH \\ HT \\ TH \\ TT \end{matrix}$$

Note Not all 4-dim probability vectors can be written in the above form.

$$p(x_1 x_2) \neq p(X_1)\, p(X_2)$$

e.g. $\frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{matrix} HH \\ \\ \\ TT \end{matrix} \neq \begin{pmatrix} p \\ q \end{pmatrix} \otimes \begin{pmatrix} r \\ s \end{pmatrix}$

$$\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{matrix} HH \\ \\ TH \\ \\ \end{matrix}$$

dependent
correlated

Multiple Devices with Incomplete Information

$$A \otimes B = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \otimes \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1 B & a_2 B \\ a_3 B & a_4 B \end{pmatrix}$$

$$= \begin{pmatrix} \begin{matrix} a_1 b_1 & a_1 b_2 \\ a_1 b_3 & a_1 b_4 \end{matrix} & \begin{matrix} a_2 b_1 & a_2 b_2 \\ a_2 b_3 & a_2 b_4 \end{matrix} \\ a_3 B & a_4 B \end{pmatrix}$$

There exist 4×4 transformations $C \neq A \otimes B$.

Multiple Devices with Incomplete Information

$$
\begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array}
$$

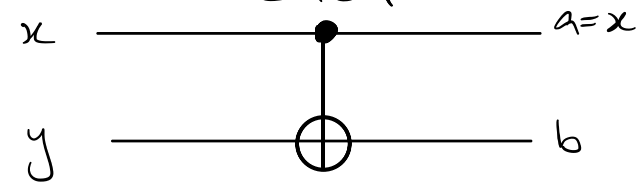
$$
\left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\}
$$

Controlled-NOT

CNOT

control

target



$b = x \oplus y$

x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

$$
N := \begin{array}{c c} & \overbrace{\begin{array}{cccc} 00 & 01 & 10 & 11 \end{array}}^{xy} \\ \underbrace{\begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array}}_{ab} & \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \end{array}
$$

$N \neq A \otimes B$

How can a Biased Coin Simulate a Fair Coin?

Assume we have a coin $\begin{pmatrix} p \\ 1-p \end{pmatrix}$, $p \neq 0$, $p \neq \frac{1}{2}$.

How can we simulate a fair coin?

Quantum Bit (Qubit)

Modelling a quantum system X with $\Sigma = \{0,1\}$

Central Claim of Quantum Physics

To describe an isolated quantum system we need to give an amplitude ($\alpha \in \mathbb{C}$) for each possible state $\sigma \in \Sigma$.

Classical

$$\hat{v} = \underbrace{p\begin{pmatrix}1\\0\end{pmatrix} + (1-p)\begin{pmatrix}0\\1\end{pmatrix}}_{\text{convex combination}}$$

$p \geq 0$

Quantum

$$\hat{v} = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix}$$

$\alpha, \beta \in \mathbb{C}$

What are the restrictions on α, β

Moving forward assume $\alpha, \beta \in \mathbb{R}$

Getting Probabilities from Amplitudes

Born Rule

Probability to observe a particular outcome,

e.g., Prob(X is in state 0) is given by

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longrightarrow \begin{matrix} \alpha^2 \longrightarrow \Pr(X=0) \\ \beta^2 \longrightarrow \Pr(X=1) \end{matrix}$$

assuming
$\alpha, \beta \in \mathbb{R}$

$$\hat{v} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\alpha^2 + \beta^2 = 1$$

Quantum Bit (Qubit)

