ØKSENDAL $\boxed{4.1}$, $\boxed{4.2}$

4.1 a $\quad Y_t = B_t^2$

$$dY_t = 2B_t\,dB_t + 2\,dt\,.$$

b $\quad Y_t = 2 + t + e^{B_t}\,.$

$$dY_t = e^{B_t}\,dB_t + \left(\tfrac{1}{2}e^{B_t} + 1\right)$$

c $$dY_t = 2\Big[B_t^{(1)}dB_t^{(1)} + B_t^{(2)}dB_t^{(2)} +$$

d $$d\underline{Y}_t = \begin{bmatrix} t\,dt \\ dB_t \end{bmatrix}.$$

e $$d\underline{Y}_t = \begin{bmatrix} dB_t^{(1)} + dB_t^{(2)} + dB_t^{(3)} \\ 2B_t^{(2)}dB_t^{(2)} + 2dt - B_t^{(1)}dB_t^{(3)} - B_t^{(3)}dB_t^{(1)} \end{bmatrix}.$$

4.2 $$d(B_t^3) = 3B_t^2\,dB_t + 3B_t\,dt\,.$$

$$\Longrightarrow \quad B_t^3 = \int_0^t 3B_t^2\,dB_t + \int_0^t 3B_t\,dt \quad \Longrightarrow \text{Rearrange.}$$

ØKSENDAL $\boxed{4.3}$, $\boxed{4.4}$

4.3

For $f:\mathbb{R}^2 \to \mathbb{R}$, by THEOREM 4.2.1,

$$df(X_t, Y_t) = \frac{\partial f}{\partial x} dX_t + \frac{\partial f}{\partial y} dY_t + \frac{1}{2}\frac{\partial^2 f}{\partial y^2} d[Y]_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} d[X]_t + \frac{\partial^2 f}{\partial x \partial y} d[X,Y]_t .$$

When $f(x,y) = xy$.

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x, \quad \frac{\partial f}{\partial x \partial y} = 1, \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 0 .$$

Thus

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t \cdot dY_t .$$

4.4 a

$$dZ_t = Z_t\left[\theta(t,\omega)dB_t - \tfrac{1}{2}\theta^2(t,\omega)dt + \underbrace{\tfrac{1}{2}(\theta(t,\omega)dB_t)^2}_{*}\right].$$

$$= Z_t\, \theta(t,\omega)\, dB_t$$

* If you are uneasy about this bit, write $X_t = \int_0^t \theta(s,\omega)dB_s$. Then $dX_t = \theta dB_t + \frac{1}{2}\theta^2 dB_t^2$.

b Clearly holds by Proof of theorem 3.2.5, which states

$$\int_0^t \Lambda_s dB_s$$

is always a martingale.

Discussion about the remark.

In order for Z_t to be a martingale, we need to show $Z_t \theta_t^{(k)} \in \mathcal{V}[0,T]$ for each dimension $k = 1, \dots, n$.

Recall, $$\boxed{\mathcal{V}(S,T) = \left\{ \begin{array}{lll} \mathrm{I} & (t,\omega) \to \phi(t,\omega) & \mathcal{B} \times \mathcal{F} \text{ measurable} \\ \mathrm{II} & \phi(t,\omega) & \mathcal{F}_t\text{-adapted} \\ \mathrm{III} & \mathbb{E}\left[\int_S^T \phi(t,\omega)^2 dt\right] < \infty. & \end{array} \right\}.}$$

In the context of $\boxed{4.4}$, the function $\phi^{(k)}(t,\omega) = Z(t,\omega)\,\theta(t,\omega)$ clearly satisfies I and II, since :

By the claim we proved in $\boxed{3.7}$,

$$\theta_t \in \mathcal{V}(0,T) \Rightarrow \int_0^t \theta_s dB_s \in \mathcal{V}(0,T).$$

Thus

$$\phi^{(k)}(t,\omega) = \exp\Big(\underbrace{\int_0^t \theta_s dB_s}_{\substack{\in \mathcal{V}(0,T) \\ \text{By comment} \\ \text{immediately} \\ \text{above}}} - \underbrace{\frac{1}{2}\int_0^t \theta^2 ds}_{\in \mathcal{V}(0,T)} \Big) \underbrace{\theta_t}_{\in \mathcal{V}(0,T)}.$$

So Φ_t is a Borel function of things in $\mathcal{V}(0,T)$ at time t.
So I and II follow by properties of measurable functions.
Left to verify is whether III holds. The remark states that.

$$\text{NOVIKOV} \Rightarrow \text{KAZAMAKI} \Rightarrow \mathrm{III}.$$

(P.T.O)

4.5

$$d(B_t^k) = k B_t^{k-1} dB_t + \tfrac{1}{2}k(k-1) B_t^{k-2} (dB_t)^2$$

$$B_t^k = \underbrace{k\int_0^t B_s^{k-1} dB_s}_{\text{martingale starting at } 0.} + \tfrac{1}{2}k(k-1)\int_0^t B_s^{k-2} ds.$$

Hence,

$$\mathbb{E}[B_t^k] = \tfrac{1}{2}k(k-1)\int_0^t \mathbb{E}[B_s^{k-1}] ds. \qquad (*).$$

a

$$\mathbb{E}[B_t^4] = \tfrac{1}{2}4\cdot 3\int_0^t \mathbb{E}[B_s^2] ds.$$

$$= 6\int_0^t s\, ds = 3t^2.$$

b.

$\mathbb{E}[B(t)^{2k+1}] = 0$ since $\mathbb{E}[B_t^1] = 0$,

and by $(*)$, this holds for all odd numbers.

