Stochastic integral of predictable processes — examples, ~~extensions~~

Ex. (Doleanmeasure & Poisson process)

Given a L^2 cadlag mart M.

μ_M is a measure on σ-field $\mathcal{P}$ ← predictable σ-field.

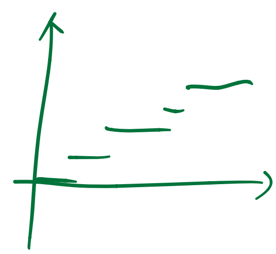
$$\mu_M(\underset{\in \mathcal{P}}{A}) = \mathbb{E}\int_0^\infty \mathbb{1}_A(t,\omega)\, d[M]_t(\omega)$$

$$\left(\underline{[0,\infty)\times\Omega},\ \underline{\mathcal{P}},\ \underline{\mu_M}\right)$$

N_t is Poisson α. $\left(N_t - N_s \sim Poi(\alpha(t-s)) \right)$

$\underline{M_t} = N_t - \alpha t$ is Mart.

compensated Poisson



$$\mu_M = \alpha\, m \otimes P$$

m: Leb on $[0,\infty)$

P: measure on Ω

Want to show $\underline{N_t}$ is not prectable.

(recall $N_t = \lim N_s$)

$t-$ $\quad s \nearrow t$

is left-cont $\Rightarrow$ predictable

(a) $\displaystyle\int_{[0,T]\times\Omega} N_s(\omega)\, \overbrace{(\alpha m \otimes P)}^{\mu_M}(ds\,d\omega)$ $\qquad \boxed{X}(t,\omega)$ $\quad [0,\infty)\times\Omega$

$$= \int_{[0,T]\times\Omega} E N_s \quad \alpha\, ds$$

$N_s \sim Poi(\alpha s)$

$E N_s = \alpha s$

$$= \int_{[0,T]\times\Omega} \alpha s \cdot \alpha\, ds$$

$$= \frac{1}{2}\alpha^2 T^2$$

$$\mu_M(\overset{f}{A}) = E\int_0^T \overset{f}{\mathbb{1}_A}\, d[M]_s$$

(b) $$E\int_0^T N_s\, d[M]_s$$

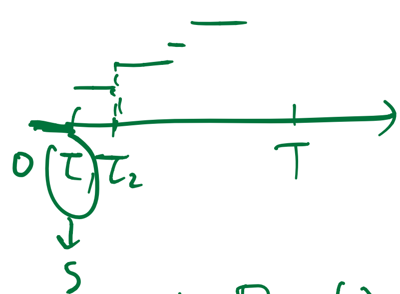
$$= E\int_0^T N_s\, dN_s$$

$[M] = N$

$$\|$$

$$1\cdot 1 + 2\cdot 1 + \dots N_T \cdot 1$$

$$= \frac{1}{2} N_T (N_T+1)$$



$$E\left[\frac{1}{2} N_T (N_T+1)\right] =$$

$X \sim Poi(\lambda)$

$EX = \lambda$

$E(X^2) = \lambda + \lambda^2$

$$= \frac{1}{2}\left(\alpha T + (\alpha T)^2\right) + \frac{1}{2}\alpha T$$

$$= \underline{\frac{1}{2}(\alpha^2 T^2)} + \alpha T$$

$$\mathcal{U}_M(N) \neq \underline{E\int_0^T N \, d[M]}$$

$\Rightarrow$ N is not predictable.