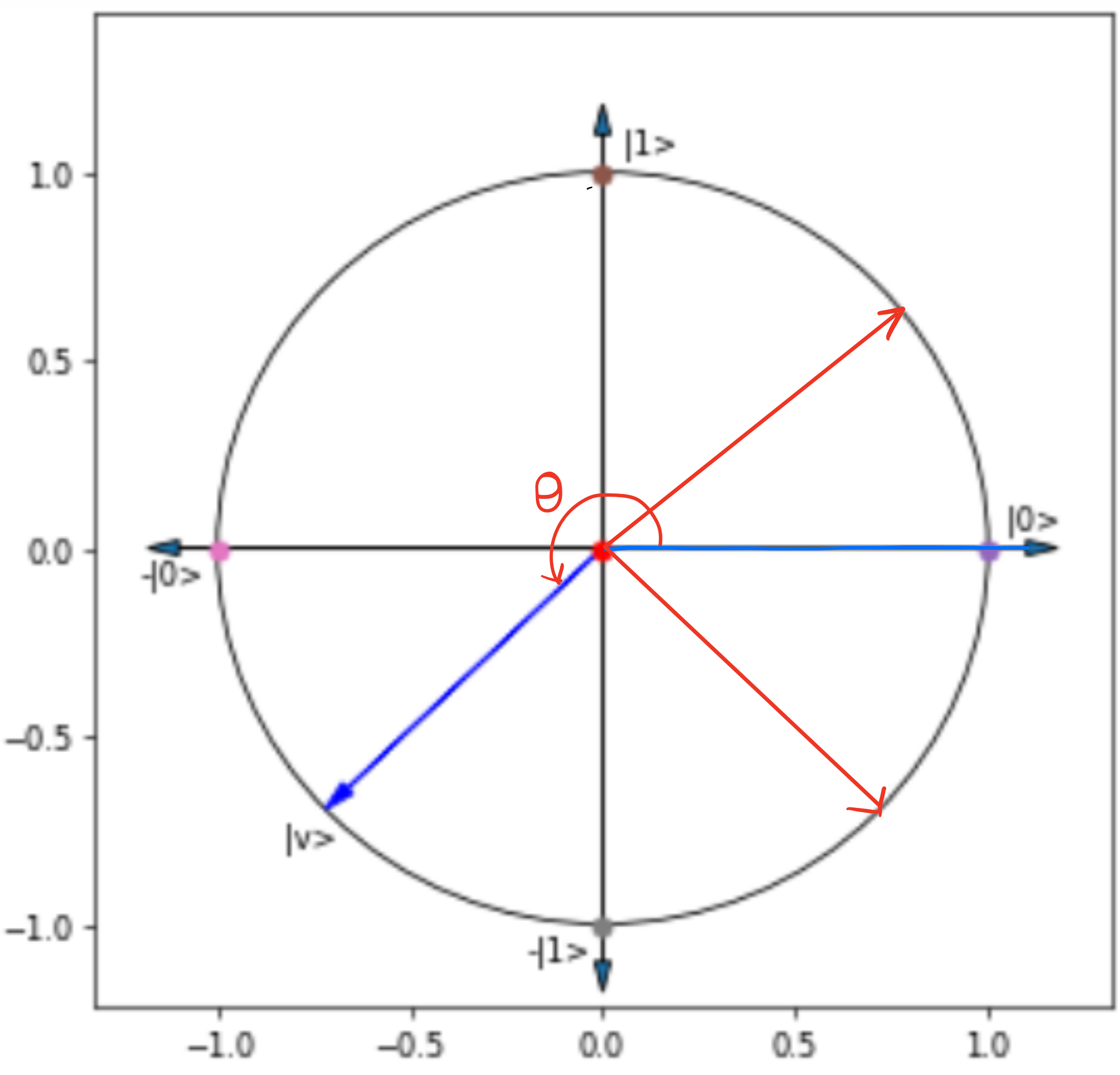
For real amplitudes, we can describe the state as

$$\hat{v} = \underbrace{\cos\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{superposition}}, \quad \theta \in [0, 2\pi)$$

$$= \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\hat{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \hat{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Qubit on a Unit Circle



Quantum Operations

$U \hat{v} = \hat{w}$

$H^T = H$

Symmetric

$U \hat{v} = \hat{w}$

$$U \begin{pmatrix} \vdots \\ \alpha_i \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \beta_i \\ \vdots \\ \vdots \end{pmatrix}$$

$$\sum_i \alpha_i^2 = 1 = \sum_i \beta_i^2$$

i) $U^T U = \mathbb{1} \rightarrow$ identity , $U^T = U^{-1}$

ii) U^T is the inverse of U,

U is reversible

In general, Unitary transformations.

