

Benefits of “concreteness fading” for children's mathematics understanding[☆]



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ARTICLE INFO

Article history:

Received 29 April 2014

Received in revised form

6 October 2014

Accepted 23 October 2014

Available online

Keywords:

Math equivalence

Concrete manipulatives

Learning and transfer

Concreteness fading

ABSTRACT

Children often struggle to gain understanding from instruction on a procedure, particularly when it is taught in the context of abstract mathematical symbols. We tested whether a “concreteness fading” technique, which begins with concrete materials and fades to abstract symbols, can help children extend their knowledge beyond a simple instructed procedure. In Experiment 1, children with low prior knowledge received instruction in one of four conditions: (a) concrete, (b) abstract, (c) concreteness fading, or (d) concreteness introduction. Experiment 2 was designed to rule out an alternative hypothesis that concreteness fading works merely by “warming up” children for abstract instruction. Experiment 3 tested whether the benefits of concreteness fading extend to children with high prior knowledge. In all three experiments, children in the concreteness fading condition exhibited better transfer than children in the other conditions. Children's understanding benefits when problems are presented with concrete materials that are faded into abstract representations.

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1. Introduction

When we teach children a procedure for solving a mathematics problem, we not only want them to learn the procedure and apply it correctly, but also want them to understand why the procedure works. Indeed, a key question in the development of children's mathematical thinking is how we can help children gain understanding of underlying concepts from the procedures they are taught, so they can transfer those procedures beyond the specific, instructed context. Unfortunately, children struggle to gain conceptual understanding from a procedure, especially when it is taught in the context of abstract mathematics symbols (e.g., McNeil

& Alibali, 2000; Rittle-Johnson & Alibali, 1999). Because understanding abstract symbols and manipulating them in meaningful ways are critical aspects of learning mathematics, the use of abstract symbols during instruction cannot and should not be avoided altogether. However, relatively minor changes to *when* and *how* abstract symbols are introduced during instruction may improve children's ability to extend their knowledge beyond the instructed procedure.

In the present study, we tested one hypothesized method for helping children extend their mathematical knowledge beyond a simple, instructed procedure: beginning with concrete examples and then explicitly fading to the abstract symbols. This “concreteness fading” technique is hypothesized to facilitate conceptual understanding by fostering knowledge that is both grounded in meaningful concrete contexts, and also generalized in a way that promotes transfer (e.g., Fyfe, McNeil, Son, & Goldstone, 2014; Goldstone & Son, 2005).

Students spend a lot of time learning and practicing mathematical procedures. For example, in representative eighth-grade math classrooms, students spent approximately two-thirds of individual work time solving problems using an instructed procedure (Hiebert et al., 2003). Unfortunately, students typically just memorize the procedure and rarely apply it as instructed. This leads to misunderstandings and failure to transfer the procedure appropriately. Indeed, children rarely benefit from procedural

* This work was based, in part, on a senior honors thesis conducted at the University of Notre Dame by Emily R. Fyfe under the supervision of Nicole M. McNeil. The work was supported by the University of Notre Dame through a grant from the Center for Undergraduate Scholarly Engagement to Fyfe and summer fellowships to Fyfe and Borjas from the Undergraduate Research Opportunity Program of the Institute of Scholarship in the Liberal Arts. It was also supported by U.S. Department of Education, Institute of Education Sciences Grant R305B070297 to McNeil. The opinions expressed are those of the authors and do not represent views of the Institute or the U.S. Department of Education. The authors would like to thank April Dunwiddie for her help with data collection and scheduling as well as Vladimir Sloutsky for his insightful comments that inspired Experiment 2.

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instruction, and correct transfer and conceptual understanding are low as a result (e.g., Matthews & Rittle-Johnson, 2009; Perry, 1991). For example, children often make “buggy” subtraction errors when they try to follow rules that they do not understand, such as changing a 7 in the tens place into a 5 in order to “borrow” two one’s (e.g., Fuson et al., 1997). Similarly, children who learn a procedure for solving math equivalence problems with a repeated addend ($3 + 4 + 5 = 3 + \underline{\hspace{1cm}}$) often fail to correctly transfer that procedure to problems without a repeated addend ($3 + 4 + 5 = 2 + \underline{\hspace{1cm}}$; McNeil & Alibali, 2000).

One factor contributing to children’s difficulties gaining conceptual understanding from procedural instruction may be premature reliance on abstract symbols. In fact, studies have shown that abstract symbols can interfere with understanding because they elicit the use of rote instructed procedures at the expense of informal, logical reasoning (e.g., Koedinger & Nathan, 2004). For example, children aged 9–15 were better able to perform math calculations when presented in a concrete context (e.g., “If I purchase four coconuts and each coconut costs \$35, how much do I owe you?”) than when presented in an abstract, symbolic context (e.g., $35 \times 4 = \underline{\hspace{1cm}}$; Carraher, Carraher, & Schliemann, 1985). In the concrete context, children used mental calculations of relevant quantities, whereas in the abstract context, children tried (unsuccessfully) to employ school-taught procedures. More generally, research suggests that abstract symbols can lead to inflexible application of learned procedures (McNeil & Alibali, 2005) and illogical errors (Carraher & Schliemann, 1985), neither of which indicate deep, transferable knowledge of mathematics.

One solution for helping children gain conceptual understanding from procedural instruction is to use concrete materials during instruction (e.g., blocks, balance scale). Such materials have several potential benefits. For example, concrete materials can activate intuitive, real-world knowledge during learning (Baranes, Perry, & Stigler, 1989; Kotovsky, Hayes, & Simon, 1985), enable learners to construct their own knowledge of abstract concepts (Brown, McNeil, & Glenberg, 2009), and prompt physical action, which has been shown to improve understanding and retention (e.g., Martin & Schwartz, 2005). However, the mere use of concrete materials does not guarantee success (McNeil & Jarvin, 2007). Indeed, although concrete materials may facilitate initial understanding, learners often struggle to apply that understanding beyond the instructed context. For example, numerous studies have shown that concrete materials hinder transfer to new, dissimilar situations (e.g., Gentner, Ratterman, & Forbus, 1993; Goldstone & Sakamoto, 2003; Kaminski, Sloutsky, & Heckler, 2008; Son, Smith, & Goldstone, 2011).

A promising alternative to using concrete materials alone to support conceptual understanding is to use them in combination with abstract symbols in a fading sequence. Specifically, many theorists recommend beginning with concrete examples and slowly fading to the more abstract (e.g., Bruner, 1966; Fyfe et al., 2014; Gravemeijer, 2002; Lehrer & Schauble, 2002). For example, Bruner (1966) proposed that new concepts and procedures should be taught using three progressive forms: (1) an *enactive* form, which is a physical, concrete model, (2) an *iconic* form, which is a graphic or pictorial representation, and finally (3) a *symbolic* form, which is an abstract, symbolic representation. The goal is to promote a rich, grounded understanding from instruction that is tied to conventional abstract symbols. In the present study, we use “concreteness fading” to refer to the specific three-step progression by which the physical instantiation of a concept becomes increasingly abstract over time.

Recently, McNeil and Fyfe (2012) provided support for concreteness fading. Undergraduates learned modular arithmetic using abstract symbols (e.g., pictures of solid black-on-white

shapes), concrete examples (e.g., pictures of measuring cups), or a concreteness fading progression. In the fading condition, the concrete elements and abstract elements were explicitly linked with an intermediate instantiation. Specifically, after learning with the concrete measuring cups, participants were told that the measuring cups would be represented by simpler symbols: I, II, III. After learning via Roman numerals, participants were told that any symbols could be used, and they were presented with the abstract elements. Students then completed a transfer test immediately, one week later, and three weeks later. Importantly, undergraduates in the fading condition exhibited the best transfer performance at all three time points.

Research by Goldstone and Son (2005) also provided some initial support for the “fading” hypothesis. Undergraduate students learned the scientific principle of competitive specialization via computer simulations that differed in their perceptual concreteness. The elements in the display either remained concrete (e.g., realistic ants and fruit), remained abstract (e.g., simple black dots and green shapes), switched from concrete to abstract, or switched from abstract to concrete. Going from concrete to abstract resulted in the most optimal transfer.

Despite these promising results, more research is needed. Neither of these studies included physical, concrete objects during the enactive stage as originally recommended by Bruner (1966). Further, neither progression fully explored the scope of abstractness. The concrete and abstract elements were both relatively concrete (i.e., pictorial images); the abstract elements were simply stripped of perceptual detail. It remains unclear if concreteness fading can improve understanding of conventional abstract symbols, of which learners may already have some misunderstanding. Finally, this prior research was limited to undergraduate students learning upper-level mathematical and scientific concepts. One study tested the benefits of concreteness fading for teaching fractions, but was limited to middle school students with mathematics disabilities. Butler, Miller, Crehan, Babbitt, and Pierce (2003) compared a full concrete-representational-abstract (CRA) sequence to a representational-abstract (RA) sequence. The CRA group used concrete manipulatives for the first few lessons, while the RA group used representational drawings. Students in the CRA group exhibited higher learning than students in the RA group.

Importantly, the “fading” method has yet to be applied in mathematics with typically-developing children in elementary school learning a specific procedure. Further, no study to date has tested the full three-stage fading method and included a rigorous control in which learners are exposed to the three stages in the reverse order. Finally, no study to date has explicitly examined whether the benefits of concreteness fading extend to learners with varying prior knowledge of the target concept. We addressed these gaps in the literature by examining the fading hypothesis in the context of children learning a procedure to solve math equivalence problems.

Math equivalence is the idea that two sides of an equation represent the same quantity, and it is critical to developing algebraic thinking (e.g., Falkner, Levi, & Carpenter, 1999; Knuth, Stephens, McNeil, & Alibali, 2006). Thus, it is an educationally-relevant and developmentally-appropriate topic for elementary school students. For example, the Common Core State Standards have included math equivalence as a first-grade standard (e.g., www.corestandards.org/Math/Content/1/OA/D/7), and numerous studies suggest that children in first-through third-grade can learn to understand math equivalence (e.g., Baroody & Ginsburg, 1983; Davydov, 1969/1991; Jacobs, Franke, Carpenter, Levi, & Battey, 2007).

This domain is an apt domain in which to investigate whether concreteness fading can help children extend their knowledge

beyond a simple, instructed procedure. Math equivalence problems are difficult for elementary school children in numerous countries, including the U.S., Mexico, Canada, the UK, Turkey, and Australia (e.g., DeCorte & Verschaffel, 1981; Filloy & Rojano, 1989; Humberstone & Reeve, 2008; Jones, Inglis, Gilmore, & Evans, 2013; Li, Ding, Capraro, & Capraro, 2008; Molina, Castro, & Castro, 2009; Oksuz, 2007; Saenz-Ludlow & Walgamuth, 1998; Sherman & Bisanz, 2009). For example, when asked to solve the problem $3 + 4 + 5 = 3 + \underline{\quad}$, most children misinterpret the equal sign as an operator symbol meaning “find the total,” and they add up all the numbers (answering 15; e.g., Falkner et al., 1999; Rittle-Johnson, 2006). Further, instruction on the problems runs the risk of eliciting rote manipulation of symbols without conceptual understanding or transfer (Alibali, 1999; Matthews & Rittle-Johnson, 2009; McNeil & Alibali, 2000; Perry, 1991). For example, children who learn a procedure for solving math equivalence problems (e.g., $3 + 4 + 5 = 3 + \underline{\quad}$) in which they are taught to “cancel the two 3’s and add 4 + 5 to get 9” often fail to transfer that procedure to novel problems (e.g., McNeil & Alibali, 2000). Thus, math equivalence is an appropriate domain in which to investigate how children gain understanding from learning procedures.

