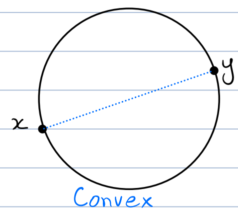
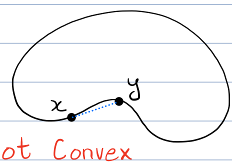
Convex Sets : Polytopes

Linear optimization is a special case of convex optimization. It helps to see things in the general context.





Intuition line joining any two points is contained entirely within set.

Def A set $S \subseteq \mathbb{R}^n$ is convex if for all points $x, y \in S$ the convex combination

$px + (1-p)y$ is also in S for all $p \in [0,1]$.

Conv(x,y) is all the points on the straight line in $\mathbb{R}^n$ between x & y.

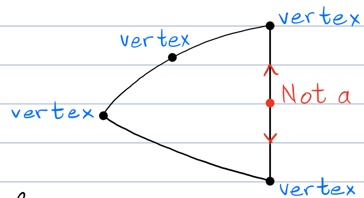
Def The set of points $x \in \mathbb{R}^n$ that satisfy $a \cdot x \leq b$ for $b \in \mathbb{R}$ & non-zero $a \in \mathbb{R}^n$ are called a half-space.
Think of them as one-side of a plane in $\mathbb{R}^n$.

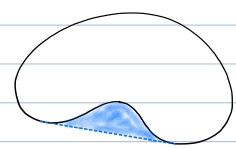
Fact Intersection of convex sets is a convex set.

Def A polytope is the intersection of a finite number of half-spaces.

Note that the feasible region of an LP is a polytope & is convex.

Def The convex hull of a set $S \subseteq \mathbb{R}^n$ is the set of all convex combinations of points from S.





Shaded region included in convex hull.
Dotted line identifies boundary.

Def

An extreme point or vertex of a convex set $S \subseteq \mathbb{R}^n$ is a point $q \in S$, s.t., there do not exist $p, r \in S$ & $\lambda \in \mathbb{R}$, $0 < \lambda < 1$, $p \neq r$ for which

$q = \lambda p + (1-\lambda) r$, i.e.,

q cannot be written as a proper convex combination of points in S.

Thm (Minkowski–Weyl) The following are equivalent:

1. P is a polytope
2. P is the convex hull of a finite set S, i.e., P is convex hull of S.

This gives us a dual representation of polytopes as either
a set of vertices OR a set of inequalities.

Def A point q is a vertex or node of a polytope P if:

i) $q \in P$
ii) For any $v \in \mathbb{R}^n$, $v \neq 0$, then either $q+v$ OR $q-v$ is not in P.

This means that if we are at a point for which we can move in two opposite directions & still stay inside the polytope, then we are not at a vertex.

This leads to the main idea of the simplex algorithm.

1. Start at 'some' vertex of the feasible region.
2. Look at neighbouring vertices to improve objective function value. Stop when at optimal.

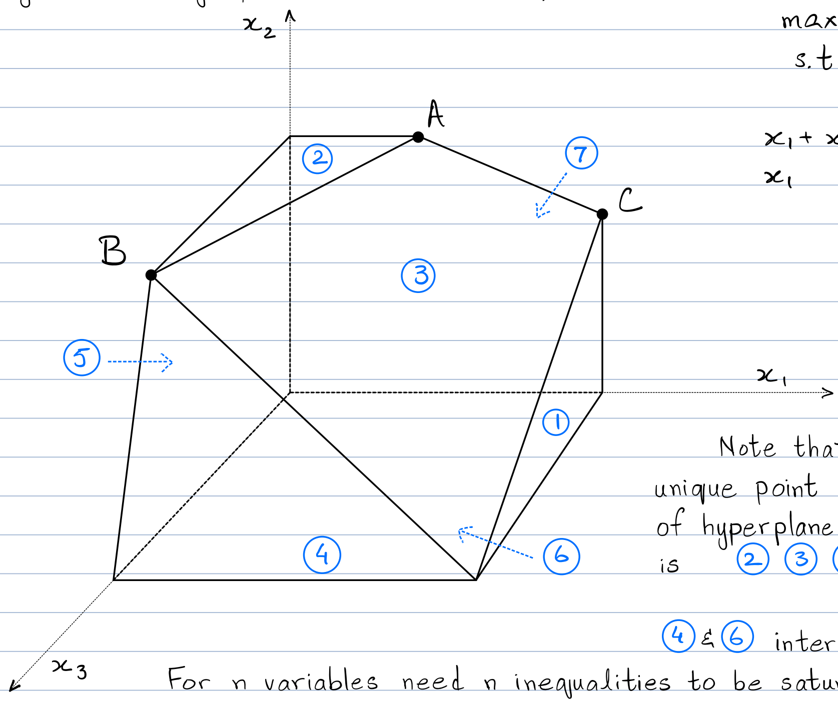
Def

feasible region — region that satisfies all constraints
optimal solution — feasible point 'farthest' in the objective function direction within the feasible region.

