Investigating The Chaotic Behaviour of a Nonlinear Circuit

Zamir Said*

Dept. of Physics, Simon Fraser University, Burnaby BC

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Abstract

In this report we investigate the behaviour of chaotic systems. To study a chaotic system we used a circuit that exhibits chaotic behaviour and can be approximately modelled by a nonlinear, third-order differential equation. This circuit exhibited chaotic behaviour and the measured voltage amplitudes exhibited period-doubling. The behavior of this circuit was consistent with our simulations of the model. A constant of period-doubling systems, the Feigenbaum constant, was also measured with an experimental value of $\delta = 4.6 \pm 0.9$.

INTRODUCTION

A chaotic system is one that has unpredictable and seemingly random behaviour but is described by deterministic laws [1]. One of the interesting characteristics of this behaviour is that the system does not repeat past behaviour making it hard to predict the long-term behaviour of the system [2]. Being able to understand this chaotic behaviour is important because chaotic systems are prevalent in many topics, from the rhythms in a persons heartbeat to fluctuations in the weather [1].

In this experiment an electronic circuit containing a nonlinear element was the system used for studying chaotic behaviour [3]. This "chaotic circuit" has behaviour that can be approximated by solutions of a nonlinear, third-order differential equation known as a "jerk" equation [3]. By varying one of the constants in the equation we observed the different ways the circuit behaved, one of which was chaotic behaviour. The point at which the behaviour changes is known as a bifurcation point and plotting the solutions of the equation against the varying constant is known as a bifurcation diagram. Using these bifurcation points we can estimate the Feigenbaum constant, which is known to be a constant in all period-doubling systems [4].

To compare our results with theory we simulated the jerk equation and compared the behaviour to the experimental data. We also created a simulated bifurcation diagram to superimpose on our experimental one to visually see how the behaviour of the circuit deviates from the simulations.

THEORY

The chaotic circuit is made up of elements that all contain a component called an operational amplifier or op-amp for short. Op-amps are high-gain differential amplifiers and they are often present in the context of feedback systems [4]. Op-amps have two "golden rules" that can help approximately describe their behaviour [4]:

I. The output attempts to do whatever is necessary to make the voltage difference between the inputs zero.

II. The inputs draw no current.

Using these two rules, the input and output voltages for all the op-amp sub-circuits in Fig. 1 can be predicted and used in finding the equation that describes this circuit.

The circuit's behaviour can be approximately described by a third-order, nonlinear, ordinary differential equation,

$$\dddot{x} = -A\,\ddot{x} - \dot{x} + D(x) - \alpha. \tag{1}$$

In this equation A and α are constants, $D(x)$ represents the nonlinear element in Fig. 1, and the dots above the variable are derivatives with respect to a dimensionless time, $\tilde{t} = \frac{t}{RC}$, where R is resistance and C is capacitance [3]. By using Kirchoff's rules on the circuit, the equation becomes

$$\dddot{x} = -\left(\frac{R}{R_v}\right)\ddot{x} - \dot{x} + D(x) - \left(\frac{R}{R_0}\right)V_0, \tag{2}$$

where R is all of the unlabelled resistors in Fig. 1, and R_v is the variable resistor [3]. Using Kirchoff's rules on the nonlinear element in Fig. 1, we can find an expression that approximates the the nonlinear element $D(x)$ [3],

$$D(x) = -\left(\frac{R_2}{R_1}\right)\min(x, 0), \tag{3}$$

where R_2 and R_1 are the resistance values of the two resistors in the non-linear element [3].

We found the solution to Eq. (2) by separating the third-order equation into coupled first-order equations and solving them numerically. This is the method we used to make simulations of the circuits behaviour to compare to after we collect the data from the actual circuit. Equation (2) has solutions of different types depending on the value of R_v which can be seen in Fig. 2. Figure 2(a) shows an example of the solution in the region where there is one stable solution, which can be seen from the figure by noting the number of different distinct voltage amplitudes. Figures 2(b) and (c) show the regions where there is two and four distinct voltage amplitudes respectively and Fig. 2(d) shows the region where it exhibits chaotic behaviour. In this region there are an infinite number of distinct voltage amplitudes.

Using the simulated solution to Eq. (2) we also plotted two dimensional projections of phase portraits. Phase portraits are a plot in three dimensional space where x, $\dot{x}$, and $\ddot{x}$ are plotted on the three orthogonal axes. The curve on this graph, when projected on the x-$\dot{x}$ plane, crosses over itself when the period of the system doubles. On this projection the curve crossing itself looks like "rings" and the number of distinct voltage amplitudes that exist in that region corresponds to the number of rings on the plot. Figure 2(a) has one ring and it corresponds to a R_v value in the region where there is one voltage amplitude. Figures 2(b) and (c) show two and four distinct rings and they are in the regions where there is two and four distinct voltage amplitudes. Similarly, Fig. 2(d) shows the region of chaotic behaviour.