We conducted three experiments to test a new method for helping children extend their knowledge beyond a simple instructed procedure. In Experiment 1, we tested our hypothesis that concreteness fading will foster a greater understanding of math equivalence than concrete, abstract, or “reverse fading” methods for children with low prior knowledge. Experiment 2 was included as a follow-up to rule out an alternative hypothesis in favor of the “fading” hypothesis. In Experiment 3, we tested the generalizability of concreteness fading to children with high prior knowledge.

2. Experiment 1

In Experiment 1, we examined children’s transfer after receiving instruction with concrete materials, abstract materials, or concrete materials that faded to more abstract representations. Additionally, we included a fourth instruction condition that presented abstract materials that increased in concreteness to determine whether the progression from concrete to abstract is important. We expected children to develop a better understanding of math equivalence after receiving instruction with concrete materials that faded to more abstract representations.

2.1. Method

2.1.1. Participants

Participants were 179 second- and third-grade children from public and private elementary schools in the midwestern United States. Of those children, 64 met criteria for participation because they could not solve any math equivalence problems correctly. We screened for children who could not solve these problems correctly because our goal was to test which instruction condition was best for children with low prior knowledge of math equivalence. The screening measure included four math equivalence problems taken from McNeil, Fyfe, Petersen, Dunwiddie, and Brletic-Shipley’s (2011) equation solving measure of math equivalence understanding (see Appendix). Children were given feedback about correctness after each problem, and they were included in the study if they solved all four problems incorrectly. Similar measures of math equivalence understanding have been used in many previous studies of children in this age range (e.g., Chesney et al., 2014; Cook, Mitchell, & Goldin-Meadow, 2008; Matthews & Rittle-Johnson, 2009; McNeil et al., 2012). One child had to be excluded due to experimenter error during instruction. Thus, the final sample

contained 63 children (ages 7–9; M age = 8 yrs, 4 mo; 34 girls, 29 boys; 46% white, 32% African American or black, 21% Hispanic or Latino, 2% Asian).

2.1.2. Design

The experiment had a between-subjects design with condition as the independent variable and number correct on the transfer test as the dependent variable. The experiment consisted of two phases: an instruction phase and a test phase. For the instruction phase, children were randomly assigned to one of four conditions: (a) concrete ($n = 16$), (b) abstract ($n = 19$), (c) concreteness fading ($n = 13$), or (d) concreteness introduction ($n = 13$). For the test phase, children solved five transfer problems. Both phases occurred during a single 30-min session. There were no significant differences between children in the four conditions in terms of age, $F(3, 59) = 1.11, p = .35$, gender, $\chi^2(3, N = 63) = 0.19, p = .98$, grade, $\chi^2(3, N = 63) = 0.74, p = .86$, or ethnicity, $\chi^2(9, N = 63) = 14.76, p = .10$.

2.1.3. Instruction phase

All children received one-on-one instruction on the same six reflexive, right-blank math equivalence problems (i.e., $a + b = a + \underline{\quad}$; see Appendix for the exact problems used). Thus, amount of instruction was equated across conditions. The only difference between conditions was the format of the problem, which varied in its concreteness versus abstractness. The instruction was scripted, and included prompts to elicit responses from the children. After children stated an answer to each problem, they were given feedback about correctness. If children provided an incorrect response, they received more explicit instruction as well as a second chance to solve the problem. If children were incorrect a second time, they were given the correct answer and an explanation for why it was correct. The script was designed to be parallel across conditions (see Appendix for the complete scripts).

2.1.3.1. Concrete. Children received instruction with concrete materials, first in the context of Monkey and Frog sharing stickers equally (3 problems) and then in the context of balancing objects on a scale (3 problems). See Fig. 1 for pictures of the objects used. In the Monkey and Frog context, each puppet had two sticker collectors (small red squares of paper in the web version). In the balance scale context, children were presented with a balance scale with two empty bins.

2.1.3.2. Abstract. Children received instruction with six abstract, symbolic math equivalence problems written on paper (e.g. $2 + 3 = 2 + \underline{\quad}$). After a simple introduction to the right and left side of the equation, children were asked to determine what number they should write in the blank so that the right side had the same amount as the left side.

2.1.3.3. Concreteness fading. Children received instruction in three formats: first with concrete materials, then with “fading” worksheets, and finally with abstract problems. See Fig. 2 for pictures of this progression. The first three problems were taught in the context of Monkey and Frog sharing stickers equally. We first used Monkey puppet, Frog puppet, and their sticker collectors. Second, we used a “fading” worksheet with pictures of a monkey and a frog above a blank equivalence problem (e.g., $\underline{\quad} + \underline{\quad} = \underline{\quad} + \underline{\quad}$). The worksheet was introduced as a similar sharing game, but on paper as opposed to with puppets. Third, we used a written, symbolic math equivalence problem. The last three problems followed this sequence in regard to balancing objects on a balancing scale. Thus, we first used the balancing scale with small toy bears. Second, we used a “fading” worksheet with a picture of a balancing scale below



Fig. 1. Concrete materials used during instruction in the concrete condition.

a blank equivalence problem. Third, we used a written, symbolic math equivalence problem.

2.1.3.4. Concreteness introduction. Children were instructed in the same formats as the concreteness fading condition, but in the reverse order. Thus, the formats increased in concreteness. Children were taught first with abstract, symbolic problems, then with worksheets, and finally with concrete materials. The concrete context (e.g., sharing stickers or balancing a scale) was first introduced with the worksheets. Again, the first three problems were taught in the context of Monkey and Frog sharing stickers equally and the last three problems were taught in the context of balancing objects on a balancing scale. The concreteness introduction condition was included in order to determine whether the inclusion of an intermediate stage is sufficient to benefit learning or whether the specific progression from concrete to abstract matters.

2.1.4. Transfer test phase

Following instruction, all children completed a transfer test with five written, novel math equivalence problems (see [Appendix](#)). We designed the transfer items to be more structurally complex than the instruction items. We based this on previous studies of math equivalence that had used transfer tests with similar-aged children (e.g., [Alibali, 1999](#); [Fyfe, Rittle-Johnson, & DeCaro, 2012](#); [Matthews & Rittle-Johnson, 2009](#); [Perry, 1991](#)). Problems differed from instructional problems in terms of addends on the left side (e.g., three instead of two), the position of the blank ($a + b + c = \underline{\quad} + c$), and the presence/absence of a repeated addend ($a + b + c = d + \underline{\quad}$). One problem was a multiple-choice story-to-equation translation

problem. All children were told: "These problems are just like the problems we just solved together. You should think about them the same way." We specifically chose transfer problems that were more structurally complex than the instruction problems in order to challenge children's thinking about math equivalence and see if they gained knowledge that they could extend beyond the instructed context. Because of that, and because of children's low prior knowledge, we did not expect high performance on the transfer test overall. We were more interested in relative differences across conditions.

2.1.5. Coding

Children's strategies were coded based on a system developed by [Perry, Church, and Goldin-Meadow \(1988\)](#). Strategies were assigned based on the solutions that children wrote in the blank. For example, for the problem $4 + 8 + 9 = 4 + \underline{\quad}$, 17 would be coded as a correct strategy. However 25 (adding all numbers), 21 (adding the numbers before the equal sign), or 8 (copying a number from the left side) would be coded as incorrect strategies. As in prior work ([McNeil, 2007](#)), solutions were coded as reflecting a correct strategy if they were within 1 of the number that would be achieved with a correct strategy (e.g., 16 or 18 for $4 + 8 + 9 = 4 + \underline{\quad}$).

2.2. Results

2.2.1. Instruction phase

All children could commit one error per problem, for a total of six errors in the instruction phase. Children made few errors across the six problems ($M = 0.57$, $SD = 0.93$). Trial-by-trial data revealed

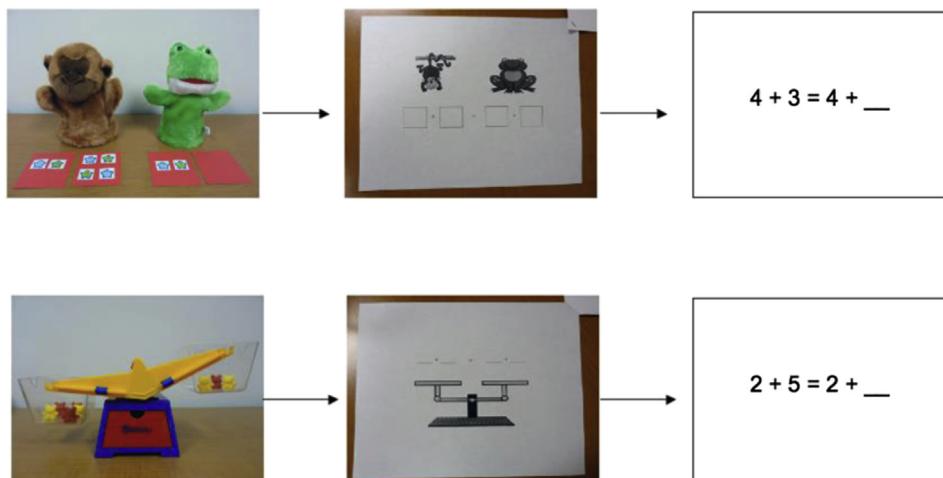


Fig. 2. Progression of materials used during instruction in the concreteness fading condition.

that 29% of children made an error on the first problem, 10% made an error on the second problem, 5% on the third problem, 8% on the fourth problem, 2% on the fifth problem, and 5% on the final problem. Thus, the first problem was the only problem with which children had some difficulty. We examined the percentage of children in each condition who committed an error on the first problem and we found that it differed by condition, $\chi^2 (3, N = 63) = 27.13, p < .001$. None of the children in the concrete (0 of 16) and concreteness fading (0 of 14) conditions made an error on the first problem, whereas children in both the abstract (13 of 19 [68%]) and concreteness introduction (5 of 14 [36%]) conditions did. Thus, children who solved a concrete problem as their first problem (concreteness fading and concrete) were less likely to make an error than were children who solved an abstract problem first (concreteness introduction and abstract), $\chi^2 (1, N = 63) = 22.91, p < .001$. This is consistent with previous research that suggests initial learning and performance may benefit more from concrete materials relative to abstract materials (Kaminski, Sloutsky, & Heckler, 2009).

