Thm: $X_i \sim \text{Bernoulli}(p_i)$ $\quad 1 \le i \le n$ independent

$S = X_1 + \dots + X_n$

$Y \sim \text{Poisson}(\lambda) \qquad \lambda = \sum_{i=1}^{n} p_i$

Then

$$\|Q_S - Q_Y\|_{TV} = \sum_{i=0}^{\infty} |P(S=i) - P(Y=i)| \le 2\sum_{\ell=1}^{n} p_\ell^2$$

$$\sup_{A \subset \mathbb{Z}} |P(S \in A) - P(Y \in A)| \le \sum_{\ell=1}^{n} p_\ell^2$$

Proof: μ_i is a distribution on $\{-1, 0, 1, \dots\}$

$$\mu_i(-1) = e^{-p_i} - (1-p_i) \qquad \mu_i(0) = 1 - p_i$$

$$\mu_i(\ell) = \frac{p_i^\ell}{\ell!} e^{-p_i} \qquad \ell \ge 1$$

$$Y_i(\omega) = \omega^+ \sim \text{Poisson}(p_i)$$

$$X_i(\omega) = \mathbb{1}(\omega \ne 0) \sim \text{Bernoulli}(p_i)$$

$$P(X_i \ne Y_i) = 1 - \mu_i(\{0, 1\}) \le p_i^2$$

Consider $\mu_1 \times \mu_2 \times \dots \times \mu_n$

$S = X_1 + \dots + X_n \qquad Y_1 = Y_{1_1} + \dots + Y_{1_n} \sim \text{Poisson}(\lambda)$

$$P(S \neq Y_1) = \sum_{i=1}^{n} P(X_i \neq Y_{1_i}) \leq \sum_{i=1}^{n} p_i^2$$

$$\sum_{i=0}^{\infty} \left| P(S=i) - P(Y_1=i) \right| = \sum_{i=0}^{\infty} \left| P(S=i, Y_1 \neq i) - P(S \neq i, Y_1 = i) \right|$$

$$\leq \underbrace{\sum_{i=0}^{\infty} P(S=i, Y_1 \neq i)}_{= P(S \neq Y)} + \underbrace{\sum_{i=0}^{\infty} P(Y=i, S \neq i)}_{= P(S \neq Y)} = 2P(S \neq Y)$$

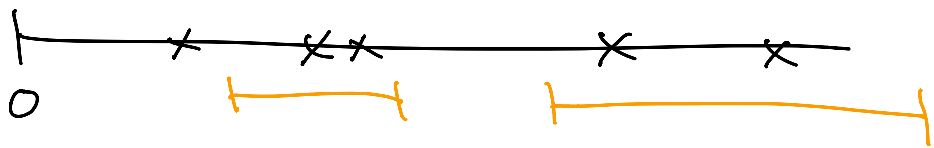
Similar statement for independent X_i which are taking values from $\{0, 1, 2, \dots\}$

We will have an extra term

$$\sum_i P(X_i \geq 2).$$

Poisson process

Setup: random points on $[0, \infty)$



Assume:

- \# of arrivals in disjoint intervals are independent

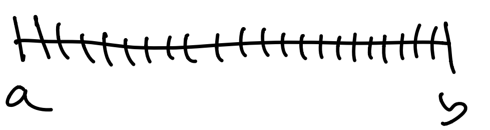
$N(a,b) = \#$ of points in $(a,b]$ $\quad a<b$

- $N(s,t) \stackrel{d}{=} N(0,t-s)$
- $P(N(0,h)=1) = \lambda \cdot h + o(h)$
- $P(N(0,h) \geq 2) = o(h)$

Then we must have

$$N(a,b) \sim \text{Poisson}(\lambda(b-a))$$

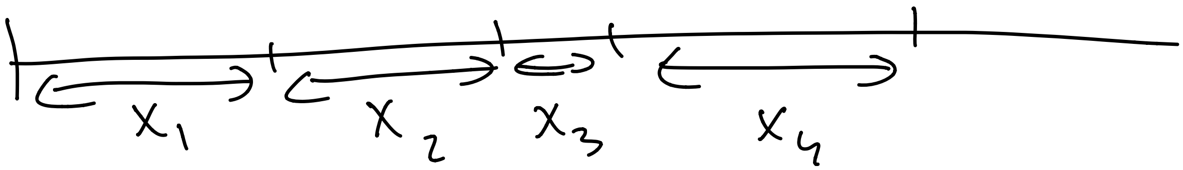
Proof



Apply Poisson approximation to the counting functions of the small intervals.

Is there a process like this??

YES!



$X_1, X_2, \ldots$ iid $\qquad X_i \sim \mathrm{Exp}(\lambda)$

For this process we have

- independent increments for the counting function
- # of points in $(a,b] \sim \mathrm{Poisson}(\lambda(b-a))$

Limits in distribution in $\mathbb{R}^d$

CDF of $(X_1, \ldots, X_d)$

$$F(x_1, \ldots, x_d) = P(X_1 \le x_1, \ldots, X_d \le x_d)$$

We can define convergence in distribution in $\mathbb{R}^d$ the same way as in $\mathbb{R}$

$F_n \Rightarrow F$ if $F_n(x) \to F(x)$ for all $x \in \mathbb{R}^d$ where F is cont.

There are analogues of $\Rightarrow$ in $\mathbb{R}^d$.

E.g. $X_n \Rightarrow X$ if $E g(X_n) \to E g(X)$ for all $g : \mathbb{R}^d \to \mathbb{R}$ cont, bounded

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Tightness: for any $\varepsilon > 0$ there is $M > 0$ so that $P(|X_n| > M) \leq \varepsilon$ for all n.