Quantum Operations

Ket Notation

$\underbrace{|\cdot\rangle}_{\text{ket}} \equiv$ column vector

$|0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hat{v}$

$|1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\underbrace{\langle\cdot|}_{\text{bra}} \equiv$ row vector

$\langle 0| \equiv \begin{pmatrix} 1 & 0 \end{pmatrix}$

$\langle 1| \equiv \begin{pmatrix} 0 & 1 \end{pmatrix}$

$\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow \alpha|0\rangle + \beta|1\rangle = |\psi\rangle$

$\gamma|0\rangle + \delta|1\rangle = |\varphi\rangle$

$\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \longrightarrow \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$

$\underbrace{\langle\psi|\varphi\rangle}_{\hat{v}\cdot\hat{w}} \longrightarrow \langle\psi|\,|\varphi\rangle$

$(\alpha\langle 0| + \beta\langle 1|)(\gamma|0\rangle + \delta|1\rangle)$

$(\alpha\gamma\langle 0|0\rangle + \alpha\delta\langle 0|1\rangle + \beta\gamma\langle 1|0\rangle + \beta\delta\langle 1|1\rangle)$

$= \alpha\gamma + \beta\delta$

$\{|0\rangle, |1\rangle\} \longrightarrow$ Computational Basis

Standard

Quantum Operations

Hadamard Matrix

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H^T H = \mathbb{1}$$

$$H^T = H$$

$$F = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

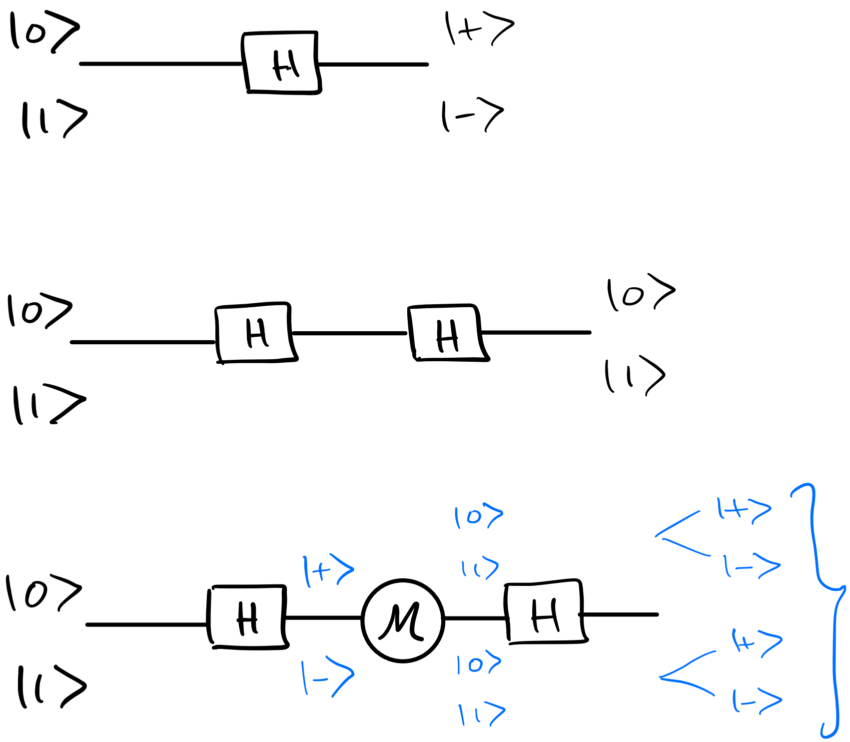
$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle \right) = |+\rangle$$