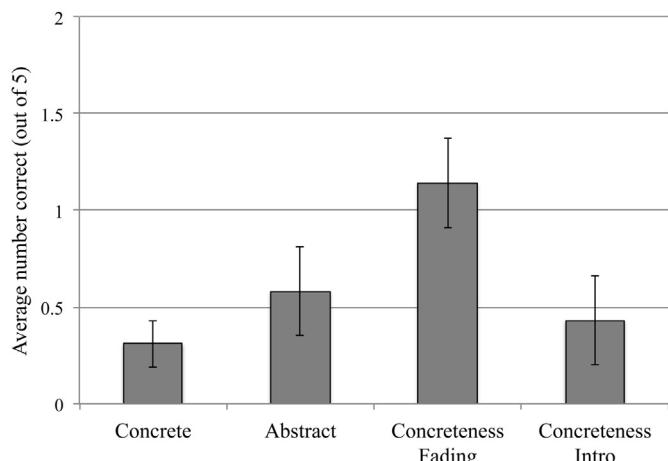
Children who stated an incorrect answer on any of the given problems received additional instruction and got a second chance to answer. Thus, it was possible for some children to commit up to two errors on each problem. However, few children (18%) actually committed two errors on any problem, and even fewer (2%) committed two errors on more than one problem.

Thus, all children received instruction on six relatively simple math equivalence problems and learned to solve them accurately with very few errors. This is not surprising given the basic and repetitive nature of the problems as well as the explicit instruction and feedback. Of particular interest, however, was children's performance on the subsequent transfer test, which revealed what children actually learned from the instruction and whether or not they could apply it to more varied and complex math equivalence problems. More generally, the transfer test allowed us to determine which condition helped children achieve deep, transferable knowledge of mathematical equivalence from a simple instructed procedure.

2.2. Transfer test phase

As expected, performance on the transfer test was poor overall, with children solving 0.60 ($SD = 0.87$) problems correctly on average (out of 5). Recall, our goal was not to see if children could master these transfer problems, but rather to determine which instructional condition facilitated transfer relative to the others. Fig. 3 displays the number of transfer problems solved correctly by children in each condition. To examine the effect of instruction condition on transfer performance, we performed a one-way ANOVA with condition (concrete, abstract, concreteness fading or concreteness introduction) as the independent variable and number correct on the transfer test (out of 5) as the dependent variable. There was a significant main effect of instruction condition, $F(3, 59) = 2.80, p = .048, \eta_p^2 = 0.13$ (Fig. 3).

To test our specific *a priori* predictions of interest, we performed the set of orthogonal Helmert contrasts. Specifically, we contrasted (1) concreteness fading versus the three comparison conditions, (2) concreteness introduction versus the non-fading conditions, and (3) abstract versus concrete. As predicted, children in the concreteness fading condition solved more transfer problems correctly ($M = 1.14, SD = 0.86$) than did children in the other conditions ($M = 0.45, SD = 0.82$), $F(1, 59) = 7.68, p = .007, d = 0.76$. Additionally, the transfer performance of children in the concreteness introduction condition ($M = 0.43, SD = 0.85$) was not statistically different from the transfer performance of children in the two non-fading conditions ($M = 0.46, SD = 0.82$), $F(1, 59) = 0.004, p = .95, d = 0.03$. Finally, the transfer performance of



Note. Errors bars represent standard errors.

Fig. 3. Children's performance on the transfer test by condition (Experiment 1).

children in the abstract condition ($M = 0.58, SD = 1.02$) was not statistically different from the transfer performance of children in the concrete condition ($M = 0.31, SD = 0.48$), $F(1, 59) = 0.88, p = .35, d = 0.31$. A trial-by-trial examination revealed that accuracy was similar across the transfer problems.

Given that scores on the transfer test were low and not normally distributed, we also performed a nonparametric analysis to ensure that the effects did not depend on the method of analysis. Across conditions, 41% of children solved at least one problem correctly. We used binomial logistic regression to predict the log of the odds of solving at least one transfer problem correctly (see Agresti, 1996). In line with the ANOVA, three Helmert contrast codes were used to represent the four levels of condition: (1) concreteness fading versus the three comparison conditions, (2) concreteness introduction versus the non-fading conditions, and (3) abstract versus concrete. As predicted, participants in the concreteness fading condition were more likely than participants in the other three conditions to solve at least one problem correctly (10 of 14 [71%] versus 16 of 49 [31%]), $\hat{\beta} = 1.66, z = 2.49$, Wald (1, $N = 63$) = 6.21, $p = .01$, OR = 5.28. There were no statistical differences between the concreteness introduction and non-fading conditions (4 of 14 [29%] versus 12 of 35 [34%]), $\hat{\beta} = -0.25, z = -0.36$, Wald (1, $N = 49$) = 0.13, $p = .72$, OR = 0.78, or between the abstract and concrete conditions, (7 of 19 [37%] versus 5 of 16 [31%]), $\hat{\beta} = -0.25, z = -0.35$, Wald (1, $N = 35$) = 0.12, $p = .73$, OR = 0.78.

The most common incorrect strategy across the four equation-solving problems was to carry a number from the left side of the equation to the blank (e.g., write 8 in the blank for $4 + 8 + 9 = 4 + \underline{\hspace{1cm}}$). This "carry" strategy was used on 36% of all trials, and 63% of children used it at least once. This was more prevalent than we would expect based on their performance on the screening measure (used on 17% of all trials, 43% of children used it at least once). We suspect this is a result of negative transfer from the instruction phase, in which all of the problems could be solved using this narrow procedure. However, the percentage of children who used the carry strategy on the transfer test did not differ by condition, $\chi^2 (3, N = 63) = 5.42, p = .14$.

Other incorrect strategies included adding all the numbers, adding the numbers before the equal sign, or adding two numbers. These three strategies were each used on about 10% of trials. We were particularly interested in the "add all" strategy because children who have not received instruction on math equivalence commonly use it to solve the problems (McNeil & Alibali, 2000).

The “add all” strategy was common on the screening measure for children who pretested into the study (25% of trials, 65% of children used it at least once; similar across conditions). However, after receiving instruction, none of the children in the concreteness fading used the “add all” strategy on the test, whereas children in all other conditions did (7% concreteness introduction, 26% abstract, and 31% concrete), $\chi^2 (3, N = 63) = 7.14, p = .06$. A similar pattern emerged on the multiple-choice story problem. The most common incorrect answer was the “add all” option (i.e., $5 + 4 + 7 = \underline{\hspace{2cm}}$). Fewer children in the concreteness fading condition (21%) selected the “add all” option compared to children in the concreteness introduction (57%), abstract (58%), and concrete (75%) conditions, $\chi^2 (3, N = 63) = 8.99, p = .03$.

2.3. Discussion

In Experiment 1, our hypotheses were supported. Children in the concreteness fading condition solved more transfer problems correctly than children in the other conditions. The fact that concreteness fading resulted in better transfer than abstract alone is quite compelling given that the transfer problems were more similar to the abstract problems. Additionally, children in the concreteness introduction condition did not perform better than children in the abstract and concrete conditions, ruling out a number of alternative explanations for why concreteness fading is superior. For example, children in both the concreteness fading and concreteness introduction conditions solved problems in three different formats (i.e., concrete, abstract, worksheet), received instruction in multiple contexts (i.e., puppets sharing stickers and balancing a scale), actively constructed sets (of stickers and toy bears) of a given cardinality, and were encouraged to map concrete examples to abstract representations. Thus, it appears it was the specific progression of formats (i.e., starting with concrete and fading to abstract) that influenced children’s understanding of math equivalence. These results provide evidence in support of concreteness fading.

However, there is reason to be cautious when interpreting these results. One could argue that the abstract portion of the concreteness fading condition was in fact driving the transfer benefits. The concreteness fading condition may have promoted greater transfer than the abstract condition simply because the concrete and fading stages of the instruction “warmed up” the children, so they were ready to attend to abstract instruction. When children in the concreteness fading condition entered the experiment, they were introduced to monkey and frog and participated in the engaging task of handing out stickers. In contrast, children in the abstract condition were immediately instructed on a written math problem, which may have led them to regard the situation as a typical, boring math exercise or to “zone out” and thus not attend to the beneficial instruction. According to this alternative “warm-up” hypothesis, abstract problem formats may in fact result in superior learning and transfer as long as children are warmed up, or ready to attend to the instruction. Importantly, this hypothesis suggests the math components of the concrete and fading stages were irrelevant; rather, the fact that they “warmed up” the children’s attention for the ensuing abstract instruction is what mattered. To address this concern, we conducted a brief follow-up experiment with a new instructional condition—the play-to-abstract condition. This condition included two stages of “play” to engage the children before receiving instruction on abstract math equivalence problems. The two “play” stages were matched as closely as possible to the concrete and fading stages of the concreteness fading condition, but included no instruction.

3. Experiment 2

Experiment 2 was designed to rule out an alternative interpretation of the results from Experiment 1 and provide further evidence in support of concreteness fading. Specifically, we tested if concreteness fading is beneficial for transfer because of the specific progression from concrete to abstract formats, or if it merely works by “warming up” children’s attentional focus for abstract instruction. In the concreteness fading condition, children received instruction on math equivalence with concrete materials that faded into abstract materials, whereas children in the play-to-abstract condition engaged in two stages of “play” in the same concrete contexts and then received instruction with abstract problems. We expected children to develop a better understanding of math equivalence after receiving instruction in the concreteness fading condition.

3.1. Method

3.1.1. Participants

Participants were 61 first- and second-grade children from public and private elementary schools in a mid-sized city in the midwestern United States. To improve generalizability and to maximize the number of children who could participate, we used a more lenient inclusion criteria than we had in the previous experiment. Instead of excluding children who solved any math equivalence problems correctly on the screening measure, we excluded only those children who solved three or four (out of 4) math equivalence problems correctly. Thirty-nine children were excluded based on this criterion. The final sample included 22 children (M age = 7 yrs, 5 mo; 10 girls, 12 boys; 68% white, 32% African American).

3.1.2. Design

The design and procedure were identical to Experiment 1, but included two instructional conditions instead of four. Children were randomly assigned to the concreteness fading condition ($n = 10$) or to the play-to-abstract condition ($n = 12$). Children in these two conditions solved a comparable number of math equivalence problems correctly on the screening measure, $t(20) = 0.19, p = .85$. Additionally, children in the two conditions were well matched in terms of age, $F(1, 20) = 0.56, p = .47$, gender, $\chi^2 (1, N = 22) = 0.22, p = .64$, grade, $\chi^2 (1, N = 22) = 1.56, p = .21$, and ethnicity, $\chi^2 (1, N = 22) = 1.18, p = .28$.