To prove $\mathbb{E}[B(t)^{2k}] = \frac{(2k)!\, t^k}{2^k k!}$ $k = 1, 2, \dots$

we proceed by induction. Note it is true when $k = 1$,

and

$$\mathbb{E}[B(t)^{2(k+1)}] = \tfrac{1}{2}(2k+2)(2k+1)\int_0^t \frac{(2k)!}{k!}\frac{s^k}{2^k} ds.$$

$$= \tfrac{1}{2}\frac{(2k+2)!}{k!}\frac{1}{2^k}\cdot\int_0^t s^k ds.$$

$$= \frac{(2(k+1))!}{(k+1)!}\frac{1}{2^{k+1}}\cdot t^{k+1}.$$

4.6 a $\quad X_t = e^{ct + \alpha B_t}$.

$$dX_t = \left[c\, dt + \alpha\, dB_t + \tfrac{1}{2}\alpha^2\, dt \right] X_t .$$ Rearrange

b $\quad X_t = \exp\left(ct + \sum_{j=1}^{n} \alpha_j B_j(t) \right)$.

$$dX_t = \left[c\, dt + \sum \left(\alpha_j dB_t + \tfrac{1}{2}\alpha_j^2\, dt \right) \right] X_t$$ Rearrange.

Note that if $c = -\frac{1}{2}\sum \alpha_j^2$, then X_t is a martingale.

4.7 **a** $V \equiv 1$. Then $X_t = B_t$, and X_t^2 is not a martingale.

b Prove if v bounded

Then $\Longrightarrow$ $$M_t = X_t^2 - \int_0^t |v_s|^2\,ds$$

is a martingale.

I M_t clearly adapted.

II $$\mathbb{E}[|M_t|] = \mathbb{E}\left[|X_t^2 - \int_0^t |v_s|^2\,ds\right]$$

$$\leq \mathbb{E}\left[\int_0^t |v_s|\,dB_s\right] + \mathbb{E}\left[\int_0^t |v_s|^2\,ds\right].$$

$$\leq M\,\mathbb{E}[|B_t|] + tM^2.$$

where $|v| \leq M$.

$$\leq M\sqrt{\mathbb{E}[B_t^2]} + tM^2.$$

$$\leq M\sqrt{t} + tM^2 < \infty.$$

III Let $s < t$.

$$\mathbb{E}[M_t \mid \mathcal{F}_s] = \mathbb{E}\left[X_t^2 - \int_0^t |v_s|^2\,ds \,\middle\|\, \mathcal{F}_s\right]$$

$$= \mathbb{E}\left[X_t^2 - X_s^2 - \int_s^t |v_s|^2\,ds \,\middle|\, \mathcal{F}_s\right] + X_s^2$$

$$= \mathbb{E}\Big[\underbrace{2X_s(X_t - X_s)}_{=0} + \underbrace{(X_t - X_s)^2}_{=\mathbb{E}\left[\int_s^t |v_s|^2 ds \,\middle|\, \mathcal{F}_s\right]} - \int_s^t |v_s|^2\,ds \,\Big\|\, \mathcal{F}_s\Big] + X_s^2.$$

since X martingale. by ITO ISOMETRY.

$$= X_s^2.$$

4.8a (In THEOREM 4.2.1)

By the Multidimensional Itô Formula, if

$$g:[0,\infty)\times\mathbb{R}^n \to \mathbb{R}^p.$$

Then $\forall\ k=1\cdots p$

$$d(g(t,\underline{X})) = dY_t^{(k)} = \frac{\partial g^{(k)}}{\partial t}dt + \sum_{i=1}^{n}\frac{\partial g^{(k)}}{\partial x_i}dX_t^{(i)} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 g^{(k)}}{\partial x_i\partial x_j}dX_t^{(i)}dX_t^{(j)}$$

With $p=1$,

$$df(B_t) = \sum_{i=1}^{n}\frac{\partial f^{(k)}}{\partial x^{(i)}}dB_t^{(i)} + \frac{1}{2}\sum_i\sum_j \underbrace{\frac{\partial^2 f^{(k)}}{\partial x^{(i)}\partial x^{(j)}}dB_t^{(i)}dB_t^{(j)}}_{=\mathbb{1}\{i=j\}\,dt\,\frac{\partial^2 f}{(\partial x^{i})^2}}$$

$$= \underbrace{(\nabla f)}_{\nabla f=\left(\frac{\partial f}{\partial x^{(1)}},\ldots,\frac{\partial f}{\partial x^{(n)}}\right)}\cdot\,\underline{dB_t} + \frac{1}{2}\underbrace{\Delta f}_{\Delta f=\sum_{i=1}^{n}\frac{\partial^2 f}{(\partial x^{(i)})^2}}\,dt$$

Result follows.

4.8b Let $g:\mathbb{R}\to\mathbb{R}\in\mathcal{C}^1(\mathbb{R})\cap\mathcal{C}^2(\mathbb{R}-\{z_1\cdots z_N\})$.

Now, $\exists\ f_k \in\mathcal{C}^2(\mathbb{R})$:
- $f_k \rightrightarrows g$
- $f_k' \rightrightarrows g'$
- $|f_k''|\le M$.
- $f_k''\to g''$ on $\mathbb{R}-\{z_1\cdots z_N\}$.

