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Math 419
Comparison of Conceptual Math Topics

An Exploration and Comparison of Laplace/Fourier Transformations and Cavalieri's Principle

Peering down the timeline of the development of mathematics, we see a plethora of points in time where significant contributions were made by remarkable mathematicians. The eponymous ideas of mathematics will forever etch its pioneer(s) in the annals of history, each contributing a drop of novel fact to the sea of mathematical knowledge. Of course, some of these ideas have contributed to the overall development of mankind in more tangible or meaningful ways than others. We will explore two prominent concepts, Cavalieri's Principle and Fourier/Laplace transformations. Although Cavalieri's Principle was conceived nearly a century prior to Laplace and Fourier's work, we shall see that Cavalieri's Principle was more integral to the entire body of mathematics.

Bonaventura Cavalieri was born just as fervor of the French Rennaissance began to subside, in 1598, Milan. The afterglow of the Rennaissance created a nurturing and rich world for those with curious minds. In the final years of his education, Cavalieri's renowned instructor Benedetto Antonio Castelli, a lecturer in mathematics at the University of Pisa, facilitated a correspondence between Galileo and Cavalieri. It is through these letters we see the formation and reasoning of Cavalieri's namesake notion. The principle states: "If two solids are included between a pair of parallel planes, and if the areas of the two sections cut by them on any plane parallel to the including planes are always equal, then the volumes of the two solids are also equal" (Eves, 1991, p. 118-124). The same is true for a two-dimension scenario, but with the

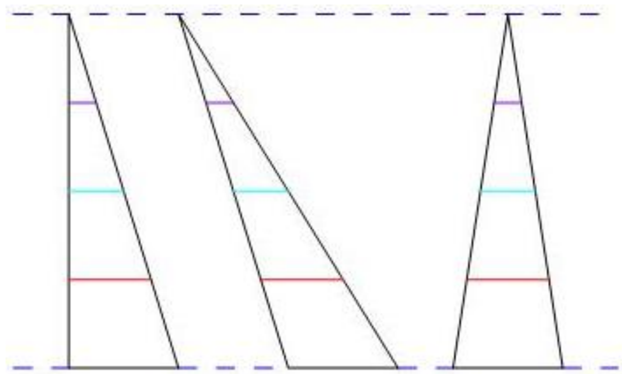

Figure 1: Three non-congruent triangles with equal area.

shapes having equal area. Figure 1 illustrates this with line segments of same color having equal length and parallel to the dashed lines. Cavalieri reasoned that if each line parallel to the dashed line and bound by its triangle was equal, then the triangles would have the same area. This principle was closely tied with Cavalieri's method of indivisibles which posited that two-dimensional shapes could be thought of as a set of infinitely many parallel lines (see Figure 2). Cavalieri did acknowledge this notion of "space occupied by 'all the lines'" (Anderson, 1984, p.

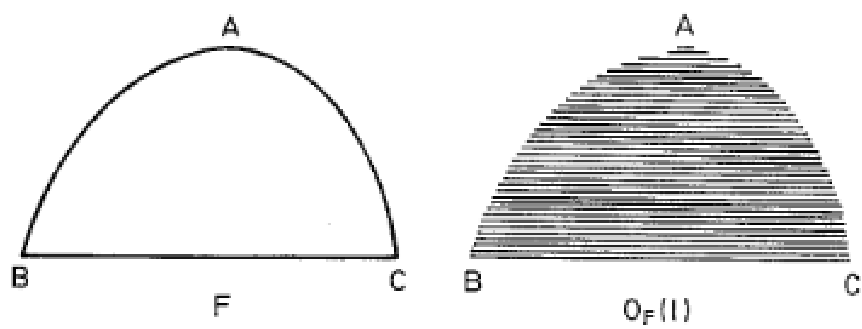

Figure 2: Area of F is equal to the sum of the set of all parallel lines contained in $O_F(l)$ (Anderson, 1984, p.301)

305) came with conceptual difficulties, an issue that irked countless mathematicians all the way back to the ancient Greeks. As if conceptual quandaries weren't enough, the method of indivisibles was criticized for not being sufficiently rigorous, specifically from Swiss mathematician Paul Guldin. These notions were published in Cavalieri's magnum opus, *Geometria Indivisibilibus Continuorum Nova Auadam Ratione Promota*, in 1635. With our modern perspective, we can see how the method of indivisibles acted as a forerunner for integral

calculus. In fact, Newton would extrapolate from Cavalieri's ideas, considering those infinitely many lines as dynamic points in motion, moving along the "line" they represent. Nonetheless, Cavalieri's concepts would lend themselves to what Newton and Leibniz would come to contribute with integral calculus.

