Review of Probability

A prob space $(\Omega, \mathcal{F}, P)$

Ω is set of "outcomes".

$\mathcal{F}$: σ-algebra of "events", i.e. subsets of Ω.

P: $\mathcal{F} \to [0, \infty)$ assigns a positive number (non-neg) to each event.

A random variable: X is a real-valued function defined on Ω. s.t. for every Borel set $B \in \mathcal{B}(\mathbb{R})$ we have $X^{-1}(B) = \{\omega : X(\omega) \in B\} \in \mathcal{F}$

$\omega \in \Omega \xrightarrow{X} \mathbb{R} \supset B$

(Rmk: can also talk about separable metric space-valued R.V.'s) e.g. $\mathbb{R}^d$.

prob distribution of X

$$\mu_X(B) = P(X \in B) = P(X^{-1}(B))$$

We say X is discrete r.v. / has discrete distribution if its range is countable.

namely, $\exists$ a sequence $\{x_i\}$ such that $\{x_i\} = \text{range}(X)$, $\left(\sum_i P(X = x_i) = 1\right)$

Expection for discrete r.v.

$$\mathbb{E}X = \sum_i x_i P(X = x_i)$$

for non-discrete r.v. X:

When $X \in \left(\frac{k}{n}, \frac{k+1}{n}\right]$, then $Y_n = \frac{k}{n}$, $Z_n = \frac{k+1}{n}$

s.t. $Y_n < X \le Z_n$ $\quad |Z_n - Y_n| \le \frac{1}{n}$

$$\mathbb{E}X \overset{\text{def}}{=} \lim_{n\to\infty} \mathbb{E}(Y_n) = \lim_{n\to\infty} \mathbb{E}(Z_n)$$

If the limits exist.

When $\mathbb{E}X$ exists, we say X is integrable.

properties of $\mathbb{E}$.

Linearity: $\mathbb{E}(aX+bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$

Monotonicity: $X \ge Y$ a.s. $\Rightarrow \mathbb{E}X \ge \mathbb{E}Y$.

Convergence of random variables.

(a) $X_n \to X$ a.s $\iff P\left\{w: \lim_{n\to\infty} X_n(w) = X(w)\right\} = 1$

(b) $X_n \to X$ in prob. $\iff \lim_{n\to\infty} P\left(|X_n - X| > \varepsilon\right) = 0$ $\quad \forall \varepsilon > 0$.

(c) $X_n \to X$ in distribution (in law) $\left(\begin{matrix} X_n \Rightarrow X \\ X_n \xrightarrow{\text{law}} X \end{matrix}\right)$: $\left\{ \lim_{n\to\infty} P(X_n \le x) = P(X \le x) = F_X(x) \right\}$

for all x at which F_X is continuous.

Thm. (a) $\Rightarrow$ (b) $\Rightarrow$ (c)

proof. (b) $\Rightarrow$ (c) need to show

$$\begin{cases} \limsup P(X_n \le x) \le P(X \le x+\varepsilon) \leftarrow \\ \liminf P(X_n \le x) \ge P(X \le x - \varepsilon) \end{cases}$$

$$P(X_n \le x) - P(X \le x+\varepsilon) = P(X_n \le x;\ X > x+\varepsilon) - P(X \le x+\varepsilon;\ X_n > x)$$
$$\le P(|X_n - X| > \varepsilon) \xrightarrow{n\to\infty} 0$$

properties 1) $X_n \to X$ in prob, $Y_n \to Y$ in prob $\Rightarrow aX_n + bY_n \to aX + bY$ in prob

false for conv. in law

also hold for conv. in law

2) $f: \mathbb{R} \to \mathbb{R}$ is continuous and $X_n \to X$ in prob $\Longrightarrow f(X_n) \to f(X)$ in prob.

3) If $X_n \to X$ in prob. and $X_n - Y_n \to 0$ in prob then $Y_n \to X$ in prob.

Thm: (Bounded convergence) If $X_n \Rightarrow X$ and $P(|X_n| \le a) = 1$. Then $\mathbb{E}X_n \to \mathbb{E}X$.

Thm: Suppose $X \ge 0$. a.s. Then $\lim_{M\to\infty} \mathbb{E}(X \wedge M) = \mathbb{E}X$

min.

Thm: (monotone convergence) $0 \le X_n \le X$, $X_n \to X$ in prob. Then $\boxed{\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X}$

pf: send $M \to \infty$

$$\mathbb{E}(X \wedge M) \le \liminf_{n\to\infty} \mathbb{E}X_n \le \limsup_{n\to\infty} \mathbb{E}X_n \le \mathbb{E}X$$

$$\mathbb{E}X \ge \mathbb{E}(X_n) \ge \mathbb{E}(X_n \wedge M) \xrightarrow{n\to\infty} \boxed{\mathbb{E}(X \wedge M)}$$

Thm (Fatou's lemma) If $X_n \ge 0$, and $X_n \Rightarrow X$

then $\liminf \mathbb{E} X_n \geq \mathbb{E} X$

Thm (Dominated convergence) Assume $X_n \Rightarrow X$, $Y_n \Rightarrow Y$.

$|X_n| \leq Y_n$. and $\mathbb{E} Y_n \to \mathbb{E} Y < \infty$

Then $\mathbb{E}(X_n) \to \mathbb{E}(X)$

Markov inequality: $P(|X| > a) \leq \dfrac{\mathbb{E}(X)}{a}$ $\forall a \geq 0$