METHODS

The setup for this lab consisted of an implementation of the circuit in Fig 1. The circuit was constructed with a breadboard and the voltage measurements were taken using an oscilloscope and voltage probes. After being recorded by the oscilloscope, the data was transferred to a USB to perform the analysis.

All of the unlabelled resistors and capacitors in Fig. 1 have nominal values of $R = 47$ kΩ and $C = 1$ μF respectively. The resistor labelled R_0 had a value of $R_0 = (147 \pm 6.72)$ kΩ. The resistors R_1 and R_2, in the nonlinear element, should have resistance values such that $\frac{R_2}{R_1} \approx 6$ so that the output voltage, D(x), is on the order of 1 mV [3]. Our values for R_1 and R_2 were: $R_1 = (32.4 \pm 0.5)$ kΩ and $R_2 = (192.45 \pm 3.4)$ kΩ. This results in a ratio of $\frac{R_2}{R_1} = 5.94 \pm 0.13$.

Using a variable resistor, the value for R_v was manually set to each value of R_v that we took data for. Starting at $R_v = 40$ kΩ we increased the resistance by 1 kΩ until $R_v = 140$ kΩ, noting when the first few bifurcation points occurred. Then we went back to the R_v values that the bifurcation points occurred and took more detailed data near those points. We did

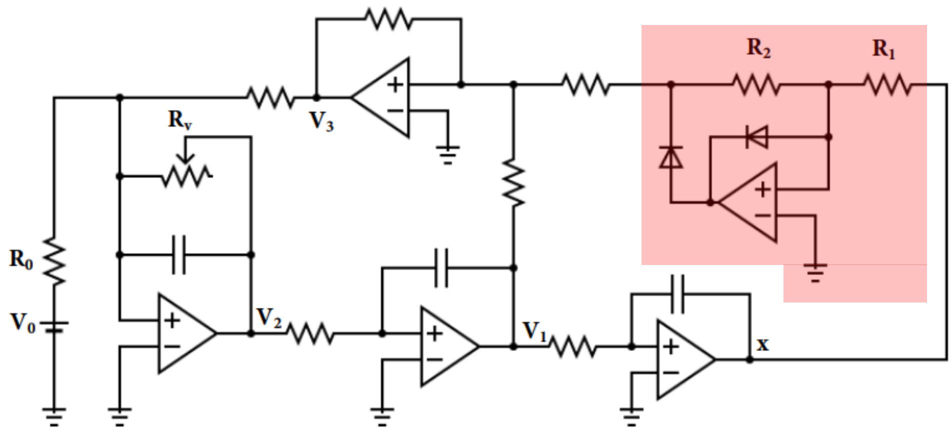

FIG. 1: Schematic diagram of the chaotic circuit. The shaded section of the circuit is the nonlinear element. This schematic was adapted from Kiers et al. [3].

this by going to the resistance value right before the bifurcation took place and taking 100 Ω steps until reaching the bifurcation point. This way we were able to cover many R_v values but also get detailed data near the bifurcation points since those points are more important for analysis.

RESULTS AND DISCUSSION

Figure 3 shows the voltage measurements for four values of R_v. The behaviour of the circuit can be inferred from the voltage data by counting the number of distinct voltage amplitudes there are in the waveform. Figure 3(a) shows the stable region with one distinct voltage amplitude. Figure 3(b) and (c) show the regions where there are two and four distinct voltage amplitudes respectively and (d) shows one of the regions where the circuit exhibits chaotic behaviour. Figure 3 also shows the phase diagrams from the same regions as the voltage measurements.

Comparing Figs. 2 and 3, we can see that the values of R_v are not all the same between the simulations and the experimental data. This is because while our data follows the general patterns that we would expect, it is slightly shifted towards higher R_v values. We think that one of the reasons for this is that the value for the ratio $\frac{R_2}{R_1}$ is not exactly 6. Although this results in a deviation from the simulations, since all of the major patterns that we expect are preserved we do not think this is a big issue.

