FROM RIEMANN TO LEBESGUE

Contents

1. Review: The Riemann integral

We recall some definitions. In what follows $a, b \in \mathbb{R}$, with $a < b$ are given. In this section we recall basic definitions which lead to the definition of Riemann integrable functions on $[a, b]$, and the Riemann integral of such functions.

Definition 1.1. (i) A partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ is a finite subset of $[a, b]$ which includes the points a and b and is ordered in the following way:

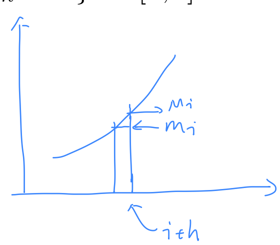
$$a = x_0 < \ldots < x_i < x_{i+1} < \ldots < x_n = b.$$

P'是P的细化

(ii) If P, P' are partitions of $[a, b]$ with $P \subset P'$ then P' is called a refinement of P.

Definition 1.2. Given a partition $P = \{a = x_0 < \cdots < x_n = b\}$ of $[a, b]$ and a *bounded* function $f : [a, b] \to \mathbb{R}$ define

$$m_i(f) = \inf_{t \in [x_{i-1}, x_i]} f(t),$$
$$M_i(f) = \sup_{t \in [x_{i-1}, x_i]} f(t).$$



(i) The expression

$$L(f, P) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1})$$

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is called the lower sum of f with respect to the partition P.

(ii) The expression

$$U(f, P) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1})$$

is called the upper sum of f with respect to the partition P.

Lemma 1.3. *Let P, P' be partitions of $[a, b]$, let $f : [a, b] \to \mathbb{R}$ be bounded, and let P' be a refinement of P. Then*

$$(b - a) \inf_{[a,b]} f \leq L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P) \leq (b - a) \sup_{[a,b]} f.$$

Corollary 1.4. *Let P_1, P_2 be partitions of $[a, b]$. Then $L(f, P_1) \leq U(f, P_2)$.*

Definition 1.5. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. The numbers

$$\underline{\mathcal{I}}_a^b(f) := \sup_P L(f, P), \qquad \overline{\mathcal{I}}_a^b(f) := \inf_P U(f, P)$$

are called the *lower and upper Riemann-Darboux integrals of f* on the interval $[a, b]$, respectively. Here the sup and inf are taken over all partitions of $[a, b]$.

Lemma 1.6. *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then*

$$(b-a)\inf_{[a,b]} f \le \underline{\mathcal{I}}_a^b(f) \le \overline{\mathcal{I}}_a^b(f) \le (b-a)\sup_{[a,b]} f.$$

We are now ready to define the concept of Riemann integrable functions and the Riemann integral of such functons.

Definition 1.7. (i) Let $f : [a, b] \to \mathbb{R}$ be bounded. f is called Riemann integrable if $\underline{\mathcal{I}}_a^b(f) = \overline{\mathcal{I}}_a^b(f)$.

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(ii) If f is Riemann integrable the number $\underline{\mathcal{I}}_a^b(f) = \overline{\mathcal{I}}_a^b(f)$ is called the Riemann integral of f, denoted by $\int_{[a,b]} f$ or by $\int_a^b f$ (or even by $\int_a^b f(t)dt$ ...)

Lemma 1.8. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable if and only if for every $\varepsilon > 0$ there is a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$.*

Proof. Suppose f is Riemann integrable. Then there are partitions P_1, P_2 of $[a, b]$ such that $L(f, P_1) \ge \int_a^b f - \varepsilon/2$, $U(f, P_2) \le \int_a^b f + \varepsilon/2$ and thus $U(f, P_2) - L(f, P_1) < \varepsilon$. Let P be the refinement $P_1 \cup P_2$. Then $U(f, P_2) \ge U(f, P) \ge L(f, P) \ge L(f, P_1)$ and hence $U(f, P) - L(f, P) < \varepsilon$.

