

$(\Omega, \mathcal{F}, \mathbb{P})$ probability space

(real valued) random variable:

$X: \Omega \rightarrow \mathbb{R}$ Borel measurable

Distribution of X : prob measure on $(\mathbb{R}, \mathcal{B})$
with $\underline{Q_X(B)} = \underline{\mathbb{P}(X^{-1}(B))} = \underline{\mathbb{P}(X \in B)}$

cumulative distribution function of X :

$$F_X(y) = \mathbb{P}(X \leq y) = Q_X((-\infty, y]) \quad y \in \mathbb{R}$$

If $f(y) = \int_{-\infty}^y f(x) dx \quad (f = \frac{dQ}{dx})$ then

f is called the probability density function
and $\underline{\mathbb{P}(X \in B)} = \underline{\int_B f(y) dy}$ for B Borel.

$$\underline{f \geq 0 \quad \int_{-\infty}^{\infty} f(x) dx = 1}$$

If X is supported on a countable
(distribution f_X) set (There is a countable
set $S \subset \mathbb{R}$ so that $\mathbb{P}(X \in S) = 1$.)

then Q_X can be described by

the function $\underline{p_X(a) = \mathbb{P}(X=a)}$
 $a \in S$

Probability mass function of X
(we say that X is a discrete r.v.)

Ex: Indicator random variable: I_A

More general framework:

we can have \mathbb{R}^d -valued functions
or we can consider a measure
space $(\mathcal{X}, \mathcal{S})$

$X: \Omega \rightarrow \mathcal{X}$
we need $X^{-1}(A) \in \mathcal{F}$ for $A \in \mathcal{S}$

Q_X is a probability measure on
 $(\mathcal{X}, \mathcal{S})$

If μ is a measure on $(\mathcal{X}, \mathcal{S})$

and $Q_X \ll \mu$ (Q_X is abs. cont.
with respect to μ) then

R-N derivative $\frac{dQ_X}{d\mu}$

$$\underline{P(X \in A)} = \int_A \frac{d\lambda_x}{d\mu} \cdot d\mu$$

Def: If X is a r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$
 then we denote by $\sigma(X)$ the
 smallest σ -field with respect to
 which X is measurable.

$$\sigma(X) \subset \mathcal{F}$$

$\sigma(X)$ = σ -field generated by the
 sets $\tilde{X}^{-1}(B)$ with $B \in \mathcal{B}$

$$\underline{\text{Ex: } X = 1_A}$$

$$\sigma(X) = \{\emptyset, \Omega, A, A^c\}$$

Notation: $\boxed{X \in G} \in \sigma$ -field
 if $\sigma(X) \subset G$.

Recall: measurable functions are
 pointwise limits of "simple" functions.
 simple functions: finite linear combination
 of indicators

$$X = \sum_{i=1}^n c_i \mathbb{1}_{B_i}, \quad c_i \in \mathbb{R}$$

Claim (HW2)

Suppose that X and Y are random variables on (Ω, \mathcal{F}, P) . Y is measurable wrt $\mathcal{G}(X)$ ($Y \in \mathcal{G}(X)$) if and only if $Y = f(X)$ for some measurable function f .

If X is a r.v. and $f: \mathbb{R} \rightarrow \mathbb{R}$ measurable then $f(X)$ is also r.v. and is $\mathcal{G}(X)$ -measurable.

$f: \mathbb{R}^d \rightarrow \mathbb{R}$ measurable

X_1, X_2, \dots, X_n random variables on the same prob. space (Ω, \mathcal{F}, P)

then $f(X_1, \dots, X_n)$ is also a r.v.
sum, products of random variables
min, max

$\inf, \sup, \liminf, \limsup$ of countably many random variables
 \Leftrightarrow also a random variable.

If X_1, X_2, \dots is a sequence of r.v.
 on (Ω, \mathcal{F}, P)

then the following sets are all events:

$$\left\{ \limsup_{n \rightarrow \infty} X_n \leq c \right\}, \left\{ \lim_{n \rightarrow \infty} X_n \text{ exists} \right\}$$

$$\left\{ \lim_{n \rightarrow \infty} X_n = c \right\}$$

$$\left\{ \lim_{n \rightarrow \infty} X_n \text{ exists} \right\} = \left\{ \omega \in \Omega : \lim_{n \rightarrow \infty} X_n^{(\omega)} \text{ exists} \right\}$$

So we can talk about

$$P\left(\lim_{n \rightarrow \infty} X_n \text{ exists} \right)$$

Def: $\{X_n\}_{n \geq 1}$ converges almost surely

$$\text{if } P\left(\lim_{n \rightarrow \infty} X_n \text{ exists} \right) = 1 \boxed{X_n \xrightarrow{\text{a.s.}} X}$$

