Strong Law of Large Numbers

$X_1, X_2, \ldots$ iid $\quad E|X_1| < \infty$

$S_n = X_1 + \ldots + X_n$

$$\frac{S_n}{n} \xrightarrow{a.s.} E[X_1]$$

In the proof we only used pairwise independence

If $X_1, X_2, \ldots$ iid and $E|X_1| = \infty$ then $P(\lim \frac{S_n}{n}$ exists, finite$) = 0$. $\Big] \; E$

Thm: If $X_1, X_2, \ldots$ iid $E[X_1^+] = \infty$, $E[X_1^-] < \infty$.

Then $\frac{S_n}{n} \xrightarrow{a.s.} \infty$.

Proof: The average of the negative parts converges to $E[X_1^-]$.

It's enough to show that $\frac{\sum_{i=1}^{n} X_i^+}{n} \xrightarrow{a.s.} \infty$

We can assume $X_n \geq 0$ with $E[X_1] = \infty$

$$Y_{n,c} = X_n \mathbb{1}(X_n \leq c)$$

$$Y_{n,c} \le X_n$$

$$\frac{\sum_{i=1}^{n} Y_{i,c}}{n} \le \frac{S_n}{n}$$

$$n \to \infty \quad \Big\downarrow \text{a.s.}$$

$$E[X_1 \mathbb{1}(X_1 \le c)]$$

$$\liminf_{n \to \infty} \frac{S_n}{n} \ge E[X_1 \mathbb{1}(X_1 \le c)]$$

With $c \to \infty$ this will $\to \infty$.

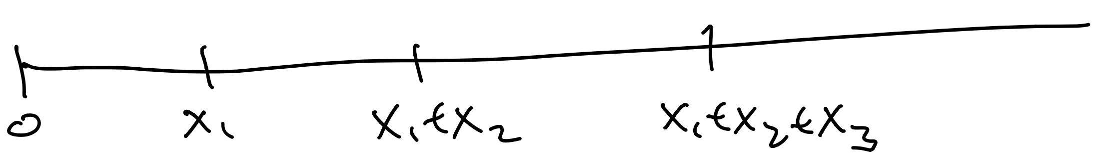
What if $EX_1^+ = \infty$, $EX_1^- = \infty$.

