Stopping time theorem

- (Discrete-time) martingale transform theorem.

$\{M_n\}$ is mart. wrt $\{\mathcal{F}_n\}$

$$\tilde{M}_n \stackrel{\text{def}}{=} M_0 + A_1 (M_1 - M_0) + A_2 (M_2 - M_1) + \cdots + A_n (M_n - M_{n-1})$$

If $\{A_n\}$ is non-anticipating,

(meaning: $A_n \in \mathcal{F}_{n-1}$ $\forall n$)

Then, $\{\tilde{M}_n\}$ is mart.

Pf: $\mathbb{E}[\tilde{M}_n - \tilde{M}_{n-1} \mid \mathcal{F}_{n-1}] \stackrel{?}{=} 0$

$$\| \quad \mathbb{E}[A_n (M_n - M_{n-1}) \mid \mathcal{F}_{n-1}] \quad // \quad \checkmark$$

- Uniform integrability.

Def: We say a collection C of random var's is unif integrable if

$$\sup_{Z \in C} \mathbb{E}\left[|Z| \mathbb{1}_{|Z|>x} \right] \xrightarrow{x \to \infty} 0$$

<u>Lemma</u>: If $\{Z_n\}$ is unif integrable, and $Z_n \to Z$ a.s. then

$$\mathbb{E}|Z_n - Z| \to 0 \quad n \to \infty$$

"How to tell if C is unif integrable?"

<u>Lemma</u>. If $\frac{\phi(x)}{x} \to \infty$ $x \to \infty$

and C is a collection of r.v.'s

$$\mathbb{E}[\phi(|Z|)] \le \underline{B} < \infty \quad \text{for } \forall Z \in C$$

then C is unif integrable.

"Any L^1 random variable is in <u>some sense</u> look like L^p for $p>1$"

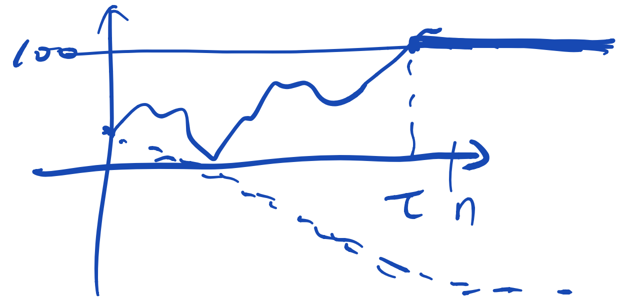
<u>Lemma</u>: If Z is r.v. $\mathbb{E}|Z| < \infty$

Then $\exists \phi$ s.t $\frac{\phi(x)}{x} \xrightarrow{x \to \infty} \infty$

and $\mathbb{E}[\phi(|Z|)] < \infty$

Stopping time thm.

(Discrete-time version)



Thm: $\{M_n\}$ mart $\Rightarrow$ $M_{n\wedge\tau}$ is mart
τ is stopping.

pf. Assume $M_0 = 0$ (otherwise $\overline{M}_n = M_n - M_0$)

$$M_{n\wedge\tau} = \mathbb{1}_{\tau\geq 1}(M_1 - M_0) + \mathbb{1}_{\tau\geq 2}(M_2 - M_1) + \cdots + \mathbb{1}_{\tau\geq n}(M_n - M_{n-1})$$

$(\{\tau \geq k\} \in \mathcal{F}_{k-1})$ non-ancipating

So, by mart. transf. $M_{n\wedge\tau}$ is mart.

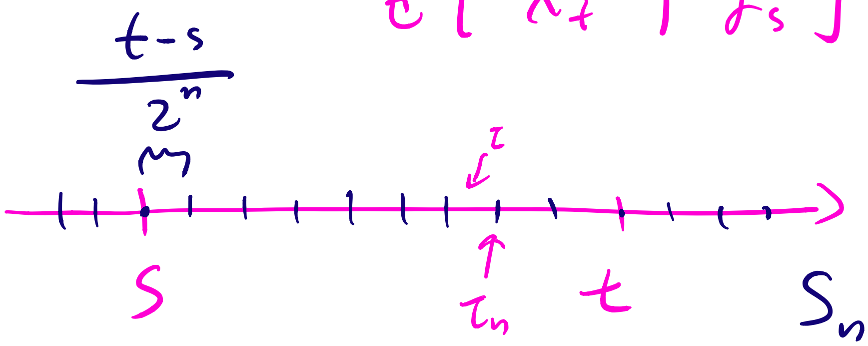
(Continuous-time)

thm $\{M_t\}$ continuous mart.
τ is stopping time

Then $X_t \overset{def}{=} M_{t\wedge\tau}$ is cont. mart.

pf. (1) $\mathbb{E}|X_t| < \infty \quad \forall t$

(2) $\mathbb{E}[X_t \mid \mathcal{F}_s] \overset{?}{=} X_s \quad s<t$



$\tau_n \geq \tau$ nearest to τ.

as $n \to \infty$

$\tau_n \searrow \tau$

Consider $M_u \quad u \in S_n$

$$\mathbb{E}[|M_{t\wedge\tau_n}|] \leq \underbrace{\mathbb{E}[|M_t|]}_{<\infty}$$

Send $n \to \infty$

$$\mathbb{E}[|M_{t\wedge\tau}|] < \infty \quad \forall t \quad \Rightarrow (1)$$

By discrete-time stopping time thm.

$$\boxed{\mathbb{E}[M_{t\wedge\tau_n} \mid \mathcal{F}_s]} = M_{s\wedge\tau_n}$$

Send $n \to \infty$

$M_{t\wedge\tau}$ a.s.

$M_{s\wedge\tau}$ a.e. $\omega \in \Omega$

(recall if we have uniform integrability then the cond exp on LHS $\to \mathbb{E}[M_{t\wedge\tau} \mid \mathcal{F}_s]$)

$\mathbb{E}|M_t| < \infty$ $\exists \phi$. convex $\frac{\phi(x)}{x} \xrightarrow{x\to\infty} \infty$

$$\mathbb{E}[\phi(|M_t|)] < \infty.$$

$$\mathbb{E}[\phi(|M_{t\wedge\tau_n}|)] \le \underline{\mathbb{E}[\phi(|M_t|)]}$$

$\Rightarrow \{M_{t\wedge\tau_n}\}_n$ is unif. integrable.

$\Rightarrow L^1$ convergence. $\blacksquare$

Timo Thm 3.6: If M is right-cont. submart. and σ, τ stopping times.

Then $\mathbb{E}[M_{\tau\wedge t} \mid \mathcal{F}_\sigma] \geq M_{\sigma\wedge\tau\wedge t}$

<u>Cor 3.8</u> If M right-cont. submart.

Then $M_{t\wedge\tau}$ is submart.

(If M mart. then $M_{t\wedge\tau}$ mart)