Duality for LPs

A matching min-cut solution provides proof of optimality for max-flow found by Ford Fulkerson.
Similarly, for any LP, we can find a related dual problem. How do we find it in general?
e.g.,

$$\begin{aligned} \max\quad & x_1 + 6x_2 + 13x_3 \\ \text{s.t.}\quad & x_1 \le 200 \quad (1) \\ & x_2 \le 300 \quad (2) \\ & x_1 + x_2 + x_3 \le 400 \quad (3) \\ & x_2 + 3x_3 \le 600 \quad (4) \\ & x_1, x_2, x_3 \ge 0 \\ & (5)\ \ (6)\ \ (7) \end{aligned}$$

Given a solution $(0, 300, 100)$ how could we prove its optimal? Note that if we take the following linear combination of the constraints:

$(2) + (3) + 4\,(4)$

we obtain

$x_1 + 6x_2 + 13x_3 \le 3100.$

Note that the LHS is exactly the objective function we want to maximize & the RHS gives us a bound on it. This bound is saturated by our solution & so the solution must be optimal. Any addition to x_i's will violate this bound. But, how do we figure out the coefficients $(0,1,1,4)$ for the linear combination in general? And how do we guarantee there isn't a tighter bound possible?

e.g.,

$$\begin{aligned} \max\quad & x_1 + 6x_2 \\ & x_1 \le 200 \quad (1) \\ & x_2 \le 300 \quad (2) \\ & x_1 + x_2 \le 400 \quad (3) \\ & x_1, x_2 \ge 0 \\ & (4)\ \ (5) \end{aligned}$$

Optimal solution is $(100, 300)$
Objective value is $1900.$

$(1) + 6\,(2) \longrightarrow x_1 + 6x_2 \le 2000$

This linear combination doesn't match the objective value of our solution. We can find a better combination.

$0\,(1) + 5\,(2) + (3) \longrightarrow x_1 + 6x_2 \le 1900$

Let us write a general linear combination of the constraints (other than positivity)

Multiplier	Inequalities
y_1	$(x_1 \le 200)$
y_2	$(x_2 \le 300)$
$+\ y_3$	$(x_1 + x_2 \le 400)$

Note that the y_i's must be ≥ 0, otherwise the inequality sign will flip.

Need LHS to equal objective function. $\longleftarrow$ $(y_1+y_3)x_1 + (y_2+y_3)x_2 \le 200y_1 + 300y_2 + 400y_3$

This implies constraints $y_1 + y_3 \ge 1$ & $y_2 + y_3 \ge 6$

$\ge$ Inequality since then the obtained bound will be stronger.

We can easily find large enough y's that satisfy constraints, but then may obtain a relaxed bound on the expression $200y_1 + 300y_2 + 400y_3$. Would like to minimize this expression. This results in the LP:

$$\begin{array}{ll} \min & 200y_1 + 300y_2 + 400y_3 \\ & y_1 + y_3 \geq 1 \\ & y_2 + y_3 \geq 6 \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

Dual LP for Primal LP

$$\begin{array}{ll} \max & x_1 + 6x_2 \\ & x_1 \leq 200 \\ & x_2 \leq 300 \\ & x_1 + x_2 \leq 400 \\ & x_1, x_2 \geq 0 \end{array}$$

Verify dual solution $(0, 5, 1)$ for primal solution $(100, 300)$.

Note that applying this process on dual results in primal LP.

e.g.

Primal

$$\begin{array}{ll} \max & (2x_1 + 3x_2) \\ \text{s.t.} & 4x_1 + 8x_2 \leq 12 \\ & 2x_1 + x_2 \leq 3 \\ & 3x_1 + 2x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{array}$$

Dual

$$\begin{array}{ll} \min & (12y_1 + 3y_2 + 4y_3) \\ & 4y_1 + 2y_2 + 3y_3 \geq 2 \\ & 8y_1 + y_2 + 2y_3 \geq 3 \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

Verify solutions $(1/2, 5/4)$ $(5/16, 0, 1/4)$

<u>Exercise:</u>

$$\begin{array}{ll} \max & (7x_1 - x_2 + 5x_3) \\ & x_1 + x_2 + 4x_3 \leq 8 \\ & 3x_1 - x_2 + 2x_3 \leq 3 \\ & 2x_1 + 5x_2 - x_3 \leq -7 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Convert to dual.

So, in general we have:

PRIMAL

$$\begin{array}{ll} \max & c \cdot x \\ \text{s.t.} & Ax \leq b \quad (\oplus_2) \\ & x \geq 0 \end{array}$$

$\Longleftrightarrow$

DUAL

$$\begin{array}{ll} \min & y \cdot b \\ \text{s.t.} & y^T A \geq c^T \quad (\oplus_1) \\ & y \geq 0 \end{array}$$

Note that m constraints in primal are replaced by m variables in dual.
Similarly, n variables in primal are replaced by n constraints in dual.

