Matlab Practical 2

GEOL647

Zach Vig

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Problem 1.1: What is the analytic expression for the frequency response function of a simple seismometer with mass **m**, spring constant **k**, and coefficient of friction **D**?

Solution:In the Fourier domain, the ODE for seismometer motion is:

$$\omega^2 \tilde{u}(\omega) = \frac{k}{m}\tilde{x}(\omega) + i\omega\frac{D}{m}\tilde{x}(\omega) - \omega^2\tilde{x}(\omega)$$

Where D is the damping term, u is the relative position of the Earth (input) and x is the relative position of the seismometer (output). So the frequency response function (FRF) is:

$$\tilde{X}(\omega) = \frac{\tilde{x}(\omega)}{\tilde{u}(\omega)} = \frac{\omega^2}{\frac{k}{m} + \frac{D}{m}i\omega - \omega^2}$$

Problem 1.2: Derive the amplitude response function of the seismometer from the FRF:

Solution:

$$ARF = \sqrt{|\tilde{X}(\omega)|} = \sqrt{\frac{\omega^2}{\frac{k}{m} + \frac{D}{m}i\omega - \omega^2}\frac{\omega^2}{\frac{k}{m} - \frac{D}{m}i\omega - \omega^2}}$$

$$ARF = \frac{\omega^2}{\sqrt{\left(\frac{k}{m} - \omega^2\right)^2 + \left(\frac{D}{m}\omega\right)^2}}$$

Problem 1.3: Derive the phase delay function of the seismometer from the FRF:

Solution:

$$\Theta(\omega) = \arctan\left(\frac{Im(\tilde{X}(\omega))}{Re(\tilde{X}(\omega))}\right)$$

To find this, we must rewrite $\tilde{X}(\omega)$ as:

$$\tilde{X}(\omega) = \frac{\omega^2}{\frac{k}{m} + \frac{D}{m}i\omega - \omega^2}\frac{\frac{k}{m} - \frac{D}{m}i\omega - \omega^2}{\frac{k}{m} - \frac{D}{m}i\omega - \omega^2} = \frac{-\omega^4 + \omega^2\frac{k}{m} - i\omega^3\frac{D}{m}}{\left[\frac{k}{m} - \omega^2\right]^2 + \left(\frac{D}{m}\omega\right)^2}$$

$$Im(\tilde{X}(\omega)) = \frac{-\omega^3\frac{D}{m}}{\left[\frac{k}{m} - \omega^2\right]^2 + \left(\frac{D}{m}\omega\right)^2}, \quad Re(\tilde{X}(\omega)) = \frac{-\omega^4 + \omega^2\frac{k}{m}}{\left[\frac{k}{m} - \omega^2\right]^2 + \left(\frac{D}{m}\omega\right)^2}$$

Plugging this in, we have:

$$\Theta(\omega) = \arctan\left(\frac{-\omega^3\frac{D}{m}}{-\omega^4 + \omega^2\frac{k}{m}}\right) = \arctan\left(\frac{-\omega\frac{D}{m}}{\frac{k}{m} - \omega^2}\right) = \arctan\left(\frac{-\omega D}{k - m\omega^2}\right)$$

Problem 2.1: Write the frequency response function for a seismometer with this type of feedback system. To do this, think about inputs and outputs. Because voltage is output by the system and ground acceleration is input, compute $V/\ddot{u}$.

$$(1)\ \ddot{y} = \ddot{u} - \ddot{x} \qquad (2)\ \ddot{x} = \beta V \qquad (3)\ V = \frac{KA\omega^2}{\omega^2 - 2i\epsilon\omega - \omega_0^2}\ddot{y}$$

Solution: To begin, we will rewrite certain terms to make things clearer:

$$\begin{aligned} V &= KA\tilde{X}_0\ddot{y} \\ \ddot{u} &= \ddot{y} + \ddot{x} = \ddot{y} + \beta KA\tilde{X}_0\ddot{y} \end{aligned}$$

Where $\tilde{X}_0$ is the FRF of an inertial seismometer. Now, solving for the FRF, we have:

$$\begin{aligned} \tilde{X}(\omega) = \frac{V}{\ddot{u}} &= \frac{KA\tilde{X}_0\ddot{y}}{\ddot{y} + \beta KA\tilde{X}_0\ddot{y}} \\ \tilde{X}(\omega) &= \frac{KA\tilde{X}_0}{1 + \beta KA\tilde{X}_0} \\ \tilde{X}(\omega) &= \frac{KA\omega^2}{\omega^2 - 2i\epsilon\omega - \omega_0^2 + \beta KA\omega^2} \\ \tilde{X}(\omega) &= \frac{KA\omega^2}{(1 + \beta KA)\omega^2 - 2i\epsilon\omega - \omega_0^2} \end{aligned}$$

Problem 2.2: Solve for the amplitude response function of the system (I'm going to call $ARF(\omega)$, $\xi(\omega)$ because I think it's cool):

Solution:

$$\begin{aligned} \xi(\omega) &= |\tilde{X}(\omega)| \\ \xi(\omega) &= \sqrt{\frac{KA\omega^2}{(1 + \beta KA)\omega^2 - 2i\epsilon\omega - \omega_0^2}\frac{KA\omega^2}{(1 + \beta KA)\omega^2 + 2i\epsilon\omega - \omega_0^2}} \\ \xi(\omega) &= \frac{KA\omega^2}{\sqrt{\left[(1 + \beta KA)\omega^2 - \omega_0^2\right]^2 + 4\epsilon^2\omega^2}} \end{aligned}$$