CSE 317: Design and Analysis of Algorithms
Divide and Conquer Algorithms

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SMCS, IBA

Divide and Conquer Algorithms

Divide and Conquer Algorithms

- We start with an *original problem*
- **Divide** it into two ore more *subproblems* (ideally of approximately equal sizes)
- **Solve** each subproblem *recursively*
- And **combine** the solutions of subproblems to form a solution for the original problem

Mergesort Algorithm

- We consider MERGESORT as a first example of divide and conquer algorithms
- Let $\langle a_1, \ldots, a_n \rangle$ be the input sequence to be sorted
- MERGESORT divides the array into two (almost) equal parts as:

$$\langle a_1, \ldots, a_{\lfloor n/2 \rfloor} \rangle \quad \text{and} \quad \langle a_{\lfloor n/2 \rfloor + 1}, \ldots, a_n \rangle$$

- We use MERGESORT to sort these two subsequences (subproblems)
- Let α and β be two the sorted sequences we receive after recursive calls
- Now we combine (merge) these lists to form a new list
- We compare first element of α with first element of β and transfer the smaller element to the new sequence and move the pointer where we take element from
- If at any point if one of the sequences α or β becomes empty we concatenate the other sequence to the new sequence.

Mergesort Algorithm

Algorithm: MERGESORT

Input: A sequence $A = \langle a_1, a_2, \ldots, a_n \rangle$

Output: The sorted sequence $A' = \langle a_{i_1}, a_{i_2}, \ldots, a_{i_n} \rangle$ s.t. $a_{i_1} \leq a_{i_2} \leq \cdots \leq a_{i_n}$

1. **if** $n > 1$ **then**
2. $\quad \alpha \leftarrow \text{MERGESORT}(\langle a_1, \ldots, a_{\lfloor n/2 \rfloor} \rangle)$
3. $\quad \beta \leftarrow \text{MERGESORT}(\langle a_{\lfloor n/2 \rfloor + 1}, \ldots, a_n \rangle)$
4. $\quad \text{MERGE}(\alpha, \beta)$
5. **return** A

Mergesort Algorithm: Merge

Algorithm: MERGE

Input: Two sorted lists $A = \langle a_1, \ldots, a_k \rangle$ and $B = \langle b_1, \ldots, b_l \rangle$

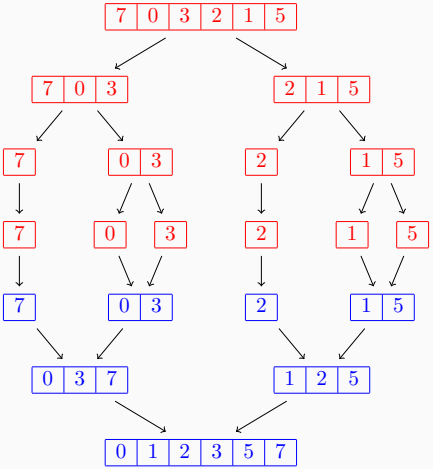
Output: Combined sorted sequences of A and B

1. **if** $k = 0$ **then return** $\langle b_1, \ldots, b_l \rangle$
2. **if** $l = 0$ **then return** $\langle a_1, \ldots, a_k \rangle$
3. **if** $a_1 \leq b_1$ **then return** $\langle a_1 \rangle \circ \text{MERGE}(\langle a_2, \ldots, a_k \rangle, \langle b_1, \ldots, b_k \rangle)$
4. **else return** $\langle b_1 \rangle \circ \text{MERGE}(\langle a_1, \ldots, a_k \rangle, \langle b_2, \ldots, b_l \rangle)$

[Here $\circ$ denotes concatenation.]

Mergesort Algorithm: Visualization

Sorting the sequence $\langle 7, 0, 3, 2, 1, 5 \rangle$ using MERGESORT.