We need to establish ①, ②, ③, ④.

$$f_k(B_t) = f_k(B_0) + \int_0^t f_k'(B_s)\,ds + \frac{1}{2}\int_0^t f_k''(B_s)\,ds$$

$$\Downarrow ① \qquad \Downarrow ② \qquad \Downarrow ③ \qquad \Downarrow ④$$

$$g(B_t) = g(B_0) + \int_0^t g'(B_s)\,ds + \frac{1}{2}\int_0^t g''(B_s)\,ds$$

Well, ① and ② follow by pointwise convergence of $f_k \to g$.

③ Now, since $f_k' \rightrightarrows g$,

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} : \forall x \in \mathbb{R}, \ \forall k \geq N.$$

$$|g'(x) - f_k'(x)| \leq \epsilon.$$

Thus $$\left| \int_0^t f_k'(B_s)\,ds - \int_0^t g'(B_s)\,ds \right| \leq \int_0^t |(f_k' - g')(B_s)|\,ds \leq \epsilon t. \to 0.$$

④. This follows from bounded convergence theorem on the measure space $(\text{Leb}, \mathcal{B}[0,t], [0,t])$

4.9 Let $T_n = \inf\{s > 0 : |X_s(\omega)| \geq n\}$. Prove

$$\int_0^t v \cdot \frac{\partial g_n}{\partial x}(s, X_s)\, \mathbb{I}\{s \leq T_n\}\, dB_s = \int_0^{t \wedge T_n} v \frac{\partial g_n}{\partial x}(s, X_s)\, dB_s$$

$$= \int_0^{t \wedge T_n} v \cdot \frac{\partial g}{\partial x}(s, X_s)\, dB_s .$$

Well, this follows since $\forall\, s \leq t \wedge T_n$, the integrands are identical, and when / if $T_n < s \leq t$, the first integrand is 0.

Then

$$g(t \wedge T_n, X_{t \wedge T_n}) = g(0, X_0) + \int_0^{t \wedge T_n} \left(\frac{\partial g}{\partial s} + u \frac{\partial g}{\partial x} + \frac{1}{2} v^2 \frac{\partial^2 g}{\partial x^2} \right) ds + \int_0^{t \wedge T_n} v \frac{\partial g}{\partial x}\, dB_s .$$

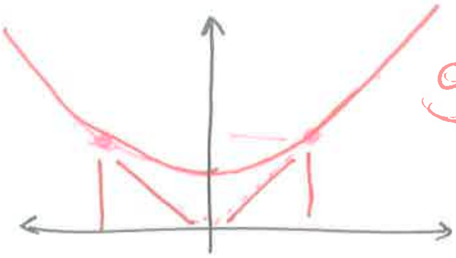
Finally, $\forall\, t$ fixed, X_t is a finite random variable.

Thus $1 = \mathbb{P}(X_t \in \mathbb{R}) = \mathbb{P}\left(\limsup_{n \to \infty} \{|X_t| \leq n\} \right)$

$$\Rightarrow \quad \mathbb{P}(|X_t| \leq n) \uparrow 1 \quad \text{as} \quad n \to \infty .$$

Thus taking $n \to \infty$ holds a.s on both sides.

4.10 TANAKA'S FORMULA & LOCAL TIME



$$g_\epsilon(x) = \begin{cases} |x| & |x| \geq \epsilon \\ \frac{1}{2}\left(\epsilon^2 + \frac{x^2}{\epsilon}\right) & |x| < \epsilon \end{cases}$$

a By 4.8 b), we note that g_ϵ is $\mathcal{C}^2(\mathbb{R} - \{-1, 1\})$.

$$g_\epsilon''(x) = \frac{1}{\epsilon} \mathbb{1}\{|x| < \epsilon\}.$$

Thus,

$$g_\epsilon(B_t) = g_\epsilon(B_0) + \int_0^t g_\epsilon'(B_s)\, dB_s + \frac{1}{2}\int_0^t g''(B_s)\, ds.$$

$$= g_\epsilon(B_0) + \int_0^t g_\epsilon'(B_s)\, dB_s + \frac{1}{2\epsilon}\int_0^t \mathbb{1}\{B_s \in (-\epsilon, \epsilon)\}\, ds.$$

$$= g_\epsilon(B_0) + \int_0^t g_\epsilon'(B_s)\, dB_s + \frac{1}{2\epsilon} \mathrm{Leb}\{s : B_s \in (-\epsilon, \epsilon)\}.$$

b $$\mathbb{E}\left[\left(\int_0^t \frac{B_s}{\epsilon} \mathbb{1}\{B_s \in (-\epsilon,\epsilon)\}\, dB_s\right)^2\right]$$

$$= \mathbb{E}\left[\int_0^t \frac{B_s^2}{\epsilon^2} \mathbb{1}\{B_s \in (-\epsilon, \epsilon)\}\, ds\right]$$

$$\leq \mathbb{E}\left[\int_0^t \mathbb{1}\{B_s \in (-\epsilon, \epsilon)\}\, ds\right]$$

$$\left[\text{Now } \mathbb{P}(|B_s| < \epsilon) = \int_{-\epsilon}^{\epsilon} \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}x^2}\, dx \leq \frac{2\epsilon}{\sqrt{2\pi t}}.\right]$$

$$\leq \int_0^t \frac{2\epsilon}{\sqrt{2\pi s}}\, ds = \sqrt{\frac{2}{\pi}}\,\epsilon \int_0^t s^{-1/2}\, ds = \frac{1}{\sqrt{2\pi}}\left[s^{1/2}\right]_0^t \epsilon = \frac{t^{1/2}}{\sqrt{2\pi}}\,\epsilon \to 0$$

c Thus as $\epsilon \to 0$,

$$\lim_{\epsilon\downarrow 0} g_\epsilon(B_t) = \lim_{\epsilon\downarrow 0}\left[g_\epsilon(B_0) + \int_0^t g_\epsilon'(B_s)\,dB_s + \frac{1}{2\epsilon}\,\mathrm{Leb}\{s : |B_s| < \epsilon\}\right]$$

$$= |B_0| + \int_0^t g_\epsilon'(B_s)\,\mathbb{1}\{|B_s| \geq \epsilon\}\,dB_s + \underbrace{\frac{1}{2}\int_0^t g_\epsilon'(B_s)\,\mathbb{1}\{|B_s| < \epsilon\}}_{\to 0 \text{ in } L^2} + L_t$$

$$= |B_0| + \int_0^t \mathrm{sign}(B_s)\,\mathbb{1}\{|B_s| \geq \epsilon\}\,dB_s + L_t .$$

