MATH 733 - Fall 2020

Homework 4

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1. $X_n \Rightarrow X$ means

$$\lim_{n\to\infty} P(X_n \leqslant k) = P(X \leqslant k)$$

Since all X and X_n are integer random variables. We have , for all k

$$\begin{aligned}\lim_{n\to\infty} P(X_n = k) &= \lim_{n\to\infty} P(X_n \leqslant k) - \lim_{n\to\infty} P(X_n \leqslant k-1)\\ &= P(X \leqslant k) - P(X \leqslant k-1)\\ &= P(X = k)\end{aligned}$$

If $\lim_{n\to\infty} P(X_n = k) = P(X = k)$ for all $k \in \mathbb{Z}$, for finite a and b, the sum of $P(X_n = m)$ is $P(a \leqslant X_n \leqslant b)$. It is also finite. We get the following equation immediately

$$\lim_{n\to\infty} P(a \leqslant X_n \leqslant b) = P(a \leqslant X \leqslant b)$$

Consider small enough $\epsilon > 0$, there exists L s.t.

$$P(-L \leqslant X \leqslant L) \geqslant 1 - \epsilon$$

Pick big enough N so that for $n \geqslant N$, we have

$$P(-L \leqslant X_n \leqslant L) \geqslant 1 - 2\epsilon$$

Thus $P(X_n < -L) < 2\epsilon$ for $n > N$. By triangle inequality,

$$\begin{aligned}|P(X_n \leqslant b) - P(X \leqslant b)| &= |P(X_n < -L) + P(-L \leqslant X_n \leqslant b) - P(-L \leqslant X \leqslant b) - P(X < -L)|\\ &\leqslant |P(-L \leqslant X_n \leqslant b) - P(-L \leqslant X \leqslant b)| + 3\epsilon\end{aligned}$$

$$\limsup_{n\to\infty} |P(X_n \leqslant b) - P(X \leqslant b)| \leqslant 3\epsilon$$

Therefore,

$$X_n \Rightarrow X$$

2. Let F_n be the CDF of X_n

$$\begin{aligned}F_n(t) &= P(X_n \leqslant t)\\ &\geqslant P(X_n \leqslant t, X \leqslant t - \epsilon)\\ &\geqslant P(X \leqslant t - \epsilon) - P(|X_n - X| > \epsilon)\end{aligned}$$

Thus, we have

$$\lim_{n\to\infty} F_n(t) \geqslant P(X \leqslant t - \epsilon) = F(t - \epsilon)$$

Similarly,

$$F(t + \epsilon) \geqslant \lim_{n\to\infty} F_n(t)$$

Therefore,

$$X_n \Rightarrow X$$

If $X_n \Rightarrow c$, then $\lim_{n\to\infty} F_n(x) = \mathbb{1}_{x\geqslant c}$. For all $\epsilon > 0$,

$$\begin{aligned} P(|X_n - c| \geqslant \epsilon) &= 1 - P(X_n < c + \epsilon) + P(X_n \leqslant c - \epsilon) \\ &= 1 - F_n(c+\epsilon) + F_n(c-\epsilon) \\ &\to 0 \end{aligned}$$

So,

$$X_n \xrightarrow{P} X$$

3. Let F_n be the distribution function of X_n and F the distribution function of X. For fixed x and small enough ϵ.

$$\begin{aligned} P(X_n + Y_n \leqslant x) &= P(X_n + Y_n \leqslant x, |Y_n - c| \leqslant \epsilon) + P(X_n + Y_n \leqslant x, |Y_n - c| > \epsilon) \\ &\leqslant P(X_n \leqslant x - c + \epsilon) + P(|Y_n - c| > \epsilon) \end{aligned}$$

We know that

$$P(X_n \leqslant x - c + \epsilon) = F_n(x - c + \epsilon) \to F(x - c + \epsilon)$$

$$\limsup_{n\to\infty} P(X_n + Y_n \leqslant x) \leqslant F(x - c)$$

Similarly, the lower bound is

$$\liminf_{n\to\infty} P(X_n + Y_n \leqslant x) \geqslant F(x - c)$$

This implies

$$\lim_{n\to\infty} P(X_n + Y_n \leqslant x) = F(x - c)$$

Therefore, this shows that

$$X_n + Y_n \Rightarrow X + c$$

4. Consider $X_n = \xi$, $Y_n = (-1)^n \xi$, where $\xi \sim N(0,1)$ then

$$P(X_n \leqslant x, Y_n \leqslant y) = \begin{cases} P(\xi \leqslant \min(x,y)), \text{ if } n \text{ is even} \\ P(-y \leqslant \xi \leqslant x), \text{ if } n \text{ is odd} \end{cases}$$

$X_n + Y_n$ does not converge in distribution.