$$\limsup \frac{S_n}{n} = \infty$$

$$\liminf \frac{S_n}{n} = -\infty$$

Applications of SLLN

Ex: $X_1, X_2, \ldots$ iid $\quad X_n \ge 0$



$S_n = X_1 + \cdots + X_n \qquad S_0 = 0$

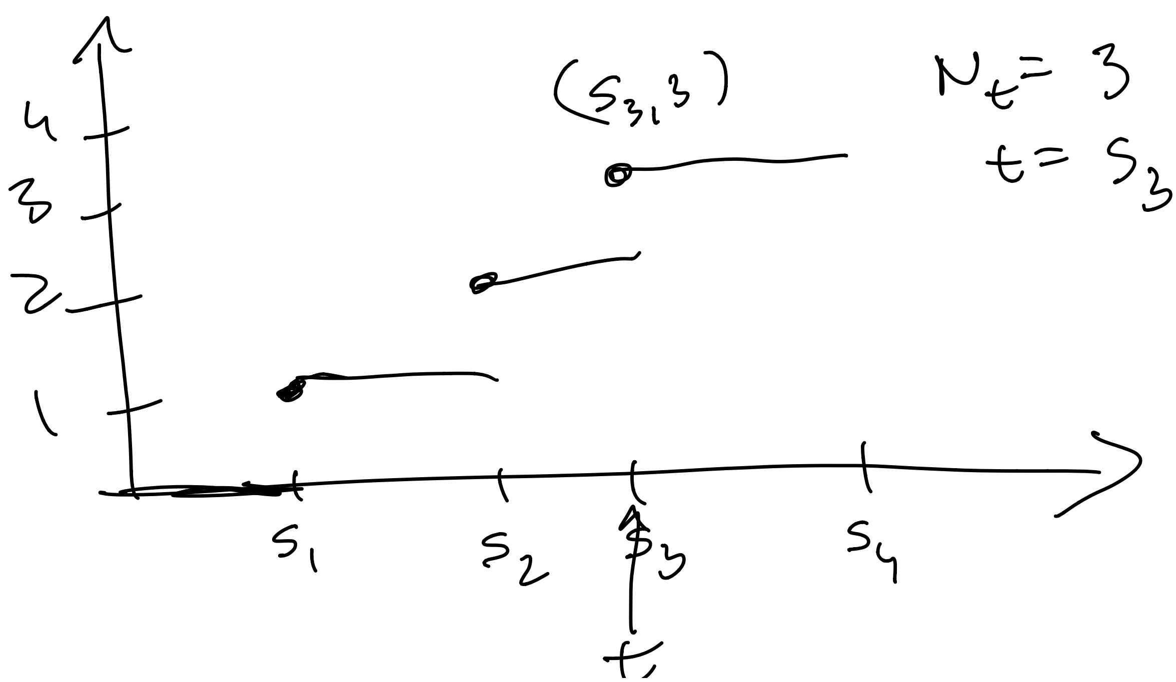
For $t \geq 0$ define

$$N_t = \sup\{n : S_n \leq t\}$$

Assume that $E[X_1] = \mu < \infty$,

then $\dfrac{N_t}{t} \xrightarrow{a.s.} \dfrac{1}{\mu}$

Strong LLN: $\underbrace{\dfrac{S_n}{n} \xrightarrow{a.s.} \mu}$



$$S_{N_t} \le t \le S_{N_t+1}$$

$$\boxed{\frac{S_{N_t}}{N_t}} \le \boxed{\frac{t}{N_t}} \le \frac{S_{N_t+1}}{N_t} = \boxed{\frac{S_{N_t+1}}{N_t+1}} \cdot \boxed{\frac{N_t+1}{N_t}}$$

$$\frac{S_{N_t}}{N_t} \to \mu \qquad \frac{S_{N_t+1}}{N_t+1} \to \mu \qquad \frac{N_t+1}{N_t} \to 1$$

Claim: $N_t \xrightarrow{a.s.} \infty$.

$\frac{S_n}{n} \xrightarrow{a.s.} \mu$ hence $S_n \xrightarrow{a.s.} \infty$

If $t > S_n$ then $N_t \geq n$

thus $N_t \to \infty$ as $t \to \infty$.

$$\to \frac{t}{N_t} \xrightarrow{a.s.} \mu$$

∎

Suppose that $X_1, X_2, \ldots$ iid

Goal: say something about the distribution of X_1 in terms of $(X_1, \ldots, X_n)$.

SLLN: $\frac{\sum_{i=1}^{n} X_i}{n} \xrightarrow{a.s.} E[X_1]$

(if this is finite)

Empirical distribution from $(X_1, \ldots, X_n)$:

$$\underbrace{\mu_n = \sum_{i=1}^{n} \frac{1}{n} \delta_{X_i}}$$

$2, 1, 2, 3, 4 \rightsquigarrow$ $P(\{1\}) = \frac{1}{5}$

$P(\{2\}) = \frac{2}{5}$

$P(\{3\}) = P(\{4\}) = \frac{1}{5}$

Q: what can we say about the sequence $\mu_n, n \geq 1$?

Thm: F_n : CDF of μ_n

Then $\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \xrightarrow{a.s.} 0$

where F is the CDF of X_1.

$$F_n(x) = \mu_n((-\infty, x]) = \frac{1}{n} \# \{1 \le j \le n : X_j \le x\} = \frac{1}{n} \sum_{i=1}^{n} \underbrace{\mathbb{1}(X_i \le x)}$$

Claim: for a fixed $x \in \mathbb{R}$

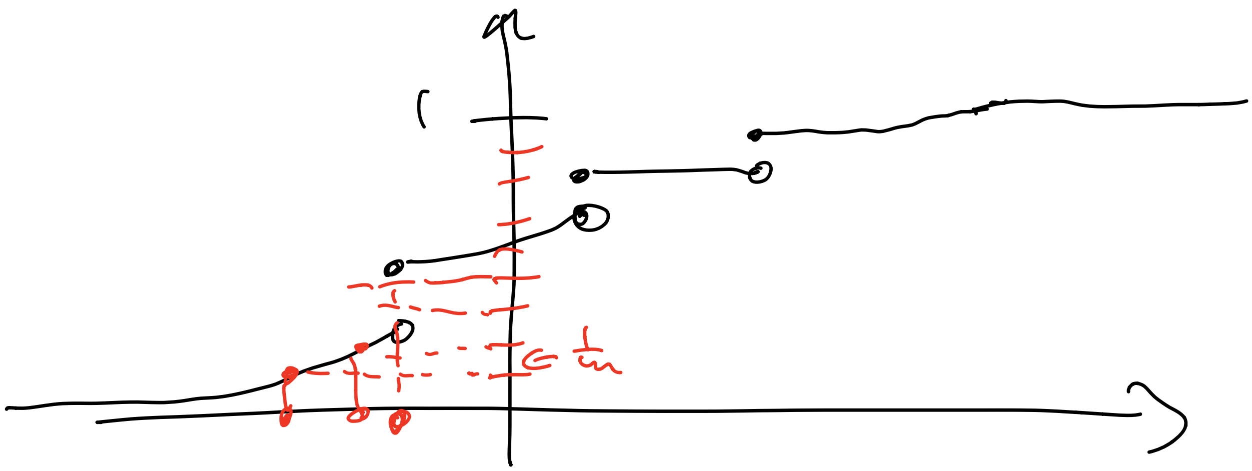
$$F_n(x) \xrightarrow{a.s.} F(x).$$

By the SLLN (applied to $Y_n = \mathbb{1}(X_n \le x)$)

