CS 760: Machine Learning - Fall 2020

Homework 1: Review

Due : 09/24/2020

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Problem 1

Proof. By the definition, $\mathbb{R}^D$ is a subspace if for every $a,\ b \in \mathbb{R}$ and every $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^D$, $a\boldsymbol{u} + b\boldsymbol{v} \in \mathbb{R}^D$.
It is trivial. □

Problem 2

Proof.

(a) $\boldsymbol{x} = (-1, \cdots, -1) \in \mathbb{R}^D$. The element-wise square roots is $(i, \cdots, i) \notin \mathbb{R}^D$.
$\mathbb{R}^D$ is not closed under element-wise square roots.

(b) $\mathbb{C}^D$.

□

Problem 3

Proof. For every element $\boldsymbol{x},\ \boldsymbol{y} \in \mathbb{U}$. They can be represented by a linear combination of $\boldsymbol{u}_1, \cdots, \boldsymbol{u}_R$.

$$\boldsymbol{x} = \alpha_1 \boldsymbol{u}_1 + \cdots + \alpha_R \boldsymbol{u}_R$$

$$\boldsymbol{y} = \beta_1 \boldsymbol{u}_1 + \cdots + \beta_R \boldsymbol{u}_R$$

For every $a,\ b \in \mathbb{R}$,

$$a\boldsymbol{x} + b\boldsymbol{y} = (a\alpha_1 + b\beta_1)\boldsymbol{u}_1 + \cdots + (a\alpha_R + b\beta_R)\boldsymbol{u}_R$$

$a\boldsymbol{x} + b\boldsymbol{y} \in \mathbb{U}$, so $\mathbb{U}$ is a subspace. □

Problem 4

Proof.

(a) By the *Bayes rule*,

$$\begin{aligned} &\mathbb{P}(\text{Have diabetes}|\text{These genes inactive}) \\ &= \frac{\mathbb{P}(\text{These genes inactive}|\text{Have diabetes}) \cdot \mathbb{P}(\text{Have diabetes})}{\mathbb{P}(\text{These genes inactive})} \end{aligned}$$

(b) I need to know $\mathbb{P}(\text{These genes inactive})$.

(c) If the probability that *these three genes inactive* is very small, I should be concerned.

□

Problem 5

Proof.

$$\mathbb{P}(x|\theta) = \begin{cases} \theta e^{-\theta(x-t_0)} & , x \geqslant t_0 \\ 0 & , x \leqslant t_0 \end{cases}$$

- *t_0 is the minimal time delay*: $\mathbb{P}(x \leqslant t_0|\theta) = 0$
- *Larger delays are rarer than shorter ones*: For all $x,\ y \in [t_0, \infty]$, $x < y$,

$$\begin{aligned} \frac{\mathbb{P}(y|\theta)}{\mathbb{P}(x|\theta)} &= \frac{\theta e^{-\theta(x-t_0)}}{\theta e^{-\theta(y-t_0)}} \\ &= e^{-\theta(x-y)} \\ &< 1 \end{aligned}$$

- This probabilistic model delays with a single free parameter θ.
- $\mathbb{E}(x|\theta) = t_0 + \frac{1}{\theta}$.
- *Markov property*:
  Each delay has nothing to do with the previous delay.

$$\begin{aligned} \mathbb{P}(x - t_0 > s + t | x - t_0 > s) &= \frac{\mathbb{P}(x - t_0 > s + t \cap x - t_0 > s)}{\mathbb{P}(x - t_0 > s)} \\ &= \frac{\mathbb{P}(x - t_0 > s + t)}{\mathbb{P}(x - t_0 > s)} \\ &= \frac{e^{-\theta(s+t)}}{e^{-\theta s}} \\ &= e^{-\theta t} \\ &= \mathbb{P}(x - t_0 > t) \end{aligned}$$

□

Problem 6

Proof.

(a) It doesn't look fairly uniform.

