Itô formula (general form)

$\int_0^t X \, dM$

predictable $\quad$ L^2-mart

$X \in \mathcal{L}_2 \qquad M \in \mathcal{M}_2$

Recall: a cadlag process M is a local L^2-mart if $\exists$ nondecreasing seq of stopping times $\{\sigma_k\}$ $\quad \sigma_k \nearrow \infty$ a.s. as $k \to \infty$

$$M^{\sigma_k} = \{ M_{\sigma_k \wedge t}, \ t \in \mathbb{R}_+ \}$$

is L^2-mart. for each k

Write $\mathcal{M}_{2,loc}$ space of cadlag local L^2-mart's

$\int X \, dM \qquad M \in \mathcal{M}_{2,loc}.$ $\qquad \left(\begin{matrix} \int X \, dM \\ X \in \mathcal{L}_2(M) \end{matrix} \right)$

Def: Given $M \in \mathcal{M}_{2,loc}$.

Let $\mathcal{L}(M)$ denote the class of predictable processes X s.t.

$\exists$ seq of stopping ...

$\exists$ seq of stopping times

$0 \le \tau_1 \le \tau_2 \le \cdots \le \tau_k \le \cdots$,

localizing seq. for (X, M)

(1) $P(\tau_k \uparrow \infty) = 1$

(2) M^{τ_k} is L^2 mart $\forall k$.

(3) $X \cdot \mathbb{1}_{[0,\tau_k]} \in \mathcal{L}_2(M^{\tau_k})$

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for each k.

$$Y^k = \int \underline{\mathbb{1}_{[0,\tau_k]} X} \cdot d\underline{\underline{M^{\tau_k}}}$$

Def. $\int_0^t X \, d\underline{M}$ is the cadlag $\underline{\text{local}}$ L^2-mart

$\hat{\mathcal{M}}_{2,loc}$.

defined as. ① on Ω_0:

$$\left(\int_0^t X \, dM\right)_{(\omega)} = \left(\int_0^t \mathbb{1}_{[0,\tau_k]} X \, dM^{\tau_k}\right)_{(\omega)}$$

for any k s.t. $\tau_k(\omega) \ge t$.

② on $\Omega \setminus \Omega_0$, $\int_0^t X \, dM = 0$

Rmk. $P(\Omega_0) = 1$. $\leftarrow$ Timo's notes Lem 5.22

$\Omega_0 = \{ \omega : \tau_k(\omega) \uparrow \infty$ as $k \uparrow \infty$.

and for $\forall (k,m)$

$$Y^k_{t\wedge \tau_k \wedge \tau_m}(\omega) = Y^m_{t\wedge\tau_k\wedge\tau_m}(\omega)$$

(on Ω_0. if $t \le \tau_k \wedge \tau_m$, then $Y^k_t = Y^m_t$)

Fact: $\left(\int_0^t X\, dM\right)^2 - \int_0^t X_u^2\, d[M]_u$ is mart.

(If $X=1$, $M^2 - [M]$ is mart)

A cadlag semi-mart.

$$Y = \underline{Y_0} + M_t + V_t$$

M_t: cadlag local mart

V_t: cadlag FV process

$M_0 = V_0 = 0.$

Def

$$\int_0^t X_s\, dY_s = \int_0^t X_s\, dM_s + \int_0^t X_s\, dV_s$$

semi-mart (Y)

(recall: $\int N_{s-}\, d\underline{M_s}$, where $M = N - \alpha s$

$$= \boxed{\int N_{s-}\, dN_s} - \alpha \int N_{s-}\, ds$$

$\underline{N_t} = \underline{M_t} + \alpha t$

Leb-Stieltjes integrals

Itô formula.

Let Y be a cadlag semi-mart. with QV $[Y]$.

$$f(Y_t) = f(Y_0) + \int_0^t f'(\underbrace{Y_{s-}}_{\lim_{u \uparrow s} Y_u}) \, dY_s + \frac{1}{2} \int_0^t f''(Y_{s-}) \, d[Y]_s$$

$$+ \sum_{s \in [0,t]} \left\{ f(Y_s) - f(Y_{s-}) - f'(Y_{s-}) \Delta Y_s - \frac{1}{2} f''(Y_{s-})(\Delta Y_s)^2 \right\}$$

$$\Delta Y_s \overset{\text{def}}{=} Y_s - Y_{s-}$$

Special cases:

① Y is continuous. $\Rightarrow$ 2nd line disappear.

$Y_{s-} = Y_s$

(2) Y is FV. then

$$\begin{cases} f(Y_t) = f(Y_0) + \int_0^t f'(Y_{s-})\,dY_s \\ \quad + \sum_{s\in(0,t]} \left\{ f(Y_s) - f(Y_{s-}) - f'(Y_{s-})\,\Delta Y_s \right\} \end{cases}$$

Pf: Timo's notes Cor A.11

If f is FV.

$$[f]_T = \sum_{s\in(0,T]} \big(\underbrace{f(s) - f(s-)}_{\Delta f}\big)^2$$

$$\Rightarrow \frac{1}{2}\int_0^t f''(Y_{s-})\,d[Y]_s = \sum_{s\in(0,t]} \frac{1}{2} f''(Y_{s-})(\Delta Y_s)^2$$

(3) For Poisson N_t

$$f(N_t) = f(0) + \int_0^t \big(f(N_s) - f(N_{s-})\big)\,dN_s$$

$$\begin{cases} \int f'(N_{s-})\,dN_s \\ - \sum_s f'(N_{s-})\,\Delta N_s \end{cases}$$

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$\tau_1 \le \tau_2 \le \cdots$

$$f(0) + \cancel{f(N_{\tau_1})} - f(0)$$

$$+ f(N_{\tau_2}) - \cancel{f(N_{\tau_1})}$$

$$+ \cdots$$