5. M_n is the maximum of the first n element, then $F^n(x) = F_{M_n}(x)$ Thus

$$F^n\left(n^{\frac{1}{\alpha}}x\right) = F_{M_n}\left(n^{\frac{1}{\alpha}}x\right) = P\left(M_n \leqslant n^{\frac{1}{\alpha}}x\right) = P\left(n^{-\frac{1}{\alpha}}M_n \leqslant x\right)$$

Replace $x = n^{\frac{1}{\alpha}}x$ in

$$\lim_{x\to\infty} x^\alpha(1 - F(x)) = b$$

$$\lim_{x\to\infty} nx^\alpha\left(1 - F\left(n^{\frac{1}{\alpha}}x\right)\right) = b$$

Thus we have,

$$\begin{aligned} \lim_{x\to\infty} F^n\left(n^{\frac{1}{\alpha}}x\right) &= \lim_{x\to\infty}\left(1 - \frac{b}{nx^\alpha}\right)^n \\ &= \lim_{x\to\infty}\left(\left(1 - \frac{b}{nx^\alpha}\right)^{\frac{nx^\alpha}{b}}\right)^{\frac{b}{x^\alpha}} \end{aligned}$$

If $n \to \infty$,

$$n^{-\frac{1}{\alpha}}M_n \Rightarrow \exp\left\{-\frac{b}{x^\alpha}\right\}$$

6. Consider $X_i \sim$ Bernoulli$(-1, 1)$, let
$$Y = \sum_{k=1}^{\infty} X_k 2^{-k}$$
Note that the characteristic function of right hand side is
$$\prod_{k=1}^{\infty} \cos\left(t 2^{-k}\right)$$
Recall the Problem 6 in Homework 2, $Y \sim$ Uniform$(-1, 1)$, then its characteristic function is
$$\int \frac{e^{itx}}{2} dx = \frac{\sin(t)}{t}$$
Therefore,
$$\frac{\sin(t)}{t} = \prod_{k=1}^{\infty} \cos\left(\frac{t}{2^k}\right)$$

7. First consider the expectations of S_n
$$\begin{aligned} E[S_n] &= E[X_1 + \cdots + X_n] \\ &= E[X_1] + \cdots + E[X_n] \\ &= 0 \end{aligned}$$
Since $E[X_m] = 0 + 0 = 0$.
Then consider the variance of S_n
$$\begin{aligned} E[S_n^2] &= E\left[(X_1 + \cdots + X_n)^2\right] \\ &= E\left[X_1^2\right] + \cdots + E\left[X_n^2\right] + \sum_{i \neq j} E[X_i]E[X_j] \\ &= 2n - \sum_{k=1}^{n} \left(\frac{1}{k^2}\right) \end{aligned}$$
Thus
$$\frac{\mathsf{Var}(S_n)}{n} = 2 - \frac{1}{n} \sum_{k=1}^{n} \left(\frac{1}{k^2}\right) \to 2 \ (n \to \infty)$$
Consider the characteristic function of X_k,
$$\begin{aligned} \varphi_{X_k}(t) &= E[e^{itX_k}] \\ &= \frac{\cos(tk)}{k^2} + \cos(t)\left(1 - \frac{1}{k^2}\right) \end{aligned}$$
$$\varphi_{\frac{X_k}{\sqrt{n}}}(t) = \frac{\cos\left(\frac{tk}{\sqrt{n}}\right)}{k^2} + \cos\left(\frac{t}{\sqrt{n}}\right)\left(1 - \frac{1}{k^2}\right)$$
When $k \to \infty$, $\varphi = \left(1 - \frac{t^2}{2n}\right)$ (by Taylor Series).
Thus
$$\varphi_{\frac{S_n}{\sqrt{n}}}(t) \to \varphi^n = \left(1 - \frac{t^2}{2n}\right) = e^{-\frac{t^2}{2}}$$
Therefore,
$$\frac{S_n}{\sqrt{n}} \Rightarrow N(0, 1)$$