3.1.3. Instruction phase

Instruction in the *concreteness fading* condition was identical to that in Experiment 1. Children progressed through the three stages (i.e., concrete, fading, abstract) in the context of Monkey and Frog sharing stickers equally and then in the context of balancing objects on a scale (see Fig. 2). Instruction in the *play-to-abstract* condition progressed through the same three stages, but children only received mathematics instruction during the abstract stage. Thus, amount of abstract instruction was equated across conditions, but total amount of instruction differed. This design decision allowed us to test whether the first two steps in the concreteness fading sequence were necessary components of instruction or whether they merely “warmed” children up to working with the tutor so that they were more receptive and attentive to the abstract instruction. During the concrete and fading stages children played a pattern matching game to “ready” them for later instruction during the abstract stage. See the Appendix for the scripts used during the two “play” stages.

The first stage of the play-to-abstract condition involved placing stickers on Monkey and Frog's sticker collectors. The experimenter created a pattern by placing a set of stickers on Monkey's collector and then asked the child to make the same pattern on Frog's collector. The second stage included worksheets that were identical to the worksheet used in the concreteness fading condition, with the exception that the math symbols were not used. The experimenter created a pattern on one worksheet by drawing a set of circles in the empty boxes and then asked the child to draw the same pattern on an identical worksheet. Finally, the third stage involved solving a written, symbolic math equivalence problem. Children received the same instruction as children in the abstract condition in Experiment 1 (i.e., the same instruction in the abstract phase of the concreteness fading condition). In the context of the balance scale, there were also two stages of "play" followed by abstract instruction. Children matched the experimenter's pattern of toy bears on a balance scale, then matched a drawn pattern on the worksheets, and finally solved a written, symbolic math equivalence problem.

3.2. Results

3.2.1. Instruction phase

We examined errors on the two abstract math equivalence problems solved by children in both conditions. All children could commit one error per problem, for a total of two errors. Children made few errors on the two abstract problems ($M = 0.18$, $SD = 0.40$). On the first abstract problem, more children committed an error in the play-to-abstract condition (4 of 12, [33%]) than in the concreteness fading condition (0 of 10), $\chi^2 (1, N = 22) = 4.07$, $p = .04$. On the final abstract problem, no errors were committed. This is consistent with Experiment 1; children who receive instruction on abstract problems first tend to make more errors initially than children who receive instruction on concrete problems first. Children who stated an incorrect answer were given a second chance. Only one child committed an error on a second attempt. This child was in the play-to-abstract condition.

3.2.2. Transfer test phase

Similar to Experiment 1, performance on the transfer test was poor overall, with children solving 0.73 ($SD = 1.39$) problems correct on average (out of 5). We performed an analysis of covariance (ANCOVA) with instruction condition (concreteness fading or play-to-abstract) as the independent variable, number correct on the prior knowledge screening measure (out of 4) as a covariate, and number correct on the transfer test (out of 5) as the dependent variable. The number correct on the prior knowledge measure was not related to transfer performance, $F(1, 19) = 1.83$, $p = .19$, $\eta_p^2 = 0.09$. As predicted, there was a significant main effect of instruction condition, $F(2, 19) = 5.64$, $p = .028$, $\eta_p^2 = 0.23$. Children in the concreteness fading condition solved more transfer problems correctly ($M = 1.40$, $SD = 1.84$) than did children in the play-to-abstract condition ($M = 0.17$, $SD = 0.39$). The effect of condition remained when the covariate was removed from the model, $F(2, 19) = 5.18$, $p = .034$, $\eta_p^2 = 0.21$.

Given that transfer scores were low and not normally distributed, we also performed a chi-square test to ensure that the observed effects did not depend on the method of analysis. Across conditions, 32% of children solved at least one problem correctly. Results from the chi-square test were consistent with the ANOVA. Participants in the concreteness fading condition were more likely than participants in the play-to-abstract condition to solve at least one problem correctly (5 out of 10 [50%] versus 2 of 12 [17%]), though the effect was only marginally significant, $\chi^2 (1, N = 22) = 2.79$, $p = .09$, presumably because of low power.

3.3. Discussion

Results from Experiment 2 were consistent with Experiment 1 and supported our hypothesis. Children in the concreteness fading condition solved more transfer problems correctly than children in the play-to-abstract condition. These results provide evidence against the alternative "warm-up" hypothesis and evidence for the "fading" hypothesis. The first two steps in the concreteness fading sequence were necessary components of instruction; they did not just "warm up" children so they were ready to attend to abstract instruction. Although we used a more lenient inclusion criteria in Experiment 2 relative to Experiment 1, error patterns and transfer scores in the concreteness fading condition were similar across Experiments 1 and 2, indicating that the fading instructional method created similar learning environments for the two distinct samples of children. Thus, Experiment 2 replicated the benefits of concreteness fading found in Experiment 1 and ruled out an alternative explanation.

Despite the positive effects of the fading method demonstrated in Experiments 1 and 2, it remains unclear how concreteness fading affects the transfer performance of individuals with high prior knowledge. In Experiments 1 and 2, we only tested children with relatively low knowledge of math equivalence. However, a relevant study by Goldstone and Sakamoto (2003) indicated that individuals with high knowledge in the target domain are less influenced by superficial, concrete representations than individuals with low knowledge. This suggests that individuals with high knowledge may do well regardless of whether concrete materials are faded or not. To address this concern, we conducted a third experiment with children who demonstrated higher prior knowledge of math equivalence on the screening measure. We included the four instructional conditions from Experiment 1 and taught an advanced problem-solving procedure.

4. Experiment 3

Experiment 3 was designed to test whether the results from Experiment 1 extend to children with higher prior knowledge in the target domain and to provide further evidence in support of concreteness fading. Specifically, we focused on children who already knew one correct procedure for solving math equivalence problems. We examined their transfer performance after receiving instruction on a new, advanced procedure with concrete materials, abstract materials, concrete materials that faded to more abstract representations, or abstract materials that increased in concreteness. We expected children to develop a better understanding of math equivalence after receiving instruction with concrete materials that faded to abstract representations.

4.1. Method

4.1.1. Participants

Participants were 149 second- and third-grade children from public and private elementary schools in the midwestern United States. Of those children, 50 met criteria for participation because they solved at least three of the four math equivalence problems correctly on the screening measure. We screened for children who could solve math equivalence problems correctly because our goal was to test which instruction condition was best for children with higher prior knowledge. One child had to be excluded due to experimenter error during instruction. Thus, the final sample contained 49 children (ages 7–9; M age = 7 yrs, 8 mo; 29 girls, 20 boys; 71% white, 8% African American or black, 14% Hispanic or Latino, 6% Asian).

4.1.2. Design

The design and procedure were identical to Experiment 1, but included instruction on a novel, advanced problem-solving procedure. Children were randomly assigned to one of four conditions: (a) concrete ($n = 14$), (b) abstract ($n = 14$), (c) concreteness fading ($n = 9$), or (d) concreteness introduction ($n = 12$). All children solved either three or four math equivalence problems correctly on the screening measure. Despite random assignment, the likelihood of solving all four problems correctly was not the same across conditions, $\chi^2(3, N = 49) = 7.21, p = .07$; however, it did not favor the concreteness fading condition. The percentage of children who solved all four correctly in each condition was 93% concrete, 57% abstract, 67% concreteness fading, and 92% concreteness introduction. Conditions also differed in terms of the percentage of girls (versus boys), $\chi^2(3, N = 49) = 11.96, p = .007$. The percentage of girls in each condition was 85% concrete, 43% abstract, 33% fading, and 75% concreteness introduction. We controlled for these initial differences across conditions in our analyses. Random assignment was better in terms of matching children in the four conditions in terms of age, $F(3, 45) = 1.89, p = .14$, grade, $\chi^2(3, N = 49) = 2.80, p = .42$, and ethnicity, $\chi^2(9, N = 49) = 8.54, p = .48$.

4.1.3. Instruction phase

All children received one-on-one instruction on the same six reflexive, right-blank math equivalence problems (i.e., $a + b = a + \underline{\hspace{1cm}}$). Thus, amount of instruction was equated across conditions. The instruction phase was identical to that of Experiment 1, except the instruction focused on teaching an advanced procedure for solving math equivalence problems called the “cancel” or grouping strategy. Importantly, although children in this study could solve math equivalence problems correctly, none of them specifically used the cancel strategy to solve the screening problems. The cancel strategy can be used on reflexive math equivalence problems (e.g., $3 + 4 = 3 + \underline{\hspace{1cm}}$) by “canceling” the equivalent addends on the opposite sides of the equal sign (e.g., the two 3's in the previous example) and putting the remaining addend (e.g., 4) in the blank. See the [Appendix](#) for the complete set of scripts. In the *concrete* condition, children received instruction with concrete materials, first in the context of Monkey and Frog sharing stickers equally and then in the context of balancing objects on a scale. In the *abstract* condition, children received instruction with six abstract, symbolic math equivalence problems written on paper. In the *concreteness fading* condition, children progressed through the three stages (i.e., concrete, fading, abstract) in the context of Monkey and Frog sharing stickers equally and then in the context of balancing objects on a scale. In the *concreteness introduction* condition, children progressed through the three stages in reverse order (i.e., abstract, fading, concrete) in the context of Monkey and Frog sharing stickers equally and then in the context of balancing objects.

4.1.4. Transfer test phase

The transfer test included six written, novel math equivalence problems (see [Appendix](#)). Most problems were adapted from the transfer items in Experiment 1. For example, the item from Experiment 1 (i.e., $3 + 9 + 5 = 7 + \underline{\hspace{1cm}}$), was adapted (i.e., $3 + 4 + 5 = 7 + \underline{\hspace{1cm}}$) so that children could cancel the “3 + 4” on the left side of the equal sign and the “7” on the right side of the equal sign. The fifth problem on the transfer test ($258 + 29 + 173 = 29 + \underline{\hspace{1cm}} + 258$) was a “challenge” problem specifically designed to elicit the cancel strategy. We designed this problem based on [Siegler and Jenkins' \(1989\)](#) “carrot and stick” approach for eliciting a desired strategy. Specifically, the problem could be solved quite easily and quickly with the cancel strategy, but other

correct strategies (e.g., add-subtract) could not be as easily used on the problem.

4.2. Results

4.2.1. Instruction phase

All children could commit one error per problem, for a total of six errors. Children made almost no errors across the six problems ($M = 0.04, SD = 0.20$). Indeed, only two children ever made an error (one child in the concrete condition and one child in the abstract condition). Of particular interest, however, was children's performance on the subsequent transfer test and whether they used the advanced cancel strategy.