The second topic of interest is Fourier/Laplace transformations. This takes us to the mid-18th century, nearly one hundred years since the advent of calculus and on the heels of the French Revolution. Two French mathematicians would rise through academia with much acclaim and their paths would intersect on several occasions. The driving force of both Pierre-Simon Laplace and Joseph Fourier was their fascination with using mathematics to describe natural phenomena, such as heat flow between two molecules (Fourier) and describing the dynamic movement of the earth's tides (Laplace). Fourier's contribution provided a novel method to transform a non-

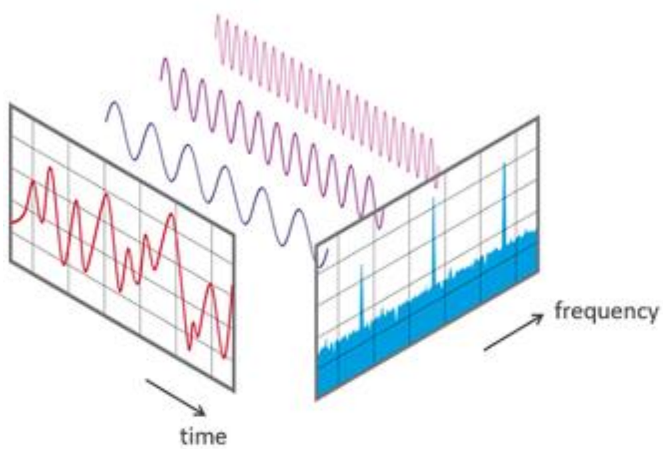

Figure 3: Fourier transform of a function with its corresponding Fourier series

periodic function from the "amplitude versus time domain, to the amplitude versus frequency domain" (van Biezen, 2016). This transformation is closely related to the Fourier series which is a linear combination of trigonometric functions used to approximate another function. Figure 3 shows the original function can be approximated by the sum of several sine or cosine functions.

Figure 3 also illustrates how the amplitude of each of those trigonometric functions relates to the frequency domain. Laplace generalized this idea, hypothesizing functions could be transformed from a function dependent on time, to a function dependent on complex values. Similar to the Fourier transformation, the Laplace transformation decomposes a function into sums of moments, rather than specific trigonometric functions. The Laplace transform can be conceptualized as a tool that converts a differential equation into an algebraic equation that can be more readily solved. The resulting algebraic equation can then be converted back into its original form with the inverse Laplace transformation, now with a solution in the time domain. Fourier and Laplace could not have imagined the profound implications of their work. Our modern electronic era is powered by their findings. Take digital signal processing for example; an audible sound is recorded and represented as a function of time. The Laplace transform allows one to convert this signal into a series of ones and zeros that can then be interpreted and manipulated by a computer. From electrical circuitry analysis to modeling radioactive decay, these methods are ubiquitous in science and math fields.

It is clear these two concepts have certainly progressed mankind's understanding of the mathematical realm. However, if one reverses the timeline, it is evident that Fourier and Laplace's work could not had been done without the foundations of calculus available to them. The innovative mathematics from Leibniz and Newton would not have been possible without Cavalieri's notion of a set of infinite parallel lines or infinite parallel planes. In fact, Cavalieri's ideas bridge the gap between those of the ancient Greeks and the fathers of calculus, pushing the concept of "infinitely many" forward. Although Cavalieri himself did not rigorously flesh out the meaning of "infinitely many," Newton did acknowledge his predecessors accomplishments lending themselves to his own stating, "If I have seen further it is by standing on the shoulders of giants," ("Issac Newton." Wikiquote, 2018). The problem with how to approach concepts like infinity plagued many of Greek's thinkers and was never fully put to rest. Cavalieri was instrumental in providing a modern (at the time) framework that attempted to resolve the problem. The weight and scope of the implication from Cavalieri's principle cannot be overstated nor forgotten.

Cavalieri, Fourier, and Laplace each deserve their place in the history of mathematics on their own right. Ultimately, their contributions have progressed the well of knowledge and provided a foundation for modern scientists and mathematicians to use. There is no doubt that the ideas from Fourier and Laplace have shaped our modern world, but it was Cavalieri that helped to lay the groundwork for the development of calculus and acted as a conduit between the Greeks of antiquity to the fathers of calculus.

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