the limit is $E[Y_1] = E[\mathbb{1}(X_1 \le x)]$

$= P(X_1 \le x) = F(x).$

Fix $m \geq 1$, consider the m^{th} quantiles of F.



$$q_{m,i} = \inf\{y : F(y) \geq \tfrac{i}{m}\}$$

$0 \leq i \leq m$

$$F(q_{m,i}-) \leq \frac{i}{m} \leq F(q_{m,i})$$

$$F_n(q_{m,i}) \xrightarrow{a.s.} F(q_{m,i})$$

$$F_n(q_{m,i}-) \xrightarrow{a.s.} F(q_{m,i}-)$$

For any $x \in \mathbb{R}$

$$|F_n(x) - F(x)| \leq \frac{2}{m}$$

if n is large enough

$$|F_n(q_{m,i}) - F(q_{m,i})| \leq \frac{1}{m}$$

$$|F_n(q_{m,i}^-) - F(q_{m,i}^-)| \leq \frac{1}{m}$$

$$\limsup_{n\to\infty} \|F_n - F\|_\infty \leq \frac{2}{m} \quad \text{a.s.}$$

Hence $\lim_{n\to\infty} \|F_n - F\|_\infty = 0$ a.s.

Glivenko–Cantelli thm. ∎

SLLN $\quad X_1, X_2, \ldots$ iid $E|X_1|<\infty$

$$\underbrace{P\left(\frac{S_n}{n} \to E[X_1]\right)}_{\text{this event depends on } X_1, X_2, \ldots\ldots} = 1.$$

Actually: the event does not depend on X_1 !

$$\frac{S_n}{n} = \frac{X_1 + (X_2 + \cdots + X_n)}{n} = \underset{\downarrow \atop 0}{\frac{X_1}{n}} + \frac{X_2 + \cdots + X_n}{n}$$

In ~~fact~~ fact $\left\{\frac{S_n}{n} \to c\right\}$ does not depend on $X_1, \ldots, X_k$ for any fixed k.

Def: $X_1, X_2, \ldots$ random variables

$\mathcal{G}_k = \sigma(X_{k+1}, X_{k+2}, \ldots)$

k^{th} tail σ-field

$\mathcal{G}_1 \supset \mathcal{G}_2 \supset \mathcal{G}_3 \supset \ldots$

$T = \bigcap_{k=1}^{\infty} \mathcal{G}_k$ this is a σ-field

"tail σ-field"

$A \in T$ if A does not depend on $X_1, \ldots, X_k$ for any fixed k.

Ex: $\{\lim \frac{S_n}{n}$ exists and finite$\}$

is in $\mathcal{T}$.

$\{S_n \text{ converges}\} \in \mathcal{T}$

$\{\limsup S_n > 0\} \notin \mathcal{T}$

Thm (Kolmogorov's 0-1 law)

If $\mathcal{F}_1, \mathcal{F}_2, \dots$ are independent, $\mathcal{G}_k = \sigma\left(\bigcup_{\ell \geq k+1} \mathcal{F}_\ell\right)$ then $\bigcap_{k=1}^{\infty} \mathcal{G}_k = \mathcal{G}_\infty$ is a "trivial" σ-field in the sense that $A \in \bigcap_{k=1}^{\infty} \mathcal{G}_k$ implies $P(A) = 0$ or $P(A) = 1$.

If $X_1, X_2, \dots$ independent then $\bigcap_{k=1}^{\infty} \sigma(X_{k+1}, X_{k+2}, \dots)$ is trivial.

Proof: $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k, \mathcal{G}_k$ are independent. Then $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k, \mathcal{G}_\infty$ are independent. $(\mathcal{G}_\infty \subset \mathcal{G}_k)$

This is true for any k:

$\mathcal{G}_\infty$ is independent of $\mathcal{F}_1, \mathcal{F}_2, \ldots$

$\rightarrow$ $\mathcal{G}_\infty$ is independent of $\mathcal{G}_\infty$.

If A is independent of A then

$$P(A) \cdot P(A) = P(A \cap A) = P(A)$$

and $P(A) = 0$ or 1.

Hence $\mathcal{G}_\infty$ is trivial. ∎