Vice versa assume that for every ϵ there is a partition P_ε of $[a, b]$ such that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon.$$

Then $\overline{\mathcal{I}}_a^b(f) - \underline{\mathcal{I}}_a^b(f) \le U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$, and since ε was arbitrary we conclude $\overline{\mathcal{I}}_a^b(f) = \underline{\mathcal{I}}_a^b(f)$. Hence f is Riemann integrable. □

Theorem 1.9. *If $f : [a, b] \to \mathbb{R}$ is continuous in $[a, b]$ then f is Riemann integrable.*

Proof. Recall that a continuous function on a compact set is uniformly continuous. Hence given $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that $|f(x) - f(\tilde{x})| < \varepsilon/(b - a)$ provided that $|x - \tilde{x}| < \delta$. Let N be such that $(b-a)/N < \delta$ and choose the partition $P = \{x_j := a + j\frac{b-a}{N}, j = 0, \dots N\}$. Let $I_j = [x_{j-1}, x_j]$, $j = 1, \dots, N$. Then

$$(M_i(f) - m_i(f) = (\sup_{I_j} f - \inf_{I_j} f) < \varepsilon/(b-a)$$

for $i = 1, \dots, N$ so that

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$$U(f, P) - L(f, P) = \sum_{i=1}^N M_i f(x_j - x_{j-1}) - \sum_{i=1}^N m_i(f)(x_j - x_{j-1})$$

$$= \sum_{i=1}^N (M_j(f) - m_i(f))(x_j - x_{j-1}) \le \sum_{i=1}^N \frac{\varepsilon}{b-a}(x_j - x_{j-1}) = \frac{\varepsilon}{b-a}(b-a) = \varepsilon.$$

We can apply Lemma 1.8 to see that f is Riemann integrable. □

Exercise 1.10. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Under each of the following hypotheses on f show that f is Riemann integrable.

(i) There is a point $c \in [a, b]$ such that f is continuous on $[a, b] \setminus \{c\}$.

(ii) f is continuous except possibly at a finite number of points in $[a, b]$.

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(iii) f is continuous in $[a, b] \setminus \{c_k : k \in \mathbb{N}\}$, where $(c_k)_{k\in\mathbb{N}}$ is a *convergent* sequence of points in $[a, b]$,

Exercise 1.11. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \in [0,1] \setminus \mathbb{Q}. \end{cases}$$

Compute $\overline{\mathcal{I}}_0^1(f)$ and $\underline{\mathcal{I}}_0^1(f)$.

2. Lebesgue null sets

For a compact interval $I = [c, d]$ we call $d - c$ the length of I, also denoted by $\ell(I)$.

Definition 2.1. A set $E \subset [a, b]$ is called a Lebesgue null set if for every $\varepsilon > 0$ there is a sequence $(I_n)_{n\in\mathbb{N}}$ of intervals such that $E \subset \cup_{n\in\mathbb{N}} I_n$ and $\sum_{n=1}^{\infty} \ell(I_n) < \varepsilon$.

Lemma 2.2. *Countable unions of Lebesgue null sets are Lebesgue null sets.*

Proof. Let E_k, $k \in \mathbb{N}$ be Lebesgue null sets. For each k find a countable family of intervals $\{I_{k,n}\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \ell(I_{k,n}) < \varepsilon 2^{-k-1}$ and $E_k \subset \cup_{n=1}^{\infty} I_{k,n}$.

The family of intervals $\{I_{k,n}\}_{(k,n)\in\mathbb{N}^2}$ is countable (and thus cn be arranged in a series) and we have

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \ell(I_{k,n}) < \sum_{k=1}^{\infty} \varepsilon 2^{-k-1} = \varepsilon/2. \qquad \square$$

Exercise 2.3. Which theorems about series with nonnegative terms have been used in this proof?

Definition 2.4. A set $E \subset [a, b]$ has content zero if for every $\varepsilon > 0$ there is a finite set of intervals $I_1, \dots, I_N$ such that $E \subset \cup_{n=1}^{N} I_n$ and $\sum_{n=1}^{N} \ell(I_n) < \varepsilon$.

