Math 735 - Fall 2020

Homework 1

Due: 10/14

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Question 1

Proof. If Y has a single continuous path, then it means there exists a ω s.t. $Y_t(\omega) = 0$ for all t. This is equivalent to

$$\exists\ \omega \in [0,1] \text{ s.t. } t \neq \omega\ \forall t \in \mathbb{R}_+$$

This is impossible, so Y does not have a single continuous path.
X and Y are modifications of each other means

$$\mathbb{P}(X_t = Y_t) = \mathbb{P}(Y_t = 0) = \mathbb{P}(t \neq \omega) = 1$$

For any fixed ω, this is true. So, X and Y are modifications of each other. □

Question 2

Proof. We need prove $\widetilde{B_t}$ is a Gaussian process.

$$\widetilde{\mathbb{P}}(t,x) = \mathbb{P}(\lambda^2 t, \lambda x) = \frac{1}{\sqrt{2\pi\lambda^2 t}} e^{-\frac{(\lambda x)^2}{2\lambda^2 t}} = \frac{1}{\lambda} \cdot \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} = \frac{1}{\lambda}\mathbb{P}(t,x)$$

So, $\widetilde{B_t} \sim \frac{1}{\lambda} N(0,t) = N(0, \frac{t}{\lambda^2})$ □

Question 3

Proof.

$$\begin{aligned} E[B_t^k] &= \int_{-\infty}^{\infty} x^k \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dx \\ &= \frac{1}{k+1} x^{k+1} \cdot \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \Big|_{-\infty}^{\infty} + \frac{1}{t(k+1)} \int_{-\infty}^{\infty} x^{k+2} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dx \\ &= \frac{1}{t(k+1)} E[B_t^{k+2}] \end{aligned}$$

$$E[B_t^{k+2}] = t(k+1)E[B_t^k]$$

We know $E[B_t^0] = 1$, $E[B_t^1] = 0$, so we have
When $k = 2j$,

$$E[B_t^k] = t^j \prod_{i=1}^{j} (2i-1)$$

When $k = 2j-1$

$$E[B_t^k] = 0$$

□

Question 4

Proof. Stopping time

1. Play until there is no money or play 500 games before stopping.
2. Play N games to stop,$\{\tau = N\}$
3. Flip a coin N times, stop at the first appearance of *head, tail, head.*

Not stopping time

1. Stopping gambling when the gambler gets the maximum amount of money he can win is not a stopping time.
   Explanation: Because it needs not only the information of the present and the past, but also the information of the future.
2. Stopping when the gambler doubles his wager is not a stopping time.
   Explanation: Because there is a positive probability that he will never double his money.
3. Sold the day before the stock fell is not a stopping time.
   Explanation: Because we don't know the future information.

□

Question 5

Proof. □

Question 6

Proof. □

Question 7

Proof. □