4.2.2. Transfer test phase

Unlike Experiments 1 and 2, performance on the transfer test was relatively good overall, with children solving 3.78 ($SD = 1.78$) problems correctly (out of 6). We performed an analysis of covariance (ANCOVA) with instruction condition (concrete, abstract, concreteness fading, or concreteness introduction) as the independent variable, correctness on the screening measure (0 = three correct, 1 = four correct), and gender (0 = boy, 1 = girl) as covariates, and number correct on the transfer test (out of 6) as the dependent variable. Correctness on the screening measure was related to transfer performance, $F(1, 43) = 5.23, p = .03, \eta_p^2 = 0.11$, but gender was not, $F(1, 43) = 1.32, p = .26, \eta_p^2 = 0.03$. There was not a significant effect of instruction condition, $F(3, 43) = 0.31, p = .82, \eta_p^2 = 0.02$. Thus, children in all four conditions scored similarly on the transfer test (concrete: $M = 3.38, SE = 0.50$; abstract: $M = 3.91, SE = 0.49$; concreteness fading: $M = 3.91, SE = 0.62$; concreteness introduction: $M = 3.98, SE = 0.52$). Result did not change when we performed a nonparametric analysis.

Next we examined children's correct use of the cancel strategy. The cancel strategy was relatively infrequent, accounting for only 18% of the correct strategies used across all problems. However, 41% of children did use it at least once. We used binomial logistic regression to predict the log of the odds of using the cancel strategy correctly at least once. As in the previous analyses, we used three Helmert contrast codes to represent the four levels of condition and controlled for correctness on the screening measure and gender. There was not a statistical difference between the concreteness fading and other three conditions (5 of 9 [56%] versus 15 of 40 [38%]), $\hat{\beta} = 1.15, z = 1.34$, $Wald(1, N = 49) = 1.77, p = .18$, $OR = 3.15$. Nor was there a statistical difference between the concreteness introduction and non-fading conditions (3 of 12 [25%] versus 12 of 28 [43%]), $\hat{\beta} = -1.03, z = -1.27$, $Wald(1, N = 40) = 1.60, p = .21$, $OR = 0.36$. However, the difference between the abstract and concrete conditions was marginally significant (8 of 14 [57%] versus 4 of 14 [29%]), $\hat{\beta} = 1.79, z = 1.87$, $Wald(1, N = 28) = 3.50, p = .06$, $OR = 5.98$. Neither control variable was associated with use of the cancel strategy, both $p > 0.2$.

Finally, we performed a similar logistic regression analysis to examine children's performance on the challenge problem that was specifically designed to elicit the cancel strategy (i.e., $258 + 29 + 173 = 29 + \underline{\hspace{1cm}} + 258$). We first classified children according to whether they solved this problem correctly (see [Fig. 4](#)). Across conditions, 27% of children solved the challenge problem correctly. As predicted, participants in the concreteness fading condition were significantly more likely than participants in the other three conditions to solve the problem correctly (5 of 9 [56%] versus 8 of 40 [20%]), $\hat{\beta} = 2.24, z = 2.32$, $Wald(1, N = 49) = 5.36, p = .02$, $OR = 9.44$. There were no statistical differences between the concreteness introduction and non-fading conditions (2 of 12 [17%] versus 6 of 28 [21%]), $\hat{\beta} = -0.52, z = -0.55$, $Wald(1, N = 40) = 0.30, p = .58$, $OR = 0.60$, or between the abstract and

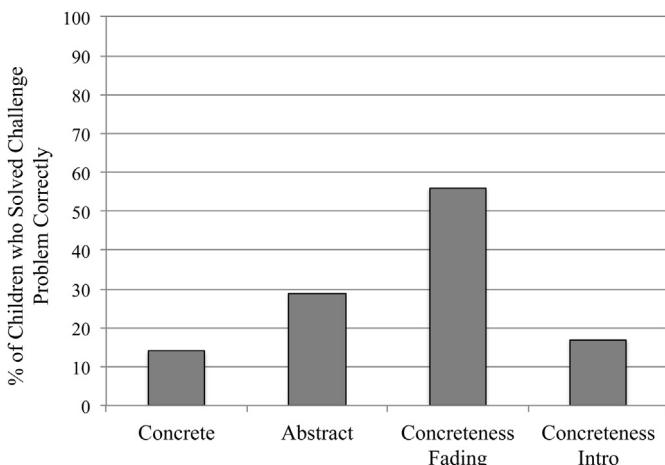


Fig. 4. Children's performance on the challenge transfer problem by condition (Experiment 3).

concrete conditions, (4 of 14 [29%] versus 2 of 14 [14%]), $\hat{\beta} = 1.56$, $z = 1.45$, $\text{Wald } (1, N = 28) = 2.09$, $p = .15$, $OR = 4.75$. Neither control variable was associated with correctness on the challenge problem, both $p's > .2$.

Next, we classified children according to whether they used the cancel strategy on this problem. Across conditions, 22% of children used the cancel strategy to solve this problem. Consistent with our predictions, children in the concreteness fading condition were more likely than participants in the other conditions to use the cancel strategy on the challenge problem (5 of 9 [56%] versus 6 of 40 [15%]), $\hat{\beta} = 2.24$, $z = 2.32$, $\text{Wald } (1, N = 49) = 5.36$, $p = .02$, $OR = 10.33$. There were no statistical differences between the concreteness introduction and non-fading conditions (2 of 12 [17%] versus 6 of 28 [21%]), $\hat{\beta} = 0.16$, $z = 0.16$, $\text{Wald } (1, N = 40) = 0.30$, $p = .58$, $OR = 0.60$, or between the abstract and concrete conditions, (3 of 14 [21%] versus 1 of 14 [7%]), $\hat{\beta} = 1.67$, $z = 1.27$, $\text{Wald } (1, N = 28) = 1.63$, $p = .20$, $OR = 5.29$. Neither control variable was associated with correctness on the challenge problem, both $p's > 0.2$.

4.3. Discussion

Results from Experiment 3 were mostly consistent with Experiments 1 and 2 and showed some benefits of concreteness fading. Although children in all conditions performed well on the transfer test, the challenge problem revealed meaningful differences. Children in the concreteness fading condition were more likely than children in the other three conditions to use the advanced, cancel strategy on the challenge problem. They were also more likely than children in the other conditions to solve the challenge problem correctly. These results suggest that the benefits of concreteness fading extend to children with higher prior knowledge. Thus, Experiment 3 provided additional support for the fading hypothesis, indicating that the benefits of concreteness fading are not limited to children with low prior knowledge in the target domain.

5. General discussion

In the present study, we tested the effects of concreteness fading on children's ability to extend their knowledge beyond a simple instructed procedure. In Experiment 1, low-knowledge children received instruction on a procedure in one of four conditions: concrete, abstract, concreteness fading, or concreteness introduction. In Experiment 2, we addressed an alternative hypothesis by

including a condition in which children played through the concrete and fading stages to "warm them up" for the abstract instruction. Finally, in Experiment 3, we examined the effects of concreteness fading for teaching an advanced procedure to children with higher prior knowledge. In all three experiments, children in the concreteness fading condition exhibited better transfer than children in the other conditions. Overall, results suggest that children benefit more from instructed procedures when those procedures are taught with concrete materials that fade into abstract representations.

Extending one's knowledge appropriately beyond an instructed procedure can be quite difficult (e.g., Rittle-Johnson & Alibali, 1999; Singley & Anderson, 1989). Learners either fail to see that the procedure applies in a novel situation, or they readily transfer the procedure to the situation, but without adapting it appropriately. For example, before children receive instruction on math equivalence problems, they often "transfer in" an incorrect add-all strategy from their experience with typical arithmetic problems (e.g., $3 + 7 = \underline{\hspace{2cm}}$) on which the strategy works successfully (McNeil & Alibali, 2005). Even after children receive instruction, correct transfer is often limited to problems similar to those presented during instruction. For example, when children are presented with $3 + 4 + 5 = 3 + \underline{\hspace{2cm}}$ and are taught to "cancel the two 3's and add 4 + 5 to get 9," they subsequently solve $6 + 3 + 7 = 6 + \underline{\hspace{2cm}}$ correctly by adding $3 + 7$, but they also add $3 + 7$ when presented with $6 + 3 + 7 = 4 + \underline{\hspace{2cm}}$ (McNeil & Alibali, 2000). Similarly, in the current study, all children learned how to apply a correct procedure to solve a specific type of math equivalence problem after explicit instruction and feedback. However, most children had difficulty applying and adapting the procedure to novel problem types. Importantly, concreteness fading resulted in superior performance relative to the other conditions and reduced children's incorrect transfer of the "add-all" strategy. These results suggest the fading technique helped children extract a better understanding of the underlying concept from the instructed procedure.

The benefits of concreteness fading may be explained by the fact that it starts with a well-understood concrete format, and explicitly links and fades it to the abstract symbols. Concrete materials are advantageous initially because they allow the math concept to be grounded in easily understood, real-world scenarios (Baranes et al., 1989; Kotovsky et al., 1985). In the present study, children learned equivalence in terms of sharing stickers and balancing a scale, which enabled them to evaluate their answers in a meaningful context. Indeed, previous research suggests that children understand math equivalence in concrete contexts (Sherman & Bisanz, 2009). Thus, when concrete materials are presented first, they activate relevant, helpful knowledge. In support of this point, children who solved a concrete problem first were less likely to make an error than children who solved an abstract problem first. However, if given only concrete instruction, children's knowledge may become too tied to the concrete context and may not transfer to dissimilar situations (e.g., Kaminski et al., 2009). Indeed, children in the concrete condition did not transfer their situated knowledge of math equivalence to the symbolic context.

Conversely, the advantage of abstract, symbolic problem formats is that they promote more context-independent knowledge, which is applicable in a variety of learning situations (Goldstone & Sakamoto, 2003; Kaminski et al., 2008; Sloutsky, Kaminski, & Heckler, 2005). The abstract, context-free symbols highlight the problem's relational structure by eliminating all extraneous or superficial information. However, abstract symbols often encourage the use of rote, procedures without understanding (Carraher et al., 1985). For example, in the context of math equivalence, abstract symbols often activate children's knowledge of arithmetic and the oft-used "add-all" strategy, which previous work suggests is

unhelpful (McNeil & Alibali, 2005). Thus, when abstract symbols are presented first, they can activate unhelpful knowledge that makes it harder to benefit from instruction. In the current study, children in the abstract condition learned a specific procedure, but did not understand the underlying concept sufficiently to adapt the procedure to new problems.

The key to the concreteness fading method appears to be the explicit link between the grounded, concrete materials, and the context-free abstract materials. The fading technique likely promoted correct transfer and prevented incorrect transfer of the “add-all” strategy because it started with understanding and then linked it to the abstract problem. By the time children solved abstract problems, their relevant, helpful knowledge had already been activated and the instructor had specifically linked that helpful knowledge to the abstract symbols. The linking between concrete and abstract appears to be key. For example, Kaminski et al. (2008) found that students who learned a math concept via abstract examples only exhibited better transfer than students who learned via a concrete example followed by an abstract one. However, the examples in this study were presented one after another *in isolation*, rather than in a linked, fading progression.

According to Goldstone and Son (2005), the key fading stage provides unique benefits in terms of the learner's interpretation of the problem elements. People have a natural tendency to interpret ambiguous objects (e.g., arbitrary symbols) in terms of unambiguous objects with which they are familiar (Leeper, 1935; Medin, Goldstone, & Gentner, 1993). Thus, if the abstract problem is preceded by and linked to a concrete problem, the learner can successfully interpret the abstract symbols in terms of the well-understood concrete objects. The key “fading” stage allows learners to more explicitly link the concrete and abstract materials as mutual referents.