Mergesort Algorithm: Time Complexity

- Let the two sorted lists A and B are of sizes k and l, respectively
- We can show that the procedure MERGE can merge the lists A and B in time $O(k+l)$
- So if $|A|+|B|=n$ then running time of MERGE is $O(n)$
- Let $T(n)$ represents the running time of MERGESORT then:

$$T(n) = 2T\left(\frac{n}{2}\right) + n \quad \text{with } T(1) = 1$$

- This is a *divide and conquer recurrence* and we show that

$$T(n) = O(n \log n)$$

Mergesort recurrence

- Solving the recurrence $T(n) = 2T\left(\frac{n}{2}\right) + O(n)$ with $T(1) = 1$ and $n = 2^k$

$$
\begin{aligned}
T(n) &= 2T\left(\frac{n}{2}\right) + n \\
&= 2\left(2T\left(\frac{n}{4}\right) + \frac{n}{2}\right) + n = 2^2 T\left(\frac{n}{2^2}\right) + n + n \\
&= 2^2\left(2T\left(\frac{n}{8}\right) + \frac{n}{4}\right) + n + n = 2^3 T\left(\frac{n}{2^3}\right) + n + n + n \\
&\vdots \\
&= 2^k T\left(\frac{n}{2^k}\right) + kn \qquad \text{since } T(1) = 1 \\
&= 2^k T(1) + kn = n + kn = n(1 + k) \\
&= n(1 + \log_2 n) = O(n \log n)
\end{aligned}
$$

Divide and Conquer Recurrences

- Divide and conquer recurrences are of the form:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

where $f(n)$ is the time taken to divide the problem into subproblems and combine the solutions of subproblems

Theorem (Master Theorem)

Let $T(n) = aT(n/b) + f(n)$ be a divide and conquer recurrence with $a \geq 1$, $b > 1$ and $f(n)$ is asymptotically positive. Then $T(n)$ has the following asymptotic bounds:

$$T(n) = \begin{cases} \Theta(n^{\log_b a}) & \text{if } f(n) = O(n^{\log_b a - \epsilon}) \text{ for some } \epsilon > 0 \\ \Theta(n^{\log_b a} \log n) & \text{if } f(n) = \Theta(n^{\log_b a}) \\ \Theta(f(n)) & \text{if } f(n) = \Omega(n^{\log_b a + \epsilon}) \text{ for some } \epsilon > 0 \end{cases}$$

Maximum Subarray Sum

- For a given array $A[1..n]$ of n numbers, the *maximum subarray sum*, denoted as $\mathcal{M}$ is the largest sum of contiguous subarray of $A[i..j]$ where $1 \leq i \leq j \leq n$, i.e.,

$$\mathcal{M}(A) = \sum_{x=i}^{j} A[x]$$

- For example, for $A = [-2, 3, 4, -1, 5, -3, 2, -1]$, $\mathcal{M}(A) = 11$ for the subarray $[3, 4, -1, 5]$

Maximum Subarray Sum: Trivial Solution

Algorithm: BRUTEFORCE-MAX-SUBARRAY-SUM

Input: An array $A[1..n]$ of n numbers

Output: The maximum subarray sum $\mathcal{M}(A)$

```
1. M ← −∞
2. for i ← 1 to n
3.       for j ← i to n
4.       s ← 0
5.             for k ← i to j
6.                   s ← s + A[k]
7.             if s > M then M = s
8. return M
```

The time complexity of above algorithms is $\Theta(n^3)$

Maximum Subarray Sum: Divide and Conquer

- Let us design a divide and conquer algorithm
- We divide the array into two almost equal parts and find the maximum subarray sum in the left and right parts recursively
- At this point we have three cases:
  1. The maximum subarray sum is in the left part
  2. The maximum subarray sum is in the right part
  3. The maximum subarray sum crossovers the middle point
- Thus, the algorithm returns the maximum of these three cases
- The crossover sum can be computed in $O(n)$ time by starting from the middle point and going left and right to find the maximum sum
- If $T(n)$ represents the time complexity of the algorithm then we have the following recurrence:

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

Maximum Subarray Sum: Divide and Conquer

Algorithm: DC-MAX-SUBARRAY-SUM
Input: An array $A[l..h]$ of $h - l + 1$ numbers
Output: The maximum subarray sum $\mathcal{M}(A)$

1. **if** $l = h$ **then return** $A[l]$
2. $m \leftarrow \lfloor (l + h)/2 \rfloor$
3. $\mathcal{M}_L \leftarrow$ DC-MAX-SUBARRAY-SUM$(A[l..m])$
4. $\mathcal{M}_L \leftarrow$ DC-MAX-SUBARRAY-SUM$(A[m + 1..h])$
5. $\mathcal{M}_L \leftarrow$ MAX-CROSSING-SUM$(A[l..h])$
6. **return** $\max\{\mathcal{M}_L, \mathcal{M}_R, \mathcal{M}_C\}$

Algorithm: MAX-CROSSING-SUM
Input: An array $A[l..h]$ of $h - l + 1$ numbers
Output: The maximum crossing sum