Now note

$$\mathbb{E}\left[\left(\int_0^t \mathrm{sign}(B_s)\,\mathbb{1}\{|B_s| < \epsilon\}\,dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t \mathbb{1}\{|B_s| < \epsilon\}\,ds\right] \longrightarrow 0$$

by argument in part b.

Thus $$= |B_0| + \int_0^t \mathrm{sign}(B_s)\,dB_s + L_t .$$

4.11

a

$$dX_t = \frac{1}{2}e^{\frac{1}{2}t}\cos(B_t)\,dt - e^{\frac{1}{2}t}\sin(B_t)dB_t - \frac{1}{2}e^{\frac{1}{2}t}\cos(B_t)dt$$

$$= -e^{1/2 t}\sin(B_t)\,dB_t .$$

Thus martingale.

b

$$dX_t = e^{1/2 t}\cos(B_t)\,dB_t .$$ Same.

c

$$f(x,t) = (x+t)\exp(-x - \tfrac{1}{2}t)$$

$$\frac{\partial f}{\partial t} = \exp(-x-\tfrac{1}{2}t) - \tfrac{1}{2}(x+t)\exp(-x-\tfrac{1}{2}t)$$

$$= [1-\tfrac{1}{2}x - \tfrac{1}{2}t]\exp(-x-\tfrac{1}{2}t)$$

$$\frac{\partial f}{\partial x} = [1 - x - t]\exp(-x-\tfrac{1}{2}t)$$

$$\frac{\partial^2 f}{\partial x^2} = [-1 - (1-x-t)]\exp(-x-\tfrac{1}{2}t)$$

$$dX_t = (1-B_t-t)\exp(-B_t-\tfrac{1}{2}t)\,dB_t$$

$$+ \Big[\underbrace{\big(1-\tfrac{1}{2}B_t - \tfrac{1}{2}t\big) + \tfrac{1}{2}(-2+x+t)}_{=0}\Big]dt .$$

$$= [1-B_t-t]e^{-B_t-\frac{1}{2}t}\,dB_t$$

4.12

$$0 = \mathbb{E}\left[X_s - X_t \mid \mathcal{F}_t^{(n)}\right]$$
$$= \mathbb{E}\left[\int_t^s u(r,\omega)\,dr \mid \mathcal{F}_t^{(n)}\right]$$

Then

$$0 = \frac{d}{ds}\mathbb{E}\left[\int_t^s u(r,\omega)\,dr \mid \mathcal{F}_t^{(n)}\right]$$
$$= \mathbb{E}\left[\frac{d}{ds}\int_t^s u(r,\omega)\,dr \mid \mathcal{F}_t^{(n)}\right] \quad \forall \text{ a.a } s > t.$$
$$= \mathbb{E}\left[u(s,\omega) \mid \mathcal{F}_t^{(n)}\right]$$

Now, as $t \uparrow s$, by COROLLARY C.9,

$$0 = \mathbb{E}\left[u(s,\omega) \mid \mathcal{F}_t^{(n)}\right] \to \mathbb{E}\left[u(s,\omega) \mid \mathcal{F}_s^{(n)}\right] = u(s,\omega).$$

Thus $u(s,\omega)$ $\forall$ a.a (s,ω).

4.13

$$dX_t = u\,dt + dB_t.$$

Show $Y_t = X_t M_t$ is a martingale, where

$$M_t = \exp\left(-\int_0^t u\,dB_r - \frac{1}{2}\int_0^t u^2\,dr\right).$$

(is also a martingale).

$$dY_t = M_t\,dX_t + X_t\,dM_t + dX_t\,dM_t.$$

$$\left(\begin{aligned} \text{Now } dM_t &= M_t\left(-u\,dB_t - \tfrac{1}{2}u^2dt + \tfrac{1}{2}u^2(dB_t)^2\right) \\ &= -uM_t\,dB_t. \end{aligned}\right)$$

$$= M_t\left[u\,dt + dB_t + X_t(-u\,dB_t) + (-u\,dB_t)(u\,dt + dB_t)\right]$$

$$= M_t\left[u\,dt + dB_t - uX_t\,dB_t - u^2dB_t\,dt - u(dB_t)^2\right]$$

$$= M_t\left[1 - uX_t\right]dB_t.$$

4.13 (cont.)

Remarks.

a With $X_t = 1\,dt + 1\,dB_t$ (in $\boxed{4.11\ c}$),

Clearly by $\boxed{4.13}$

$$X_t \exp\left(-\int_0^t 1\,dB_s - \frac{1}{2}\int_0^t 1^2\,ds\right)$$

is a martingale.

