Last time: $E[X] = \int_\Omega X \, dP$

If $X \geq 0$ then $E[X] \in \mathbb{R}_+ \cup \{\infty\}$

In general $E[X]$ exists if $E[X^+]$ or $E[X^-]$ is finite.

$E[X+Y] = E[X] + E[Y]$

$E[aX] = a\,E[X] \quad a \in \mathbb{R}$

If $X \leq Y$ then $E[X] \leq E[Y]$

Ex: Pick 5 cards out of 52.

X: # of aces

$E[X] = ?$

$$X = \sum_{i=1}^{5} \mathbb{1}(i^{th} \text{ card is an ace})$$

$$E[X] = \sum_{i=1}^{5} P(i^{th} \text{ card is an ace})$$

There 4 aces

$$X = \sum_{j=1}^{4} \mathbb{1}\left(\begin{array}{c} j^{th} \text{ ace is among} \\ \text{the 5 cards} \end{array}\right)$$

$$E[X] = \sum_{j=1}^{4} P(j^{th} \text{ ace is chosen})$$

N items, N_A of them are marked

We choose $n \le N$ of them without replacement

Y_1 : # of marked items in the sample

(distribution of Y_1: hypergeometric N, N_A, n)

$$E[Y_1] = ?$$

Important inequalities

Jensen's inequality

If $\phi : \mathbb{R} \to \mathbb{R}$ is convex then

$$\Phi(E[X]) \le E[\Phi(X)]$$

(with the appropriate disclaimers)

Also: $X \in \mathbb{R}^d$, $\phi : \mathbb{R}^d \to \mathbb{R}$ convex

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Hölder's inequality

$p, q \in [1, \infty]$ $\quad \frac{1}{p} + \frac{1}{q} = 1$

then $E[|XY|] \le (E|X|^p)^{\frac{1}{p}} (E|Y|^q)^{\frac{1}{q}}$

$p = q = 2$ $\quad E[|XY|] \le \sqrt{E|X|^2} \sqrt{E|Y|^2}$

$E[X^p]$: p^{th} moment of X

$E[|X|^p]$: p^{th} absolute moment of X

Markov's inequality

If $\underbrace{X \geq 0}$ and $\underbrace{c > 0}$

$$P(X \geq c) \leq \frac{E[X]}{\underbrace{c}}$$

Proof: $\underbrace{c\,P(X \geq c)} \leq E[X]$

We construct Y with $E[Y] = c\,P(X \geq c)$ and $0 \leq Y \leq X$.

$$Y = c \cdot 1(X \geq c) = \begin{cases} c & \text{on } \{X \geq c\} \\ 0 & \text{otherwise} \end{cases}$$

$0 \leq Y$ and $Y \leq X$.

Then $\underset{c\,P(X \geq c)}{\underset{\parallel}{E[Y]}} \leq E[X]$

If $X \geq 0$, $E[X] < \infty$ then $P(X \geq c) \to 0$ at least as fast as $\frac{1}{c}$.

Generalized version: use Markov's inequality for $\varphi(X)$.

If $X \geq 0$ and $c > 0$ then

$$P(X \geq c) = P(X^p \geq c^p) \leq \frac{E[X^p]}{c^p}$$

$p > 1$

Expectation and limits

Recall that $X_n \xrightarrow{a.s.} X$ if $X_1, X_2, \dots, X$ are random variables on the same prob space and

$$P(\lim_{n \to \infty} X_n = X) = 1.$$

Q: is it true that if $X_n \xrightarrow{a.s.} X$ then $E[X_n] \to E[X]$?

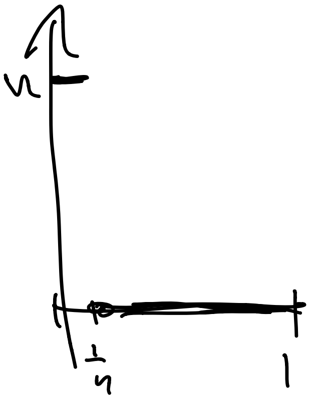
(assume that $E[X_n]$, $E[X]$ all exist)

In general the answer is NO.

$([0,1], \mathcal{B}, \mathbb{P})$

Lebesgue

$X_n = n \cdot \mathbb{1}_{[0,\frac{1}{n}]}$



$E[X_n] = n \cdot P([0,\frac{1}{n}]) = 1$

$\lim_{n\to\infty} X_n = X = 0 \quad$ a.s.

$\lim_{n\to\infty} E[X_n] \neq E[X]$

Thm (Fatou's lemma). If $X_n \geq 0$

$$\liminf_{n\to\infty} E[X_n] \geq E[\liminf_{n\to\infty} X_n]$$

Thm (Monotone convergence theorem)

If $0 \leq X_n \nearrow X$ a.s. then $E[X_n] \nearrow E[X]$

$X_1 \leq X_2 \leq \dots$

Thm (Dominated convergence thm)

If $X_n \xrightarrow{a.s.} X$ and $|X_n| \leq Y$ with

$\check{E}[Y] < \infty$ then $E[X_n] \to E[X]$.

