is

$$\mathcal{A}f(y_1,y_2) = \tfrac{1}{2}\left[y_2^2\frac{\partial^2 f}{\partial y_1^2} - 2y_1y_2\frac{\partial^2 f}{\partial y_1 \partial y_2} + y_1^2\frac{\partial^2 f}{\partial y_2^2} - y_1\frac{\partial f}{\partial y_1} - y_2\frac{\partial f}{\partial y_2}\right] .$$

This is because $dY = -\frac{1}{2}Y dt + KY dB$, where

$$K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

so that

$$dY = b(Y)dt + \sigma(Y)dB$$

with

$$b(y_1,y_2) = \begin{pmatrix} -\frac{1}{2}y_1 \\ -\frac{1}{2}y_2 \end{pmatrix} , \qquad \sigma(y_1,y_2) = \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}$$

and

$$a = \tfrac{1}{2}\sigma\sigma^T = \tfrac{1}{2}\begin{pmatrix} y_2^2 & -y_1y_2 \\ -y_1y_2 & y_1^2 \end{pmatrix} .$$

Example 7.5.6. Let D be an open subset of $\mathbf{R}^n$ such that $\tau_D < \infty$ a.s. Q^x for all x. Let ϕ be a bounded, measurable function on ∂D and define

$$\widetilde{\phi}(x) = E^x[\phi(X_{\tau_D})]$$

($\widetilde{\phi}$ is called the *X-harmonic extension of* ϕ). Then if U is open, $x \in U \subset\subset D$, we have by (7.2.8) that

$$E^x[\widetilde{\phi}(X_{\tau_U})] = E^x[E^{X_{\tau_U}}[\phi(X_{\tau_D})]] = E^x[\phi(X_{\tau_D})] = \widetilde{\phi}(x) .$$

So $\widetilde{\phi} \in \mathcal{D}_\mathcal{A}$ and

$$\mathcal{A}\widetilde{\phi} = 0 \quad \text{in } D ,$$

in spite of the fact that in general $\widetilde{\phi}$ need not even be continuous in D (See Example 9.2.1).

Exercises

7.1. Find the generator of the following Itô diffusions:

a) $dX_t = \mu X_t dt + \sigma dB_t$ (The Ornstein-Uhlenbeck process) ($B_t \in \mathbf{R}$; μ, σ constants).

b) $dX_t = rX_t dt + \alpha X_t dB_t$ (The geometric Brownian motion) ($B_t \in \mathbf{R}$; r, α constants).

c) $dY_t = r\,dt + \alpha Y_t dB_t$ ($B_t \in \mathbf{R}$; r, α constants)

d) $dY_t = \begin{bmatrix} dt \\ dX_t \end{bmatrix}$ where X_t is as in a)

e) $\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^{X_1} \end{bmatrix} dB_t \quad (B_t \in \mathbf{R})$

f) $\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & X_1 \end{bmatrix} \begin{bmatrix} dB_1 \\ dB_2 \end{bmatrix}$

g) $X(t) = (X_1, X_2, \cdots, X_n)$, where

$$dX_k(t) = r_k X_k dt + X_k \cdot \sum_{j=1}^{n} \alpha_{kj} dB_j \; ; \qquad 1 \leq k \leq n$$

$((B_1, \cdots, B_n)$ is Brownian motion in $\mathbf{R}^n$, r_k and α_{kj} are constants).

7.2. Find an Itô diffusion (i.e. write down the stochastic differential equation for it) whose generator is the following:

a) $Af(x) = f'(x) + f''(x)\,; \;\; f \in C_0^2(\mathbf{R})$

b) $Af(t,x) = \frac{\partial f}{\partial t} + cx\frac{\partial f}{\partial x} + \frac{1}{2}\alpha^2 x^2 \frac{\partial^2 f}{\partial x^2}\,; \;\; f \in C_0^2(\mathbf{R}^2)$, where c, α are constants.

c) $Af(x_1,x_2) = 2x_2\frac{\partial f}{\partial x_1} + \ln(1 + x_1^2 + x_2^2)\frac{\partial f}{\partial x_2}$
$+\frac{1}{2}(1 + x_1^2)\frac{\partial^2 f}{\partial x_1^2} + x_1\frac{\partial^2 f}{\partial x_1 \partial x_2} + \frac{1}{2} \cdot \frac{\partial^2 f}{\partial x_2^2}\,; \;\; f \in C_0^2(\mathbf{R}^2)$.

