

# CS 726 - Fall 2020

## Homework #2

Due : 10/05/2020, 5pm

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### Question 1

*Proof.*  $\Rightarrow$ , Let

$$\varphi(\alpha) = \frac{1}{\alpha} (f((1-\alpha)x + \alpha y) - f(x))$$

$f$  is  $m$ -strongly convex means,

$$\begin{aligned} f((1-\alpha)x + \alpha y) &\leq (1-\alpha)f(x) + \alpha f(y) - \frac{m}{2}\alpha(1-\alpha)\|y-x\|^2 \\ f((1-\alpha)x + \alpha y) - f(x) &\leq \alpha(f(y) - f(x)) - \frac{m}{2}\alpha(1-\alpha)\|y-x\|^2 \\ f(y) - f(x) &\geq \varphi(\alpha) + \frac{m}{2}(1-\alpha)\|y-x\|^2 \end{aligned}$$

Let  $\alpha \rightarrow 0$ , we have

$$\begin{aligned} f(y) &\geq f(x) + \varphi'(0) + \frac{m}{2}\|y-x\|^2 = f(x) + \langle \nabla f(x), y-x \rangle + \frac{m}{2}\|y-x\|^2 \\ f(x + \alpha(y-x)) &\geq f(x) + \langle \nabla f(x), \alpha(y-x) \rangle + \frac{m}{2}\alpha^2\|y-x\|^2 \end{aligned}$$

Consider, Taylor Theorem:

$$f(x + \alpha(y-x)) = f(x) + \langle \nabla f(x), \alpha(y-x) \rangle + \frac{\alpha^2}{2}(y-x)^T \nabla^2 f(x + \gamma\alpha(y-x))(y-x)$$

Combine the above two formulas, it gives

$$(y-x)^T \nabla^2 f(x)(y-x) \geq m\|y-x\|^2$$

Thus, we have

$$\nabla^2 f(x) \succeq mI$$

$\Leftarrow$ , By Taylor Theorem,

$$f(y) = f(x) + \langle \nabla f(x), y-x \rangle + \frac{1}{2}\nabla^2 f(x + \gamma(y-x))\|y-x\|^2$$

$\nabla^2 f(x) \succeq mI$  means the smallest eigenvalue of  $\nabla^2 f(x)$  is greater than  $m$ , therefore

$$\frac{1}{2}\nabla^2 f(x + \gamma(y-x))\|y-x\|^2 \geq \frac{1}{2}m\|y-x\|^2$$

That is

$$f(y) \geq f(x) + \langle \nabla f(x), y-x \rangle + \frac{m}{2}\|y-x\|^2$$

Consider

$$(1-\alpha)f(x) + \alpha f(y) - f((1-\alpha)x + \alpha y)$$

We will have

$$(1-\alpha)f(x) + \alpha f(y) - f((1-\alpha)x + \alpha y) \geq \frac{m}{2}\|y-x\|^2(\alpha - \alpha^2) = \frac{m}{2}\alpha(1-\alpha)\|y-x\|^2$$

□

## Question 2

*Proof.* Let  $x_{k+1} = x_k + \nabla f(x_k)$ , we have

$$f(x_{k+1}) - f(x_k) \geq \langle \nabla f(x_k), \nabla f(x_k) \rangle + \frac{m}{2} \|\nabla f(x_k)\|^2$$

Add them together,

$$f(x_{k+1}) - f(x_0) \geq \left(1 + \frac{m}{2}\right) \sum_{i=0}^k \|\nabla f(x_i)\|^2 \geq \left(1 + \frac{m}{2}\right) \left\| \sum_{i=0}^k \nabla f(x_i) \right\|^2 = \left(1 + \frac{m}{2}\right) \|x_{k+1} - x_0\|^2$$

The gradient of  $f$  can go to  $\infty$ , when  $\|x_{k+1} - x_0\|$  is large enough.

So,  $f$  cannot be Lipschitz continuous on the entire  $\mathbb{R}^d$ . But it is possible on the unit Euclidean ball. □

## Question 3

*Proof.* □

## Question 4

*Proof.* □

## Question 5

*Proof.* □