Def: Slightly weaker type of convergence:

if $\boxed{\text{for every } \varepsilon > 0 \text{ we have}}$

$$\lim_{n \rightarrow \infty} P(|X_n - X| > \varepsilon) = 0$$

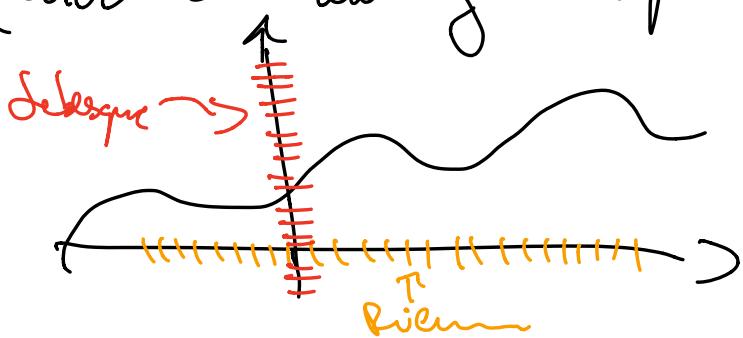
then we say that $X_n \xrightarrow{\text{a.s.}} X$.

$$X_n \xrightarrow{\text{P}} X.$$

Q: Is it possible to define a prob. space to model the choosing a positive (ergodic theory)

Often we work with random variables that can take the value $+\infty$ or $-\infty$. We can make this rigorous by extending \mathbb{R} with $\{-\infty, +\infty\}$ and change the topology accordingly.

Quick overview of integration with respect to P



0. Indicator r.v.

$$\int_{\Omega} \mathbb{1}_B dP = P(B)$$

1. Simple functions

$$X = \sum_{i=1}^n c_i \mathbb{1}_{B_i}$$

$$\int_{\Omega} X dP = \int_{\Omega} \sum_{i=1}^n c_i \mathbb{1}_{B_i} dP = \sum_{i=1}^n c_i P(B_i)$$

Properties: $X \geq 0 \rightsquigarrow \int_{\Omega} X dP \geq 0$

$X \geq Y \rightsquigarrow \int_{\Omega} X dP \geq \int_{\Omega} Y dP$
Same if $P(X \geq Y) = 1$

$$\int_{\Omega} (X+Y) dP = \int_{\Omega} X dP + \int_{\Omega} Y dP$$

$$\int_{\Omega} c \cdot X dP = c \int_{\Omega} X dP$$

$$\left| \int_{\Omega} X dP \right| \leq \int_{\Omega} |X| dP$$

2. Bounded functions

If X is bounded ($|X| \leq C$)

then $\int_X dP = \sup_{\substack{Y \leq X \\ Y \text{ simple}}} \int_Y dP$

The same properties that we had before for simple r.v.'s will extend to bounded r.v.'s.

3. Nonnegative functions

$$Y \geq 0 \text{ then}$$

$$\int_X dP = \sup_{\substack{0 \leq Y \leq X \\ \text{bounded}}} \int_Y dP$$

(could be infinite)

Same properties hold as before

4. general measurable functions

$$X = X_+ - X_-$$

↑ ↑
positive part negative part

$$\int_X dP = \int_{X_+} dP - \int_{X_-} dP$$

unless both of these are ∞
(in that case $\int_X dP$ is not defined)

The same properties hold.

$\int_X dP$ is well-defined if
 $\int_{\Omega} X_+ dP$ or $\int_{\Omega} X_- dP$ is finite.

(could be finite, ∞ , $-\infty$)

Def: The expected value of a r.v. X
on (Ω, \mathcal{F}, P) is defined as

$$E[X] = \int_{\Omega} X dP$$

The expectation exists if $E[X^+]$ or $E[X^-]$
is finite.

$$\text{Ex: } X = 1_A \quad E[X] = P(A).$$

Ex: If X is discrete with values from $A = \{a_1, a_2, \dots\}$ then

$$E[X] = \sum_i a_i P(X=a_i) \quad | \quad X = \sum_i 1_{\{a_i\}} \cdot X(a_i)$$

if this sum is well-defined

Ex: If X is abs cont with PDF f

then $E[X] = \int_{-\infty}^{\infty} x f(x) dx$

if this integral is well-defined

Properties: If $0 \leq X$ then $E[X] \geq 0$.

If $X \leq Y$ then $E[X] \leq E[Y]$
(if these are well-defined)

$$E[X+Y] = E[X] + E[Y]$$

$$E[cX] = c E[X]$$

$$(E[X])^2 \leq E[|X|^2]$$

Ex: We pick 5 cards randomly from a deck of 52. What's the expected number of aces among the chosen cards?

X : # of aces

This is discrete $\mathcal{S}(X \in \{0, 1, 2, 3, 4\}) = 1$

One can compute $E[X] = \sum_{i=0}^4 i \cdot P(X=i)$

Easier way: write X as the sum of indicators.