Note that any set of content zero is a Lebesgue null set, but the converse is not true (see Exercise 2.8 below).

Lemma 2.5. *Let $\{I_\nu\}_{\nu=1}^{N}$ be a finite collection of intervals such that $[a, b] \subset \cup_{\nu=1}^{N} I_j$. Then $\sum_{\nu=1}^{N} \ell(I_\nu) \ge b - a$. In particular $[a, b]$ does not have content zero.*

Proof. Let $J_\nu := \overline{I}_\nu \cap [a, b]$. Arrange the finite set formed by all the endpoint of those intervals in increasing order, written as $a \le x_1 \le \cdots < x_M = b$. Then every interval $[x_{i-1}, x_i]$ is contained in at least J_ν.

Define inductively sets of indices $\mathcal{J}_\nu$. For $\nu = 1$ set

$$\mathcal{J}_1 = \big\{ i \in \{1, \dots, M\} : [x_{i-1}, x_i] \subset J_1 \big\}.$$

For any $\nu > 1$ we are either in the situation that $\mathcal{J}_1 \cup \cdots \cup \mathcal{J}_{\nu-1}$ contains all $i \in \{1, \dots, M\}$ (then we stop the construction) or, if not, then we form

$$\mathcal{J}_\nu = \big\{ i \in \{1, \dots, M\} : [x_{i-1}, x_i] \subset J_\nu \text{ and } [x_{i-1}, x_i] \subsetneq J_l \text{ for } l \le \nu - 1 \ \big\}.$$

The construction stops after K steps, where $K \le N$. Note that each index i is in exactly one family $\mathcal{J}_\nu$ and also for each ν we have

$$\sum_{i\in\mathcal{J}_\nu} (x_i - x_{i-1}) \le \ell(J_\nu).$$

Consequently

$$b - a = \sum_{i=1}^{N} (x_i - x_{i-1}) = \sum_{\nu=1}^{K}\sum_{i\in\mathcal{J}_\nu} (x_i - x_{i-1}) \le \sum_{\nu=1}^{K} \ell(J_\nu) \le \sum_{\nu=1}^{N} \ell(I_\nu). \qquad \square$$

Lemma 2.6. *Let E be a compact Lebesgue null set. Then E has content zero.*

Proof. Let $\varepsilon > 0$. Since E is a null set there is a countable family $\{I_\nu\}_{\nu\in\mathbb{N}}$ of closed intervals such that $\sum_{\nu=1}^{\infty} \ell(I_\nu) < \varepsilon/2$. Write $I_\nu = [a_\nu, b_\nu]$ and form the slightly larger open intervals $\widetilde{I}_\nu = (a_\nu - \varepsilon 2^{-\nu-2}, b_\nu + \varepsilon 2^{-\nu-2})$ so that $\ell(\widetilde{I}_\nu) = \ell(I_\nu) + \varepsilon 2^{-\nu-1}$ and thus

$$\sum_{\nu=1} \ell(\widetilde{I}_\nu) \le \sum_{\nu=1}^{\infty} \ell(I_\nu) + \sum_{\nu=1}^{\infty} \varepsilon 2^{-\nu-1} < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since E is compact we may choose finitely many $\widetilde{I}_{\nu_1}, \dots, \widetilde{I}_{\nu_M}$ such that $E \subset \cup_{l=1}^{M} \widetilde{I}_{\nu_l}$ and $\sum_{l=1}^{M} \ell(\widetilde{I}_{\nu_l}) \le \sum_{\nu=1} \ell(\widetilde{I}_\nu) < \varepsilon$. Hence E has content zero. $\square$

Corollary 2.7. *Let $a < b$. Then $[a, b]$ is not a Lebesgue null set.*

Proof. This is an immediate consequence of from Lemma 2.5 together with Lemma 2.6. $\square$

Exercise 2.8. Let E be the set of rational numbers in $[a, b]$. Show that E is a Lebesgue null set but E is not of content zero.