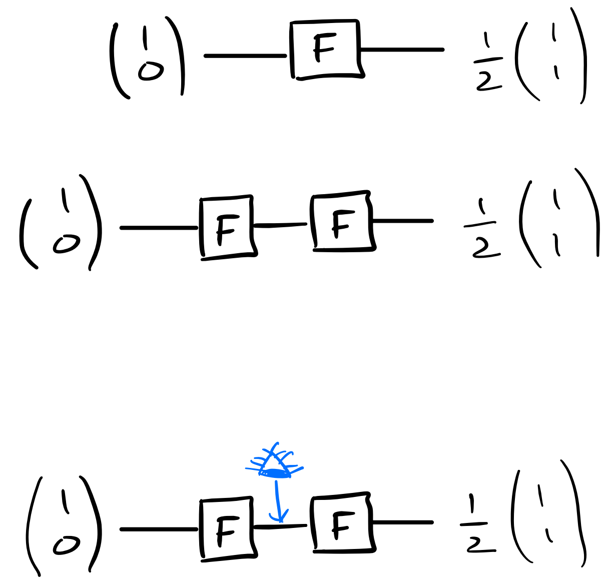
$$\frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$H|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left(|0\rangle - |1\rangle \right) = |-\rangle$$

$\{ |+\rangle, |-\rangle \}$ Hadamard basis

Classical vs Quantum Coin Flipping



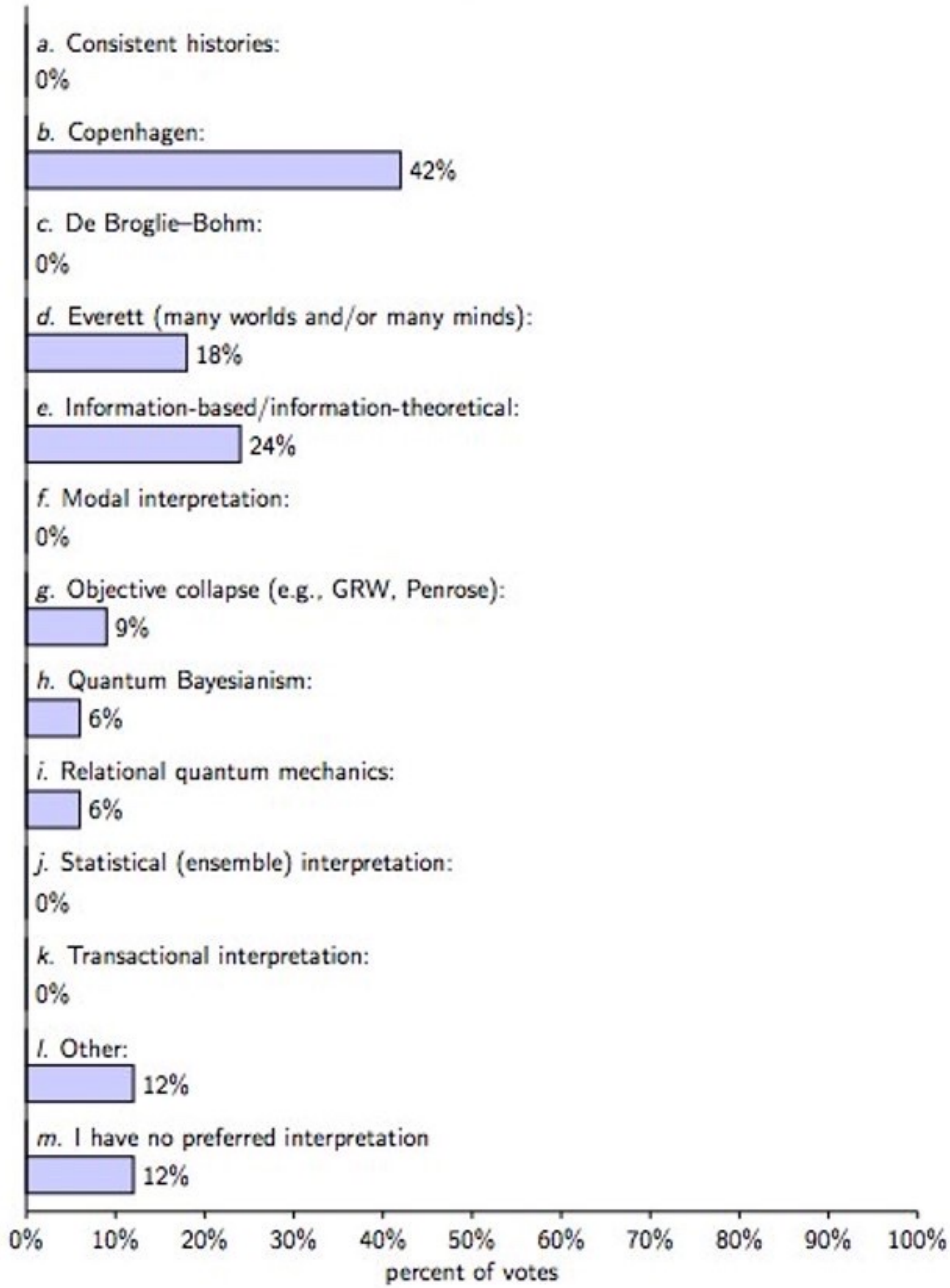


$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$F = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$F\begin{pmatrix} p \\ 1-p \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Question 12: What is your favorite interpretation of quantum mechanics?



Multiple Qubits

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, $|\varphi\rangle = \gamma|0\rangle + \delta|1\rangle$

$$|\psi\rangle \otimes |\varphi\rangle = |\psi\rangle|\varphi\rangle = |\psi\varphi\rangle = |\chi\rangle$$
$$= (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$$
$$= \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle \equiv \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix}$$

When does a quantum state admit an efficient simulation on a quantum computer?

Assume a system of n quantum bits.

The state at any given time should be of form

$$|\psi\rangle = \underbrace{(\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle) \otimes \cdots \otimes (\alpha_n|0\rangle + \beta_n|1\rangle)}_{\text{Separable}}$$

QBronze Summary

n-qubit Quantum State $|\psi\rangle = \sum_{i=0}^{2^n-1} \alpha_i |i\rangle \ , \ \sum_{i=0}^{2^n-1} \alpha_i^2 = 1, \ \alpha_i \in \mathbb{R}$

Unitary Evolution $U|\psi\rangle = |\varphi\rangle = \sum_{i=0}^{2^n-1} \beta_i |i\rangle \ , \ \sum_{i=0}^{2^n-1} \beta_i^2 = 1, \ \beta_i \in \mathbb{R}$ $\quad U^T U = \mathbb{1}$