4.14

a

$$B_T(\omega) = \int_0^T 1 \, dB_t.$$

b

$$d(tB_t) = B_t\,dt + t\,dB_t.$$

$$\Rightarrow \quad TB_T = \int_0^T B_t\,dt + \int_0^T t\,dB_t.$$

$$\int_0^T B_t\,dt = \int_0^T (T-t)\,dB_t.$$

c

$$d(B_t^2) = 2B_t\,dB_t + dt.$$

$$B_T^2(\omega) = T + \int_0^T 2B_t\,dB_t.$$

d

$$d(B_t^3) = 3B_t^2\,dB_t + 3B_t\,dt.$$

$$= 3B_t^2\,dB_t + 3d(tB_t) - 3t\,dB_t.$$

$$B_T^3(\omega) = \int_0^T 3B_t^2 + 3T - 3t \; dB_t.$$

e

$$d(e^{B_t}) = e^{B_t}\,dB_t + \underbrace{\tfrac{1}{2}e^{B_t}\,dt}.$$

$$d(te^{B_t}) = t\,d(e^{B_t}) + \underbrace{e^{B_t}\,dt}$$

So

$$d(e^{B_t}) = e^{B_t}\,dB_t + \tfrac{1}{2}\left[d(te^{B_t}) - t\,d(e^{B_t})\right]$$

$$(1+\tfrac{1}{2}t)\,d(e^{B_t}) = e^{B_t}\,dB_t + \tfrac{1}{2}d(te^{B_t}).$$

$\underline{\underline{e}}$ This one is a bit trickier to spot:

Note $d(e^{B_t - \frac{1}{2}t})$

$$= e^{B_t - \frac{1}{2}t} dB_t + \frac{1}{2} e^{B_t - \frac{1}{2}t} dB_t^2 - \frac{1}{2} e^{B_t - \frac{1}{2}t} dt.$$

$$= e^{B_t - \frac{1}{2}t} dB_t.$$

Thus $$e^{B_T - \frac{1}{2}T} - 1 = \int_0^T e^{B_t - \frac{1}{2}t} dB_t.$$

$$\Rightarrow \quad \underline{e^{B_T} = e^{\frac{1}{2}T} + \int_0^T e^{\frac{1}{2}T}(e^{B_t - \frac{1}{2}t}) dB_t.}$$

$\underline{\underline{f}}$ Again, like in $\underline{\underline{e}}$, let's resort to finding a martingale:

$$d(e^{\frac{1}{2}t} \sin B_t) = e^{\frac{1}{2}t} \cos(B_t) dB_t - \tfrac{1}{2} e^{\frac{1}{2}t} \sin dB_t^2 + \tfrac{1}{2} e^{\frac{1}{2}t} \sin B_t dt$$

$$= e^{\frac{1}{2}t} \cos(B_t) dB_t.$$

Thus $$e^{\frac{1}{2}T} \sin(B_T) = \int_0^T e^{\frac{1}{2}t} \cos(B_t) dB_t.$$

$$\Rightarrow \quad \sin(B_T) = \int_0^T e^{-\frac{1}{2}T} (e^{\frac{1}{2}t} \cos(B_t)) dB_t$$

4.15

$$X_t = (x^{1/3} + \tfrac{1}{3}B_t)^3$$

$$= f(t, B_t) = f(B_t), \qquad f(u) = (x^{1/3} + \tfrac{1}{3}u)^3$$

$$f'(u) = (x^{1/3} + \tfrac{1}{3}u)^2 = f(u)^{2/3}$$

$$f''(u) = \tfrac{2}{3}(x^{1/3} + \tfrac{1}{3}u) = \tfrac{2}{3}f(u)^{1/3}.$$

ITÔ: $$dX_t = f'(B_t)\,dB_t + \tfrac{1}{2}f''(B_t)\,dB_t^2$$

$$dX_t = X_t^{2/3}\,dB_t + \tfrac{1}{3}X_t^{1/3}\,dt \qquad \square$$

4.16 a

Let $Y \in L^2(\mathcal{F}_T)$. $M_t = \mathbb{E}[Y \mid \mathcal{F}_t]$.

a Show $\mathbb{E}[M_t^2] < \infty$ $\forall\, t \in [0, T]$.

By $\boxed{3.16}$, $\mathbb{E}\big[\, \mathbb{E}[Y \mid \mathcal{F}_t]^2 \,\big] \leqslant \mathbb{E}[Y^2]$

since $\mathcal{F}_t \subseteq \mathcal{F}_T$.

b i Find g:

$$\mathbb{E}[B_T^2 \mid \mathcal{F}_t] = \mathbb{E}[B_T^2] + \int_0^t g(s, \omega)\, dB_s .$$

Now,

$$d(B_t^2 - t) = 2B_t\, dB_t .$$

ie $$B_T^2 - T = \int_0^T 2B_s\, dB_s ,$$ so

$$\mathbb{E}[B_T^2 \mid \mathcal{F}_t] = T + \int_0^t 2B_s\, dB_s .$$

Checking:

$$\begin{aligned}
\mathbb{E}[B_T^2 \mid \mathcal{F}_t] &= \mathbb{E}[((B_T - B_t) + B_t)^2 \mid \mathcal{F}_t] \\
&= (T - t) + B_t^2 \\
&= T + (B_t^2 - t) \\
&= T + \int_0^t 2B_s\, dB_s
\end{aligned}$$

4.16 b ii

We've seen before that

$$\mathbb{E}[B_T^3 \mid \mathcal{F}_t] = B_t^3 + 3B_t(T-t)$$

And in $\boxed{4.14}$ d,

$$B_t^3 = \int_0^t 3B_s^2 + 3t - 3s \, dB_s .$$

Also,

$$3B_t(T-t) = \int_0^t 3(T-t) \, dB_s .$$

These integrands shouldn't depend on t in the final answer...

