Math 733 - Fall 2020

Homework 2

Due: 09/27, 10pm

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1. *Proof.* Assume X and Y are random variables from $(\Omega, \mathcal{F})$ to $(S, \mathcal{S})$, then

$$\{\omega : f\left(X(\omega)\right) \in B\} = \{\omega : X(\omega) \in f^{-1}(B)\} \in \mathcal{F}$$

If and only if $f^{-1}(B) \in \sigma(X)$, $Y = f(X)$ is measurable w.r.t $\sigma(X)$. □

2. *Proof.*

$$\begin{aligned} E[X^p] &= \int_0^1 y^p \cdot \mathbb{P}(X = y)dy \\ &= \int_0^{1-\varepsilon} y^p \cdot \mathbb{P}(X = y)dy + \int_{1-\varepsilon}^1 y^p \cdot \mathbb{P}(X = y)dy \ \forall \varepsilon \in [0,1] \end{aligned}$$

When $p \to \infty$,

$$\int_0^{1-\varepsilon} y^p \cdot \mathbb{P}(X = y)dy \to 0$$

$$\begin{aligned} \int_{1-\varepsilon}^1 y^p \cdot \mathbb{P}(X = y)dy &\leqslant \int_{1-\varepsilon}^1 y^p dy \\ &= \frac{1}{p+1} \cdot (1 - (1-\varepsilon)^p) \\ &\leqslant \frac{1}{1+p} \to 0 \ (as\ p \to \infty) \end{aligned}$$

So, $E\left[X^p\right] = 0$ as $p \to \infty$. □

3. *Proof.*

   (a) Consider *Derangement formula*

$$\mathbb{P}(X_n = 0) = \frac{D(n)}{n!}$$

where

$$D(n) = n! \cdot \sum_{k=2}^{n} \frac{(-1)^k}{k!}$$

Then

$$\mathbb{P}(X_n = 0) = \sum_{k=2}^{n} \frac{(-1)^k}{k!}$$

When $n \to \infty$,

$$\mathbb{P}(X_n = 0) \to \sum_{k=2}^{\infty} \frac{(-1)^k}{k!}$$

This is one of the expression of $\frac{1}{e}$.

$$\lim_{n\to\infty} \mathbb{P}(X_n = 0) = \frac{1}{e}$$

(b) Noticed that

$$\mathbb{P}(X_n = 1) = \frac{\binom{n}{1}D(n-1)}{n!} = \frac{D(n-1)}{(n-1)!} = \mathbb{P}(X_{n-1} = 0)$$

$$\mathbb{P}(X_n = 2) = \frac{\binom{n}{2}D(n-2)}{n!} = \frac{1}{2} \cdot \frac{D(n-2)}{(n-2)!} = \frac{1}{2}\mathbb{P}(X_n = 0)$$

$$\mathbb{P}(X_n = k) = \frac{\binom{n}{k}D(n-k)}{n!} = \frac{1}{k}\mathbb{P}(X_n = k-1)$$

So, we have

$$\mathbb{P}(X_n = k) = \frac{1}{k!} \cdot \mathbb{P}(X_n = 0)$$

$$\begin{aligned} E[X_n] &= \sum_{k=0}^{n} k \cdot \mathbb{P}(X_n = k) \\ &= \sum_{k=0}^{n} k \cdot \frac{1}{k!}\mathbb{P}(X_n = 0) \\ &= \mathbb{P}(X_n = 0) \cdot \sum_{k=1}^{n} \frac{1}{(k-1)!} \\ &= \sum_{k=2}^{n} \frac{(-1)^k}{k!} \cdot \sum_{k=1}^{n} \frac{1}{(k-1)!} \end{aligned}$$

By the way, when n is large enough, $E[X_n] \to 1$.

□

4. *Proof.* Consider the Integral form of *Cauchy–Schwarz inequality.*
Let $f = y \cdot \sqrt{\mathbb{P}(Y = y)}$, $g = \sqrt{\mathbb{P}(Y = y)}$.
By the non-negativity of Y

$$\left(\int \left(y \cdot \sqrt{\mathbb{P}(Y = y)}\right)^2 dy\right) \cdot \left(\int \left(\sqrt{\mathbb{P}(Y = y)}\right)^2 dy\right) \geqslant \left(\int y \cdot \mathbb{P}(Y = y) dy\right)^2$$

That is

$$\mathbb{P}(Y > 0) \geqslant \frac{(E[Y])^2}{E[Y^2]}$$

□

5. *Proof.* When $X_1 = X_2 = \cdots = X_n = S$, $\prod_{j=1}^{n} g_j = 1$
Consider

$$\mathbb{P}\left(X_1 = x_1, X_2 = S, \cdots, X_n = S\right) = g_1(x_1) \cdot C \quad \text{C is a constant}$$

So we have

$$\mathbb{P}\left(X_2 = x_2, \cdots, X_n = x_n | X_1 = x_1\right) = \frac{\prod_{j=2}^{n} g_j(x_j)}{C}$$

Thus, X_1 is independent. So does $X_2, \cdots, X_n$. On the other hand, if they are independent, we have

$$\mathbb{P}\left(X_1 = x_1, \cdots, X_n = x_n\right) = \prod_{j=1}^{n} \mathbb{P}\left(X_j = x_j\right)$$

Here, let $g_j(x) = \mathbb{P}\left(X_j = x\right)$ for all $1 \leqslant j \leqslant n$, they are non-negative functions from S to $\mathbb{R}$. □

6. *Proof.* Consider the expression of ω in binary. The probability of $X_n = 1$ means the probability that the nth decimal place is 1. By the definition and arbitrariness of ω, we know

$$\mathbb{P}\left(X_n = 0\right) = \mathbb{P}\left(X_n = 1\right) = \frac{1}{2}$$

That means,

$$X_n \sim \text{Bernoulli}(1/2),\ \forall n \in \mathbb{N}$$

And their independence is obvious. $\square$