<u>Weak Duality Theorem</u>

If $\hat{x}$ is a feasible solution to primal LP
& $\hat{y}$ is a feasible solution to dual LP, then

$$c \cdot \hat{x} \leq \hat{y} \cdot b$$

i.e., the dual value is an upper bound on the primal value.

Proof

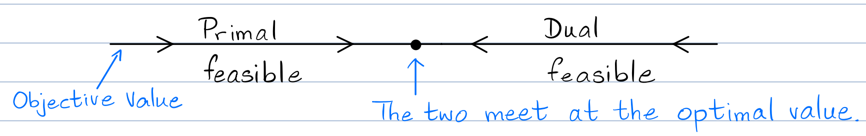
$$c\cdot\hat{x} = c^T\hat{x} \underset{\text{by } (\oplus_1)}{\le} (\hat{y}^T A)\hat{x} = \hat{y}^T(A\hat{x}) \underset{\text{by } (\oplus_2)}{\le} \hat{y}^T b = \hat{y}\cdot b$$

Strong Duality Theorem

Assume primal LP is feasible & bounded

$\Rightarrow$ dual LP is also feasible & bounded.

Moreover, $c\cdot\hat{x} = \hat{y}\cdot b$.



Dual \ Primal	Infeasible	Unbounded	Finite & Optimal
Infeasible	Possible	Possible	Impossible
Unbounded	Possible	Impossible	Impossible
Finite & Optimal	Impossible	Impossible	Possible

Interconversion when taking duals

Primal	Dual
maximization problem	minimization problem
$\le$ constraint	≥ 0 variable
$\ge$ constraint	≤ 0 variable
$=$ constraint	free variable

Handling assumptions for Simplex algorithm

1. What if origin is not a solution?

Assume LP has form $\max\ c\cdot x$ s.t. $Ax = b,\ x \ge 0$

i) Ensure RHS, i.e., b has all non-negative entries. Multiply by (-1) if a $b_i < 0$.

ii) Create m new variables $z_1, \dots, z_m \ge 0$, where m is number of constraints

iii) Add z_i to LHS of i^{th} constraint equation

iv) Define new objective function $\min\ z_1 + z_2 + \cdots + z_m$

v) Note $x = 0$ & $z = b$ is a solution for this new LP.

$$\min \sum_i z_i$$

$$\text{s.t. } \underbrace{\bar{A}\bar{x} = \bar{b}}, \quad \bar{x} \geqslant 0$$

modified constraints based on above construction.

Solve this LP to obtain:

$$\sum_i z_i = \begin{cases} 0 & \longrightarrow \text{ignore } z_i\text{'s and read off } x \text{ as starting solution of original LP} \\ > 0 & \longrightarrow \text{we need to add } z_i\text{'s to get a solution} \end{cases}$$

$\Rightarrow$ original LP is infeasible.

2. What if original LP has free variables, i.e., x is allowed to be negative.

Introduce two new variables x^+ & $x^- \geqslant 0$. Replace x with $x^+ - x^-$.

3. How can we convert an inequality constraint into an equality.

Replace $\sum_i a_i x_i \leq b$ with $\sum_i a_i x_i + \underset{\text{slack variable}}{s} = b$

4. Runtime of simplex.

Upperbound on number of vertices $\binom{m+n}{n} \rightarrow$ exponential in n.

So, in the worst case simplex is exponential, although runs well in practice.

USING FAST FOURIER TRANSFORM (FFT) FOR POLYNOMIAL MULTIPLICATION

Given polynomials

$$p(x) = a_0 + a_1 x + \cdots + a_d x^d$$
&
$$q(x) = b_0 + b_1 x + \cdots + b_d x^d$$

calculate

$$r(x) = p(x)q(x) = c_0 + c_1 x + \cdots + c_{2d} x^{2d}, \text{ where } c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0$$
$$= \sum_{i=0}^{k} a_i b_{k-i}$$

So, computing c_k takes $O(k)$ steps. Finding all $2d+1$ coefficients is $O(d^2)$.
Can we do faster?
We use an alternate representation of polynomials.

Fact A degree d polynomial is uniquely identified by its value at any $d+1$ distinct points, e.g., 2 points determine a line.

