CSE 317: Design and Analysis of Algorithms

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Recurrences: Fall 2024

Recurrences

Recurrences

- Recurrences (or recurrence relations or difference equations) arise naturally during analysis of algorithms
- For example, most recursive algorithms can be represented by a recurrence and then the time complexity of the algorithm is just the solution of the recurrence
- For example consider the following recursive algorithm that computes n! for any $n \geq 1$

Algorithm: FACTORIAL

Input: An integer $n \geq 1$

Output: The value of $n!$

1. **if** $n = 1$ **then return** 1
2. **else return** $n * \text{FACTORIAL}(n-1)$

Recurrences

- Assume that the time to compute factorial of an integer $n \geq 1$ is represented as $T(n)$
- Clearly $T(n) = 1 + T(n-1)$, furthermore, $T(1) = 1$
- We can solve this recurrence as following:

$$
\begin{aligned}
T(n) &= 1 + T(n-1) \\
&= 1 + 1 + T(n-2) \\
&= 1 + 1 + 1 + T(n-3) \\
&\quad \ldots \\
&= 1 + 1 + 1 + \cdots + 1 + T(1) \\
&= n = \Theta(n)
\end{aligned}
$$

A Simple Recurrence

- Consider the following recurrence:

$$A(n) = 2A(n-1) + 1, \qquad \text{such that} \quad A(0) = 0$$

- First few terms of the sequence are: $0, 1, 3, 7, 15, 31, 63, \ldots$
- It seems like $A(n) = 2^n - 1$
- We can prove this by induction
- Base case: $A(0) = 0 = 2^0 - 1$
- Inductive step: Assume that $A(k) = 2^k - 1$ for all $k < n$
- Then for $n = k + 1$ we have:

$$A(k+1) = 2A(k)+1 = 2(2^k-1)+1 = 2^{k+1}-2+1 = 2^{k+1}-1$$

A Simple Recurrence

- We can also solve this recurrence by expanding the recursive term as follows:

$$
\begin{aligned}
A(n) &= 2A(n-1)+1 \\
&= 2(2A(n-2)+1)+1 \\
&= 2^2A(n-2)+2+1 \\
&= 2^2(2A(n-3)+1)+2+1 \\
&= 2^3A(n-3)+2^2+2+1 \\
& \quad \dots \\
&= 2^nA(0)+2^{n-1}+2^{n-2}+\cdots+2^2+2+1 \\
&= 2^n-1=\Theta(2^n)
\end{aligned}
$$

A General Two Terms Recurrence

One Term Recurrence with Constant Coefficient

A recurrence of the following form, where a and $c > 0$ are constants and $f(n)$ is a function of n:

$$A(n) = cA(n-1) + f(n), \qquad \text{such that} \quad A(0) = a$$

has the solution:

$$A(n) = a \cdot c^n + \sum_{j=0}^{n-1} c^j \cdot f(n-j).$$

Proof for Two Terms Recurrences

$$
\begin{aligned}
A(n) &= cA(n-1) + f(n) \\
&= c(cA(n-2) + f(n-1)) + f(n) \\
&= c^2A(n-2) + cf(n-1) + f(n) \\
&= c^2(cA(n-3) + f(n-2)) + cf(n-1) + f(n) \\
&= c^3A(n-3) + c^2f(n-2) + cf(n-1) + f(n) \\
&\quad \dots \\
&= c^nA(0) + c^{n-1}f(1) + c^{n-2}f(2) + \cdots + cf(n-1) + f(n) \\
&= a \cdot c^n + \sum_{j=0}^{n-1} c^j \cdot f(n-j).
\end{aligned}
$$

Linear Homogeneous Recurrences

- A *linear homogeneous recurrence of degree k* with constant Coefficients is a recurrence relation which has k recursive terms and is of the form:

  $$A(n) = c_1 A(n-1) + c_2 A(n-2) + \cdots + c_k A(n-k)$$

  where c_i's are constants and $c_k \neq 0$ with $A(0) = C_0$, ..., $A(k-1) = C_{k-1}$ as k *initial conditions*
- We can solve this recurrence by finding the roots of the corresponding characteristic equation