1. $m \leftarrow \lfloor (l + h)/2 \rfloor$, $s \leftarrow 0$, $left\_sum \leftarrow -\infty$
2. **for** $i \leftarrow m$ **downto** l
3. $\quad s \leftarrow s + A[i]$
4. $\quad$ **if** $s > left\_sum$ **then** $left\_sum = s$
5. $s \leftarrow 0$, $right\_sum \leftarrow -\infty$
6. **for** $i \leftarrow m + 1$ **to** h
7. $\quad s \leftarrow s + A[i]$
8. $\quad$ **if** $s > right\_sum$ **then** $right\_sums$
9. **return** $left\_sum + right\_sum$

Closest-Pair Problem

- Given a set of n points $P = \{p_1 = (x_1, y_1), \ldots, p_n = (x_n, y_n)\}$
- The *closest-pair* problem asks to find the pair of points p_i and p_j that are closest to each other i.e., $d(p_i, p_j)$ is minimum, where

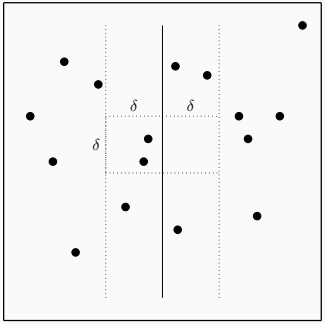
$$d(p_i, p_j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

- A brute force algorithm will find the distance between all pairs of points and return the minimum distance
- This algorithm has a time complexity of $O(n^2)$ as there are $n(n-1)/2$ different pairs of points
- We will see a divide and conquer algorithm that solves this problem in $O(n \log n)$ time

Closest-Pair Problem: Divide and Conquer

- The first step is to sort the points according to their x-coordinates
- We divide the points into two almost equal parts P_L and P_R and find the closest pair in the left and right parts recursively
- Let l be the vertical line that separates P_L and P_R
- Let δ_L and δ_R be the minimum distance between two points in P_L and P_R, respectively
- Let δ' be the minimum distance between a point in P_L and a point in P_R
- The closest pair is either in P_L or P_R or it crosses the line l, therefore the closest pair is the minimum of δ_L, δ_R and δ'

Closest-Pair Problem: Divide and Conquer



- We set $\delta = \min\{\delta_L, \delta_R\}$
- If there is a pair of points $p_i \in P_L$ and $p_j \in P_R$ such that $d(p_i, p_j) < \delta$ then this is the closest pair
- If the closest pair consists of some point in P_L and P_R then we need to check only those points that are within δ distance from the line l (both sides)
- We can now run a linear scan to find the closest pair in this region from top to bottom (see the figure)

Closest-Pair Problem: Divide and Conquer

- Let $T(n)$ be the time complexity of the algorithm
- We have the following recurrence:

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n) = O(n \log n)$$

Integer Multiplication

- Let X and Y be two n-bit integers
- Adding x and y takes $O(n)$ bit operations (linear time)
- Multiplying x and y takes $O(n^2)$ bit operations (quadratic time)
- Product of two n-bit numbers is at most a $2n$-bit number

$$
\begin{array}{cccccccc}
 & & & & 1 & 0 & 0 & 1 \\
 & & & \times & 1 & 0 & 1 & 0 \\
\hline
 & & & & 0 & 0 & 0 & 0 \\
 & & & 1 & 0 & 0 & 1 & \\
 & & 0 & 0 & 0 & 0 & & \\
 & 1 & 0 & 0 & 1 & & & \\
\hline
 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
\hline
\end{array}
$$

Integer Multiplication: Divide and Conquer

- Let $X = \langle x_1, x_2, \ldots, x_n \rangle_2$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle_2$ be two n-bit integers, assuming $n = 2^k$, $k \in \mathbb{N}$
- We can write $X = X_L X_R$ and $Y = Y_L Y_R$, where
- $X_L = \langle x_1, x_2, \ldots, x_{n/2} \rangle_2$, $X_R = \langle x_{n/2+1}, x_{n/2+2}, \ldots, x_n \rangle_2$
- $Y_L = \langle y_1, y_2, \ldots, y_{n/2} \rangle_2$, $Y_R = \langle y_{n/2+1}, y_{n/2+2}, \ldots, y_n \rangle_2$
- So, we can write $X = 2^{n/2} X_L + X_R$ and $Y = 2^{n/2} Y_L + Y_R$
- The product of X and Y is:

$$XY = (2^{n/2} X_L + X_R)(2^{n/2} Y_L + Y_R) = 2^n X_L Y_L + 2^{n/2}(X_L Y_R + X_R Y_L) + X_R Y_R$$

- So we can compute product of two n-bit integers by computing four products of $n/2$-bit integers