One of the tools we can use to see the patterns in the behaviour of the circuit is bifurcation diagrams. Bifurcation diagrams plot the long-term solutions as a function of the resistance of R_v. These plots also help us more clearly see the bifurcation points especially at lower R_v values where there are not many points. Figure 4 shows both the simulated and experimental bifurcation diagrams. The general pattern of the simulated bifurcation diagram from Fig. 4(a) is that it starts with one solution, bifurcates into two then three and so on until it reaches chaotic behaviour. Then at $R_v \approx 85$ kΩ it once again has a stable region before once again bifurcating until it reaches chaos.

Figure 4(c) shows the experimental bifurcation diagram superimposed with the simulated one for easy comparison. From this figure we can confirm our earlier conclusion that the circuit behaves similar to what we would expect but it is slightly shifted towards the higher resistance values.

Using our data we were also able to calculate the Feigenbaum constant. The Feigenbaum constant is defined as

$$\delta = \lim_{n \to \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n}, \tag{4}$$

where r_n denotes the value of R_v at the n'th bifurcation point [1]. The known value of the Feigenbaum constant is $\delta \approx 4.669$. For this experiment we were only able to take detailed data for the first three bifurcation points. Using the first three bifurcation points, we can only get one value of δ and will not be able to extrapolate as $n \to \infty$. Our experimental value for the Feigenbaum constant is $\delta = 4.6 \pm 0.9$. This is consistent with the known value within error but the error is quite large. We found the error by using error propagation for correlated data since one of the values in the numerator of Eq. (4) is the same as one of the values in the denominator [5]. Even so, we think the error is overestimated since we are treating the measurement as if we are measuring separate R_v values and subtracting them to find the width, but in reality we are measuring just a width.

CONCLUSION

In this experiment we observed the behaviour of a circuit that exhibited chaos. We made simulations of the behaviour of the circuit to compare to our data. The circuit was mostly consistent with the simulations by having the same general patterns and only deviating by a shift in R_v values. The average percent difference between the simulated R_v values and the experimental values was 2.7%. We think that this shift can be attributed to the experimental values of resistors not being exactly the same as the nominal values that the theoretical model used. Using bifurcation diagrams we also estimated the Feigenbaum constant. The value of the experimental Feigenbaum constant was consistent with theory with a value of $\delta = 4.6 \pm 0.9$.

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* zamir_said@sfu.ca

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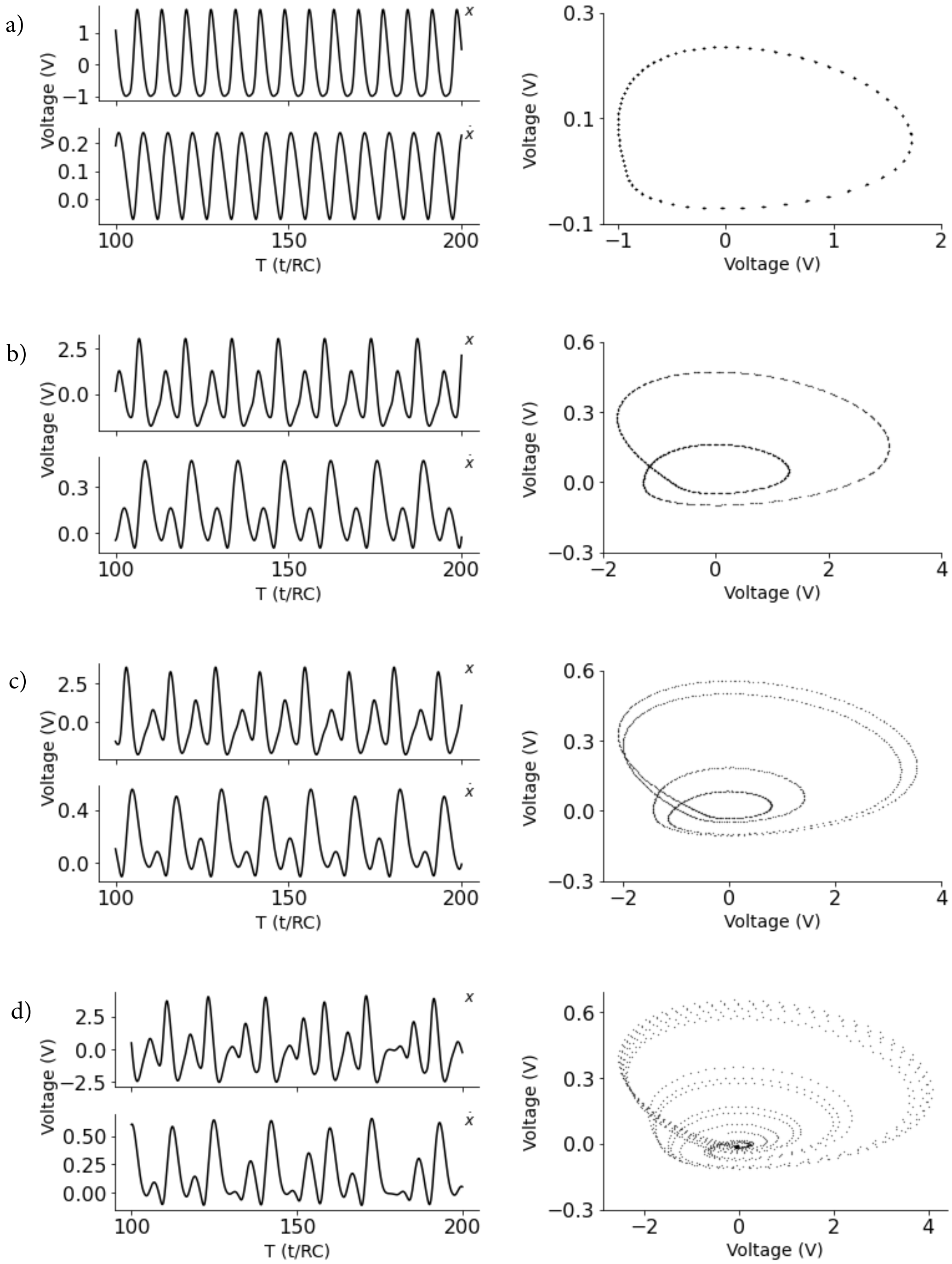