Fortunately, the t-dependence cancels:

$$B_t^3 + 3B_t(T-t) = \int_0^t 3B_s^2 + 3(T-s) \, dB_s .$$

So

$$\mathbb{E}[B_T^3 \mid \mathcal{F}_t] = \underbrace{0}_{\mathbb{E}[B_T^3]=0} + \int_0^t 3B_s^2 + 3(T-s) \, dB_s$$

4.16 b iii

$$\mathbb{E}[Y \mid \mathcal{F}_t] = \mathbb{E}\left[e^{\sigma B_T} \mid \mathcal{F}_t\right]$$
$$= e^{\frac{1}{2}\sigma^2 T}\,\mathbb{E}\left[e^{\sigma B_T - \frac{1}{2}\sigma^2 T} \mid \mathcal{F}_t\right]$$
$$= e^{\frac{1}{2}\sigma^2 T}\, e^{\sigma B_t - \frac{1}{2}\sigma^2 t}.$$

With $X_t = e^{\sigma B_t - \frac{1}{2}\sigma^2 t}$

$$dX_t = \sigma\left(e^{\sigma B_t - \frac{1}{2}\sigma^2 t}\right) dB_t$$

Thus $X_t = 1 + \int_0^t \sigma e^{\sigma B_s - \frac{1}{2}\sigma^2 s}\, dB_s.$

It follows that

$$\mathbb{E}[Y \mid \mathcal{F}_t] = e^{\frac{1}{2}T} + \int_0^t \sigma e^{\sigma B_s + \frac{1}{2}\sigma^2 (T-s)}\, dB_s$$

4.17

LEMMA 4.3.1

$$\forall \epsilon > 0, \quad \exists n, \exists t_1, \dots, t_n, \exists \phi \in \mathcal{C}_0^\infty(\mathbb{R}^n):$$
$$\| Y - \phi(B_{t_1}, \dots, B_{t_n}) \|_{\mathcal{L}^2} = \mathbb{E}\left[|Y - \phi(B_{t_1}, \dots, B_{t_n})|^2 \right]^{1/2} < \epsilon$$

Assuming this Lemma, we want to show,
$\forall Y \in \mathcal{L}^2(\mathcal{F}_T)$, there exists $f(t, \omega)$ such that
Y has representation
$$Y(\omega) = \mathbb{E}[F] + \int_0^T f(t, \omega)\, dB_t \qquad (4.3.14).$$

Now the question says

"In view of this Lemma it is enough to prove the representation holds for Y of the 'form'
$$Y = \phi(B_{t_1}, \dots, B_{t_n}) \quad \phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)"$$
(★)

Before we answer a, b, c, let us check (★) is true.

ØKSENDAL $\boxed{4.17}$: (2)

Claim

IF

Lemma 4.3.1

1. Every $Y \in \mathcal{L}^2(\mathcal{F}_T)$ can be approximated in $\mathcal{L}^2$ by function of form
$$\phi(B_{t_1}, \ldots, B_{t_n})$$ (+)

What we have to show in $\boxed{4.17}$

2. Every function of form $\Phi(B_{t_1}, \ldots, B_{t_n})$ can be represented
$$\Phi(B_{t_1}, \ldots, B_{t_n}) = \mathbb{E}[\Phi(B_{t_1} \cdots B_{t_n})] + \int_0^T f(t,\omega)\, dB_t \, .$$

Then. Every $Y \in \mathcal{L}^2(\mathcal{F}_T)$ can be represented.
$$Y = \mathbb{E}[Y] + \int_0^T f(t,\omega)\, dB_t \, .$$

Proof.

- Let $Y \in \mathcal{L}^2(\mathcal{F}_T)$. Then $\exists$ Φ_k of the form (+) such that $\Phi_k \to Y$ in $\mathcal{L}^2$.
- Without loss of generality, assume $\mathbb{E}[\Phi_k] = \mathbb{E}[Y]$.
  (otherwise replace $\hat{\Phi}_k = \Phi_k + a$, $a = \mathbb{E}[Y] - \mathbb{E}[\Phi_k] \in \mathbb{R}$).
  Then $\hat{\Phi}_k$ clearly also has the form (+)..
- Since $\Phi_k \to Y$ in $\mathcal{L}^2$, by $\boxed{2.19}$ the Φ_k are cauchy.
- Furthermore, $$\Phi_k = \mathbb{E}[Y] + \int_0^T f_k(t,\omega)\, dB_t \, . \quad \forall k \, .$$
- We now show that $f_k \to f^*$ in $\mathcal{L}^2[0,T]$, and that
$$Y = \mathbb{E}[Y] + \int_0^T f^* \, dB_t$$ KEY IDEA
  - Well, (Φ_k) cauchy $\Rightarrow$ $\forall\, \epsilon > 0$ $\exists$ N : $\forall\, k, j \geq N$
  $$\| \Phi_k - \Phi_j \|_2^2 < \epsilon \, .$$
  - By ITÔ's ISOMETRY
  $$\mathbb{E}[(\Phi_k - \Phi_j)^2] = \mathbb{E}\left[\int_0^T (f_k - f_j)^2 \, dt \right]$$
  - This implies (f_k) cauchy, and $\mathcal{L}^2(0,T)$ is complete, Thus $f_k \to f^* \in \mathcal{L}^2(0,T)$.

... continued.