So, a polynomial $p(x) = a_0 + a_1 x_1 + \cdots + a_d x^d$ is given by either

1. coefficients $a_0, a_1, \ldots, a_d$

   OR

2. values $p(x_0), p(x_1), \ldots, p(x_d)$ for distinct points $x_0, x_1, \ldots, x_d$

$$\boxed{\text{Coefficient Representation}} \underset{\text{Interpolation}}{\overset{\text{Evaluation}}{\rightleftarrows}} \boxed{\text{Value Representation}}$$

Input Coefficients of two polys $p(x)$ & $q(x)$ of degree d.

Output $r = p \cdot q$

(Linear) 1. Selection, pick points $x_0, x_1, \ldots, x_{n-1}$, $n \geq 2d+1$

? 2. Evaluation, compute $p(x_0), p(x_1) \ldots p(x_{n-1})$ & $q(x_0), q(x_1) \ldots q(x_{n-1})$

(Linear) 3. Multiplication, compute $c(x_k) = p(x_k)\, q(x_k)$ for $k = 0, 1, \ldots, n-1$

4. Interpolation, recover $c(x) = c_0 + c_1 x + \cdots c_{2d} x^{2d}$

Evaluation via Divide & Conquer

How do we pick the n points for evaluation?

Idea: Choose them in pairs $\pm x_0, \pm x_1, \dots, \pm x_{\frac{n}{2}-1}$ & use the fact that even powers of $p(x_i)$ & $p(-x_i)$ are same.

e.g.,

$$p(x) = 3 + 4x + 6x^2 + 2x^3 + x^4 + 10x^5$$
$$= (3 + 6x^2 + x^4) + x(4 + 2x^2 + 10x^4)$$
$$= p_e(x^2) + x\,p_o(x^2)$$

↳ the polynomial has x^2 as its variable.

In general, for any $p(x)$,

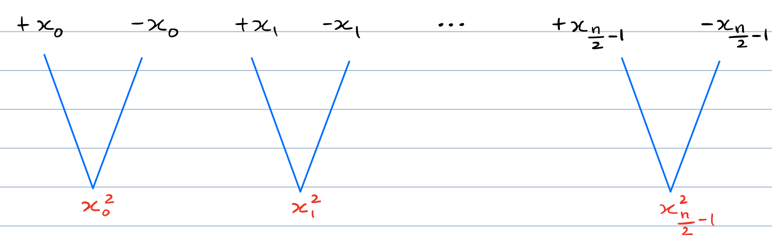
$$p(x_i) = p_e(x_i^2) + x_i\,p_o(x_i^2)$$

$$p(-x_i) = p_e(x_i^2) - x_i\,p_o(x_i^2)$$

The following division of work happens:

Evaluate $p(x)$ of degree $n-1$:

Equivalently, evaluate $p_e(x)$ & $p_o(x)$ of degree $\frac{n}{2}-1$:



This gives us recursion $T(n) = 2T\left(\frac{n}{2}\right) + O(n) = O(n \lg n)$

BUT, our $\pm$ pairing trick only works for a single level of recursion.
e.g., if we choose points

$\{1, 2, 3, 4, -1, -2, -3, -4\}$

these recursively become

$\{1, 4, 9, 16\}$ at next level.

We do not have a $\pm$ pairing to proceed to next level.
Want them to look something like

$\{1, 4, -1, -4\}$

What numbers squared give us -1 & -4?

Choose the original list to be

$\{1, 2, i, 2i, -1, -2, -i, -2i\}$

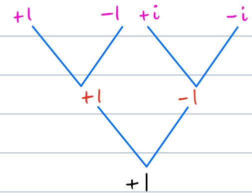
$i = \sqrt{-1}$ (review Complex numbers)

This allows us one more level of recursion, but

$\{1, 4, -1, -4\}$ is $\{1, 16\}$ at next level and we don't have a pairing again.

How do we resolve this issue in general?

At the base level of recursion we get just '+1'. Above it we have square roots of '+1', i.e., +1 & −1, and so on at each level above.



What we need are the n^{th} roots of unity, i.e., solution to the equation

$z^n = 1$, where z is a complex number.

These are given by

$1, \omega, \omega^2, \ldots, \omega^{n-1}$, where $\omega = e^{2\pi i/n}$, $i = \sqrt{-1}$

$= \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$

If n is even,

i) The n^{th} roots are $\pm$ paired, i.e., $\omega^{n/2+j} = -\omega^j$

ii) Squaring them produces the $n/2$ roots of unity.

e.g., $n = 8$, $\omega_8 = e^{2\pi i/8}$. The roots are

1, ω_8, ω_8^2, ω_8^3

$\omega_8^4 = -1$, $\omega_8^5 = -\omega_8$, $\omega_8^6 = -\omega_8^2$, $\omega_8^7 = -\omega_8^3$