Bruner (1966) provides an alternative explanation to account for the unique benefits of the fading progression. He argues that the concrete materials provide a store of images that embody the abstract symbols. Once an understanding of the abstract concept has been achieved, learners do not give up their imagery, but rather they rely on this stock of representations as a means of relating new problems to those already mastered. When the abstract symbols are forgotten or start to lose meaning, the images gained from learning with concrete materials can be retrieved to help remind learners of the relevant concepts and procedures in novel situations.

In addition to contributing to our understanding of learning and transfer, the current results also provide some practical insight into how to teach procedures. What should teachers do when instructing children on how to solve a math problem? One thing we know from previous research is that teaching about procedures should be accompanied by teaching about concepts (e.g., Hiebert & Grouws, 2007; Wittwer & Renkl, 2010). The combination is often more effective than procedural instruction alone. The current results extend this idea and suggest that when we teach a procedure, it may be best to start with a highly concrete, well-understood context and then slowly progress to teaching in the context of abstract symbols alone.

Despite the positive contributions of the current study, several questions remain. For example, do all types of learners benefit from concreteness fading? Research indicates that not all types of learners benefit from instructional techniques in the same way (Cronbach & Snow, 1977). From a developmental perspective, the fading technique may have positive effects for learners of various ages. We found benefits for elementary school children, and previous research has found similar benefits for middle-school students (Butler et al., 2003) and undergraduates (Goldstone & Son, 2005; McNeil & Fyfe, 2012). However, prior domain knowledge

may matter more than age or development per se. For example, although we found positive effects of concreteness fading for both low- and high-knowledge children, the effects were stronger for the low-knowledge learners. At least one study suggests that individuals with high knowledge are less influenced by superficial, concrete representations (Goldstone & Sakamoto, 2003). Thus, high-knowledge children may perform more similarly whether concrete materials are faded or not.

This also introduces the question of what qualifies as “low” and “high” knowledge. To be clear, children in Experiment 3 differed from children in Experiments 1 and 2 in that they *did* have prior knowledge of a correct strategy. They had some formal knowledge of math equivalence that could have allowed them to gain a deeper understanding of the taught procedure. However, our goal was to see what children gained from an instructed procedure, so we had to teach children a procedure they did not know. Thus, they had low knowledge of the advanced, cancel procedure. This could potentially explain why the challenge problem yielded the biggest difference between conditions, as it was most easily solved using the cancel strategy. Future work is needed to tease apart the differences among levels of prior knowledge and the type of knowledge that matters.

Another future inquiry is to clarify what we mean by “concrete” materials. Which types of concrete materials are best to start with and fade away from? In this study, the fading method began with actual physical objects. Several researchers suggest that the physical manipulation allowed by tangible objects is critically important for understanding (Glenberg, Brown, & Levin, 2007; Martin & Schwartz, 2005), and one study has shown that children with math difficulties perform best when the fading sequence includes the initial, physical step (Butler et al., 2003). However, it is not evident that physical manipulation is always feasible or even necessary. In fact, Bruner (1966) suggested it might be possible to by-pass the enactive stage (i.e., representing concepts with physical objects) for learners with well-developed symbolic understanding. If true, a more practical solution may involve a series of worksheets that begins with pictures or story problems, slowly integrates the abstract symbols, and ends with pure abstract notation. In addition to optimizing the implementation of the fading method, this research will help to clarify the definition of “concrete” as it is used in the learning and transfer literature.

Finally, the focus and specific aims of the study limit the scope of its practical implications. The experiments were specifically designed to test if concreteness fading helps children gain knowledge from a simple, instructed procedure, not to determine the best comprehensive intervention for teaching math equivalence. Thus, we specifically taught children a simple procedure and designed the transfer problems to be difficult enough that we could assess if children could extend their knowledge beyond the simple instructed procedure. Some prior work has found benefits of concreteness fading in remedial classrooms (Butler et al., 2003), but more research is needed to determine the practical value of concreteness fading across various educational settings and across longer periods of instructional time.

There are several avenues for future research. First, larger, more diverse samples are needed. The sample sizes in the current study were similar to those from other experiments of this kind, but small nonetheless. Second, more comprehensive instruction should be designed that contains information on concepts and procedures in the context of structurally-diverse problems. Third, the use of different concrete materials that afford different types of knowledge construction should be examined. For example, in the context of teaching equivalence, it may be important to include “weighted” materials, or materials that vary in value. In the current study, the concrete objects (e.g., bears and stickers) all had an implied value of

one and so the activities were limited to thinking of equivalence in terms of number of items. However, weighted materials (e.g., one nickel is worth five cents, one dime is worth ten cents) may allow for a more comprehensive understanding of equivalence. Finally, future work should include a wider range of problem types and problem difficulties on the transfer assessment to increase the range of outcomes. In the current study, transfer scores were low overall, which questions how much children in the concreteness fading conditions truly understood math equivalence.

Despite these limitations, the present study provides valuable insight into theories of learning and transfer. Some researchers advocate the use of concrete materials because they provide an intuitive context in which concepts can be easily learned. Others promote the use of abstract problem formats because they lead to more transferable, context-independent knowledge. The present study is one of few to include the three-step “fading” progression (Bruner, 1966), which attempts to link the concrete and abstract features in a manner conducive to learning and transfer. With concreteness fading, children can attain knowledge of an instructed procedure that is grounded in a meaningful, concrete scenario, yet still extend their knowledge beyond that specific concrete situation.

Appendix

Screening measure problems

The four problems below were used as the screening measure in Experiment 1, 2, and 3.

- 1.) $1 + 5 = \underline{\quad} + 2$
- 2.) $7 + 2 + 4 = \underline{\quad} + 4$
- 3.) $2 + 7 = 6 + \underline{\quad}$
- 4.) $3 + 5 + 6 = 3 + \underline{\quad}$

Instruction problems

The six problems below were used during the instruction in Experiment 1, 2, and 3.

- 1.) $4 + 3 = 4 + \underline{\quad}$
- 2.) $3 + 5 = 3 + \underline{\quad}$
- 3.) $2 + 4 = 2 + \underline{\quad}$
- 4.) $5 + 4 = 5 + \underline{\quad}$
- 5.) $3 + 2 = 3 + \underline{\quad}$
- 6.) $2 + 5 = 2 + \underline{\quad}$

Transfer test problems

The five problems below were used as the transfer test in Experiment 1 and 2.

- 1.) $4 + 8 + 9 = 4 + \underline{\quad}$
- 2.) $3 + 9 + 5 = 7 + \underline{\quad}$
- 3.) $9 + 2 + 7 = \underline{\quad} + 7$
- 4.) $6 + 4 + 8 = \underline{\quad} + 3$
- 5.) The boys want to have the same number of stars as the girls. Sara has 5 stars and Ashley has 4 stars. Mike has 7 stars. Dan has some stars, too. How many stars does Dan need for the boys and girls to have the same? Circle the number sentence that shows what is happening in the story problem above.
 - (a) $5 + \underline{\quad} = 4 + 7$
 - (b) $5 + 4 + 7 = \underline{\quad}$

- (c) $5 + 4 = 7 + \underline{\quad}$
- (d) $5 + 4 + 7 + 2 = \underline{\quad}$

The six problems below were used as the transfer test in Experiment 3.

- 1.) $4 + 8 + 9 = 4 + \underline{\quad}$
- 2.) $3 + 4 + 5 = 7 + \underline{\quad}$
- 3.) $9 + 2 + 7 = \underline{\quad} + 11$
- 4.) $8 + 5 + 5 = \underline{\quad} + 10$
- 5.) $258 + 29 + 173 = 29 + \underline{\quad} + 258$
- 6.) $5 + 4 - 2 = \underline{\quad} + 2$

Scripts used for instruction

Description of the conditions

Children in all conditions were instructed on the same six math equivalence problems. The conditions only differed in terms of the format of instruction as follows:

In the *abstract* condition, the problems were presented in abstract, symbolic form.

In the *concrete* condition, the problems were presented in the puppet-sharing context (first 3 problems) followed by the balance scale context (final 3 problems).

In the *concreteness fading* condition, the problems were presented in formats that faded from concrete to abstract in the three steps recommended by Bruner (1966). Fading occurred first in the puppet-sharing context (1 concrete puppet-sharing problem → 1 puppet-sharing worksheet → 1 problem presented in abstract form), then in the balance scale context (1 concrete balance scale problem → 1 balance scale worksheet → 1 problem presented in abstract form).

In the *concreteness introduction* condition, the problems were presented in formats that started abstract and became progressively more concrete, reversing the three steps in the concreteness fading condition. Concreteness introduction occurred first in the context of puppets sharing (1 problem presented in abstract form → 1 puppet-sharing worksheet → 1 puppet-sharing problem), then in the context of the balance scale (1 problem presented in abstract form → 1 balance scale worksheet → 1 balance scale problem).

In the *play-to-abstract* condition (Experiment 2), children only solved two problems. A pattern matching game was presented in a concrete format and then a worksheet format. Then a problem was presented in abstract form. Play occurred first in the puppet context (1 pattern game with puppets → 1 pattern game with puppet worksheet → 1 problem presented in abstract form), and then in the context of the balance scale (1 pattern game with scale → 1 pattern game with scale worksheet → 1 problem presented in abstract form).

Script for problems presented in abstract form (Experiments 1 and 2)

We're going to solve a math problem together.¹ (Show child the math equivalence problem [e.g., $4 + 3 = 4 + \underline{\quad}$]). This is the left side of the problem (circle the left side with finger) and this is the right side of the problem (circle the right side with finger).

¹ In the concreteness fading condition, instruction on problems in symbolic form came right after children had worked on problems in the worksheet contexts, so this sentence was replaced with: “Now I want you to think about that game to help you solve this math problem.”

I want you to figure out how many more you should put on the right side (*point to the blank*), so the right side (*circle right side*) will have the same amount as the left side (*circle left side*).

If the child is correct: That's correct. Go ahead and write a 3 in the blank.

If the child is incorrect: Good try, but that won't make them equal. The left side has 7 (*point to the 4 while counting one, two, three, four and then point to the 3 while counting five, six, seven*). The right side only has 4 (*point to the 4 while counting one, two, three, four*). How many more does the right side need to have 7?

If the child is correct: That's correct. Go ahead and write a 3 in the blank.

If the child is still incorrect: Actually the right side needs 3 more. Write a 3 in the blank.

Let's check to make sure they're equal. How many does the left side have? (*Child should say 7*.) How many does the right side have? (*Child should say 7*.) Great! You figured it out. Both sides have 7! Let's do the next problem.

Script for concrete puppet-sharing context (Experiments 1 and 2)

We're going to play a game in which you share some of these stickers with Monkey and Frog.² I'll tell you how many stickers to give them, and you put the stickers on their sticker collectors.