The Lebesgue null sets are usually called *sets of Lebesgue measure zero* . We avoid this terminology here because we have not defined any Lebesgue measure here and indeed have not identified the class of sets on which it can be defined (the so called Lebesgue measurable sets). A substitute for Lebesgue measure which can be defined on *all* subsets of $\mathbb{R}$ is Lebesgue outer measure:

Definition 2.9. For a subset of $\mathbb{R}$ the Lebesgue outer measure $\lambda_*(E)$ of E is defined as the quantity $\lambda_*(E) = \inf \sum_{n=1}^{\infty} \ell(I_n)$ where the infimum is taken over all countable collections $\{I_n\}_{n\in\mathbb{N}}$ of intervals which have the property that $E \subset \cup_{n=1}^{\infty} I_n$.

With this definition, the Lebesgue null sets are simply the sets of Lebesgue outer measure zero.

3. Oscillation as a quantification of discontinuity

In this section let (X, d) be a metric space and $f : X \to \mathbb{R}$ be a function.

Definition 3.1. (i) Let $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$. For each $x \in X$ and $\delta > 0$ we form the expressions

$$M_{f,\delta}(x) = \sup\{f(y) : d(x, y) < \delta, \quad y \in X\}$$
$$m_{f,\delta}(x) = \inf\{f(y) : d(x, y) < \delta, \quad y \in X\}$$

Observe that, for fixed $x \in X$, $m_{f,\delta}(x)$ increases in δ as δ increases. Moreover $M(f, \delta)(x)$ decreases in δ as δ decreases. Thus $M_{f,\delta}(x) - m_{f,\delta}(x)$ is a nonnegative quantity which decreases as δ decreases. Hence the limit as $\delta \to 0+$ exists.

Definition 3.2. We call the quantity

$$\operatorname{osc}_f(x) := \lim_{\delta\to 0+} M_{f,\delta}(x) - m_{f,\delta}(x)$$

the oscillation of f at x.

The number $\operatorname{osc}_f(x)$ can be used to quantify discontinuities:

Lemma 3.3. *Let $f : X \to \mathbb{R}$ be a bounded function. Then f is continuous at x if and only if $\operatorname{osc}_f(x) = 0$.*

Proof. This is a consequence of the definition of continuity. $\square$

Lemma 3.4. *Let $f : X \to \mathbb{R}$ be a bounded function. Then for every $\gamma \geq 0$ the set $\{x : \operatorname{osc}_f(x) \geq \gamma\}$ is closed.*

Proof. The conclusion is shown by proving that the complement

$$\Omega_\gamma = \{x : \operatorname{osc}_f(x) < \gamma\}$$

is open. Let $x \in \Omega_\gamma$ and choose ε such that $0 < \varepsilon < \gamma - \operatorname{osc}_f(x)$. By the definition of $\operatorname{osc}_f(x)$ we can pick $\delta > 0$ such that $M_{f,\delta}(x) - m_{f,\delta}(x) < \operatorname{osc}_f(x) + \varepsilon$. If $d(y, x) < \delta/2$ and $d(z, y) < \delta/2$ then $d(z, x) < \delta$ and thus $M_{f,\delta/2}(y) \leq M_{f,\delta}(x)$ and $m_{f,\delta/2}(y) \geq m_{f,\delta}(x)$. Hence

$$\operatorname{osc}_f(y) \leq M_{f,\delta/2}(y) - m_{f,\delta/2}(y) \leq M_{f,\delta}(x) - m_{f,\delta}(x) < \operatorname{osc}_f(x) + \varepsilon < \gamma$$

so that $B(x, \delta/2) \subset \Omega_\gamma$. Hence x is an interior point of Ω_γ and since x was chosen arbitrarily in Ω_γ this set is open. $\square$

Exercise 3.5. Define $f : [-10, 10] \to \mathbb{R}$ by $f(x) = -4x$ for $x \leq 0$, $f(x) = \sin(\pi/x)$ for $0 < x < 3/2$, $f(x) = \cos(\pi/x)$ for $x \geq 3/2$. Determine $\operatorname{osc}_f(x)$ for all $x \in [-10, 10]$.