7.3. Let B_t be Brownian motion on $\mathbf{R}, B_0 = 0$ and define

$$X_t = X_t^x = x \cdot e^{ct + \alpha B_t} \; ,$$

where c, α are constants. Prove directly from the definition that X_t is a Markov process.

7.4. Let B_t^x be 1-dimensional Brownian motion starting at $x \in \mathbf{R}^+$. Put

$$\tau = \inf\{t > 0;\, B_t^x = 0\} \; .$$

a) Prove that $\tau < \infty$ a.s. P^x for all $x > 0$. (Hint: See Example 7.4.2, second part).

b) Prove that $E^x[\tau] = \infty$ for all $x > 0$. (Hint: See Example 7.4.2, first part).

7.5. Let the functions b, σ satisfy condition (5.2.1) of Theorem 5.2.1, with a constant C independent of t, i.e.

$$|b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|) \qquad \text{for all } \; x \in \mathbf{R}^n \text{ and all } \; t \geq 0 \; .$$

Let X_t be a solution of

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \; .$$

Show that

$$E[|X_t|^2] \leq (1 + E[|X_0|^2])e^{Kt} - 1$$

for some constant K independent of t.
(Hint: Use Dynkin's formula with $f(x) = |x|^2$ and $\tau = t \wedge \tau_R$, where $\tau_R = \inf\{t > 0;\ |X_t| \geq R\}$, and let $R \to \infty$ to achieve the inequality

$$E[|X_t|^2] \leq E[|X_0|^2] + K \cdot \int_0^t (1 + E[|X_s|^2])ds\ ,$$

which is of the form (5.2.9).)

7.6. Let $g(x, \omega) = f \circ F(x, t, t+h, \omega)$ be as in the proof of Theorem 7.1.2. Assume that f is continuous.

a) Prove that the map $x \to g(x, \cdot)$ is continuous from $\mathbf{R}^n$ into $L^2(P)$ by using (5.2.9).

For simplicity assume that $n = 1$ in the following.

b) Use a) to prove that $(x, \omega) \to g(x, \omega)$ is measurable. (Hint: For each $m = 1, 2, \ldots$ put $\xi_k = \xi_k^{(m)} = k \cdot 2^{-m}$, $k = 1, 2, \ldots$ Then

$$g^{(m)}(x, \cdot) := \sum_k g(\xi_k, \cdot) \cdot \mathcal{X}_{\{\xi_k \leq x < \xi_{k+1}\}}$$

converges to $g(x, \cdot)$ in $L^2(P)$ for each x. Deduce that $g^{(m)} \to g$ in $L^2(dm_R \times dP)$ for all R, where dm_R is Lebesgue measure on $\{|x| \leq R\}$. So a subsequence of $g^{(m)}(x, \omega)$ converges to $g(x, \omega)$ for a.a. (x, ω).)

7.7. Let B_t be Brownian motion on $\mathbf{R}^n$ starting at $x \in \mathbf{R}^n$ and let $D \subset \mathbf{R}^n$ be an open ball centered at x.

a) Use Exercise 2.15 to prove that the harmonic measure μ_D^x of B_t is rotation invariant (about x) on the sphere ∂D. Conclude that μ_D^x coincides with normalized surface measure σ on ∂D.

b) Let ϕ be a bounded measurable function on a bounded open set $W \subset \mathbf{R}^n$ and define

$$u(x) = E^x[\phi(B_{\tau_W})] \qquad \text{for } x \in W\ .$$

Prove that u satisfies the classical mean value property:

$$u(x) = \int_{\partial D} u(y) d\sigma(y)$$

for all balls D centered at x with $\overline{D} \subset W$.

7.8. Let $\{\mathcal{N}_t\}$ be a right-continuous family of σ-algebras of subsets of Ω, containing all sets of measure zero.

a) Let τ_1, τ_2 be stopping times (w.r.t. $\mathcal{N}_t$). Prove that $\tau_1 \wedge \tau_2$ and $\tau_1 \vee \tau_2$ are stopping times.

b) If $\{\tau_n\}$ is a decreasing family of stopping times prove that $\tau := \lim\limits_n \tau_n$ is a stopping time.

c) If X_t is an Itô diffusion in $\mathbf{R}^n$ and $F \subset \mathbf{R}^n$ is closed, prove that τ_F is a stopping time w.r.t. $\mathcal{M}_t$. (Hint: Consider open sets decreasing to F).

