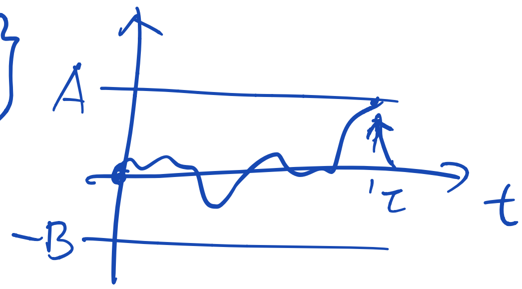
Some applications of Martingale theories

Ex1 ruin probability for BM $\quad B_0 = 0$

$A, B > 0$

$$\tau = \inf \{t : B_t = -B \text{ or } A\}$$



Prove: $\mathbb{P}(\tau < \infty) = 1$

Calculate $\mathbb{P}(B_\tau = A) = ?$

$\mathbb{E}(\tau) = ?$

$$\mathbb{P}(\tau > n+1) \leq$$

$$\mathbb{P}\left(|B_1 - B_0| < A+B, \ |B_2 - B_1| < A+B, \cdots, |B_{n+1} - B_n| < A+B\right)$$

$$= \mathbb{P}(|B_1 - B_0| < A+B) \cdots \mathbb{P}(|B_{n+1} - B_n| < A+B)$$

$\alpha \in (0,1)$

$$= \alpha^n$$

$$\mathbb{P}(\tau < \infty) = \mathbb{P}\left(\bigcup_{n=1}^{\infty} \{\tau \leq n\}\right)$$

$$= \lim_{n \to \infty} 1 - \alpha^n = 1$$

— Now we do $\mathbb{P}(B_\tau = A) = \mathbb{P}$

$$\mathbb{E}(B_\tau) = A \cdot \mathbb{P}(B_\tau = A) - B\, \underline{\mathbb{P}(B_\tau = -B)}$$

$$= A \cdot \mathbb{P} - B(1-\mathbb{P}) = 0$$

Stopping thm:

$B_{t\wedge\tau}$ is mart.

$$\mathbb{E}(B_\tau) = \lim_{t\to\infty} \mathbb{E}(B_{t\wedge\tau}) = 0$$

(Dominated convergence: $|B_{t\wedge\tau}| \le A+B$.

$$\lim_{t\to\infty}\mathbb{E}(B_{t\wedge\tau}) = \mathbb{E}\left(\lim_{t\to\infty} B_{t\wedge\tau}\right) = \mathbb{E}(B_\tau)$$)

$$\mathbb{P} = \frac{B}{A+B}.$$

— Now we consider $\mathbb{E}\tau$.

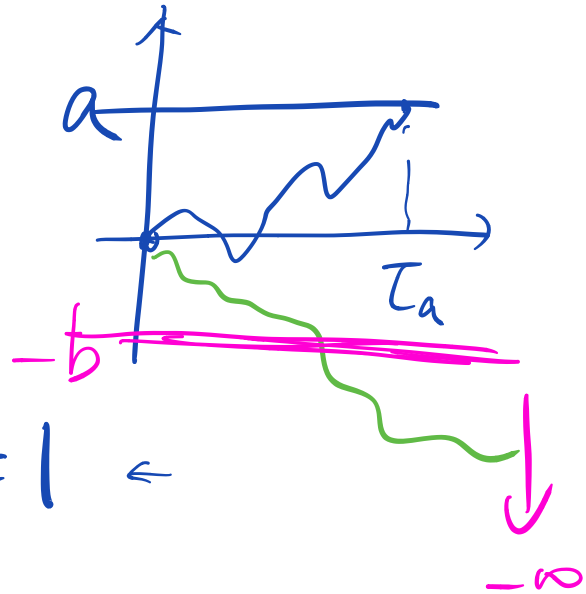
$B_t^2 - t$ is mart.

$$\mathbb{E}\tau = \underline{\mathbb{E}(B_\tau^2)} = A^2 \cdot \frac{B}{A+B} + B^2 \cdot \left(1 - \frac{B}{A+B}\right)$$

exercise

$$= AB.$$

Ex 2. Hitting a level.