Linear Homogeneous Recurrences

- We can rewrite the recurrence as follows:

$$A(n) - c_1 A(n-1) - c_2 A(n-2) - \cdots - c_k A(n-k) = 0$$

- The corresponding characteristic equation is:

$$x^k - c_1 x^{k-1} - c_2 x^{k-2} - \cdots - c_k = 0$$

- The roots of this equation $r_1, r_2, \ldots, r_k$ will be used to form the general solution of the recurrence
- The general solution of the recurrence is:

$$A(n) = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

The Fibonacci Recurrence

- Let $f_n = f_{n-1} + f_{n-2}$ with $f_0 = 0$ and $f_1 = 1$ $(n \geq 0)$
- We can rewrite it as: $f_n - f_{n-1} - f_{n-2} = 0$
- The characteristic equation is: $x^2 - x - 1 = 0$
- The two real roots of this equation are: $\frac{1 \pm \sqrt{5}}{2}$
- Thus the general solution of the Fibonacci recurrence is:

$$f_n = \alpha_1 \left(\frac{1 + \sqrt{5}}{2} \right)^n + \alpha_2 \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

- We can find the exact values of α_1 and α_2 by using the initial conditions, $\alpha_1 = \frac{1}{\sqrt{5}}$ and $\alpha_2 = -\frac{1}{\sqrt{5}}$

Another Three Term Recurrence

- Consider the recurrence: $a_n = a_{n-1} - a_{n-2}$ with $a_0 = 0$ and $a_1 = 1$ $(n \geq 2)$
- The first few terms of the sequence are: $0, 1, 1, 0, -1, -1, 0, \ldots$
- The characteristic equation is: $x^2 - x + 1 = 0$ which has the roots: $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$,
- These roots can written in polar coordinates as:

$$x_1 = re^{-i\theta} \quad \text{and} \quad x_2 = re^{i\theta}$$

- Here $r = \sqrt{(1/2)^2 + (\sqrt{3}/2)^2} = 1$ and $\theta = \tan^{-1}\sqrt{3} = \pi/3$

Another Three Term Recurrence

- Let us find x_1^n and x_2^n:

$$\begin{aligned}
x_1^n &= (re^{-i\theta})^n = r^n e^{-in\theta} = e^{-i\pi n/3} \\
&= \cos\left(\frac{-\pi n}{3}\right) + i\sin\left(\frac{-\pi n}{3}\right) \\
&= \cos\left(\frac{\pi n}{3}\right) - i\sin\left(\frac{\pi n}{3}\right) \\
x_2^n &= (re^{i\theta})^n = r^n e^{in\theta} = e^{i\pi n/3} \\
&= \cos\left(\frac{\pi n}{3}\right) + i\sin\left(\frac{\pi n}{3}\right)
\end{aligned}$$

- We can thus find the general solution as

$$\begin{aligned}
a_n &= \alpha_1 x_1^n + \alpha_2 x_2^n \\
&= \alpha_1\left[\cos\left(\frac{\pi n}{3}\right) - i\sin\left(\frac{\pi n}{3}\right)\right] \\
&\quad +\alpha_2\left[\cos\left(\frac{\pi n}{3}\right) + i\sin\left(\frac{\pi n}{3}\right)\right]
\end{aligned}$$

Another Three Term Recurrence

- We know that $a_0 = 0$ therefore:

$$\begin{aligned} 0 &= \alpha_1 \left[\cos 0 - i \sin 0\right] + \alpha_2 \left[\cos 0 + i \sin 0\right] \\ 0 &= \alpha_1 + \alpha_2 \qquad \Longrightarrow \qquad \alpha_1 = -\alpha_2 \end{aligned}$$

- Similarly, we know that $a_1 = 1$ therefore:

$$\begin{aligned} 1 &= \alpha_1 \left[\cos\left(\frac{\pi}{3}\right) - i \sin\left(\frac{\pi}{3}\right)\right] \\ &\quad + \alpha_2 \left[\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)\right] \\ &= \alpha_1 \left[\cos\left(\frac{\pi}{3}\right) - i \sin\left(\frac{\pi}{3}\right)\right] \\ &\quad - \alpha_1 \left[\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)\right] \\ 1 &= -2i\alpha_1 \sin\left(\frac{\pi}{3}\right) = -2i\alpha_1 \frac{\sqrt{3}}{2} \qquad \Longrightarrow \\ \alpha_1 &= -\frac{1}{i\sqrt{3}} \qquad \text{and} \qquad \alpha_2 = \frac{1}{i\sqrt{3}} \end{aligned}$$

Another Three Term Recurrence

- Therefore the general solution of the recurrence is:

$$
\begin{aligned}
a_n &= \frac{1}{i\sqrt{3}}\left[\cos\left(\frac{\pi n}{3}\right) + i\sin\left(\frac{\pi n}{3}\right)\right] \\
&\quad -\frac{1}{i\sqrt{3}}\left[\cos\left(\frac{\pi n}{3}\right) - i\sin\left(\frac{\pi n}{3}\right)\right] \\
a_n &= \frac{2}{\sqrt{3}}\sin\left(\frac{\pi n}{3}\right)
\end{aligned}
$$