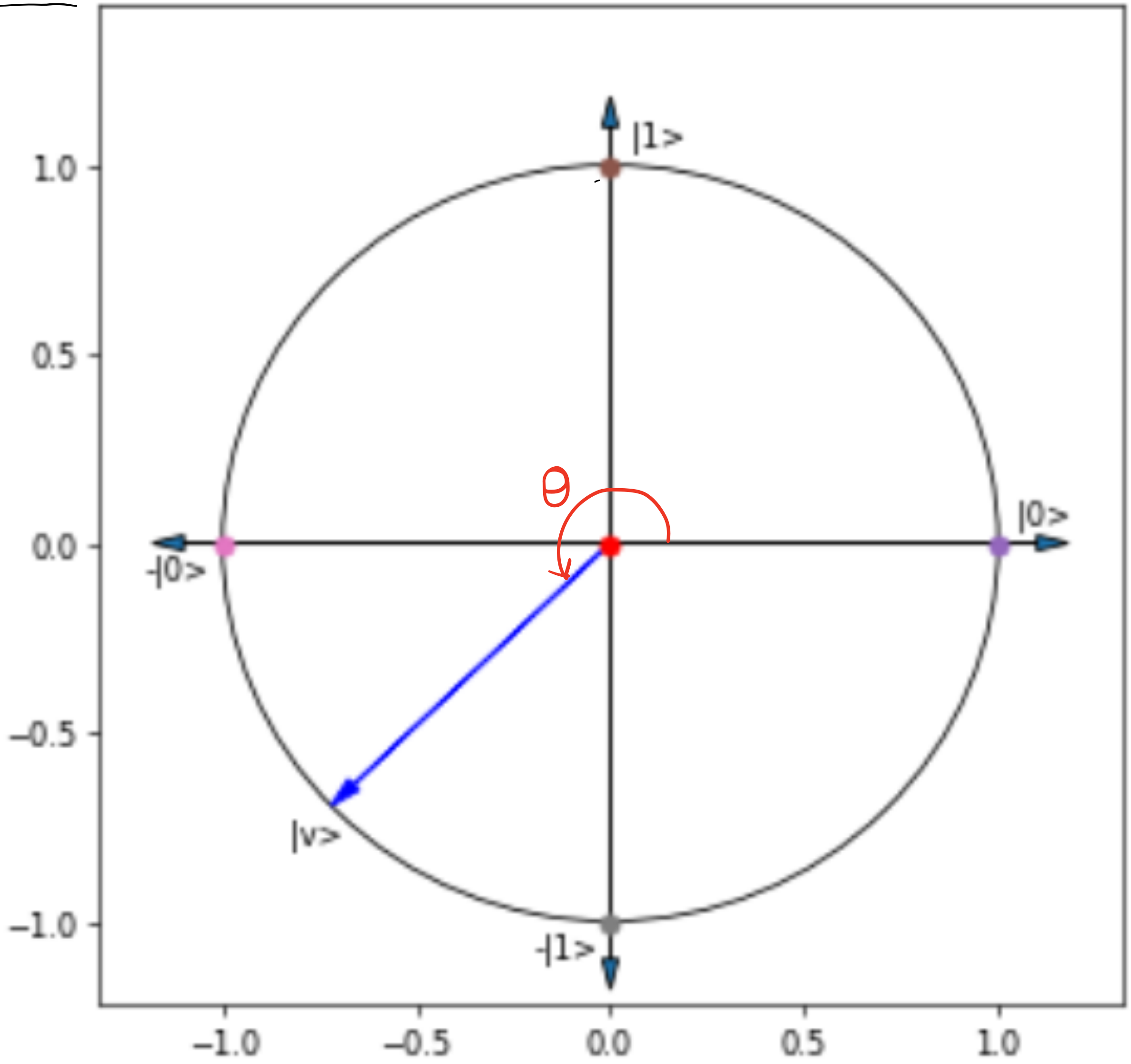
Measurement Probability to observe particular outcome i on measuring $|\psi\rangle$ is given by α_i^2

Qubit on a Unit Circle

$|\psi\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$



$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

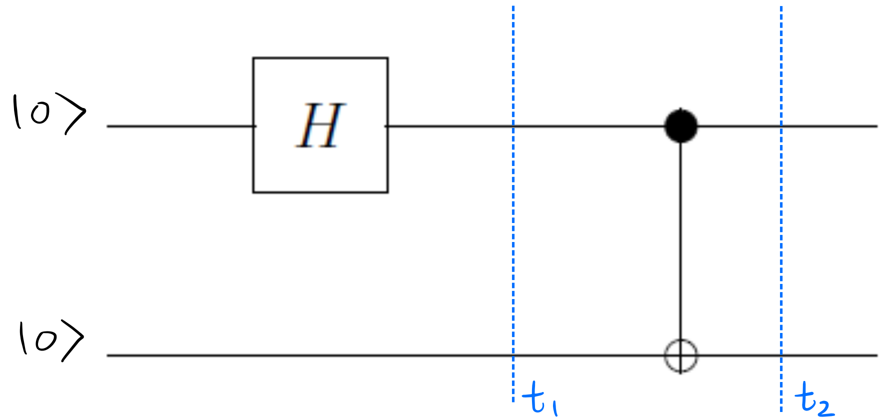
$X|0\rangle = |1\rangle$
$X|1\rangle = |0\rangle$