7.9. Let X_t be a geometric Brownian motion, i.e.

$$dX_t = rX_t dt + \alpha X_t dB_t \ , \qquad X_0 = x > 0$$

where $B_t \in \mathbf{R}$; r, α are constants.

a) Find the generator A of X_t and compute $Af(x)$ when $f(x) = x^\gamma$; $x > 0$, γ constant.

b) If $r < \frac{1}{2}\alpha^2$ then $X_t \to 0$ as $t \to \infty$, a.s. Q^x (Example 5.1.1). But what is the probability p that X_t, when starting from $x < R$, ever hits the value R ? Use Dynkin's formula with $f(x) = x^{\gamma_1}$, $\gamma_1 = 1 - \frac{2r}{\alpha^2}$, to prove that

$$p = \left(\frac{x}{R}\right)^{\gamma_1} .$$

c) If $r > \frac{1}{2}\alpha^2$ then $X_t \to \infty$ as $t \to \infty$, a.s. Q^x. Put

$$\tau = \inf\{t > 0; X_t \geq R\} .$$

Use Dynkin's formula with $f(x) = \ln x$, $x > 0$ to prove that

$$E^x[\tau] = \frac{\ln \frac{R}{x}}{r - \frac{1}{2}\alpha^2} .$$

(Hint: First consider exit times from (ρ, R), $\rho > 0$ and then let $\rho \to 0$. You need estimates for

$$(1 - p(\rho)) \ln \rho \ ,$$

where

$$p(\rho) = Q^x[X_t \text{ reaches the value } R \text{ before } \rho] \ ,$$

which you can get from the calculations in a), b).)

7.10. Let X_t be the geometric Brownian motion

$$dX_t = rX_t dt + \alpha X_t dB_t \ .$$

Find $E^x[X_T|\mathcal{F}_t]$ for $t \le T$ by
a) using the Markov property
and
b) writing $X_t = x\, e^{rt} M_t$, where

$$M_t = \exp(\alpha B_t - \tfrac{1}{2}\alpha^2 t) \qquad \text{is a martingale} \ .$$

7.11. Let X_t be an Itô diffusion in $\mathbf{R}^n$ and let $f: \mathbf{R}^n \to \mathbf{R}$ be a function such that

$$E^x\Big[\int_0^\infty |f(X_t)|dt\Big] < \infty \qquad \text{for all } x \in \mathbf{R}^n \ .$$

Let τ be a stopping time. Use the strong Markov property to prove that

$$E^x\Big[\int_\tau^\infty f(X_t)dt\Big] = E^x[g(X_\tau)] \ ,$$

where

$$g(y) = E^y\Big[\int_0^\infty f(X_t)dt\Big] \ .$$

7.12. (Local martingales)
An $\mathcal{N}_t$-adapted stochastic process $Z(t) \in \mathbf{R}^n$ is called a *local martingale* with respect to the given filtration $\{\mathcal{N}_t\}$ if there exists an increasing sequence of $\mathcal{N}_t$-stopping times τ_k such that

$$\tau_k \to \infty \quad \text{a.s. as } k \to \infty$$

and

$$Z(t \wedge \tau_k) \qquad \text{is an } \mathcal{N}_t\text{-martingale for all } k \ .$$

a) Show that if $Z(t)$ is a local martingale and there exists a constant $T \le \infty$ such that the family $\{Z(\tau)\}_{\tau \le T}$ is uniformly integrable (Appendix C) then $\{Z(t)\}_{t \le T}$ is a martingale.
b) In particular, if $Z(t)$ is a local martingale and there exists a constant $K < \infty$ such that
$$E[Z^2(\tau)] \le K$$
for all stopping times $\tau \le T$, then $\{Z(t)\}_{t \le T}$ is a martingale.
c) Show that if $Z(t)$ is a *lower bounded* local martingale, then $Z(t)$ is a supermartingale (Appendix C).