Left to verify is that this f^* actually does represent Y, ie that

$$Y(\omega) = \mathbb{E}[Y] + \int_0^T f^*(t,\omega)\, dB_t(\omega).$$

Well

$$\left\| Y - \left(\mathbb{E}[Y] + \int_0^T f^*\, dB_t \right) \right\|_{L^2(\mathcal{F}_T)}$$

$$\leq \| Y - \phi_k \|_{L^2(\mathcal{F}_T)} + \left\| \phi_k - \left(\mathbb{E}[Y] + \int_0^T f^*\, dB_t \right) \right\|_{L^2(\mathcal{F}_T)}$$

$$= \| Y - \phi_k \|_{L^2(\mathcal{F}_T)} + \left\| \int_0^T f_k - f^*\, dB_t \right\|_{L^2(\mathcal{F}_T)}.$$

ITO ISOMETRY.

$$= \underbrace{\| Y - \phi_k \|_{L^2(\mathcal{F}_T)}}_{\substack{\longrightarrow 0 \\ \text{as } k \longrightarrow \infty}} + \underbrace{\| f_k - f^* \|_{L^2(0,T)}}_{\substack{\longrightarrow 0 \\ \text{as } k \longrightarrow \infty}}.$$

$\blacksquare$

ØKSENDAL $\boxed{4.17}$: (4)

$\underline{\underline{a}}$ Let $v(t, B(t)) = w(t, B(t_1), \cdots, B(t_{k-1}), B(t))$

By ITO'S FORMULA, for $t \in [t_{k-1}, t_k]$

$$v(t, B(t)) = v(t, B(t_{k-1})) + \int_{t_{k-1}}^{t} \frac{\partial v}{\partial x}(s)\, dB_s + \int_{t_{k-1}}^{t} \left(\frac{\partial v}{\partial t}(s) + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(s) \right) ds .$$

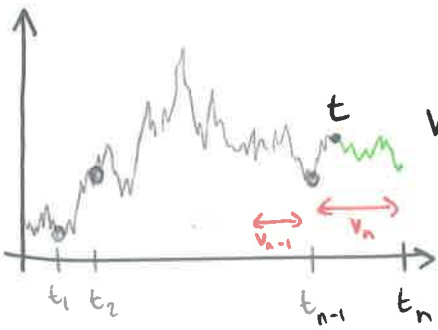
That is,

$$w(t, B(t_1), \cdots, B(t_{k-1}), B(t))$$

$$= W(t, B(t_1) \cdots B(t_k)) + \int_{t_{k-1}}^{t} \frac{\partial w}{\partial x_k}(s)\, dB_s + \int_{t_{k-1}}^{t} \frac{\partial w}{\partial t}(s) + \frac{1}{2}\frac{\partial^2 w}{\partial x_k^2}(s)\, dt .$$

$\underline{\underline{b}}$

(Preamble)



$V_n(t_n, x_1, \cdots, x_n) = \Phi(x_1, \cdots, x_n)$

For $t \in [t_{n-1}, t_n]$.

$$\frac{\partial V_n}{\partial t} + \frac{1}{2}\frac{\partial^2 V_n}{\partial x_n^2} = 0 .$$

So think of $V_n(t, x_1, \cdots, x_n) = \mathbb{E}\left[\Phi(B_{t_1}, \cdots, B_{t_n}) \mid B_{t_1} = x_1, \cdots, B_{t_{n-1}} = x_{n-1} \text{ and } B_t = x_n \text{ currently}\right]$

And think of

$t \in [t_{k-1}, t_k]$, $V_k(t, x_1, \cdots, x_k) = \mathbb{E}\left[V_{k+1}(t_k, x_1, \cdots, x_k, x_k) \mid B_{t_1} = x_1, \cdots, B_{t_k} = x_k\right]$

etc. $= \mathbb{E}\left[\Phi(B_{t_1}, \cdots, B_{t_n}) \mid B_{t_1} = x_1, \cdots, B_{t_k} = x_k\right]$.

So when $t \in [t_{k-1}, t_k]$, $V_k(t)$ is the "best guess" of $\Phi(B_{t_1}, \cdots, B_{t_n})$ given knowledge up to t.

ØKSENDAL $\boxed{4.17}$: (5)

4.17 b

Verify

$$v_k(t, x_1, \dots, x_k) = \frac{1}{\sqrt{2\pi(t_k - t)}} \int_{\mathbb{R}} v_{k+1}(t_k, x_1, \dots, x_k, y) \exp\left[-\frac{(x_k-y)^2}{2(t_k-t)}\right] dy.$$

is the solution to

$$\left[\begin{array}{ll} \text{I.} & v_k(t_k, x_1, \dots, x_k) = v_{k+1}(t_k, x_1, \dots, x_k, x_k) \\ \text{II.} & \dfrac{\partial v_k}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 v_k}{\partial x_k^2} \end{array}\right].$$

I.

Well, as $t \to t_k$, the RHS refers to integration over dirac at x_k: $\delta_{x_k}(y)$, hence I. is satisfied.

II.