$X|\psi\rangle = \cos\theta\, X|0\rangle + \sin\theta\, X|1\rangle$
$= \cos\theta|1\rangle + \sin\theta|0\rangle$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$Z|0\rangle = |0\rangle$
$Z|1\rangle = -|1\rangle$

Preparing a Bell State



$$H|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = |+\rangle$$

$$\begin{aligned} CNOT\ |00\rangle &= |00\rangle \\ |01\rangle &= |01\rangle \\ |10\rangle &= |11\rangle \\ |11\rangle &= |10\rangle \end{aligned}$$

$$|00\rangle \xrightarrow{H\otimes \mathbb{1}} |+\rangle|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|0\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right)$$

$$\xrightarrow{CNOT} \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = |\varphi^+\rangle$$ a Bell state

Entangled!

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$$

$$\{|\varphi^+\rangle, |\varphi^-\rangle, |\psi^+\rangle, |\psi^-\rangle\}$$

Entanglement vs Perfect Correlation

$$\hat{V} = \frac{1}{2}\begin{array}{c}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\end{array}\begin{array}{l}00\\01\\10\\11\end{array}$$

Perfect
Classical
Correlation

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{array}{c}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\end{array}\begin{array}{l}00\\01\\10\\11\end{array}$$

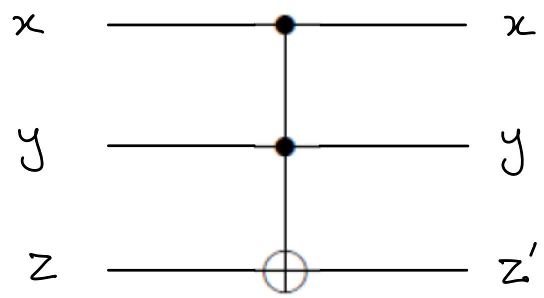
Maximally
Entangled
State

Toffoli Gate

$$\left.\begin{array}{l} |000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \end{array}\right\} \text{identity}$$

$|110\rangle \longrightarrow |111\rangle$

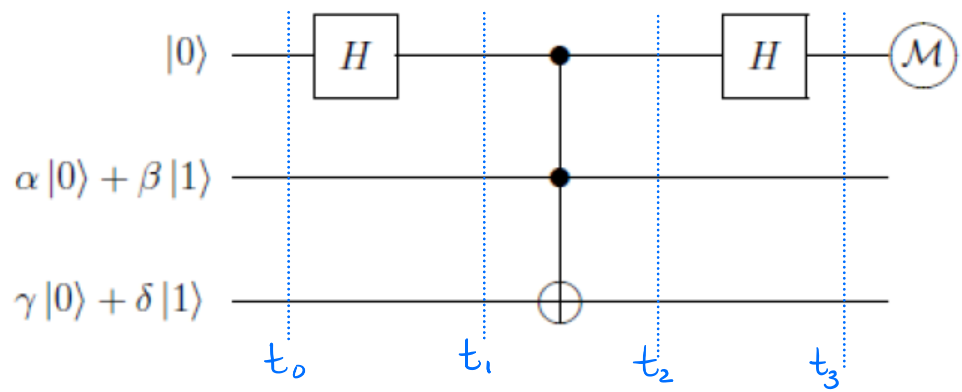
$|111\rangle \longrightarrow |110\rangle$