First, give Monkey 4 stickers (*point to Monkey's first sticker collector*). Good!

Now give Monkey 3 more stickers (*point to Monkey's second sticker collector*). OK!

Give Frog 4 stickers (*point to Frog's first sticker collector*). Great!

Monkey and Frog want to have an equal amount of stickers (*Stress the word equal and gesture towards each character with hands up like a scale*). So, I want you to figure out how many more stickers you should give to Frog (*point to Frog's second sticker collector*), so Frog will have the same amount of stickers as Monkey (*point to Monkey's sticker collectors*).

If the child is correct: That's correct. Go ahead and give Frog 3 stickers.

If the child is incorrect: Good try, but that won't make them equal. Monkey has 7 stickers (*point to each of Monkey's stickers as you count them aloud*). Frog only has 4 stickers (*point to each of Frog's stickers as you count aloud*). How many more does Frog need so he can have 7 stickers?

If the child is correct: That's correct. Go ahead and give Frog 3 stickers.

If the child is still incorrect: Actually Frog needs 3 more stickers. Give Frog 3 stickers.

Let's check to make sure they're equal. How many stickers does Monkey have? (*Child should say 7*.) How many stickers does Frog have? (*Child should say 7*.) Great! You figured it out. Monkey and Frog both have 7 stickers!

² In the concreteness introduction condition, instruction on the concrete puppet-sharing problem came right after children had worked on problems in the worksheet contexts, so this sentence was preceded by: "We're going to play the same game but this time without the paper! Think about the math problem to help you play this game."

³ In the concreteness introduction condition, the worksheets came right after children had already worked on symbolic problems, so we replaced the first sentence with the following: "Now we're going to play a game on paper. This is similar to the math problem we just did together."

Script for puppet-sharing worksheet (Experiments 1 and 2)

Now we're going to play the same game on paper.³ (*Show child worksheet*). This stands for Monkey (*point to monkey*) and this stands for Frog (*point to frog*). Here are their sticker collectors (*point to the four boxes*). I'm going to tell you how many stickers to give them, and you will write the number in their sticker collectors, OK?

First, give Monkey 3 stickers (*point to Monkey's first collector, wait for child to write 3*). Good!

Give Monkey 5 more stickers (*point to Monkey's second collector, wait for child to write 5*).

Now give Frog 3 stickers (*point to Frog's first sticker collector, wait for child to write 3*). Great!

Monkey and Frog want to have an equal amount of stickers (*stress the word equal and gesture towards each character with hands up like a scale*). So, I want you to figure out how many more stickers you should give to Frog (*point to Frog's second sticker collector*), so Frog will have the same amount of stickers as Monkey.

If the child is correct: That's correct. Go ahead and give frog 5 stickers.

If the child is incorrect: Good try, but that won't make them equal. Monkey has 8 stickers (*point to each of Monkey's stickers as you count them aloud*). Frog only has 3 stickers (*point to each of Frog's stickers as you count aloud*). How many more does Frog need so he can have 8 stickers?

If the child is correct: That's correct. Go ahead and give Frog 5 stickers.

If the child is still incorrect: Actually Frog needs 5 more stickers. Give Frog 5 stickers.

Let's check to make sure they're equal. How many stickers does Monkey have? (*Child should say 8*.) How many stickers does Frog have? (*Child should say 8*.) Great! You figured it out. Monkey and Frog both have 8 stickers!

Script for concrete balance scale context (Experiments 1 and 2)

We're going to play a new game with this scale and these bears.⁴ You are going to use the bears to make the scale balance. I'll tell you how many bears to put on each side, OK?

First, put 5 bears on the left side (*point to the left side of the scale*). Good job!

Now put 4 more bears on the left side (*point to the left side of the scale*). OK!

Put 5 bears on the right side (*point to the right side of the scale*). Great!

We want the scale to balance (*stress the word balance and gesture towards each side of the scale with hands up like a scale*). So, I want you to figure out how many more bears you need to put on the right side (*point to the right side*), to make the right side (*point to the right side*) the same amount as the left side (*point to the left side of the scale*).

If the child is correct: That's correct. Go ahead and put 4 bears on the right side.

If the child is incorrect: Good try, but that won't make the scale balance. The left side has 9 bears (*point to each one of the bears as you count them aloud*). The right side only has 5 bears (*point to each one of the bears as you count them aloud*). How many more bears

⁴ In the concreteness introduction condition, instruction on the concrete balance scale problem came right after children had worked on problems in the worksheet contexts, so this sentence was preceded by: "We're going to play the same game but this time without the paper! Think about the math problem to help you play this game."

does the right side (*point to right side of the scale*) need to have 9 bears?

If the child is correct: That's correct. Go ahead and put 4 bears on the right side.

If the child is still incorrect: Actually it needs 4 more bears. Put 4 bears on the right side.

Let's check to make sure it's balanced. How many bears does the left side have? (*Child should say 9.*) How many bears does the right side have? (*Child should say 9.*) Great! You figured it out. Both sides have 9 bears!

Script for balance scale worksheet (Experiments 1 and 2)

Now we're going to play the same game on paper.³ (*Show child worksheet*). This stands for the left side of the scale (*point to left side*) and this stands for the right side of the scale (*point to right side*). I'm going to tell you how many bears to put on each side, and you will write the number that I tell you in the blanks, OK?

First, put 3 bears on the left (*point to the first blank on the left, wait for child to write 3*). Good!

Now put 2 more bears on the left (*point to the second blank on the left, wait for child to write 2*).

Put 3 bears on the right side (*point to the first blank on the right, wait for child to write 3*). Great!

We want the scale to balance (*Stress the word balance and gesture towards each character with hands up like a scale*). So, I want you to figure out how many more bears you need to put on the right side (*point to the second blank on the right side*), to make the right side (*circle the right side*) the same amount as the left side (*circle the left side*).

If the child is correct: That's correct. Go ahead and put 2 bears on the right side.

If the child is incorrect: Good try, but that won't make the scale balance. The left side has 5 bears (*point to each one of the bears as you count them aloud*). The right side only has 3 bears (*point to each one of the bears as you count them aloud*). How many more bears does the right side (*point to right side*) need to have 5 bears?

If the child is correct: That's correct. Go ahead and put 2 bears on the right side.

If the child is still incorrect: Actually it needs 2 bears. Put 2 more bears on the right side.

Let's check to make sure it's balanced. How many bears does the left side have? (*Child should say 5.*) How many bears does the right side have? (*Child should say 5.*) Great! You figured it out. Both sides have 5 bears!

Script for pattern game with puppets (Experiment 2)

We're going to play a memory game with Monkey and Frog. Monkey likes to make patterns based on the colors of his stickers. And frog likes to try to copy Monkey's pattern. I'll show you one of Monkey's favorite patterns on his sticker collector. (*Point to Monkey's sticker collector*.) Try your best to remember what it looks like. Next, you help Frog copy the pattern on his sticker collector. (*Point to Frog's sticker collector*.) OK?

At the top of his sticker collector, Monkey likes to make this pattern: blue, green, blue. (*As you name each color, place the appropriate sticker in the appropriate place*.) At the bottom of his sticker collector, Monkey likes to make this pattern: green, blue, green. (*As you name each color, place the appropriate sticker in the appropriate place*.) There! That's Monkey's favorite pattern.

Take a good look at the pattern because you get to help Frog copy that pattern. (*Give child about 5–10 s to study the pattern. Then,*

cover the pattern.) Now you help Frog copy that pattern on his sticker collector (*point to Frog's sticker collector*.)

If the child copied the pattern correctly: Way to go! You did a good job helping Frog.

If the child copied the pattern incorrectly: Good try! You were close to copying Monkey's pattern! Let's look at what is different. (*Point out differences, explain how they could be fixed*.)

Script for pattern game with puppet worksheet (Experiment 2)

We're going to play the game on paper. (*Show worksheet*.) This stands for Monkey and this stands for Frog. These are their sticker collectors. I'll draw one of Monkey's favorite patterns using colored pencils. Try to remember what it looks like so you can help Frog copy the pattern.

Monkey likes this pattern: red in the corners and green in the middle. So I'll draw a red circle in each of the four corners. (*Draw the circles*.) Now, I'll draw a green circle in the center. (*Draw the circle*.) That's one of Monkey's favorite patterns.

Take a good look at the pattern because you get to help Frog copy that pattern. (*Give child about 5–10 s to study the pattern. Then cover the pattern*.) Now it's your turn to help Frog copy that pattern on his sticker collector.

If the child copied the pattern correctly: Way to go! You did a good job helping Frog.

If the child copied the pattern incorrectly: Good try! You were close to copying Monkey's pattern! Let's look at what is different. (*Point out differences, explain how they could be fixed*.)

Script for pattern game with balance scale (Experiment 2)

Now we're going to play a new memory game with these bins and these bears. I'll show you a pattern inside these bins. I want you to try your very best to remember exactly what it looks like. Next, you will try to copy the pattern in new empty bins. OK?

In the left bin, I'm going to make this pattern: 2 yellow bears in the center. (*Place the two yellow bears in the left bin*.) In the right bin, I'm going to make this pattern: 3 red bears in a diagonal line. (*Place the three red bears in the bin*.) That's my pattern.

Take a good look at the pattern because you get to copy that pattern next. (*Give child about 5–10 s to study the pattern. Then, hide bins on the floor nearby. Then get out two new empty bins*.) Now it's your turn to copy the exact pattern that you just saw in my bin.

If the child copied the pattern correctly: Way to go! You copied my pattern exactly.

If the child copied the pattern incorrectly: Good try! You were close to copying my pattern! Let's look at what is different. (*Point out differences, explain how they could be fixed*.)

Script for pattern game with balance scale worksheet (Experiment 2)

We're going to play the game on paper. (*Show worksheet*.) This is the scale and these are the empty bins. I'll draw a pattern using colored pencils. Try your best to remember what that pattern looks like. Then, you will try to copy that exact pattern.

In the left bin, I'm going to make this pattern: two red squares, and 1 tall green rectangle in between them. (*Draw the shapes in the bin*.) In the right bin, I'm going to make this pattern: a red square in each corner. (*Draw the squares*.) There, that's my pattern.

Take a good look at the pattern because you get to copy that pattern next. (*Give child about 5–10 s to study the pattern and then cover it with a new worksheet*.) It's your turn to copy that exact pattern on this new worksheet.

If the child copied the pattern correctly: Way to go! You copied my pattern exactly.

If the child copied the pattern incorrectly: Good try! You were close to copying my pattern! Let's look at what is different. (*Point out differences, explain how they could be fixed.*)

Script for problems presented in abstract form for the cancel strategy (Experiment 3)

We're going to solve a math problem together.¹ (*Show child the math equivalence problem [e.g., $4 + 3 = 4 + \underline{\hspace{1cm}}$]. This is the left side of the problem (circle the left side with finger) and this is the right side of the problem (circle the right side with finger).*) I want to see if you can figure out how to make each side have the same amount without adding or subtracting.