Characteristic function: $X \in \mathbb{R}^d$

$$\varphi_X(t) = E\left[e^{it \cdot X}\right] \quad t \in \mathbb{R}^d$$

Similar inversion formula as in $\mathbb{R}$.

$X_1, X_2, \ldots, X_d$ are independent

$$\varphi_{(X_1, \ldots, X_d)}(t_1, \ldots, t_d) = \varphi_{X_1}(t_1) \cdot \ldots \varphi_{X_d}(t_d)$$

Convergence in distribution $\Leftrightarrow$ pointwise conv of φ_{X_n}

Lemma (Cramer-Wald device)

If $\vartheta \cdot X_n \Rightarrow \vartheta \cdot X$ for all $\vartheta \in \mathbb{R}^d$ then

$X_n \Rightarrow X$.

Multivariate CLT

Multivariate normal distribution

$Z = (Z_1, \dots, Z_d)^T$ iid $N(0,1)$ coordinates

$$Y = \underset{n \times d}{A} \cdot \underset{d}{Z} + \underset{n}{\mu} \sim \text{multivariate normal}$$

$E[Y] = \mu$

Covariance matrix $\left[Cov(Y_i, Y_j) \right]_{1 \le i,j \le n} =$

$$= E\left[(Y - E[Y])(Y - E[Y])^T \right]$$

$$= E\left[A Z Z^T A^T \right]$$

$$= A \underbrace{E[Z Z^T]}_{I_d} A^T = A A^T$$

If AA^T is invertible then the resulting distribution has PDF

$$\frac{1}{(2\pi)^{\frac{d}{2}}} \frac{1}{\sqrt{\det(AA^T)}} \exp\left[-\frac{1}{2}(x-\mu)^T (AA^T)^{-1} (x-\mu)\right]$$

$$\Sigma = AA^T \qquad \text{"}\sigma^2\text{"}$$

Characteristic function:

$$\varphi_{Y_1}(t) = \exp\left(t\cdot\mu - \frac{1}{2}t^T \Sigma t\right)$$

Multivariate CLT:

$X_1, X_2, \ldots$ iid in $\mathbb{R}^d$ with $E[X_1] = \mu \in \mathbb{R}^d$

covariance matrix: Σ.

Then
$$\frac{\sum_{j=1}^n X_j - n\mu}{\sqrt{n}} \Rightarrow$$
multivariate normal with mean 0, covariance Σ.

CLT: $X_1, \ldots$ iid $\quad \dfrac{S_n - n\mu}{\sqrt{n\sigma^2}} \Rightarrow N(0,1)$

Q: can we get a limit by choosing a_n, b_n : $\frac{S_n - b_n}{a_n} \Rightarrow$?

If $EX_i^2 < \infty$ then we can only get a normal in the limit.

If $EX^2 = \infty$ then we can see different limits!

Thm: $X_1, X_2, \ldots$ iid with

1) $\lim_{x \to \infty} \frac{P(X_i > x)}{P(|X_i| > x)} = \theta \in [0,1]$

2) $P(|X| > x) = x^{-\alpha} L(x)$ with $\alpha < 2$ and L is slowly varying $\lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1$ for $t > 0$

Set $a_n = \inf\{x : P(|X| > x) \le \frac{1}{n}\}$

$b_n = n\, E\left[X \mathbb{1}(|X| \le a_n)\right]$.

Then $\frac{S_n - b_n}{a_n}$ converges in distribution to a non-trivial distribution.

The limit has char. function:

$$\varphi(t) = \exp\left(itc - b \cdot |t|^{\alpha}\left(1 + ik\, \mathrm{sgn}(t)\, \omega_\alpha(t)\right)\right)$$

$$\omega_\alpha(t) = \begin{cases} \tan\left(\frac{\pi\alpha}{2}\right) & \text{if } \alpha \neq 1 \\ \frac{2}{\pi} \log|t| & \text{if } \alpha = 1 \end{cases}$$

with $c \in \mathbb{R}$, $b > 0$, $|k| \leq 1$

If $\alpha = 1$, $k = 0$, $c = 0$, $b = 1$

$$\varphi(t) = e^{-|t|}$$ which is

the char function of Cauchy distrib.

These distributions are called

stable distributions.

(normal is also included)

Another characterization: Y is stable

if $\dfrac{Y_1 + Y_2 + \ldots + Y_k - b_k}{a_k} \stackrel{d}{=} Y_1$

Another nice family of distributions:

Y is infinitely divisible if there are iid $Z_1, Z_2, \ldots$ so that $Y \overset{d}{=} Z_1 + \ldots + Z_n$

Eg: $Y \sim \text{Poisson}(\lambda)$

$Y \overset{d}{=} Z_1 + \ldots + Z_n \qquad Z_i \sim \text{Poisson}(\frac{\lambda}{n})$

$\varphi_Y(t) = \varphi_{Z_1}(t)^n$

the char function has an n^{th} root which is also a characteristic function.

Levy-Khinchin formula: Y is inf. div. if and only if there is a measure μ on $\mathbb{R}$ with $\mu(\{0\}) = 0$ and $\int_{\mathbb{R}} \frac{x^2}{1+x^2} d\mu < \infty$

and

$$\log \varphi_Y(t) = ict - \frac{\sigma^2 t^2}{2} + \int_{\mathbb{R}} \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) \mu(dx)$$

HW Try to find μ for Poisson(λ) !

Next week: Conditional expectation and

martingales