Math 733 – Fall 2020

Homework 3.5

This problem set is for practice only! You don't have to submit your solutions.

1. Let $X_1, X_2, X_3, \dots$ be i.i.d. with a a distribution which is exponential with parameter λ. (This means that they have a density function $\lambda e^{-\lambda x}$ on $[0, \infty)$.)

   Show that
   $$P(\limsup_{n\to\infty} \frac{X_n}{\log n} = \frac{1}{\lambda}) = 1.$$

2. Let $X_1, X_2, \dots$ be an i.i.d. sequence of Bernoulli random variables with parameter p. (I.e. $P(X_i = 1) = p = 1 - P(X_i = 0)$.) Consider the random variable $Y = \sum_{i=1}^{\infty} \frac{X_i}{2^i}$, a random number in $[0, 1]$ with binary digits given by the X_i's. Show that if $p \neq 1/2$ then the distribution of Y is *singular* to the Lebesgue measure, meaning that there is a set A of Lebesgue measure 0 so that $P(Y \in A) = 1$.

3. Let $X_1, X_2, \dots$ be i.i.d. with $E|X_1| < \infty$. Show that
   $$\frac{\max(X_1, X_2, \dots, X_n)}{n} \to 0 \qquad \text{a.s.}$$

4. Show that if $X_1, X_2, \dots$ is a given sequence of random variables (we do not assume anything about their distribution!), then there a deterministic sequence $c_n \to \infty$ so that $\frac{X_n}{c_n} \to 0$ with probability one.

5. Find a sequence of independent, non-negative, mean one random variables $X_1, X_2, \dots$ so that
   $$\limsup_{n\to\infty} \frac{X_1 + ... + X_n}{n} = \infty \qquad \text{a.s.}$$

   (Note that we did not assume that these are identically distributed!)