

Probability space  $(\Omega, \mathcal{F}, P)$

$\Omega$ : sample space (set of possible outcomes)

$\mathcal{F}$ : set of events (subsets of  $\Omega$  that we can "measure")  
σ-field

$P$ : probability measure on  $\mathcal{F}$

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Word problem:

1, identify  $(\Omega, \mathcal{F}, P)$

2, identify the event in question

3, Try to compute  $P(\text{event})$

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Ex: We roll two dice.

What's the prob that the sum is even?

$$\Omega = \{ (a_1, a_2) : 1 \leq a_i \leq 6 \}$$

$$\mathcal{F} = 2^\Omega \quad P: \mathcal{F} \rightarrow [0, 1]?$$

$$\mathbb{P}(\{a_1, a_2\}) = \frac{1}{36}$$

$$\# \Omega = 36$$

$$A = \{ \text{the sum is even} \}$$

$$= \{ (a_1, a_2) : (1 \leq a_i \leq 6, a_1 + a_2 \text{ is even}) \}$$

$$P(A) = \sum_{(a_1, a_2) \in A} P(\{a_1, a_2\}) = \frac{\#A}{36}$$


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$$A \cup B \approx \text{"A or B"}$$

$$A \cap B \approx \text{"A and B"}$$

$$A^c \approx \text{"not A"}$$


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Examples

1. If  $\Omega$  is finite or countably infinite (nonempty). If all singletons are events then  $\mathbb{P}$  can be described via the probabilities of the singletons.

$$\Omega = \{\omega_1, \omega_2, \dots\} \quad \mathbb{P}_k = P(\{\omega_k\}) \in [0, 1]$$

$$\sum_{\varepsilon} P_{\varepsilon} = 1$$

$$A \subset \Omega \quad P(A) = \sum_{\omega \in A} P_{\varepsilon}$$

2. A uniformly order metric on  $[0,1]$ .

$$\Omega = [0,1]$$

$\mathcal{F} = ?$  intervals should be included

$$P([a,b]) = b-a$$

$$0 \leq a < b \leq 1$$

$\mathcal{F} =$   $\sigma$ -field generated by the intervals  
(set of Borel sets on  $[0,1]$ )

$P$  : Lebesgue measure

$$([0,1], \mathcal{B}, P)$$

$$\uparrow$$
  
 $\Omega$

$$\uparrow$$
  
 $\mathcal{F}$

$$\uparrow$$
  
 Lebesgue measure

More general version:

Let  $\Omega \subset \mathbb{R}^n$  be a Borel set  
with a finite Lebesgue measure.

$$(\Omega, \underbrace{\mathcal{B}}_{\substack{\uparrow \\ \text{Borel subsets of } \Omega}}, \mathbb{P}) \quad \mathbb{P}(A) = \frac{|A|}{|\Omega|}$$

"Uniformly ~~close~~ point from  $\Omega$ "

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Two more examples

3, Product spaces

$(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$  probability spaces  
 $i \in I \subseteq \text{finite/infinite}$

$$\Omega = \prod_i \Omega_i$$

$$\mathcal{F} = \sigma(\prod_i \mathcal{F}_i)$$

$$\mathbb{P} = \prod \mathbb{P}_i$$

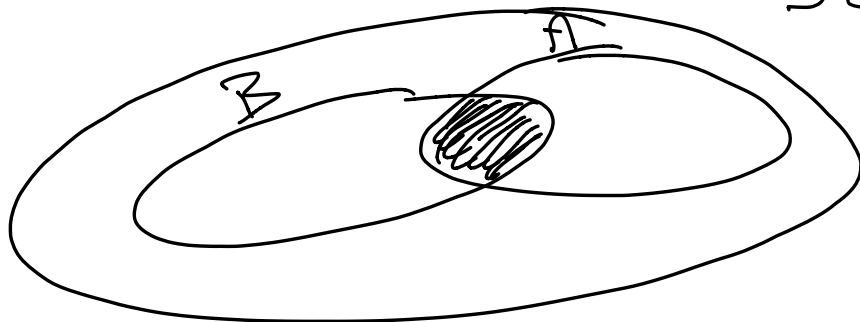
4.  $(\Omega, \mathcal{F}, \mathbb{P})$  is a prob space

fixed  $\rightarrow B \in \mathcal{F}$ ,  $P(B) > 0$   $AB = A \cap B$

$$Q(A) = \frac{P(AB)}{P(B)} = P(A|B)$$

$A \in \mathcal{F}$  "conditional prob of A given B"

$(\Omega, \mathcal{F}, Q)$  is also a  
prob space.