$B_0 = 0$

$\tau_a = \inf\{t : B_t = a\}$

prove: $\mathbb{P}(\tau_a < \infty) = 1$ $\leftarrow$

Compute $\begin{cases} \mathbb{E}\,\tau_a = \infty \\ \mathbb{E}\left(\frac{1}{\tau_a}\right) \end{cases}$

$$\mathbb{P}(\tau_a < \infty) \geq \mathbb{P}\left(B_{\tau_a \wedge \tau_{-b}} = a\right)$$

$$= \frac{b}{a+b} \qquad \text{Send } b \to \infty$$

$$\to 1$$

$$\mathbb{E}\, e^{-\lambda \tau_a} = ?$$

$M_t = e^{\alpha B_t - \frac{1}{2}\alpha^2 t}$ is mart.

$$\boxed{\mathbb{E}\, M_{\tau_a}} = \lim_{t\to\infty} \mathbb{E}\left(M_{t\wedge\tau_a}\right) = \lim_{t\to\infty} 1 = 1$$

(dom conv, $|M_{t\wedge\tau_a}| \leq e^{\alpha a}$) Stopping thm

$$M_{\tau_a} = e^{\alpha a - \frac{1}{2}\alpha^2 \tau_a}$$

$$\mathbb{E}\, e^{-\frac{1}{2}\alpha^2 \tau_a} = e^{-\alpha a} \qquad \alpha = \sqrt{2\lambda}$$

$$\mathbb{E}\, e^{-\lambda \tau_a} = e^{-\sqrt{2\lambda}\, a}$$

$$\Rightarrow \mathbb{E}\,\tau_a = -\lim_{\lambda \downarrow 0} \frac{d}{d\lambda} e^{-\sqrt{2\lambda}\, a}$$

$$= \lim_{\lambda \downarrow 0} e^{-\sqrt{2\lambda}\, a} \cdot 2a (2\lambda)^{-1/2} \to \infty$$

$$\mathbb{E}\left(\frac{1}{\tau_a}\right) = \int_0^\infty \mathbb{E}\, e^{-\lambda \tau_a}\, d\lambda = \int_0^\infty e^{-\overbrace{\sqrt{2\lambda}\, a}^{u}}\, d\lambda$$

$$= \int_0^\infty e^{-u} \cdot u \frac{du}{a^2} = \frac{1}{a^2}.$$

Ex 3. BM inversion $\boxed{B_0 = 0}$ ("scaling property")

$$Y_t = \begin{cases} 0 & t = 0 \\ t \cdot B_{1/t} & t > 0 \end{cases} \Rightarrow Y \text{ is BM.}$$

Pf: Gaussian $\mathbb{E} Y_t = t\, \mathbb{E} B_{1/t} = 0$

$s<t \quad \mathbb{E}[Y_t Y_s] \overset{?}{=} s$ ✓

$$\mathbb{E}[t B_{1/t}\, s B_{1/s}] = ts\, \mathbb{E}[B_{1/t} B_{1/s}] = ts \cdot \frac{1}{t} = s$$

Continuity at $t=0$??

$$\mathbb{P}\left(\lim_{t\to 0} Y_t = 0\right) \overset{?}{=} 1$$

$$\shortparallel$$

$$\mathbb{P}\left(\lim_{t\to 0} t B_{1/t} = 0\right) \geq \mathbb{P}\left(\lim_{u\to\infty} \frac{1}{u}|B_u| = 0\right) \overset{?}{=} 1$$

$$\left(\begin{array}{l} \operatorname{Var} B_u = u \\ \quad \text{"} \mathbb{E} B_u^2 \\ \text{"} B_u \sim \sqrt{u} \text{"} \end{array}\right)$$

Doob max ineq.

$$\lambda^p \mathbb{P}\left(\sup_{t\in[S,T]} |B_t| > \lambda\right) \leq \mathbb{E}\left[|B_T|^p\right]$$

$p=2$

$$\mathbb{P}\left(\sup_{t\in[S,T]} |B_t| > \lambda\right) \leq T/\lambda^2$$

Take $S = \frac{T}{2} \qquad \lambda = b \cdot S$

$$\mathbb{P}\left(\sup_{t\in[\frac{T}{2},T]} \frac{|B_t|}{t} > \frac{bS}{S}\right) \leq \frac{T}{b^2 S^2} = \frac{4}{b^2 T}$$

$t > S$

0

as $T \to \infty$