I want you to figure out how many more you should put on the right side (*point to the blank*), so the right side (*circle right side*) will have the same amount as the left side (*circle left side*). Remember to do it without adding or subtracting any of the numbers!

If the child is correct: That's correct. How did you figure it out without adding or subtracting?

**If the child gives a report that indicates the cancel strategy: Great!*

**If the child gives a report that indicates he/she doesn't know: Because the left side of the problem has 4 (point to the 4 on the left side of the problem and keep point there) and the right side has 4 (point to the 4 on the right side of the problem and keep point there), you can just ignore those ("crossing out" gesture at each 4) and pay attention to how many the left side has here (point to the 3). The left side has 3 here (point again to the 3 and keep point there), so the right side still needs 3 more (point to the blank on the right side of the problem) so it will have the same amount as the left side. Go ahead and write a 3 (point again to the blank).*

If the child is incorrect: Good try, but that won't make them equal. The left side has 7 (point to the 4 while counting one, two, three, four and then point to the 3 while counting five, six, seven). The right side only has 4 (point to the 4 while counting one, two, three, four). How many more does the right side need to have 7?

If the child is correct: That's correct. How did you figure it out without adding or subtracting? (*continue with the same procedure as above**)

If the child is still incorrect: Actually the right side needs 3 more. Write a 3 in the blank. (*continue with the same procedure as above**)

Let's check to make sure they're equal. How many does the left side have? (*Child should say 7.*) How many does the right side have? (*Child should say 7.*) Great! You figured it out. Both sides have 7! Let's do the next problem.

Script for concrete puppet-sharing context for the cancel strategy (Experiment 3)

We're going to play a game in which you share some of these stickers with Monkey and Frog.² I'll tell you how many stickers to give them, and you put the stickers on their sticker collectors.

First, give Monkey 4 stickers (*point to Monkey's first sticker collector*). Good!

Now give Monkey 3 more stickers (*point to Monkey's second sticker collector*). OK!

Give Frog 4 stickers (*point to Frog's first sticker collector*). Great!

Monkey and Frog want to have an equal amount of stickers (*Stress the word equal and gesture towards each character with hands up like a scale*), but Frog doesn't want to add or subtract. So, I want you to see if you can figure out how to make them have the same amount without adding or subtracting. How many more stickers

should you give to Frog (*point to Frog's second sticker collector*), so Frog will have the same amount of stickers as Monkey.

If the child is correct: That's correct. How did you figure that out without adding or subtracting?

**If the child gives a report that indicates the cancel strategy: Great!*

**If the child gives a report that indicates he/she doesn't know: Because Monkey has 4 stickers (*point to Monkey's first sticker collector and keep point there*) and Frog has 4 stickers (*point to Frog's first sticker collector and keep point there*), you can just ignore those ("crossing out" gesture at Monkey and Frog's first sticker collectors) and pay attention to how many stickers Monkey has here (*point to Monkey's second sticker collector*). Monkey has 3 stickers here (*point to Monkey's second sticker collector and keep point there*), so Frog still needs 3 more stickers (*point to Frog's second sticker collector*) so he can have the same amount as Monkey. Go ahead and give Frog 3 stickers (*point again to Frog's second sticker collector*).*

If the child is incorrect: Good try, but that won't make them equal. Monkey has 7 stickers (*point to each of Monkey's stickers as you count them aloud*). Frog only has 4 stickers (*point to each of Frog's stickers as you count aloud*). How many more does Frog need so he can have 7 stickers?

If the child is correct: That's correct. How did you figure it out without adding or subtracting? (*continue with the same procedure as above**)

If the child is still incorrect: Actually Frog needs 3 more stickers. Give Frog 3 stickers. (*continue with the same procedure as above**)

Let's check to make sure they're equal. How many stickers does Monkey have? (*Child should say 7.*) How many stickers does Frog have? (*Child should say 7.*) Great! You figured it out. Monkey and Frog both have 7 stickers!

Script for puppet-sharing worksheet for the cancel strategy (Experiment 3)

Now we're going to play the same game on paper.³ (*Show child worksheet*). This stands for Monkey (*point to monkey*) and this stands for Frog (*point to frog*). Here are their sticker collectors (*point to the four boxes*). I'm going to tell you how many stickers to give them, and you will write the number in their sticker collectors, OK?

First, give Monkey 3 stickers (*point to Monkey's first collector, wait for child to write 3*). Good!

Give Monkey 5 more stickers (*point to Monkey's second collector, wait for child to write 5*).

Now give Frog 3 stickers (*point to Frog's first sticker collector, wait for child to write 3*). Great!

Monkey and Frog want to have an equal amount of stickers (*stress the word equal and gesture towards each character with hands up like a scale*), but Frog doesn't want to add or subtract. So, I want you to see if you can figure out how to make them have the same amount without adding or subtracting. How many more stickers should you give to Frog (*point to Frog's second sticker collector*), so Frog will have the same amount of stickers as Monkey.

If the child is correct: That's correct. How did you figure that out without adding or subtracting?

**If the child gives a report that indicates the cancel strategy: Great!*

**If the child gives a report that indicates he/she doesn't know: Because Monkey has 3 stickers (*point to Monkey's first sticker collector and keep point there*) and Frog has 3 stickers (*point to Frog's first sticker collector and keep point there*), you can just ignore those ("crossing out" gesture at Monkey and Frog's first*

sticker collectors) and pay attention to how many stickers Monkey has here (*point to Monkey's second sticker collector*). Monkey has 5 stickers here (*point to Monkey's second sticker collector and keep point there*), so Frog still needs 3 more stickers (*point to Frog's second sticker collector*) so he can have the same amount as Monkey. Go ahead and give Frog 5 stickers (*point again to Frog's second sticker collector*).

If the child is incorrect: Good try, but that won't make them equal. Monkey has 8 stickers (*point to each of Monkey's stickers as you count them aloud*). Frog only has 3 stickers (*point to each of Frog's stickers as you count them aloud*). How many more does Frog need so he can have 8 stickers?

If the child is correct: That's correct. How did you figure it out without adding or subtracting? (*continue with the same procedure as above**)

If the child is still incorrect: Actually Frog needs 5 more stickers. Give Frog 5 stickers. (*continue with the same procedure as above**)

Let's check to make sure they're equal. How many stickers does Monkey have? (*Child should say 8.*) How many stickers does Frog have? (*Child should say 8.*) Great! You figured it out. Monkey and Frog both have 8 stickers!

Script for concrete balance scale context (Experiment 3)

We're going to play a new game with this scale and these bears.⁴ You are going to use the bears to make the scale balance. I'll tell you how many bears to put on each side, OK?

First, put 5 bears on the left side (*point to the left side of the scale*). Good job!

Now put 4 more bears on the left side (*point to the left side of the scale*). OK!

Put 5 bears on the right side (*point to the right side of the scale*). Great!

We want the scale to balance (*stress the word balance and gesture towards each side of the scale with hands up like a scale*), but we don't want to add or subtract. So, I want you to figure out how to make them have the same amount without adding or subtracting. How many more bears do you need to put on the right side (*point to the right side*), to make the right side (*point to the right side*) the same amount as the left side (*point to the left side of the scale*)?

If the child is correct: That's correct. How did you figure it out without adding or subtracting?

**If the child gives a report that indicates the cancel strategy:* Great!
**If the child gives a report that indicates he/she doesn't know:* Because the left side has 5 bears (*point to the 5 red bears and keep point there*) and the right side has 5 bears (*point to the 5 red bears and keep point there*), you can just ignore those bears (*crossing out gesture with both hands*) and pay attention to how many more bears are on the left side (*point to 4 yellow bears on left*). There are 4 more bears on the left side (*point to 4 yellow bears on right and keep point there*), so the right side still needs 4 more bears (*point to right side of scale*) to be equal to the left side. Go ahead and put 4 bears on the right side (*point again to right side of scale*).

If the child is incorrect: Good try, but that won't make the scale balance. The left side has 9 bears (*point to each one of the bears as you count them aloud*). The right side only has 5 bears (*point to each one of the bears as you count them aloud*). How many more bears does the right side (*point to right side of the scale*) need to have 9 bears?

If the child is correct: That's correct. How did you figure it out without adding or subtracting? (*continue with the same procedure as above**)

If the child is still incorrect: Actually it needs 4 more bears. Put 4 bears on the right side. (*continue with the same procedure as above**)

Let's check to make sure it's balanced. How many bears does the left side have? (*Child should say 9.*) How many bears does the right side have? (*Child should say 9.*) Great! You figured it out. Both sides have 9 bears!

Script for balance scale worksheet for the cancel strategy (Experiment 3)

Now we're going to play the same game on paper.³ (*Show child worksheet*). This stands for the left side of the scale (*point to left side*) and this stands for the right side of the scale (*point to right side*). I'm going to tell you how many bears to put on each side, and you will write the number that I tell you in the blanks, OK?

First, put 3 bears on the left (*point to the first blank on the left, wait for child to write 3*). Good!

Now put 2 more bears on the left (*point to the second blank on the left, wait for child to write 2*).

Put 3 bears on the right side (*point to the first blank on the right, wait for child to write 3*). Great!

We want the scale to balance (*stress the word balance and gesture towards each character with hands up like a scale*), but we don't want to add or subtract. So, I want you to figure out how to make them have the same amount without adding or subtracting. How many more bears do you need to put on the right side (*point to the second blank on the right side*), to make the right side (*circle the right side*) the same amount as the left side (*circle the left side*).

If the child is correct: That's correct. How did you figure it out without adding or subtracting?

**If the child gives a report that indicates the cancel strategy:* Great!

**If the child gives a report that indicates he/she doesn't know:* Because the left side has 3 bears (*point to the 3 red bears and keep point there*) and the right side has 3 bears (*point to the 3 red bears and keep point there*), you can just ignore those bears (*crossing out gesture with both hands*) and pay attention to how many more bears are on the left side (*point to 2 yellow bears on left*). There are 2 more bears on the left side (*point to 2 yellow bears on right and keep point there*), so the right side still needs 2 more bears (*point to right side of scale*) to be equal to the left side. Go ahead and put 4 bears on the right side (*point again to right side of scale*).

If the child is incorrect: Good try, but that won't make the scale balance. The left side has 5 bears (*point to each one of the bears as you count them aloud*). The right side only has 3 bears (*point to each one of the bears as you count them aloud*). How many more bears does the right side (*point to right side*) need to have 5 bears?

If the child is correct: That's correct. How did you figure that out without adding or subtracting? (*continue with the same procedure as above**)

If the child is still incorrect: Actually it needs 2 bears. Put 2 more bears on the right side. (*continue with the same procedure as above**)

Let's check to make sure it's balanced. How many bears does the left side have? (*Child should say 5.*) How many bears does the right side have? (*Child should say 5.*) Great! You figured it out. Both sides have 5 bears!

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