Convergence in distribution

For CDFs

$F_n \Rightarrow F$ if $F_n(x) \to F(x)$ for all $x \in \mathbb{R}$ where F is continuous

For random variables

$X_n \Rightarrow X$ if the corresponding CDFs converge in distribution

---

Central Limit Theorem (CLT)

$X_1, X_2, \ldots$ i.i.d $\quad E[X_i] = \mu \quad Var(X_i) = \sigma^2 < \infty$

$$\frac{S_n - n\mu}{\sqrt{n} \cdot \sigma} \Rightarrow N(0,1)$$

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

DeMoivre-Laplace

$P(X_i = +1) = P(X_i = -1) = \frac{1}{2} \quad \mu = 0 \quad \sigma^2 = 1$

CLT: $\quad \frac{S_n}{\sqrt{n}} \Rightarrow N(0,1)$

$$\lim_{n\to\infty} P\left(\frac{S_n}{\sqrt{n}} \leq x\right) = \int_{-\infty}^{x} \varphi(y)\,dy \quad \text{for all } x$$

$$P(S_{2n} = 2k) = \binom{2n}{n+k} \frac{1}{2^{2n}}$$

Enough: $\displaystyle \lim_{n\to\infty} P\left(a \leq \frac{S_n}{\sqrt{n}} \leq b\right) = \int_a^b \varphi(y)\,dy$ for all $a<b$

We will look at limit along the even numbers

$$P\left(a \le \frac{S_{2n}}{\sqrt{2n}} \le b\right) = \sum_{\frac{2k}{\sqrt{2n}} \in [a,b]} P(S_{2n} = 2k)$$

We need to understand the PMF on a scale of $\sqrt{n}$ near 0.

Stirling's formula: $k! \sim k^k e^{-k} \sqrt{2\pi k}$

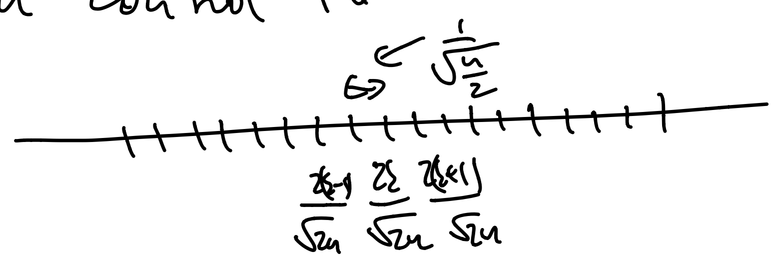
$$\left(1 + \frac{c}{k}\right)^k \longrightarrow e^c$$

$$P(S_n = 2k) \sim \frac{1}{\sqrt{\pi n}} e^{-\frac{1}{2}\left(\frac{2k}{\sqrt{2n}}\right)^2} = \frac{1}{\sqrt{\frac{n}{2}}} \varphi\left(\frac{2k}{\sqrt{2n}}\right)$$

if $|k| < C\sqrt{n}$.

One can control the error terms

$$\frac{S_{2n}}{\sqrt{2n}}$$



$$P\left(a \le \frac{S_{2n}}{\sqrt{2n}} \le b\right) = P\left(a\sqrt{2n} \le S_{2n} \le b\sqrt{2n}\right)$$

$$= \sum_{\frac{2k}{\sqrt{2n}} \in [a,b]} P(S_{2n} = 2k) \approx \sum_{\frac{2k}{\sqrt{2n}} \in [a,b]} \frac{1}{\sqrt{\frac{n}{2}}} \varphi\left(\frac{2k}{\sqrt{2n}}\right)$$

$$\approx \int_a^b \varphi(y)\, dy$$

How can we prove convergence in distribution?

Direct approach (via the CDF / PMF / PDF)

1) DeMoivre - Laplace thm

Note: if $X_1, X_2, \dots, X$ are integer valued then $X_n \Rightarrow X$ is equivalent to $\lim P(X_n=k)=P(X=k)$ for all $k \in \mathbb{Z}$

---

2) Counting rare events

iid trials with success probability $\frac{\lambda}{n}$

$S_n = X_1 + \dots + X_n \sim \text{Binomial}(n, \frac{\lambda}{n})$

# of successes among the first n trials

$E S_n = n \cdot \frac{\lambda}{n} = \lambda$

$$S_n \Rightarrow \text{Poisson}(\lambda)$$

Need: $\lim_{n\to\infty} P(S_n = k) = \frac{\lambda^k}{k!} e^{-\lambda}$

$$P(S_n=k) = \binom{n}{k}\left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n(n-1)\cdots(n-k+1)}{k!\ n^k} \lambda^k \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-k}$$

$$= \frac{\lambda^k}{k!} \underbrace{\frac{n(n-1)\cdots(n-k+1)}{n^k}}_{\downarrow\, 1} \cdot \underbrace{\left(\left(1-\frac{\lambda}{n}\right)^{n}\right)}_{\downarrow\, e^{-\lambda}} \underbrace{\left(1-\frac{\lambda}{n}\right)^{-k}}_{\downarrow\, 1}$$

$$= \frac{\lambda^k}{k!} e^{-\lambda}$$

---

3. Waiting for rare events

$X_1, X_2, \ldots$ iid trials with success prob $\frac{\lambda}{n}$

$T_n =$ # of trials needed for the first success

$T_n \sim \text{Geometric}\left(\frac{\lambda}{n}\right)$ | $P(T_n > k) = \left(1-\frac{\lambda}{n}\right)^k$

$\frac{\lambda}{n} \cdot T_n \Rightarrow \text{Exp}(1)$

---

4. Limit theorems for the maximum of iid random variables

$X_1, X_2, \ldots$ iid $\quad M_n = \max(X_1, \ldots, X_n)$

What can we say about M_n as $n \to \infty$?