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"Hat - problem"

Suppose that  $n$  guests arrive to a party (each with a hat). They take off their hats. At the end of the party everybody chooses a hat randomly.

What's the probability that nobody gets their own hat?  $n \rightarrow \infty$ ?

Simple properties of probability measures

$$\mathcal{P} : \mathcal{F} \rightarrow [0, 1] \quad \mathcal{P}(\Omega) = 1$$

$A_1, A_2, \dots \in \mathcal{F}$  disjoint

$$\mathcal{P}\left(\bigcup_i A_i\right) = \sum_i \mathcal{P}(A_i)$$

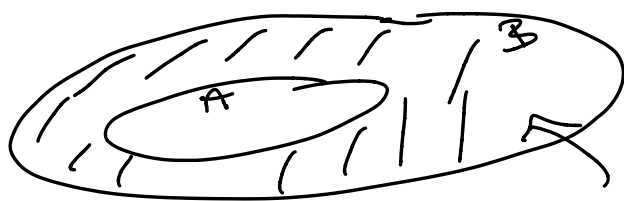
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$$1) \mathcal{P}(\emptyset) = 0$$

$$\mathcal{P}(A) + \mathcal{P}(A^c) = 1$$

$$A \cup A^c = \Omega$$

2) If  $A \subset B$  then  $\mathcal{P}(A) \leq \mathcal{P}(B)$



$$B \setminus A = B A^c$$

$$B = A \cup B A^c$$

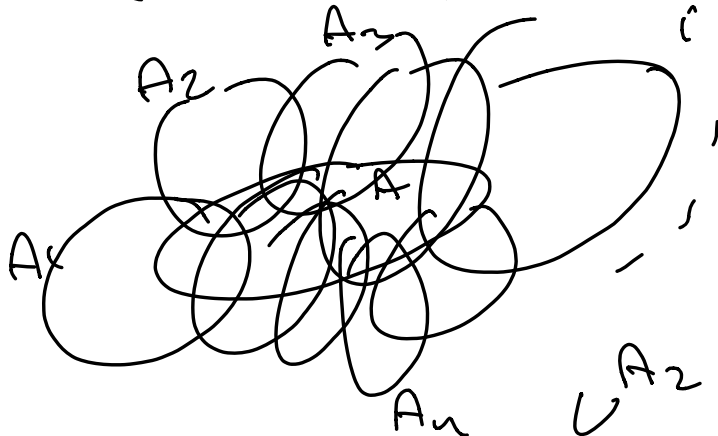
↑     ↑  
disjoint

$$\mathcal{P}(B) = \mathcal{P}(A) + \underbrace{\mathcal{P}(B A^c)}_{\geq 0} \geq \mathcal{P}(A)$$

3, Subadditive property

$A, A_1, A_2, \dots$  with  $A \subset \bigcup_i A_i$

Then  $P(A) \leq \sum_i P(A_i)$



$$UA_i = A_1 \cup (A_2 A_1^c) \cup (A_3 (A_1 \cup A_2)^c) \cup \dots$$

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$   
 $B_1 \quad B_2 \quad B_3$

