# Math 733 - Fall 2020

## Homework 3

**Due: 10/11, 10pm**

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1. (a) *Proof.*

$$X \sim B(n,p) \Rightarrow P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$Y \sim B(m,p) \Rightarrow P(X=k) = \binom{m}{k} p^k (1-p)^{m-k}$$

Then

$$\begin{aligned}
P(X+Y=k) &= \sum_{i=0}^{k} P(X=i, Y=k-i) \\
&= \sum_{i=0}^{k} P(X=i) \cdot P(Y=k-i) \\
&= \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} \cdot \binom{m}{k-i} p^{k-i} (1-p)^{m-k+i} \\
&= p^k (1-p)^{m+n-k} \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i} \\
&= \binom{n+m}{k} p^k (1-p)^{m+n-k}
\end{aligned}$$

Thus,

$$X+Y \sim B(n+m, p)$$

□

(b) *Proof.*

$$X \sim \text{Poisson}(\lambda) \Rightarrow P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$Y \sim \text{Poisson}(\mu) \Rightarrow P(Y=k) = \frac{\mu^k}{k!} e^{-\mu}$$

Then

$$\begin{aligned}
P(X+Y=k) &= \sum_{i=0}^{k} P(X=i, Y=k-i) \\
&= \sum_{i=0}^{k} P(X=i)\cdot P(Y=k-i) \\
&= \sum_{i=0}^{k} \frac{\lambda^i}{i!}e^{-\lambda}\cdot\frac{\mu^{k-i}}{(k-i)!}e^{-\mu} \\
&= e^{-(\lambda+\mu)}\sum_{i=0}^{k}\frac{\lambda^i}{i!}\frac{\mu^{k-i}}{(k-i)!} \\
&= \frac{(\lambda+\mu)^k}{k!}e^{-(\lambda+\mu)}
\end{aligned}$$

Thus,

$$X+Y \sim \text{Poisson}(\lambda+\mu)$$

□

2. (a) *Proof.* Let $h(x,y) = \mathbb{1}_{\{xy\leqslant z\}}$, let $\mu,\nu$ be the probability measures with distributions $F_X$ and $F_Y$. Since for fixed $y>0$,

$$\int h(x,y)\mu(dx) = \int \mathbb{1}_{(-\infty,z/y]}(x)\mu(dx) = F_X(\frac{z}{y})$$

So

$$\begin{aligned}
F_{XY}(z) = P(XY\leqslant z) &= \iint \mathbb{1}_{\{xy\leqslant z\}}\mu(dx)\nu(dy) \\
&= \int F_X(\frac{z}{y})dF_Y(y)
\end{aligned}$$

□

(b) *Proof.* Absolutely continuous means every set of measure zeros is probability zero. Consider

$$P(XY) = P(X)P(Y)$$

$$P(XY=x_1y_1) = P(X=x_1)P(Y=y_1) = 0$$

Then $XY$ is absolutely continuous with p.d.f

$$f_{XY} = \int f_X\left(\frac{x}{t}\right)f_Y(t)\mathrm{d}t$$

□

3. *Proof.*

$$\begin{aligned}
&\lim_{n\to\infty} P\left(|X_n+Y_n-(X+Y)|>\varepsilon\right) \\
&\leqslant \lim_{n\to\infty} P\left(|X_n-X|>\frac{\varepsilon}{2}\right) + \lim_{n\to\infty} P\left(|Y_n-Y|>\frac{\varepsilon}{2}\right) \\
&= 0
\end{aligned}$$

Since $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$. So,

$$X_n+Y_n \xrightarrow{p} X+Y$$

$$
\begin{aligned}
&\lim_{n\to\infty} P\left(|X_nY_n - XY| > \varepsilon\right) \\
&= \lim_{n\to\infty} P\left(|(X_n - X)(Y_n - Y) + Y(X_n - X) + X(Y_n - Y)| > \varepsilon\right) \\
&\leqslant \lim_{n\to\infty} P\left(|(X_n - X)(Y_n - Y)| > \frac{\varepsilon}{3}\right) + \lim_{n\to\infty} P\left(|Y(X_n - X)| > \frac{\varepsilon}{3}\right) + \lim_{n\to\infty} P\left(|X(Y_n - Y)| > \frac{\varepsilon}{3}\right) \\
&= 0
\end{aligned}
$$

