CS 760: Machine Learning - Fall 2020
Homework 2: Linear Regression

Due : 10/13/2020

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Problem 1

Proof. We notice that:

$$\|\mathbf{y} - \mathbf{X}\boldsymbol{\theta}\|_2^2 = (\mathbf{y} - \mathbf{X}\boldsymbol{\theta})^T(\mathbf{y} - \mathbf{X}\boldsymbol{\theta})$$
$$= \mathbf{y^T y} - 2\boldsymbol{\theta}^T\mathbf{X}^T\mathbf{y} + \boldsymbol{\theta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\theta}$$

Compute the differential:

$$\mathrm{d}\,\mathrm{tr}\left(\mathbf{y}^T\mathbf{y} - 2\boldsymbol{\theta}^T\mathbf{X}^T\mathbf{y} + \boldsymbol{\theta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\theta}\right) = -2\,\mathrm{tr}\left((\mathrm{d}\boldsymbol{\theta})^T\mathbf{X}^T\mathbf{y}\right) + \mathrm{tr}\left((\mathrm{d}\boldsymbol{\theta})^T\mathbf{X}^T\mathbf{X}\boldsymbol{\theta}\right) + \mathrm{tr}\left(\boldsymbol{\theta}^T\mathbf{X}^T\mathbf{X}(\mathrm{d}\boldsymbol{\theta})\right)$$
$$= 2\,\mathrm{tr}\left((\mathrm{d}\boldsymbol{\theta})^T(\mathbf{X}^T\mathbf{X}\boldsymbol{\theta} - \mathbf{X}^T y)\right)$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\theta}} = 2\mathbf{X}^T\mathbf{X}\boldsymbol{\theta} - 2\mathbf{X}^T y$$

Let this derivative to zero, we have

$$\arg\min_{\theta} = \left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{y}$$

$\square$

Problem 2

Proof. We know that

$$\boldsymbol{y} \sim \mathcal{N}\left(\mathbf{X}\boldsymbol{\theta}^*, \boldsymbol{\Sigma}^*\right)$$

Then

$$\mathbb{P}\left(\mathbf{y}, \mathbf{X}|\boldsymbol{\theta}, \boldsymbol{\Sigma}^*\right) = \frac{1}{\sqrt{(2\pi)^N|\boldsymbol{\Sigma}^*|^{\frac{1}{N}}}}\exp\left\{-\frac{1}{2}\left(\mathbf{y} - \mathbf{X}\boldsymbol{\theta}\right)^T\boldsymbol{\Sigma}^{*-1}\left(\mathbf{y} - \mathbf{X}\boldsymbol{\theta}\right)\right\}$$

$$\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}}\mathbb{P}\left(\mathbf{y}, \mathbf{X}|\boldsymbol{\theta}, \boldsymbol{\Sigma}^*\right)$$

$$= \arg\max_{\boldsymbol{\theta}}\frac{1}{\sqrt{(2\pi)^N|\boldsymbol{\Sigma}^*|^{\frac{1}{N}}}}\exp\left\{-\frac{1}{2}\left(\mathbf{y} - \mathbf{X}\boldsymbol{\theta}\right)^T\boldsymbol{\Sigma}^{*-1}\left(\mathbf{y} - \mathbf{X}\boldsymbol{\theta}\right)\right\}$$

$$= \arg\max_{\boldsymbol{\theta}} - \left(\mathbf{y} - \mathbf{X}\boldsymbol{\theta}\right)^T\boldsymbol{\Sigma}^{*-1}\left(\mathbf{y} - \mathbf{X}\boldsymbol{\theta}\right)$$

$$= \arg\min_{\boldsymbol{\theta}}\left(\mathbf{y} - \mathbf{X}\boldsymbol{\theta}\right)^T\boldsymbol{\Sigma}^{*-1}\left(\mathbf{y} - \mathbf{X}\boldsymbol{\theta}\right)$$

Compute the differential:

$$\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\theta}} = 2\mathbf{X}^T\boldsymbol{\Sigma}^{*-1}\mathbf{X}\boldsymbol{\theta} - 2\mathbf{X}^T\boldsymbol{\Sigma}^{*-1}y$$

$$\hat{\boldsymbol{\theta}} = \left(\mathbf{X}^T\boldsymbol{\Sigma}^{*-1}\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Sigma}^{*-1}y$$

$\square$

Problem 3

Proof.

$$\hat{\boldsymbol{\theta}} \sim \mathcal{N}\left(\boldsymbol{\theta}^*, |\boldsymbol{\Sigma}^*|^{\frac{1}{N}} \left(\mathbf{X}^T\mathbf{X}\right)^{-1}\right)$$

□

Problem 4

Proof.

$$\hat{\mathbf{y}} = \mathbf{x}^T\hat{\boldsymbol{\theta}} = \mathbf{x}^T\left(\mathbf{X}^T\boldsymbol{\Sigma}^{*-1}\mathbf{X}\right)^{-1}\mathbf{X}^T\boldsymbol{\Sigma}^{*-1}\mathbf{y}$$

□

Problem 5

Proof.

$$\hat{\boldsymbol{y}} \sim \mathcal{N}\left(\mathbf{x}^T\boldsymbol{\theta}^*, \mathbf{x}^T|\boldsymbol{\Sigma}^*|^{\frac{1}{N}} \left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{x}\right)$$

□

Problem 6

Proof. □

Problem 7

Proof. □