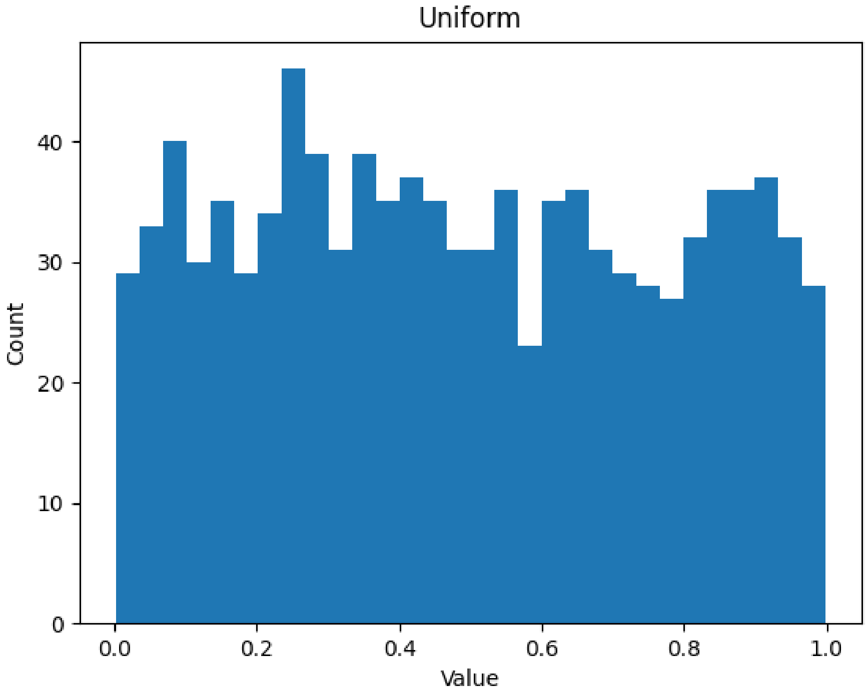
(b) $y_i \sim$ Bernoulli(p)

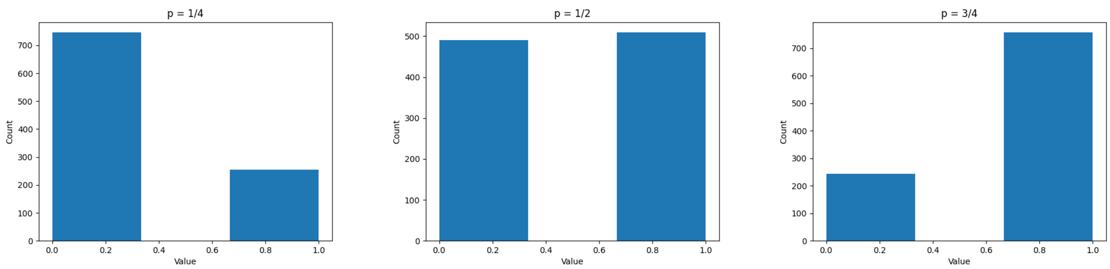
(c) Yes.

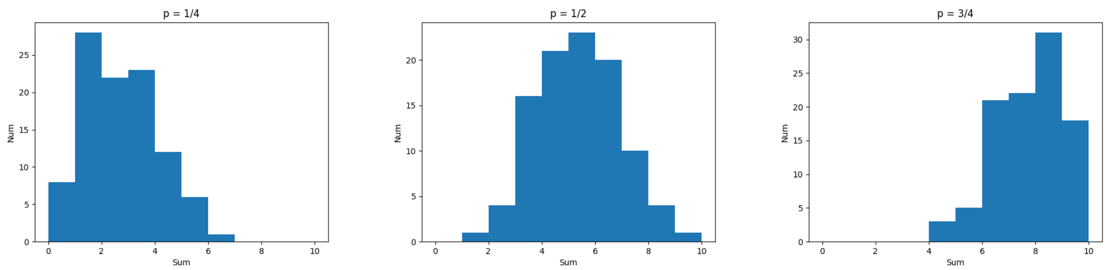
(d) $z_k \sim$ Binomial(n,p)

(e) Yes.

□







Problem 7

Proof.

(a)

$$\nabla \ell(\boldsymbol{\theta}) = \sum_{i=1}^{N} \boldsymbol{x}_i \left(y_i - \frac{1}{1 + e^{-\boldsymbol{\theta}^T \boldsymbol{x}_i}} \right)$$

(b)

$$\boldsymbol{H} = \nabla^2 \ell(\boldsymbol{\theta}) = -\sum_{i=1}^{N} \frac{e^{\boldsymbol{\theta}^T \boldsymbol{x}_i}}{\left(1 + e^{\boldsymbol{\theta}^T \boldsymbol{x_i}}\right)^2} \cdot \begin{pmatrix} \boldsymbol{x}_{i_1}^2 & \boldsymbol{x}_{i_1}\boldsymbol{x}_{i_2} & \cdots & \boldsymbol{x}_{i_1}\boldsymbol{x}_{i_n} \\ \vdots & & & \vdots \\ \boldsymbol{x}_{i_n}\boldsymbol{x}_{i_1} & \boldsymbol{x}_{i_n}\boldsymbol{x}_{i_2} & \cdots & \boldsymbol{x}_{i_n}^2 \end{pmatrix}$$

(c) $\ell(\boldsymbol{\theta})$ is a scalar.
Its gradient $\nabla \ell(\boldsymbol{\theta})$ is a vector.
Its Hessian $\nabla^2 \ell(\boldsymbol{\theta})$ is a matrix.

$\square$