FIG. 2: On the left, voltage as a function of dimensionless time for simulated solutions to Eq. (2). The equation was solved for various values of R_v to show examples of some of the different types of behaviour. On the right, simulated phase portraits of $\dot{x}$ vs. x voltages. (a) $R_v = 40$ kΩ, one stable solution. (b) $R_v = 60$ kΩ, two distinct voltage amplitudes after first bifurcation. (c) $R_v = 68$ kΩ, four distinct voltage amplitudes after second bifurcation. (d) $R_v = 78$ kΩ, chaos.

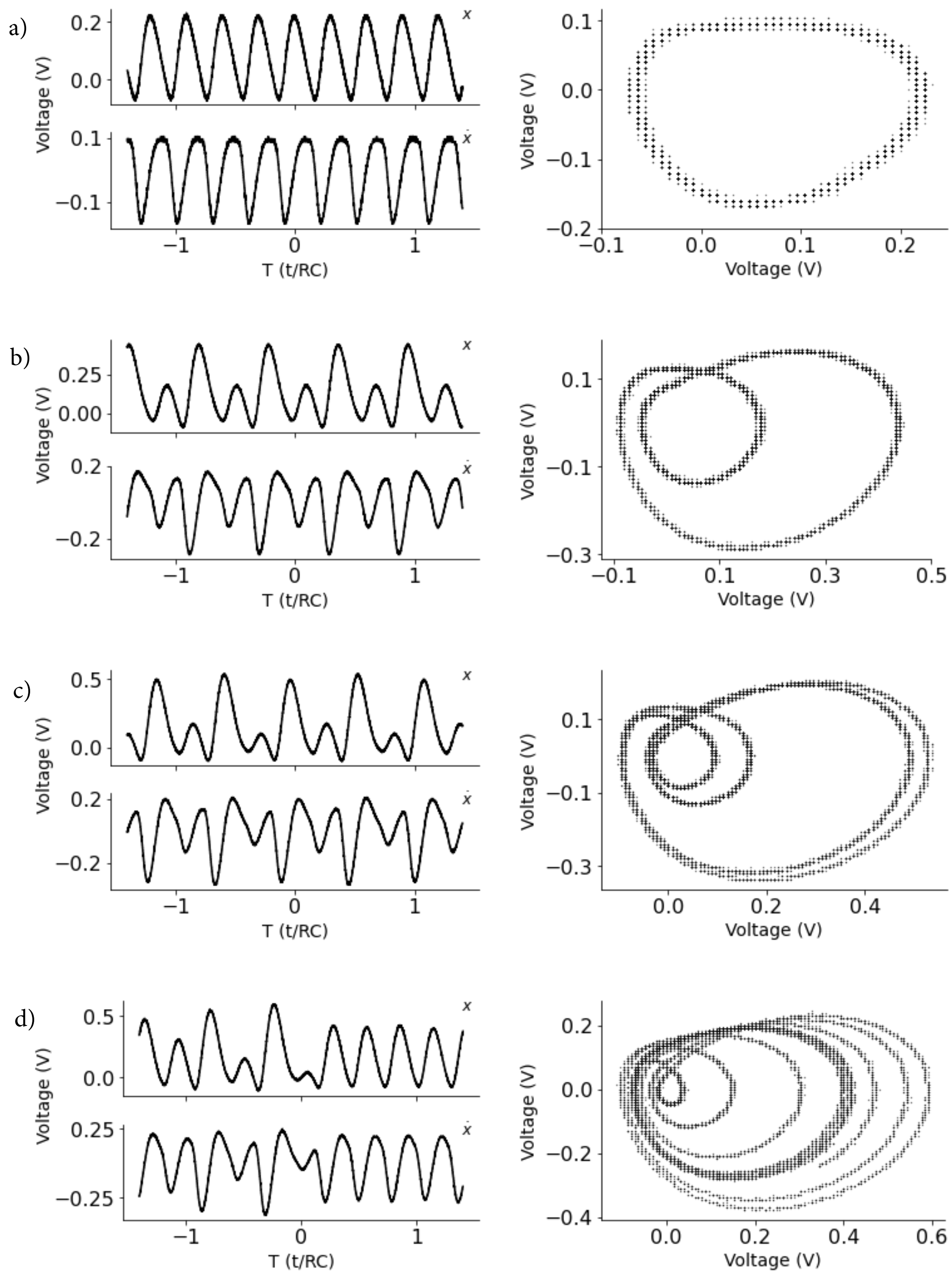