$$X_nY_n \xrightarrow{P} XY$$

□

4. *Proof.* Consider

$$x_i \sim U[0,1]\forall i \text{ and i.i.d}$$

$$
\begin{aligned}
&\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 n\left(f\left(\frac{1}{n}(x_1 + x_2 + \cdots + x_n)\right) - f\left(\frac{1}{2}\right)\right) \mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n \\
&= \lim_{n\to\infty} \mathbb{E}\left[n\left(f\left(\frac{1}{n}(x_1 + x_2 + \cdots + x_n)\right) - f\left(\frac{1}{2}\right)\right)\right] \\
&= \lim_{n\to\infty} n \cdot \mathbb{E}\left[f\left(\frac{1}{n}(x_1 + x_2 + \cdots + x_n)\right) - f\left(\frac{1}{2}\right)\right] \\
&f \text{ is continuous differentiable} \Leftrightarrow \mathbb{E}\left[f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)\right] = f\left(\mathbb{E}\left[\frac{x_1 + x_2 + \cdots + x_n}{n}\right]\right) \\
&= \lim_{n\to\infty} n \cdot \left(f\left(\mathbb{E}\left[\frac{x_1 + x_2 + \cdots + x_n}{n}\right]\right) - f\left(\frac{1}{2}\right)\right) \\
&\text{Since } \mathbb{E}\left[\frac{x_1 + x_2 + \cdots + x_n}{n}\right] = \mathbb{E}\left[x_1\right] = \int_0^1 t\mathrm{d}t = \frac{1}{2} \\
&\text{We have} \\
&= \lim_{n\to\infty} n \cdot 0 = 0
\end{aligned}
$$

□

5. (a) *Proof.* Let

$$S_n = x_1 + x_2 + \cdots + x_n$$

then

$$\mathbb{E}[\mathbb{1}_{\{S_k \leqslant x\}}] = P(S_k \leqslant x) = P(x_1 + x_2 + \cdots x_k < k) = F^{(k)}(x)$$

Let

$$N_x = \sum_{k=1}^{\infty} \mathbb{1}_{\{S_k \leqslant x\}}$$

It gives

$$\mathbb{E}[N_x] = \sum_{n=1}^{\infty} \mathbb{E}[\mathbb{1}_{\{S_n \leqslant x\}}] = \sum_{n=1}^{\infty} F^{(n)}(x)$$

□

(b) *Proof.* Let $n = \lceil x \rceil$

$$
\begin{aligned}
P(N_x = N) &= P(S_N \leqslant x < S_{N+1}) \\
&\leqslant P(S_N \leqslant n) \\
&= \binom{N}{N-n} F^{(N-n)}(0)
\end{aligned}
$$

Consider the expectation,

$$\begin{aligned}\mathbb{E}[N_x] &= \sum_{N=1}^{\infty} NP(N_x = N) \\ &\leqslant \sum_{N=1}^{\infty} N\binom{N}{N-n}F^{(N-n)}(0) \\ &\leqslant \sum_{N=1}^{\infty} N\binom{N}{n}F^{(N-n)}(0) \\ &< \infty\end{aligned}$$

□

(c) *Proof.* If$X_i$ are not integer, assume$X_i \in \mathbb{R}$. If $x_k < t$, let $x_k(t) = x_k$ else, let $x_k(t) = t$. Here we let

$$N_x(t) = \sup_n \{x_1(t) + x_2(t) + \cdots + x_n(t) \leqslant x\}$$

$$\mathbb{E}[N_x] \leqslant \mathbb{E}[N_x(t)] \leqslant \frac{\frac{x}{t}+1}{P(X \geqslant t)} < \infty$$

□

6. *Proof.* Consider $X_1 \sim$ Cauchy(0,n), $X_2 \sim$ Cauchy(0,1)

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{n\pi^2\left(1+\left(\frac{x_1}{n}\right)^2\right)(1+x_2^2)}$$

$$f_{X_1+X_2,X_2}(x_1+x_2,x_2) = \frac{1}{n\pi^2\left(1+\left(\frac{x_1+x_2}{n}\right)^2\right)(1+x_2^2)}$$

$$\begin{aligned}f_{X_1+X_2}(x_1+x_2) &= \int f_{X_1+X_2,X_2}(x_1+x_2,x_2)\mathrm{d}x_2 \\ &= \int \frac{1}{n\pi^2\left(1+\left(\frac{x_1+x_2}{n}\right)^2\right)(1+x_2^2)} \\ &= \int \frac{n}{\pi^2\left(n^2+(x_1+x_2)^2\right)(1+x_2^2)} \\ &= \frac{1}{(n+1)\pi\left(1+\left(\frac{X_1+X_2}{n+1}\right)\right)}\end{aligned}$$

This means Cauchy distribution is additive.

$$\begin{aligned}S_n = X_1 + X_2 + \cdots + X_n &\sim \text{Cauchy}(0,n) \\ P(S_n < x) &= \frac{1}{\pi}\arctan\left(\frac{x}{n}\right) + \frac{1}{2} \\ P(S_n < nx) &= \frac{1}{\pi}\arctan(x) + \frac{1}{2} \\ &= P(X_1 < x)\end{aligned}$$

So,

$$\frac{S_n}{n} \sim X_1$$

Which means

$$\frac{S_n}{c_n} = \frac{S_n}{n}\frac{n}{c_n} \sim X_1\frac{n}{c_n}$$

For all $x$,

$$\lim_{n\to+\infty}\frac{S_n(x)}{c_n}=\lim_{n\to+\infty}\frac{S_n(x)}{n}\frac{n}{c_n}=0$$

So,

$$\frac{S_n}{c_n}\to 0\Rightarrow\frac{S_n}{c_n}\xrightarrow{P}0$$

□