Integer Multiplication: Divide and Conquer

- If time to multiply X and Y is $T(n)$ then we have the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + O(1) = \Theta(n^2) \quad \text{with} \quad T(1) = 0$$

- If we use *Karatsuba's algorithm* then we have the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n) = O(n^{\log_2 3}) \approx O(n^{1.5849}) \quad \text{with} \quad T(1) = 1$$

- That is if we can somehow reduce the number of multiplications from four to three we can improve the running time

Integer Multiplication: Divide and Conquer

- $XY = 2^n X_L Y_L + 2^{n/2}(X_L Y_R + X_R Y_L) + X_R Y_R$
- We observe that:

$$(X_L Y_R + X_R Y_R) = (X_L + X_R)(Y_L + Y_R) - X_L Y_L - X_R Y_R$$

- Since $X_L Y_L$ and $X_R Y_R$ is already computed to evaluate XY we get:

$$\begin{aligned} XY &= 2^n X_L Y_L + 2^{n/2}(X_L Y_R + X_R Y_L) + X_R Y_R \\ &= 2^n X_L Y_L + 2^{n/2}((X_L + X_R)(Y_L + Y_R) - X_L Y_L - X_R Y_R) + X_R Y_R \end{aligned}$$

- So we can compute XY using three multiplications of $n/2$-bit integers instead of four to get:

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n) = O(n^{\log_2 3}) \approx O(n^{1.5849}) \quad \text{with} \quad T(1) = 1$$

Integer Mulplitiplication: Example

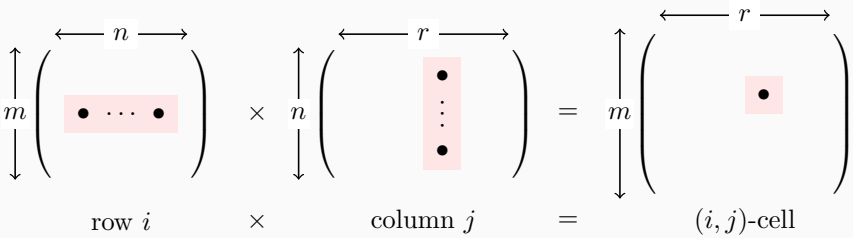
- Let $X = 1101_2$ and $Y = 1010_2$, $X_L = 11$, $X_R = 01$, $Y_L = 10$ and $Y_R = 10$
- Here $n = 2^2 = 4$ and $n/2 = 2^1 = 2$

$$\begin{aligned} XY &= 1101_2 \times 1010_2 \\ &= 2^2(11)(01) + 2^1((11+01)(10+10) - (11)(10) - (01)(10)) + (01)(10) \end{aligned}$$

- Each multiplication in above will be carried out recursively

Matrix Multiplication

- If X and Y are two matrices of sizes $m \times n$ and $n \times r$ then the product $Z = AB$ is a matrix of size $m \times r$



- In general AB is not the same as BA.

Matrix Multiplication

- Let X and Y be two matrices of size $n \times n$ such that $n = 2^k$ for some $k > 0$.
- We can break the problem into subproblems e.g.,

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \; Y = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

- Now XY can be computed as

$$XY = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

- To compute the size n product XY we now compute eight size $n/2$ products, $AE, BG, \ldots, DH$
- $T(n) = 8T(n/2) + O(n^2) = O(n^3)$

Strassen's Algorithm

- Strassen observed that XY can be computed from just seven $n/2 \times n/2$ subproblmes i.e.,

$$XY = \begin{bmatrix} P_5 + P_4 - P_2 + P_6 & P_1 + P_2 \\ P_3 + P_4 & P_1 + P_5 - P_3 - P_7 \end{bmatrix}$$

- where

$$\begin{aligned} P_1 &= A(F - H) \\ P_2 &= (A + B)H \\ P_3 &= (C + D)E \\ P_4 &= D(G - E) \end{aligned} \qquad \begin{aligned} P_5 &= (A + D)(E + H) \\ P_6 &= (B - D)(G + H) \\ P_7 &= (A - C)(E + F) \end{aligned}$$

- The new running time is

$$T(n) = 7T(n/2) + O(n^2) = O(n^{\log_2 7}) \approx O(n^{8.1})$$