Stochastic process

functions $\longrightarrow$ Calculus $\longrightarrow$ differential equation

stochastic processes $\longrightarrow$ Itô Calculus (Stoch Calculus) $\longrightarrow$ stochastic diff equ's.

$\cup$

martingales

$\cup$

Brownian motions.

Stochastic process: a collection of r.v.'s $\{\underline{X_t}\}_{t\in T}$ defined on $(\Omega, \mathcal{F}, P)$

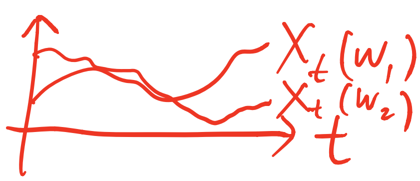
T index set.

$X_t \in \mathbb{R}$

Rmk.
- $\mathbb{R}$ can be replaced by $\mathbb{R}^n$.
- $T=[0,\infty)$. $T=\mathbb{Z}_+$ "time"

① fix t, $\omega \mapsto X_t(\omega)$ is a r.v.

② fix ω, $t \mapsto X_t(\omega)$ is a function. "a path"



③ $(t,\omega) \mapsto X_t(\omega)$.

$$\underset{\Omega}{\omega} \rightsquigarrow t\mapsto X_t(\omega)$$

$$\Omega \subset \tilde{\Omega} = \mathbb{R}^T$$

$T=[0,\infty)$

a stochastic process is a prob measure on $(\mathbb{R}^T, \mathcal{B})$

$$\mathcal{B} = \sigma\underbrace{\{\omega\in\tilde{\Omega} : \omega(t_1)\in F_1 \cdots \omega(t_k)\in F_k\}}_{F_i\in\mathcal{B}(\mathbb{R})}$$

"product topology on $\tilde{\Omega}$" $\mathcal{B}$ is Borel σ-alg.

One-point distribution $\mu_t(F) = P(X_t \in F)$ $F \in \mathcal{B}(\mathbb{R})$

finite-dim distr.

$$\mu_{t_1 \cdots t_k}(F_1 \times \cdots \times F_k) = P(X_{t_1} \in F_1, X_{t_2} \in F_2, \cdots X_{t_k} \in F_k)$$

Stoch process X $\Rightarrow$ $\mu_{t_1 \cdots t_k}$.

$\Leftarrow$ converse ?

Thm (Kolmogorov extension thm):

For all $t_1 \cdots t_k \in T$, $k \in \mathbb{N}$. Let $\mu_{t_1 \cdots t_k}$ be prob measure on $\mathbb{R}^k$. s.t

(K1) $\mu_{t_{\sigma(1)} \cdots t_{\sigma(k)}}(F_1 \times \cdots \times F_k)$

$= \mu_{t_1 \cdots t_k}(F_{\sigma^{-1}(1)} \times \cdots \times F_{\sigma^{-1}(k)})$

for all permutation σ. of $\{1 \ldots k\}$

$\mathbb{R}^d$

and

(K2) $\mu_{t_1 \cdots t_k}(F_1 \times \cdots \times F_k) = \mu_{t_1 \cdots t_k, t_{k+1} \cdots t_{k+m}}(F_1 \times \cdots \times F_k \times \mathbb{R} \times \cdots \times \mathbb{R})$

$\forall k, m.$

Then $\exists$ $(\Omega, \mathcal{F}, P)$ and a stoch process $\{X_t\}$ on Ω.

$X_t : \Omega \to \mathbb{R}$. s.t. μ is the finite dim distr of X.