FIG. 3: On the left, measured voltage as a function of time. On the right, experimental phase portraits of $\dot{x}$ vs. x voltages.(a) $R_v = 40$ kΩ, one stable solution. (b) $R_v = 60$ kΩ, two distinct voltage amplitudes after first bifurcation. (c) $R_v = 70$ kΩ, four distinct voltage amplitudes after second bifurcation. (d) $R_v = 78$ kΩ, chaos.

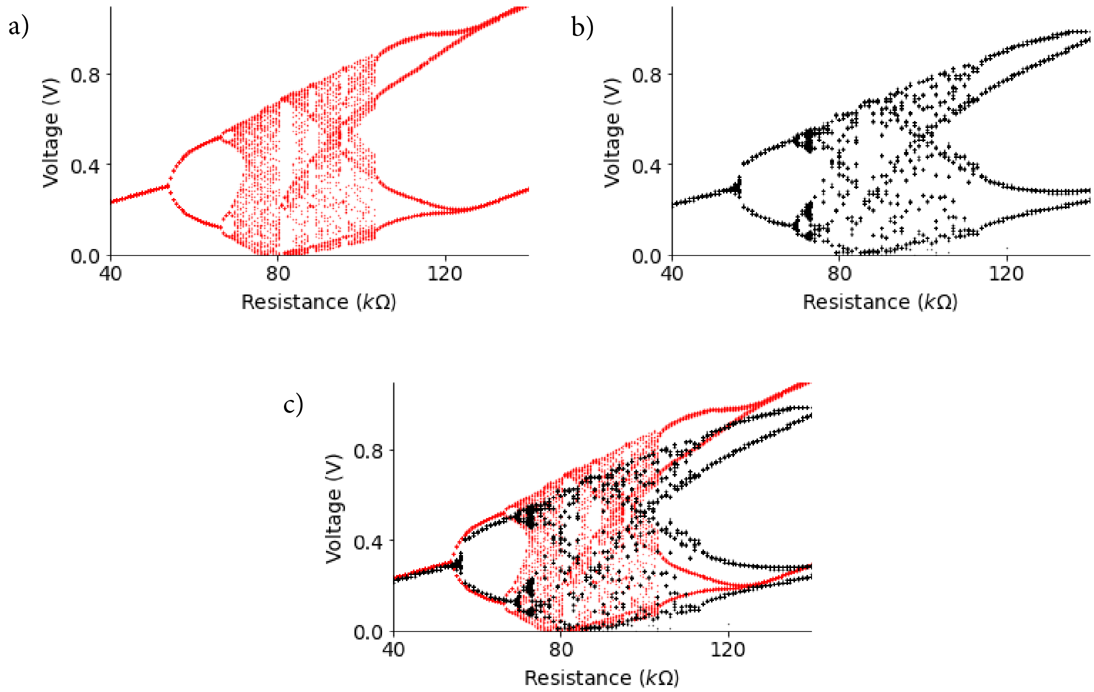

FIG. 4: Bifurcation diagrams, which plot the maxima of the x voltage as a function of resistance R_v. (a) Simulated bifurcation diagram. (b) Experimental bifurcation diagram. (c) Experimental diagram superimposed with simulated diagram.