$$\boxed{N_{t-}} \qquad \int_{(0,T]\times\Omega} N_{s-} \, \alpha(m\otimes P)(ds, d\omega)$$

$$= \underline{\frac{1}{2}\alpha^2 T}$$

$$E\int_0^T \underline{N_{s-}} \, d\underline{N_s}$$

$$= E\left[\frac{1}{2}\underline{N_T(N_T} - 1)\right]$$

$$= \frac{1}{2}\left[(\alpha T)^2 + \alpha T\right] - \frac{1}{2}\alpha T$$

$$= \underline{\frac{1}{2}(\alpha T)^2}$$

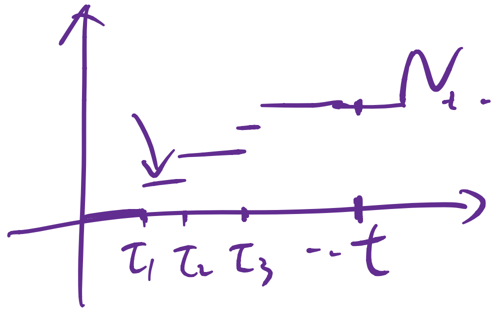
Ex. $\underline{\int_0^t N_{s-} \, dM_s}$ $\qquad$ $\underline{M_t = N_t - \alpha t}$

1). Calculate

2) Will see it's a mart.

3). $\int_0^t N_s \, dM_s$ is not a mart!

1). $\underbrace{\int_0^t N_{s-} \, dN_s}_{\frac{1}{2} N_s (N_s - 1)} - \alpha \int_0^t N_{s-} \, ds$



$\alpha \int_0^t N_{s-} \, ds = ?$

N_t

$0. \; \tau_1, \tau_2 \; --- \; \tau_{N_t}$

$$= \alpha \Big(0 \cdot \tau_1 + 1 \cdot (\tau_2 - \tau_1) + 2(\tau_3 - \tau_2) + \dots + (N_t - 1)(\tau_{N_t} - \tau_{N_t - 1}) + N_t (t - \tau_{N_t}) \Big)$$

$$= \alpha \left(-\tau_1 - \tau_2 - \tau_3 \cdots - \tau_{N_t} + t N_t \right)$$

$$= -\alpha \boxed{\sum_{i=1}^{N_t} \tau_i} + \alpha t N_t$$

$$\int_0^t N_{s-} \, dM_s = \frac{1}{2} N_t (N_t - 1) - \alpha \sum_{i=1}^{N_t} \tau_i$$

$$+\alpha t N_t$$

$$2)\quad \mathbb{E}\Big[\int_0^t N_{s-}\, d\overset{N(t)\alpha t}{M_s}\Big] = 0$$

$$= \frac{1}{2}\left((\alpha t)^2 \cancel{+\alpha t}\right) \cancel{-\frac{1}{2}\alpha t}$$

$$+\ \alpha\, \underbrace{\mathbb{E}\Big(\sum_{i=1}^{N_t} \tau_i\Big)}_{=\frac{\alpha t^2}{2}} - \alpha t \cdot \alpha t$$

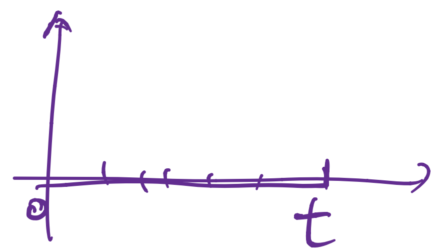
$$= 0$$

$$\mathbb{E}\Big(\sum_{i=1}^{N_t} \tau_i\Big) = \mathbb{E}\Big[\mathbb{E}\Big[\sum_{i=1}^{N_t} \tau_i \,\Big|\, N_t\Big]\Big]$$

$$= \mathbb{E} \sum_{i=1}^{N_t} \frac{t}{2}$$

$$= \mathbb{E}\Big[\frac{t}{2} N_t\Big]$$

$$= \frac{t}{2} \cdot \alpha t$$



Condition on N_t is given

the N_t points are iid uniform on $[0,t]$

not predictably.

$$3)\ \mathbb{E}\Big[\int_0^t N_s\, dM_s\Big] \neq 0.$$

$$= \mathbb{E}\int^t N_s\, dN_s - \mathbb{E}\,\alpha \int^t N_s\, ds$$

$$\underbrace{\hphantom{xxxxxxxxxxxx}}_{\frac{1}{2}N_t(N_t+1)} \qquad \underbrace{\hphantom{xxxxxxxxxxxx}}_{\neq 0}$$