7.13. a) Let $B_t \in \mathbf{R}^2$, $B_0 = x \neq 0$. Fix $0 < \epsilon < R < \infty$ and define

$$X_t = \ln |B_{t\wedge\tau}| \; ; \qquad t \geq 0$$

where

$$\tau = \inf \{t > 0; |B_t| \leq \epsilon \quad \text{or} \quad |B_t| \geq R\} \; .$$

Prove that X_t is an $\mathcal{F}_{t\wedge\tau}$-martingale. (Hint: Use Exercise 4.8.)
Deduce that $\ln |B_t|$ is a local martingale (Exercise 7.12).

b) Let $B_t \in \mathbf{R}^n$ for $n \geq 3$, $B_0 = x \neq 0$. Fix $\epsilon > 0$, $R < \infty$ and define

$$Y_t = |B_{t\wedge\tau}|^{2-n} \; ; \qquad t \geq 0$$

where

$$\tau = \inf\{t > 0; |B_t| \leq \epsilon \quad \text{or} \quad |B_t| \geq R\} \; .$$

Prove that Y_t is an $\mathcal{F}_{t\wedge\tau}$-martingale.
Deduce that $|B_t|^{2-n}$ is a local martingale.

7.14. (Doob's h-transform)
Let B_t be n-dimensional Brownian motion, $D \subset \mathbf{R}^n$ a bounded open set and $h > 0$ a harmonic function on D (i.e. $\Delta h = 0$ in D). Let X_t be the solution of the stochastic differential equation

$$dX_t = \nabla(\ln h)(X_t)dt + dB_t$$

More precisely, choose an increasing sequence $\{D_k\}$ of open subsets of D such that $\overline{D}_k \subset D$ and $\bigcup\limits_{k=1}^{\infty} D_k = D$. Then for each k the equation above can be solved (strongly) for $t < \tau_{D_k}$. This gives in a natural way a solution for $t < \tau := \lim\limits_{k\to\infty} \tau_{D_k}$.

a) Show that the generator A of X_t satisfies

$$Af = \frac{\Delta(hf)}{2h} \qquad \text{for } \; f \in C_0^2(D) \; .$$

In particular, if $f = \frac{1}{h}$ then $\mathcal{A}f = 0$.

b) Use a) to show that if there exists $x_0 \in \partial D$ such that

$$\lim_{x\to y\in\partial D} h(x) = \begin{cases} 0 & \text{if } y \neq x_0 \\ \infty & \text{if } y = x_0 \end{cases}$$

(i.e. h is a *kernel function*), then

$$\lim_{t\to\tau} X_t = x_0 \;\; \text{a.s.}$$

(Hint: Consider $E^x[f(X_T)]$ for suitable stopping times T and with $f = \frac{1}{h}$)

In other words, we have imposed a drift on B_t which causes the process to exit from D at the point x_0 only. This can also be formulated as follows: X_t is obtained by *conditioning* B_t *to exit from* D *at* x_0. See Doob (1984).

7.15. Let B_t be 1-dimensional and define

$$F(\omega) = (B_T(\omega) - K)^+$$

where $K > 0$, $T > 0$ are constants.
By the Itô representation theorem (Theorem 4.3.3) we know that there exists $\phi \in \mathcal{V}(0,T)$ such that

$$F(\omega) = E[F] + \int_0^T \phi(t,\omega)dB_t \ .$$

How do we find ϕ explicitly? This problem is of interest in mathematical finance, where ϕ may be regarded as the replicating portfolio for the contingent claim F (see Chapter 12). Using the Clark-Ocone formula (see Karatzas and Ocone (1991) or Øksendal (1996)) one can deduce that

$$\phi(t,\omega) = E[\mathcal{X}_{[K,\infty)}(B_T)|\mathcal{F}_t] \ ; \qquad t < T \ . \tag{7.5.3}$$

Use (7.5.3) and the Markov property of Brownian motion to prove that for $t < T$ we have

$$\phi(t,\omega) = \frac{1}{\sqrt{2\pi(T-t)}} \int_K^\infty \exp\left(- \frac{(x - B_t(\omega))^2}{2(T-t)} \right) dx \ . \tag{7.5.4}$$

7.16. Let B_t be 1-dimensional and let $f: \mathbf{R} \to \mathbf{R}$ be a bounded function. Prove that if $t < T$ then

$$E^x[f(B_T)|\mathcal{F}_t] = \frac{1}{\sqrt{2\pi(T-t)}} \int_{\mathbf{R}} f(x) \exp\left(- \frac{(x - B_t(\omega))^2}{2(T-t)} \right) dx \ . \tag{7.5.5}$$

(Compare with (7.5.4).)