Exercise 3.6. Consider Thomae's function $f : [0,1] \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1] \setminus \mathbb{Q}, \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ with } \gcd(m,n) = 1. \end{cases}$$

Find $\operatorname{osc}_f(x)$ for all $x \in [0,1]$.

4. Lebesgue's Characterization of the Riemann integral

We can now formulate the main theorem of this chapter.

Theorem 4.1. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable if and only if the set of discontinuities of f,*

$$D_f := \{x \in [a,b] : \ f \text{ is not continuous at } x\},$$

is a Lebesgue null set.

The following lemma linking oscillation to lower and upper sums is very helpful in the proof of Theorem 4.1.

Lemma 4.2. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function and assume that* $\operatorname{osc}_f(x) < \gamma$ *for all $x \in [a,b]$. Then there is a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \gamma(b-a)$.*

Proof. By definition of $\operatorname{osc}_f(x)$ we can find a $\delta_x > 0$ such that

$$M_{f,2\delta_x}(x) - m_{f,2\delta_x}(x) < \gamma.$$

Since $[a.b]$ is compact we find x_1, ..., x_N such that $[a,b]$ is contained in the union of the intervals $(x_i - \delta_{x_i}, x_i + \delta_{x_i})$. Consider the finite set consisting of the a, b the x_i, the corresponding point $x_i - \delta_{x_i}$ and $x_i + \delta_{x_i}$ and then discard those point which do not lie in $[a,b]$. The resulting set P is a partition of $[a,b]$ with nodes $a = t_0 < \cdots < t_M = b$ and if t_{i-1}, t_i are consecutive nodes in this partition then

$$\sup\{f(t) : t \in [t_{i-1}, t_i]\} - \inf\{f(t) : t \in [t_{i-1}, t_i]\} < \gamma.$$

Hence

$$U(f,P) - L(f,P) < \gamma \sum_{i=1}^{M} (t_i - t_{i-1}) = \gamma(b-a)$$

and the lemma is proved. ☐

Proof of Theorem 4.1. Part 1: Set of discontinuities is a null set $\implies$ f is Riemann integrable. By Lemma 1 it suffices to construct, for given $\varepsilon > 0$, a partition $\mathcal{P}$ such that

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon. \tag{4.1}$$

The function f is bounded and thus there is $C > 0$ such that $|f(x)| \le C$ for $x \in [a,b]$.

Now let $\varepsilon_1 \ll \varepsilon$ depending on ε; we will see (only at the end) that

$$\varepsilon_1 = \frac{\varepsilon}{2C + b - a}$$

is an appropriate choice. Consider the set

$$D(\varepsilon_1) = \{x \in [a,b] : \operatorname{osc}_f(x) \ge \varepsilon_1\}.$$

$D(\varepsilon_1)$ is a Lebesgue null set since $D(\varepsilon_1) \subset D_f$ and D_f is a Lebesgue null set. Also, $D(\varepsilon_1)$ is a closed subset of $[a,b]$, and thus compact and thus has *content zero.*

Thus there is a *finite* collection $\{I_\nu\}_{\nu=1}^N$ of closed intervals such that $\sum_{\nu=1}^N \ell(I_\nu) < \varepsilon_1$ and $D(\varepsilon_1) \subset \cup_{\nu=1}^N (I_\nu)^\circ$ (where $(I_\nu)^\circ$ denotes the interior of I_ν).

We may choose a partition $P = \{a = x_0 < \cdots < x_N = b\}$ such that each index i belongs to (at least) one of the following sets:

$$\mathcal{J}_1 = \{i : [x_{i-1}, x_i] \subset I_\nu \text{ for some } \nu \text{ in } [1, N]]\}$$
$$\mathcal{J}_2 = \{i : [x_{i-1}, x_i] \cap D(\varepsilon_1) = \emptyset\}.$$