CCNOT

Toffoli

Circuit Evaluation



$$|0\rangle(\alpha|0\rangle + \beta|1\rangle)(\gamma|0\rangle + \delta|1\rangle)$$

$$\xrightarrow{H\otimes 1\!\!1 \otimes 1\!\!1} \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)(\alpha|0\rangle + \beta|1\rangle)(\gamma|0\rangle + \delta|1\rangle)$$

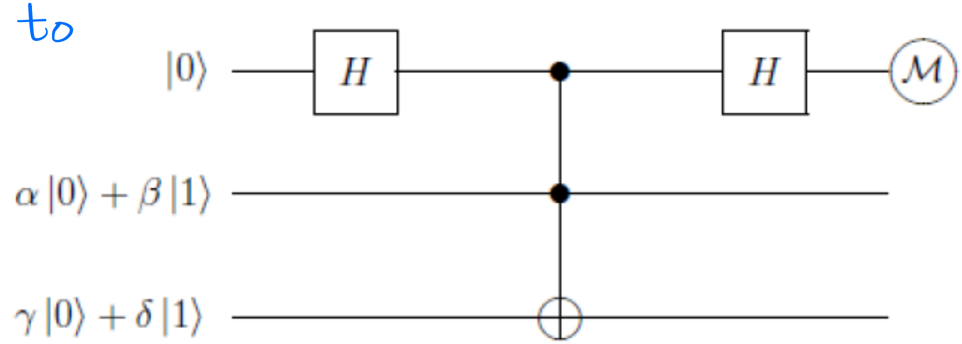
$$= \frac{1}{\sqrt{2}}(\alpha|00\rangle + \beta|01\rangle + \alpha|10\rangle)(\gamma|0\rangle + \delta|1\rangle) + \frac{\beta}{\sqrt{2}}|11\rangle(\gamma|0\rangle + \delta|1\rangle)$$

$$\xrightarrow{CCNOT} \frac{1}{\sqrt{2}}(\alpha|00\rangle + \beta|01\rangle + \alpha|10\rangle)(\gamma|0\rangle + \delta|1\rangle) + \frac{\beta}{\sqrt{2}}|11\rangle(\gamma|1\rangle + \delta|0\rangle)$$

$$\xrightarrow{H\otimes 1\!\!1 \otimes 1\!\!1} \frac{1}{2}(\alpha|00\rangle + \alpha|10\rangle + \beta|01\rangle + \beta|11\rangle + \alpha|00\rangle - \alpha|10\rangle)(\gamma|0\rangle + \delta|1\rangle) + \frac{1}{2}(\beta|01\rangle - \beta|11\rangle)(\gamma|1\rangle + \delta|0\rangle)$$

$$= \frac{1}{2}\left[2\alpha\gamma|000\rangle + 2\alpha\delta|001\rangle + \beta(\gamma+\delta)|010\rangle + \beta(\gamma+\delta)|011\rangle + \beta(\gamma-\delta)|110\rangle + \beta(\delta-\gamma)|111\rangle\right]$$

Circuit Evaluation



What is probability for first qubit to be in state |1>?

$$\frac{1}{2}\left(\beta^2(\gamma-\delta)^2\right)$$

Given that first qubit is measured in state |1>, what is the probability distribution for second qubit?

with prob. 1, second qubit is in state 1.

Does there exist a choice for α, β, γ & δ for which first qubit is measured in state |1> with probability 1?

$$\alpha = 0,\ \beta = 1,\ \gamma = \frac{1}{\sqrt{2}},\ \delta = -\frac{1}{\sqrt{2}} \longrightarrow \frac{1}{2}\left(\beta^2(\gamma-\delta)^2\right) = 1$$

$$\frac{1}{2}\Big[2\alpha\gamma|000\rangle + 2\alpha\delta|001\rangle + \beta(\gamma+\delta)|010\rangle + \beta(\gamma+\delta)|011\rangle + \beta(\gamma-\delta)|110\rangle + \beta(\delta-\gamma)|111\rangle\Big]$$