Special case: if $X_n \xrightarrow{a.s.} X$ and $|X_n| \leq c$ then $E[X_n] \to E[X]$.

"Durrett's special convergence thm"

$X_n \xrightarrow{a.s.} X$. $g, h : \mathbb{R} \to \mathbb{R}$ continuous

- $g \geq 0$, $g(x) \to \infty$ as $|x| \to \infty$
- $\dfrac{|h(x)|}{g(x)} \to 0$ if $|x| \to \infty$
- $E[g(X_n)] \leq k < \infty$

Then $E[h(X_n)] \to E[h(X)]$.

Possible use: $g(x) = |x|^p$ $p > 1$

$h(x) = x$

Hints for the proof:

$$h(X_n) = h(X_n) 1_{|h(X_n)| > c} + h(X_n) 1_{|h(X_n)| \leq c}$$

use Markov's inequality

bounded by c

Computing expectations

Important: $E f(X)$ only depends on the distribution of X and f.

Thm: $X : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$
S-valued r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$.
$f : S \to \mathbb{R}$ with $f \geq 0$ or
$E|f(X)| < \infty$.

Then
$$E[f(X)] = \int_S f(y)\, Q_X(dy)$$

distribution of X

If $S = \mathbb{R}$ then $E f(X) = \int_{-\infty}^{\infty} f(y)\, Q_X(dy)$

$$\int_{\Omega} f(X(\omega))\, dP = \int_{S} f(y)\, d(P\circ X^{-1})$$

If $\frac{dQ_X}{d\lambda}$ exists then

$$E\, f(X) = \int_S f(y) \frac{dQ_X}{d\lambda}\, \lambda(dy)$$

If X is real valued with PDF g_X

then $E[f(X)] = \int_{-\infty}^{\infty} f(y)\, g_X(y)\, dy$.

Proof: indicator (1) $\to$ simple (2) $\to$ nonnegative (3) $\to$ general (4) f

(1) $f = 1_B$

$$E\, 1_B(X) = P(X \in B) = Q_X(B)$$

$$= \int_S 1_B(y)\, Q_X(dy)$$

(2) $f = \sum_{i=1}^{n} c_i 1_{B_i}$

Follows from linearity

$$E f(X) = \int_S f(y)\, Q_x(dy)$$

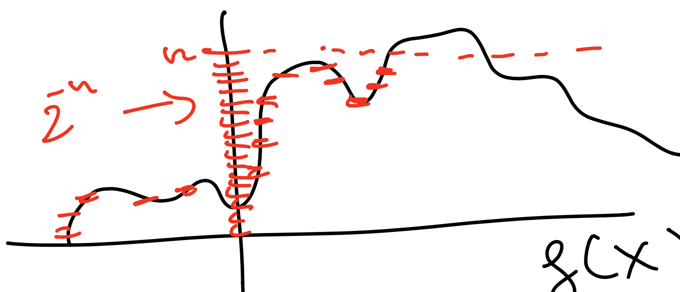
(3) $f \geq 0$ $\qquad 0 \leq f_1 \leq f_2 \leq \dots$
simple

$f_n \nearrow f$

E.g. we can do this using

$$f_n(x) = \min\left(\frac{\lfloor 2^n f(x) \rfloor}{2^n}, n\right)$$

$\lfloor x \rfloor$ is the floor function
$\lfloor x \rfloor =$
$= \max\{z \in \mathbb{Z} : z \leq x\}$



$$f_n(x) \nearrow f(x)$$

$$E f(X) = \lim_{n\to\infty} E f_n(X)$$

$$= \lim_{n\to\infty} \int_S f_n(y)\, Q_x(dy)$$

mon conv thm

$$= \int_S \lim_{n\to\infty} f_n(y)\, Q_x(dy)$$

$$= \int_S f(y)\, Q_x(dy)$$

(4) f is general $f = f^+ - f^-$.

$E|f(X)| < \infty$

Def: The variance of X is

$$\mathrm{Var}(X) = E\left[(X - E[X])^2\right]$$

(needs $E|X| < \infty$)

Properties: $\mathrm{Var}(X) \geq 0$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$

Proof: $E\left[(X - E[X])^2\right] = E\left[X^2 - 2X \cdot E[X] + (E[X])^2\right]$

$$= E[X^2] - 2\underbrace{E[X \cdot E[X]]}_{E[X \cdot E[X]] = E[X] \cdot E[X]} + \underbrace{E\left[(E[X])^2\right]}_{= (E[X])^2}$$

$$= E[X^2] - (E[X])^2$$

$$\mathrm{Var}(aX + b) = a^2 \, \mathrm{Var}(X)$$

$$E\left[\left(aX+b-\underset{\substack{\parallel \\ (aEX+b)}}{E[aX+b]}\right)^2\right]=E\left[\left(aX-aE[X]\right)^2\right]$$

$$=E\left[a^2\left(X-E[X]\right)^2\right]$$

$$=a^2\,E\left[\left(X-E[X]\right)^2\right]$$

$$=a^2\,\mathrm{Var}(X)$$

$$0\le \mathrm{Var}(X)\le E[X^2]$$

$$\left(E[X]\right)^2\le E[X^2]$$