Regarding the intervals $[x_{i-1}, x_i]$ with $i \in \mathcal{J}_1$ we have

$$\sum_{i \in \mathcal{J}_1} (x_i - x_{i-1}) \le \sum_{\nu=1}^{N} \sum_{\substack{i: \\ [x_{i-1}, x_i] \subset I_\nu}} (x_i - x_{i-1}) \le \sum_{\nu=1}^{N} \ell(I_\nu) < \varepsilon_1. \tag{4.2}$$

We observe that for all $i \in \mathcal{J}_2$ we have $\mathrm{osc}_f(x) < \varepsilon_1$ for all $x \in [x_i, x_{i+1}]$. Thus by Lemma 4.2, we find a partition P_i of $[x_{i-1}, x_i]$, labeled $\{x_i = x_{i,0}, \ldots, x_{i,N_i} := x_{i+1}\}$, such that with

$$U^i(f, P_i) := \sum_{j=1}^{N_i} (x_{i,j} - x_{i,j-1}) \sup_{[x_{i,0}, x_{i,N_i}]} f(x),$$
$$L^i(f, P_i) := \sum_{j=1}^{N_i} (x_{i,j} - x_{i,j-1}) \inf_{[x_{i,0}, x_{i,N_i}]} f(x)$$

we have

$$U^i(f, P_i) - L^i(f, P_i) < \varepsilon_1(x_i - x_{i-1}). \tag{4.3}$$

Now the desired partition is defined by

$$\mathcal{P} = \{x_i : i \in \mathcal{J}_1\} \cup \bigcup_{i \in \mathcal{J}_2} \{x_{i,0}, \ldots, x_{i,N_i}\}$$

and we can split and then estimate

$$\begin{aligned} U(f, \mathcal{P}) - L(f, \mathcal{P}) &= \sum_{i \in \mathcal{J}_1} \Big(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f \Big)(x_i - x_{i-1}) + \sum_{i \in \mathcal{J}_2} (U^i(f, P_i) - L^i(f, P_i)) \\ &\le 2C \sum_{i \in \mathcal{J}_1} (x_i - x_{i-1}) + \sum_{i \in \mathcal{J}_2} (U^i(f, P_i) - L^i(f, P_i)). \end{aligned}$$

By (4.2) and (4.3) we get

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < 2C\varepsilon_1 + \varepsilon_1 \sum_{i \in \mathcal{J}_2} (x_i - x_{i-1}) \le 2C\varepsilon_1 + (b - a)\varepsilon_1.$$

In view of our choice $\varepsilon_1 = \varepsilon/(2C + b - a)$ we have proved the desired inequality (4.1).

Part 2: f is Riemann integrable $\implies$ Set of discontinuities is a null set.

For each $n \in \mathbb{N}$ we define $D^n = \{x \in [a, b] : \mathrm{osc}_f(x) \ge 1/n\}$. Observe that $D_f = \cup_{n=1}^{\infty} D^n$, by Lemma 3.3. Thus by Lemma 2.2 it suffices to show that each D^n is a Lebesgue null set.

Fix $n \in \mathbb{N}$. Since f is Riemann integrable there exists, by Lemma 1 a partition $P = \{x_0 < \cdots < x_N\}$ of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon/n.$$

Let $\mathcal{J}$ be the set of indices i for which $I_i := [x_{i-1}, x_i]$ contains a point $\xi_i \in D_n$, so that $D^n \subset \cup_{i \in \mathcal{J}} I_i$. Clearly we have for $i \in \mathcal{J}$,

$$M_i(f) - m_i(f) \ge \mathrm{osc}_f(\xi_i) \ge \frac{1}{n}$$

Hence

$$\begin{aligned}\sum_{i\in\mathcal{J}} \ell(I_i) &= \sum_{I\in\mathcal{J}} \frac{(M_i(f)-m_i(f))(x_i-x_{i-1})}{M_i(f)-m_i(f)} \\ &\le \frac{1}{1/n}\sum_{i=1}^{N}(M_i(f)-m_i(f))(x_i-x_{i-1}) < n\cdot\frac{\varepsilon}{n} = \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary we have proved that each D^n is a Lebesgue null set, and thus D_f is a Lebesgue null set. □