LP is unbounded, if for all B, there exists a feasible x, s.t. $c \cdot x > B$.

If the feasible region is empty, then the LP is infeasible.

Polyhedron (Polytope in $n=3$ dimensions)



$$\begin{array}{lll} \max & x_1 + 6x_2 + 13x_3 & \\ \text{s.t.} & x_1 \leq 200 & (1) \\ & x_2 \leq 300 & (2) \\ & x_1 + x_2 + x_3 \leq 400 & (3) \\ & x_1 + 3x_3 \leq 600 & (4) \\ & x_1 \geq 0 & (5) \\ & x_2 \geq 0 & (6) \\ & x_3 \geq 0 & (7) \end{array}$$

Note that each vertex is a unique point at which some subset of hyperplanes meet, e.g. vertex A is (2) (3) (7) meeting.

(4) & (6) intersection is a line.

For n variables need n inequalities to be saturated to identify a vertex.

So, pick a subset of n inequalities, if there is a unique point that satisfies them with equality & it is feasible, then it is a vertex.

There can exist multiple set of n inequalities that generate a vertex v, e.g., B is given by (2) (3) (4) or (2) (4) (5).
These are called degenerate vertices.
Assume no degeneracy for now.

Notion of a neighbouring vertex

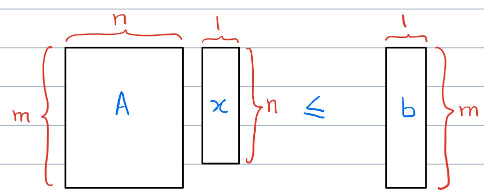
Two vertices are neighbours if they have n-1 inequalities in common, e.g. A & C sharing (3) (7) are neighbours.

Simplex Algorithm

Which vertex do we start at? Assuming positivity constraints for x_i, i.e., $x_i \geq 0$, origin is a vertex. Also assume origin is feasible, so can use as starting point.

Given LP is formulated as:

$$\begin{array}{ll} \max & c \cdot x \\ \text{s.t.} & Ax \leq b \end{array}$$



Note that origin is optimal if and only if all coefficients c_i in the objective function are ≤ 0, since otherwise could choose the positive c_i & increase $c \cdot x$ given that $x \geq 0$.

Now, we need to increase an x_i for which $c_i > 0$.

Increase by how much? Till we end up saturating a new constraint.

Note that we released saturation of a particular $x_i \geq 0$ & keep increasing this x_i till a previously loose inequality becomes saturated.

Example

$$\max \ 2x_1 + 5x_2$$

$$\begin{aligned} \text{s.t.} \quad 2x_1 - x_2 &\leq 4 && (1) \\ x_1 + 2x_2 &\leq 9 && (2) \\ -x_1 + x_2 &\leq 3 && (3) \\ x_1, \ x_2 &\geq 0 && \\ (4) \quad (5) \end{aligned}$$

$x_1 = x_2 = 0$ is feasible

Initial vertex saturates (4) (5)

Objective value = 0

Move: Increase x_2

(5) is released

(3) becomes saturated when $x_2 = 3$

So, new vertex point $x_1 = 0, x_2 = 3$ saturates (4), (3).

We now perform a change of variables, such that in terms of the new variables, the vertex $(0, 3)$ moves to origin.

This is done by setting (4), (3) equal to y_1 & y_2, i.e.,

$y_1 = x_1$ & $y_2 = 3 + x_1 - x_2$

Restate the LP in terms of y_1 & y_2.

$$\max \ 15 + 7y_1 - 5y_2$$

$$\begin{aligned} \text{s.t.} \quad y_1 + y_2 &\leq 7 && (1) \\ 3y_1 - 2y_2 &\leq 3 && (2) \\ y_2 &\geq 0 && (3) \\ y_1 &\geq 0 && (4) \\ -y_1 + y_2 &\leq 3 && (5) \end{aligned}$$

The vertex $x_1 = 0, x_2 = 3$ is now at origin in terms of y_1 & y_2. $(y_1 = 0, y_2 = 0)$

We saturate (3) (4).

Objective value = 15.

Note cannot improve value by increasing y_2.

New move: increase y_1.

Release (4). Now (2) is saturated when $y_1 = 1$.

New vertex $(y_1 = 1, y_2 = 0)$ saturates (2), (3).

New change of variables is given by z_1, z_2,

$z_1 = 3 - 3y_1 + 2y_2$ & $z_2 = y_2$.

Last Iteration

$$\max \quad 22 - \frac{7z_1}{3} - \frac{z_2}{3}$$

$$\frac{-z_1}{3} + \frac{5z_2}{3} \leq 6 \quad (1)$$

$$z_1 \geq 0 \quad (2)$$

$$z_2 \geq 0 \quad (3)$$

$$\frac{z_1}{3} - \frac{2z_2}{3} \leq 1 \quad (4)$$

$$\frac{z_1}{3} + \frac{z_2}{3} \leq 4 \quad (5)$$

Objective value = 22

Optimal, since all $c_i < 0$.

Solve (2) (3) in original LP to get solution vector.