$X_i \sim \text{Unif}[0,1] \qquad M_n \xrightarrow{a.s.} 1$ (use B.C.)

$Y_n = n(1 - M_n)$

$$P(M_n \le x) = P(X_1 \le x, \ldots, X_n \le x)$$
$$= F(x)^n = \begin{cases} x^n & x \in [0,1] \\ 1 & x > 1 \\ 0 & x < 0 \end{cases}$$

$Y_n \geq 0$

$$P(Y_n \le y) = P(n(1-M_n) \le y) \quad y \ge 0$$

$$= P\left(M_n \ge 1 - \frac{y}{n}\right) \quad \text{if } n \ge y$$

$$= 1 - \left(1 - \frac{y}{n}\right)^n \to 1 - e^{-y}$$

$$Y_n \Rightarrow \text{Exp}(1)$$

---

5. If $X_1, X_2, \ldots$ are abs cont with PDF f_n and $f_n \to f_\infty$ pointwise where f_∞ is a PDF

then $X_n \Rightarrow X$. (where the PDF of X is f_∞)

Scheffé's thm

Proof: in fact, for any Borel set B we have

$$\int_B f_n \, dx \to \int_B f_\infty \, dx.$$ This will

follow from $\displaystyle\int_{-\infty}^{\infty} |f_n - f| \, dx \to 0$

$$\int_{-\infty}^{\infty} |f_\infty - f_n| \, dx = 2 \int_{-\infty}^{\infty} \underbrace{(f_\infty - f_n)_+}_{\le f_\infty} \, dx$$

Dominated convergence theorem

Need to find new tools to prove the CLT

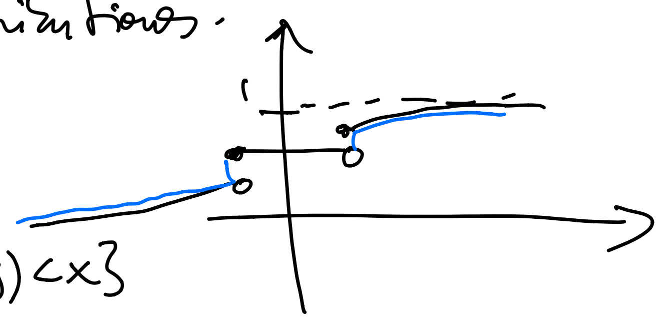
Thm: If $F_n \Rightarrow F$ then we can find random variables $X_1, X_2, \ldots$ and X on $([0,1], \mathcal{B}, \mathbb{P})$ so that $X_n \xrightarrow{a.s} X$, and the CDF of X_n is F_n, CDF of X is F.

Proof: Set $X_n(x) \overset{def}{=} \sup\{y : F_n(y) < x\}$

$x \in [0,1]$ $\quad X(x) \overset{def}{=} \sup\{y : F(y) < x\}$

We know that these random variables have the right distributions.

Claim: $X_n \xrightarrow{a.s.} X$.



let $a_x = \sup\{y : F(y) < x\}$

$b_x = \inf\{y : F(y) > x\}$ $\quad a_x \leq b_x$

$$\Omega_0 = \{x \in [0,1] : a_x = b_x\}$$

If $x \in \Omega_0$ then $X_n(x) \longrightarrow X(x)$ as $n \to \infty$.

$\mathbb{P}(\Omega_0) = 1 \qquad \Omega_0^c = \{x \in [0,1] : a_x < b_x\}$

The intervals (a_x, b_x) are disjoint hence

Ω_0^c is a countable set.

$$P(\Omega_0^c) = 0.$$

(Check proof in Durrett, or try to recreate it)

Thm: $X_n \Rightarrow X$ if and only if $E[g(X_n)] \to E[g(X)]$ for all bounded cont. function g.

Proof: $\Longrightarrow$ if $X_n \Rightarrow X$ then we can find random variables Y_n, Y on $([0,1], \mathcal{B}, \mathbb{P})$ with $Y_n \xrightarrow{a.s.} Y$, $Y_n \stackrel{d}{=} X_n$, $Y \stackrel{d}{=} X$.

For any bounded continuous g

$$g(Y_n) \xrightarrow{a.s.} g(Y), \quad Eg(Y_n) \to Eg(Y)$$

bounded conv theorem

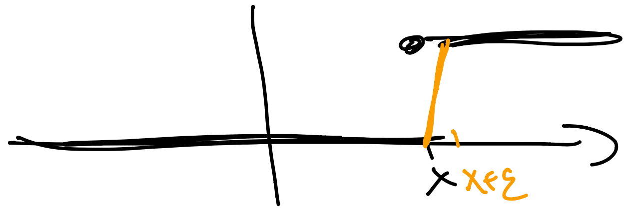
$$E\, g(Y_n) = E\, g(X_n)$$

$$E\, g(Y) = E\, g(X)$$

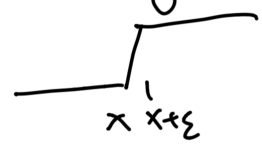
$\Longleftarrow$ need to prove $\mathbb{P}(X_n \le x) \to P(X \le x)$ if x is a cont point

$$P(X_n \le x) = E[1(X_n \le x)]$$

$\mathbb{1}(X_n \le x)$ is not a continuous function of X_n



$g_{x,\varepsilon}$:



this is a bounded cont function

$$E\, g_{x,\varepsilon}(X_n) \to E\, g_{x,\varepsilon}(X)$$

Now take $\varepsilon \to 0$.