Math 733 - Fall 2020

Homework 1

Due: 09/13, 10pm

Zijie Zhang

September 11, 2020

1. *Proof.* We noticed that $A \circ B = (A \cup B) \setminus (A \cap B)$. So it gives
$$B \circ C \cup A \circ C = (A \cup B \cup C) \setminus (A \cap B \cap C)$$
this means
$$B \circ C \cup A \circ C \supset A \circ B$$
thus
$$\mathbf{P}(B \circ C \cup A \circ C) \geqslant \mathbf{P}(A \circ B)$$
we know
$$\mathbf{P}(B \circ C) + \mathbf{P}(A \circ C) \geqslant \mathbf{P}(B \circ C \cup A \circ C)$$
so, we proved
$$\mathbf{P}(A \circ B) \leqslant \mathbf{P}(B \circ C) + \mathbf{P}(A \circ C)$$
□

2. *Proof.* $\mathcal{F}$ is a σ-algebra satisfy
   (i) if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$
   (ii) if $A_i \in \mathcal{F}$ is countable sequence of sets, then $\cup_i A_i \in \mathcal{F}$.

   On the other hand, the measure $\mathbf{P}$ is a nonnegative countably additive set function.
   (i) $\mathbf{P}(A) = 0$ if A is countable, which means A^c is not countable, $\mathbf{P}(A^c) = 1$.
   $\mathbf{P}(A) \geqslant \emptyset = 0$ for all $A \in \mathcal{F}$, $\emptyset$ is countable.
   (ii) if $A_i \in \mathcal{F}$ is countable sequence of disjoint sets, then $\mathbf{P}(A_i) = 0$ and $\cup_i A_i$ is countable.
   So, $\mathbf{P}(\cup_i A_i) = \sum_i \mathbf{P}(A_i)$

   $\emptyset$ is countable, so $\mathbf{P}(\mathbb{R}) = 1$.
   $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space. □

3. *Proof.* Let
$$\begin{aligned}
B_1 &= A_1 - A_2 \cup A_3 \cup \cdots \cup A_n \\
B_2 &= A_2 - A_1 \cup A_3 \cup \cdots \cup A_n \\
&\cdots \\
B_n &= A_n - A_1 \cup A_2 \cup \cdots \cup A_{n-1} \\
B_{n+1} &= A_1 \cap A_2 - A_3 \cup A_4 \cup \cdots \cup A_n \\
&\cdots \\
B_{n+\binom{n}{2}+1} &= A_1 \cap A_2 \cap A_3 - A_4 \cup A_5 \cup \cdots \cup A_n \\
&\cdots \\
B_{2^n-1} &= A_1 \cap A_2 \cap \cdots A_n \\
B_{2^n} &= \Omega - A_1 \cup A_2 \cup \cdots \cup A_n
\end{aligned}$$

So, the σ-field generated by $\{A_1, \cdots, A_n\}$ is exactly the set of finite unions of the sets B_i. $\square$

4. *Proof.* By the definition of σ-field.

   (i) If $A \in \cap_{j\in J}\mathcal{F}_j$, we know $A \in \mathcal{F}_j$ for all $j \in J$.
   Thus $A^c \in \mathcal{F}_j$ for all $j \in J$, then we have $A^c \in \cap_{j\in J}\mathcal{F}_j$.

   (ii) If $A_i \in \cap_{j\in J}\mathcal{F}_j$ is a countable sequence sets, then $A_i \in \mathcal{F}_j$ for all $j \in J$.
   We have $\cup_i A_i \in \mathcal{F}_j$ for all $j \in J$, thus $\cup_i A_i \in \cap_{j\in J}\mathcal{F}_j$.

   So, $\cap_{j\in J}\mathcal{F}_j$ is also a sigma-field. $\square$

5. *Proof.* Let $\Omega = \mathbb{R}[0,1]$, $\mathcal{F}$ is the σ-algebra.
   $\mathbf{P}(A)$ is the measure of the set A.
   $\mathbf{P}$ is Lebesgue measure.
   Let $X = (\mathbb{R} - \mathbb{Q}) \cap \Omega$.
   $X = q$ is null set, so $\mathbf{P}(X = q) = 0$.
   Let $A_i \in \mathbb{Q}[0,1]$ is a countable sequence of disjoint sets. By the definition, we know $\mathbf{P}(\mathbb{Q}[0,1]) = \mathbf{P}(\cup_i A_i) = \sum_i \mathbf{P}(A_i) = 0$. So $\mathbf{P}(X \text{is irrational}) = 1 - \mathbf{P}(\mathbb{Q}[0,1]) = 1$. $\square$

6. *Proof.* Set A_k as event "the coin flips $k, k+1, \cdots, 2k$ are all heads.".

$$\mathbf{P}(A_k) = \frac{1}{2^{k+1}}$$

   Set M as event "there will be no integer n so that the coin flips $n, n+1, \cdots, 2n$ are all heads.".

$$\begin{aligned}\mathbf{P}(M^c) &\leqslant \sum_{k=1}^{\infty} \mathbf{P}(A_k) \\ &= \frac{1}{2}\end{aligned}$$

$$\mathbf{P}(M) = 1 - \mathbf{P}(M^c) \geqslant \frac{1}{2}$$

   So we get the positive lower bound. $\square$