What is probability theory?

Goal: model uncertain events
quantify likelihoods

Simple examples with built in symmetry

1) Flip a fair coin.
What's the probability of tails?
Natural answer: $\frac{1}{2}$

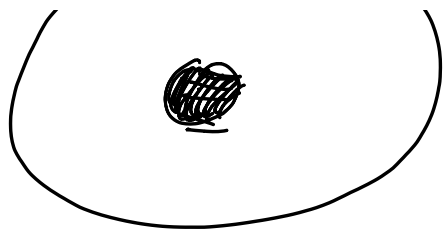
2) We flip a fair coin 10 times.
What's the probability of getting 10 tails? $\frac{1}{2^{10}}$

2^{10} outcomes

each outcome: a sequence of H/T

3) Dartboard with radius 10 in.

bull's eye:



a disk of radius 1 in

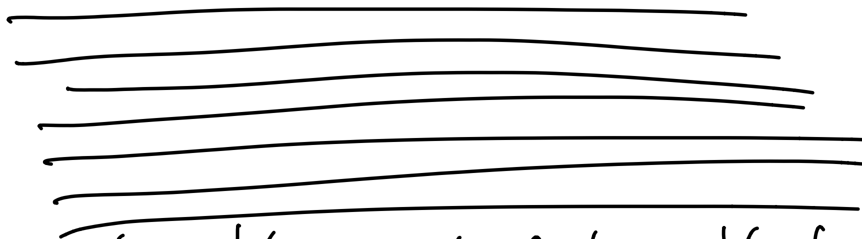
We throw a dart at the board.
What's the prob of hitting the bull's eye?

$$\frac{\text{area of } \bullet}{\text{area of board}} = \frac{1}{100}$$

Even in this simple setup you can come up with non-trivial problems.

1) We flip a fair coin 100 times. What is the probability that the # of heads $\geq$ # of tails at any given time.

2) Buffon's needle problem
needle with length $\ell < 1$.
Infinite grid of parallel lines with distance 1 from each other.



We drop the needle.

What's the probability that the needle will not intersect any of the lines?

Natural questions:

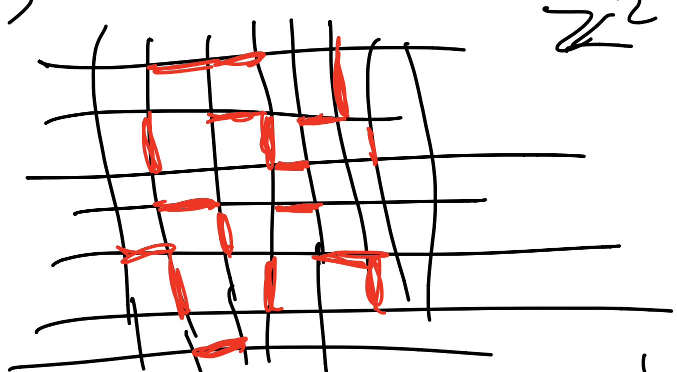
1) What do we do if the outcomes are not equally likely?
2) What if we have infinitely many outcomes?
3) How to treat all these cases together?
4) Often we are interested in a random number. How to describe this? (Random variables)

5, What if we have lots of random variables?
What can we say about the behavior of certain quantities?
Frequency of heads when we flip a coin over and over.
(Law of Large Numbers, Central Limit Theorem)

Couple of problems from modern probability theory

1, (Percolation)

$\mathbb{Z}^2$



For each edge we flip a coin, and if it is heads then color it red.

What is the probability that from $(0,0)$ we can reach ∞ using only red edges?

2) Fill a large $n \times n$ matrix using coin flips (-1: heads, $+1$: tails)
What's the probability that the matrix is singular?
What can we say about the eigenvalues as n gets large?

Quick problem for the break:
(Birthday problem)
party with n guests
Let p_n denote the probability of having a date with multiple birthdays. $p_1 = 0$ $p_{366} = 1$
(disregard leap years)
Find the smallest n for which $p_n > \frac{1}{2}$.

Rigorous framework

Kolmogorov's axioms (1933)

Probability space $(\Omega, \mathcal{F}, \mathbb{P})$

- Ω: state space
  sample space
  (set of outcomes)

- $\mathcal{F}$: set of events
  collection of subsets of Ω
  σ-field

- $\mathbb{P}$: probability measure on $\mathcal{F}$

Def of a σ-field (or σ-algebra)
(non-empty collection of sets)

1) if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$

2) if $A_i \in \mathcal{F}$, $i \in \mathbb{N}$ then
$$\bigcup_i A_i \in \mathcal{F}.$$

$\emptyset \in \mathcal{F}$, $\Omega \in \mathcal{F}$

If $\mathcal{F}_i$, $i \in I$ are σ-fields on Ω then $\bigcap \mathcal{F}_i$ is also a σ-field.
(one of the problems on HW1)

Suppose that $\mathcal{A}$ is a non-empty set of subsets of Ω. Then there is a smallest σ-field that contains $\mathcal{A}$ as a subset.

$\sigma(\mathcal{A})$: σ-field generated by $\mathcal{A}$.

Proof: Collect all σ-fields $\mathcal{F}_i$ on Ω with $\boxed{\mathcal{A} \subset \mathcal{F}_i}$.

Then $\sigma(\mathcal{A}) = \bigcap_i \mathcal{F}_i$

2^{Ω} is a σ-field so the intersection is non-empty ∎

Probability measure:

1. $\mathbb{P} : \check{\mathcal{F}} \to [0, 1]$

$\mathbb{P}(\emptyset) = 0$

2. $A_i \in \mathcal{F}$ $i \in \mathbb{N}$ disjoint

$$\mathbb{P}\left(\bigcup_i A_i\right) = \sum_i \mathbb{P}(A_i)$$

$\mathbb{P}$ is σ- additive

3. $\mathbb{P}(\Omega) = 1$

Events are elements of $\mathcal{F}$.
These are the sets that we can measure with $\mathbb{P}$.

$A_1, A_2, \ldots, A_n$ are disjoint events

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i)$$

($\emptyset$ is disjoint from any other set)

If Ω is finite or countably infinite then usually we can take $\mathcal{F} = 2^{\Omega}$.

Examples:

1) Flipping a fair coin

$\Omega = \{H, T\}$

$\mathcal{F} = \{\emptyset, \Omega, \{H\}, \{T\}\}$

$P(\{H\}) = \frac{1}{2}$ $P(\{T\}) = \frac{1}{2}$

$P(\emptyset) = 0$ $P(\Omega) = 1$

2) Rolling a fair die

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$\mathcal{F} = 2^{\Omega}$

$|\mathcal{F}| = 2^6$

$P(\{i\}) = \frac{1}{6}$ $1 \le i \le 6$

$A \in \mathcal{F}$ $P(A) = \frac{\# A}{6}$

3) Random experiment with n equally likely outcomes

$$|\Omega| = n$$

$$\Omega = \{\omega_1, \dots, \omega_n\}$$

$$\mathcal{F} = 2^{\Omega}$$

$$P(\{\omega_i\}) = \frac{1}{n} \qquad 1 \le i \le n$$

$$P(A) = \frac{|A|}{|\Omega|} \qquad A \subset \Omega$$

4) More generally:
discrete probability space
Ω is finite or countably infinite

$$\Omega = \{\omega_1, \omega_2, \dots\}$$

$$P(\{\omega_i\}) = p_i$$

$$P(A) = \sum_{\omega_i \in A} P(\{\omega_i\})$$

$$\mathcal{F} = 2^{\Omega}$$