Note $p(t, x, y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-y)^2}{2t}\right)$ satisfies the Fokker-Planck equation :

$$\left[p_t = \tfrac{1}{2} p_{xx} = \tfrac{1}{2} p_{yy} \right]$$

(This is easily verified)

Hence $q(t, x, y) = p(t_k - t, x, y)$ satisfies

$$q_t = -\tfrac{1}{2} q_{xx} = -\tfrac{1}{2} p_{yy}. \qquad t < t_k.$$

Now

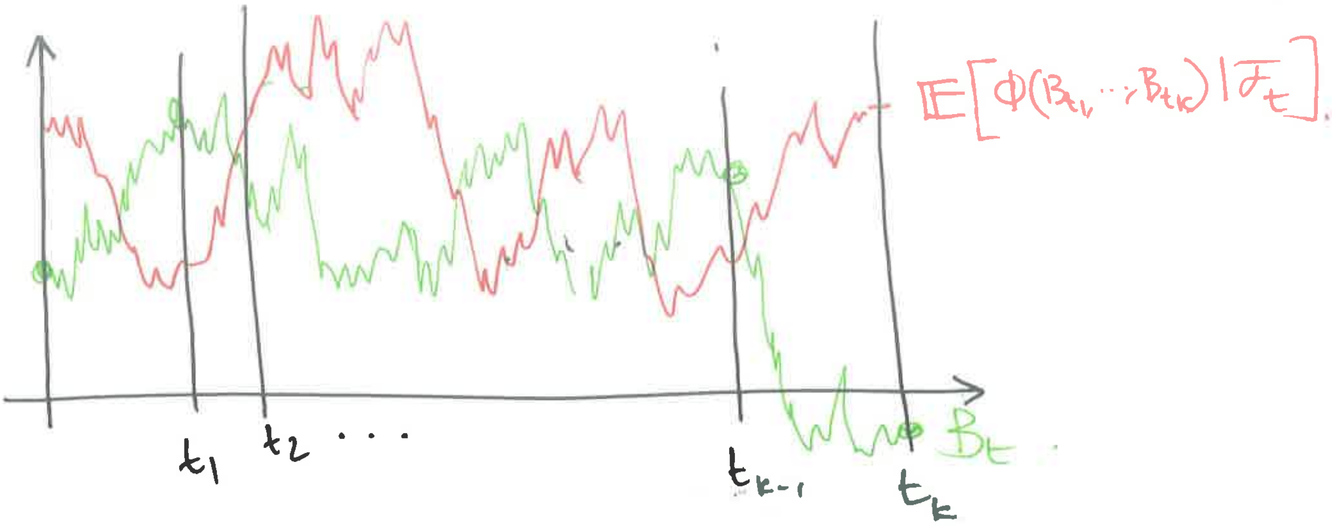
$$f(t, u) := v_k(t, x_1, \dots, u) = \int_{\mathbb{R}} \psi(y)\, q(t, u, y)\, dy.$$

where $\psi(y) := v_{k+1}(t_k, x_1, \dots, x_k, y)$

Then $$f_t = \int_{\mathbb{R}} \psi(y)\, q_t\, dy = \int_{\mathbb{R}} \psi(y)\left(-\tfrac{1}{2} q_{uu}\right) dy = -f_{uu}$$

ØKSENDAL $\boxed{14.17}$: (6)

$\underline{4.17} \subseteq$



Let $f(t, \omega) = \dfrac{\partial v_k}{\partial x_k}(t, B(t_1), \cdots, B(t_{k-1}), B(t))$.

for $t \in [t_{k-1}, t_k]$.

Show $f(t, \omega)$ satisfies the representation $\underline{4.3.15}$.

Well,

$$\Phi(B_{t_1}, \cdots, B_{t_n}) = V_n(t_n, B_{t_1}, \cdots, B_{t_n})$$

$$= V_n(t_{n-1}, B_{t_1}, \cdots, B_{t_{n-1}}, B_{t_{n-1}})$$

$$+ \int_{t_{n-1}}^{t_n} \underbrace{\frac{\partial v_n}{\partial x_n}(s, B_{t_1}, \cdots, B_{t_{n-1}}, B_s)} \, dB_s.$$

$$= V_{n-1}(t_{n-1}, B_{t_1}, \cdots, B_{t_{n-1}})$$

$$+ \int \underline{\qquad \overbrace{\qquad\qquad}^{''} \qquad} \, dB_s.$$

$$= V_{n-2}(t_{n-2}, B_{t_1}, \cdots, B_{t_{n-2}}) + \int_{t_{n-1}}^{t_n} \frac{\partial v_n}{\partial x_n}(s, B_{t_1}, \cdots, B_{t_{n-1}}, B_s) \, dB_s$$

$$+ \int_{t_{n-2}}^{t_n} \frac{\partial V_{n-1}}{\partial x_{n-1}}(s, B_{t_1}, \cdots B_{t_{n-2}}, B_s) \, dB_s.$$

By induction.

$$\Phi(B_{t_1}, \dots, B_{t_n}) = V_0(0, x_0) + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{\partial v_j}{\partial x_j}(s, B_{t_1}, \dots, B_{t_{j-1}}, B_s)\, dB_s.$$

where $v_0(0, x_0) = \mathbb{E}_{x_0}[\Phi(B_{t_1}, \dots, B_{t_n})]$ $\square$

Now this question has a lot of fiddly notation in it, but the idea is quite simple, so let's see an example.

EXAMPLE

Let $\Phi(x, y) = x + y$. Let $0 < t_1 < t_2$.

Now for $t \in [0, t_1]$,

$$\mathbb{E}[\Phi(B_{t_1}, B_{t_2}) \mid \mathcal{F}_t] = \mathbb{E}[B_{t_1} + B_{t_2} \mid \mathcal{F}_t]$$
$$= 2B_t.$$

For $t \in [t_1, t_2]$,

$$\mathbb{E}[\Phi(B_{t_1}, B_{t_2}) \mid \mathcal{F}_t] = B_{t_1} + B_t.$$

Indeed our function $f(t, \omega)$ expresses the rate of change of the expectation of $\Phi(B_{t_1}, B_{t_2})$ with changes in B_t.

ØKSENDAL $\boxed{4.17}$: (8)

Thus