$\nwarrow \quad \nearrow \quad \nearrow \quad \nearrow$   
 $\text{disjoint}$

$$B_1 = A_1 \quad B_n = A_n (A_1 \cup \dots \cup A_{n-1})^c$$

These are disjoint.  $UA_i = UB_i$

$A_2 \supset B_2$



$$\bigcup_i B_i = \bigcup_i (B_i A \cup B_i A^c)$$

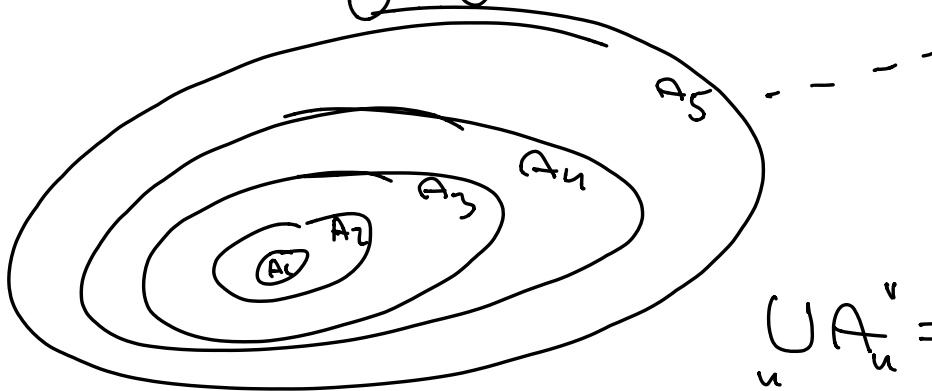
$$P(A) \leq P(\bigcup_i B_i) = \sum_i P(B_i) \leq \sum_i P(A_i)$$

$$A \subset \bigcup_i B_i$$

$$3, \quad A_1 \subset A_2 \subset A_3 \subset \dots$$

$$\lim_{n \rightarrow \infty} P(A_n) = P\left(\bigcup_n A_n\right)$$

"continuity of probability measure"



$$\bigcup_n A_n = \lim_{n \rightarrow \infty} A_n$$

$$A_n \nearrow \bigcup_n A_n$$

$$B = \bigcup_n A_n \quad \bigcup_{z=1}^n B_z = A_n$$

$$B_1 = A_1 \quad B_z = A_z \setminus A_{z-1} = A_z \setminus A_{z-1}^c$$

$B_1, B_2, \dots$  are disjoint and  $B = \bigcup_z B_z$ .

$$P(B) = \sum_{z=1}^{\infty} P(B_z)$$

$$P\left(\bigcup_n A_n\right) = \lim_{n \rightarrow \infty} \sum_{z=1}^n P(B_z)$$

$$= \lim_{n \rightarrow \infty} P\left(\bigcup_{z=1}^n B_z\right) = \lim_{n \rightarrow \infty} P(A_n)$$



$$4) \quad A_1 \supset A_2 \supset A_3 \supset \dots$$

$$\lim_{n \rightarrow \infty} P(A_n) = P\left(\bigcap_{n=1}^{\infty} A_n\right)$$

$$A_n \supset \bigcap_{n=1}^{\infty} A_n$$

We can prove this from the previous case by taking complements.

Probability measures on  $\mathbb{R}$  (or  $\mathbb{R}^d$ )

$\mathbb{P}$  is a probability measure on  $\mathbb{R}$

Set  $F(x) := \mathbb{P}((-\infty, x])$

Then  $F$  is non decreasing

— right continuous  $a < b$   
 $(-\infty, a] \subset (-\infty, b]$

—  $\lim_{x \rightarrow -\infty} F(x) = 0$

—  $\lim_{x \rightarrow \infty} F(x) = 1$

$\lim_{x \rightarrow -\infty} F(x) = \lim_{n \rightarrow \infty} F(x_n) = \lim_{n \rightarrow \infty} \mathbb{P}((-\infty, x_n])$   
 $x_n \rightarrow -\infty$  "0"

$$x_1 > x_2 > x_3 > \dots$$

$$(-\infty, x_1] \supset (-\infty, x_2] \supset \dots$$

$$\bigcap_n (-\infty, x_n] = \emptyset$$


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$$\lim_{x \rightarrow a^+} F(x) = F(a)$$

$$\lim_{x_n \rightarrow a} F(x_n) = \lim_{x_n \rightarrow a} P((-\infty, x_n]) \stackrel{=}{=} F(a)$$

$$x_1 > x_2 > x_3 > \dots$$

$$x_n > a$$

$$\bigcap_n (-\infty, x_n] = (-\infty, a]$$