7.17. Let B_t be 1-dimensional and put

$$X_t = (x^{1/3} + \tfrac{1}{3}B_t)^3 \ ; \qquad t \geq 0 \ .$$

Then we have seen in Exercise 4.15 that X_t is a solution of the stochastic differential equation

$$dX_t = \tfrac{1}{3}X_t^{1/3}dt + X_t^{2/3}dB_t \ ; \qquad X_0 = x \ . \tag{7.5.6}$$

Define

$$\tau = \inf\{t > 0; X_t = 0\}$$

and put

$$Y_t = \begin{cases} X_t & \text{for } t \leq \tau \\ 0 & \text{for } t > \tau\ . \end{cases}$$

Prove that Y_t is also a (strong) solution of (7.5.6). Why does not this contradict the uniqueness assertion of Theorem 5.2.1?
(Hint: Verify that

$$Y_t = x + \int_0^t \tfrac{1}{3} Y_s^{1/3} ds + \int_0^t Y_s^{2/3} dB_s$$

for all t by splitting the integrals as follows:

$$\int_0^t = \int_0^{t\wedge\tau} + \int_{t\wedge\tau}^t\ .\ )$$

7.18. a) Let

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t\ ; \qquad X_0 = x$$

be a 1-dimensional Itô diffusion with characteristic operator $\mathcal{A}$. Let $f \in C^2(\mathbf{R})$ be a solution of the differential equation

$$\mathcal{A}f(x) = b(x)f'(x) + \tfrac{1}{2}\sigma^2(x)f''(x) = 0\ ; \qquad x \in \mathbf{R}\ . \tag{7.5.7}$$

Let $(a,b) \subset \mathbf{R}$ be an open interval such that $x \in (a,b)$ and put

$$\tau = \inf\{t > 0;\ X_t \notin (a,b)\}\ .$$

Assume that $\tau < \infty$ a.s. Q^x and define

$$p = P^x[X_\tau = b]\ .$$

Use Dynkin's formula to prove that if $f(b) \neq f(a)$ then

$$p = \frac{f(x) - f(a)}{f(b) - f(a)}\ . \tag{7.5.8}$$

In other words, the *harmonic measure* $\mu^x_{(a,b)}$ of X on $\partial(a,b) = \{a,b\}$ is given by

$$\mu^x_{(a,b)}(b) = \frac{f(x) - f(a)}{f(b) - f(a)}\ , \quad \mu^x_{(a,b)}(a) = \frac{f(b) - f(x)}{f(b) - f(a)}\ . \tag{7.5.9}$$

b) Now specialize to the process

$$X_t = x + B_t\ ; \qquad t \geq 0\ .$$

Prove that

$$p = \frac{x-a}{b-a}\ . \tag{7.5.10}$$

c) Find p if

$$X_t = x + \mu t + \sigma B_t\ ; \qquad t \geq 0$$

where $\mu, \sigma \in \mathbf{R}$ are nonzero constants.

7.19. Let B_t^x be 1-dimensional Brownian motion starting at $x > 0$. Define

$$\tau = \tau(x,\omega) = \inf\{t > 0;\ B_t^x(\omega) = 0\}\ .$$

From Exercise 7.4 we know that

$$\tau < \infty \quad \text{a.s. } P^x \text{ and } \ E^x[\tau] = \infty\ .$$

What is the distribution of the random variable $\tau(\omega)$?

a) To answer this, first find the *Laplace transform*

$$g(\lambda) := E^x[e^{-\lambda\tau}] \qquad \text{for } \ \lambda > 0\ .$$

(Hint: Let $M_t = \exp(-\sqrt{2\lambda}\, B_t - \lambda t)$. Then

$$\{M_{t\wedge\tau}\}_{t\geq 0} \quad \text{is a bounded martingale}\ .$$

[Solution: $g(\lambda) = \exp(-\sqrt{2\lambda}\, x)$.]

b) To find the density $f(t)$ of τ it suffices to find $f(t) = f(t,x)$ such that

$$\int_0^\infty e^{-\lambda t} f(t)dt = \exp(-\sqrt{2\lambda}\, x) \qquad \text{for all } \ \lambda > 0$$

i.e. to find the *inverse* Laplace transform of $g(\lambda)$. Verify that

$$f(t,x) = \frac{x}{\sqrt{2\pi t^3}} \exp\left(-\frac{x^2}{2t}\right)\ ; \qquad t > 0\ .$$