Math 714 - Fall 2020
Homework 2

A

(a) If $v \in \text{span}\{w_1, \cdots, w_n\}$, then there exists $\{\alpha_j\}_{j=1}^n such that$

$$v = \sum_{j=1}^n \alpha_j w_j$$

$$\frac{\langle v, w_j \rangle}{\|w_j\|^2} = \frac{\alpha_j \langle w_j, w_j \rangle}{\|w_j\|^2} = \alpha_j$$

$$v = \sum_{j=1}^n \alpha_j w_j = \sum_{j=1}^n \frac{\langle v, w_j \rangle}{\|w_j\|^2} w_j$$

(b) ii. For $n = 1$, $p_1 = r_1 - \frac{\langle r_1, p_0 \rangle}{\|p_0\|^2} p_0$.

$$\begin{aligned}\langle p_1, p_0 \rangle &= \langle r_1 - \frac{\langle r_1, p_0 \rangle}{\|p_0\|^2} p_0, p_0 \rangle \\ &= \langle r_1, p_0 \rangle - \frac{\langle r_1, p_0 \rangle}{\|p_0\|^2} \|p_0\|^2 \\ &= 0\end{aligned}$$

Suppose $n = k$ is true. When $n = k + 1$, for $0 \leqslant j \leqslant k$,

$$\begin{aligned}\langle p_{k+1}, p_j \rangle &= \langle r_{k+1}, p_j \rangle - \frac{\langle r_{k+1}, p_j \rangle}{\|p_j\|^2} \langle p_j, p_j \rangle \\ &= 0\end{aligned}$$

By induction, it is true for all $0 \leqslant j < n \leqslant n^* - 1$.

(c) i

$$\begin{aligned}\langle Av, w \rangle &= \sum_{n=1}^N \langle v, \phi_n \rangle \langle A\phi_n, w \rangle \\ &= \sum_{n=1}^N \langle v, \phi_n \rangle \langle \lambda_n \phi_n, w \rangle \\ &= \sum_{n=1}^N \lambda_n \langle v, \phi_n \rangle \langle \phi_n, w \rangle\end{aligned}$$

ii By the definition of positive definite matrix.

iii $v = \sum_{n=1}^N \alpha_n \phi_n$, then $\langle Av, v \rangle = \sum_{n=1}^N \lambda_n \alpha_n^2$, $\|v\|^2 = \sum_{n=1}^N \alpha_n^2$.
By $\lambda_1 \leqslant \cdots \leqslant \lambda_N$, we know

$$\sum_{n=1}^N \lambda_1 \alpha_n^2 \leqslant \sum_{n=1}^N \lambda_n \alpha_n^2 \leqslant \sum_{n=1}^N \lambda_N \alpha_n^2$$

$$\lambda_1 \|v\|^2 \leqslant \langle Av, v \rangle \leqslant \lambda_N \|v\|^2$$

iv

$$\|Av\|^2 = \langle Av, Av\rangle = \sum_{n=1}^{N} \alpha_n^2 \lambda_n^2 \leqslant \sum_{n=1}^{N} \alpha_n^2 \lambda_N^2 = \lambda_N^2 \|v\|^2$$
$$\|Av\| \leqslant \lambda_N \|v\|$$

(d)

$$\begin{aligned} p_{n+1} &= r_{n+1} + \beta_n p_n \\ &= r_n - \alpha_n \omega_n + \beta_n p_n \\ &= r_n - \alpha_n A p_n + \beta_n p_n \\ &= p_n - \beta_{n-1} p_{n-1} - \alpha_n A p_n + \beta_n p_n \end{aligned}$$

Thus,

$$p_{n+1} = (1+\beta_n) p_n - \alpha_n A p_n - \beta_{n-1} p_{n-1}$$

(e) By Cayley-Hamilton theorem, $p(\lambda) = \det(\lambda I - A)$

$$p(\lambda) = A^N + \alpha_{N-1} A^{N-1} + \cdot + \alpha_1 A + (-1)^N \det|A| I_N = 0$$

Thus A^N is a linear combination of $I, A, A^2, \cdot, A^{N-1}$.

(f) i

$$\begin{aligned} e_n &= u_n - u \\ &= u_{n-1} + \alpha(f - A u_{n-1}) - u \\ &= (I - \alpha A)(u_{n-1} - u) \\ &= (I - \alpha A) e_{n-1} \end{aligned}$$

ii

$$\begin{aligned} \|e_{n+1}\| &= \|(I - \alpha A) e_n\| \\ &\leqslant \|(I - \alpha A)\| \cdot \|e_n\| \end{aligned}$$

Noticed that

$$\rho(A) = \max_{1 \leqslant i \leqslant N} |\lambda_i|$$

then,

$$\|e_{n+1}\| \leqslant \rho \|e_n\|$$

where $\rho = \max_{1 \leqslant j \leqslant N} |1 - \alpha \lambda_j|$

iii

$$\begin{aligned} \rho &= \max_{1 \leqslant j \leqslant N} |1 - \alpha \lambda_j| \\ &= \max(|1 - \alpha \lambda_1|, |1 - \alpha \lambda_N|) \end{aligned}$$

$$1 - \alpha \lambda_1 = -1 + \alpha \lambda_N$$
$$\alpha = \frac{2}{\lambda_1 + \lambda_N}$$

Thus, we have $\rho = 1 - \frac{2\lambda_1}{\lambda_1 + \lambda_N} = \frac{\kappa - 1}{\kappa + 1} < 1$, where $\kappa = \frac{\lambda_N}{\lambda_1}$.

iv

$$\begin{aligned} \rho &= \max_{1 \leqslant j \leqslant N} |1 - \alpha \lambda_j| \\ &= \max\left(\left|1 - \frac{2\lambda_1}{c + C}\right|, \left|1 - \frac{2\lambda_N}{c + C}\right| \right) \end{aligned}$$

Noticed that

$$
\begin{gathered}
\left|1-\frac{2\lambda_1}{c+C}\right| \leqslant \frac{C-c}{C+c} \\
1-\frac{2C}{C+c} \leqslant 1-\frac{2\lambda_1}{C+c} \leqslant 1-\frac{2c}{C+c} \\
\Leftarrow c \leqslant \lambda_1 \leqslant C
\end{gathered}
$$

Then,

$$
\rho \leqslant \frac{C-c}{C+c} = \frac{\kappa'-1}{\kappa'+1} < 1
$$

(g) i

$$
r_1 = r_0 - \alpha_0 A p_0 = r_0 - \alpha_0 A r_0
$$

ii

$$
\begin{aligned}
r_{n+1} &= r_n - \alpha_n A p_n \\
&= r_n - \alpha_n A(r_n + \beta_{n-1} p_{n-1}) \\
&= r_n - \alpha_n A r_n - \alpha_n \beta_{n-1} \frac{r_{n-1} - r_n}{\alpha_{n-1}}
\end{aligned}
$$

iii

$$
\begin{aligned}
r_0 q_0 - \delta_0 q_1 &= \frac{1}{\alpha_0} q_0 - \frac{\sqrt{\beta_0}}{\alpha_0} q_1 \\
&= \frac{1}{\alpha_0} q_0 - \frac{r_1}{\alpha_0 \|r_0\|} \\
&= \frac{\alpha_0 A r_0}{\alpha_0 \|r_0\|} \\
&= A q_0
\end{aligned}
$$

$$
\begin{gathered}
\frac{r_{n+1}}{\|r_n\|} = q_n - \alpha_n A r_n + \frac{\alpha_n \beta_{n-1}}{\alpha_{n-1}} \left(q_n - \frac{r_{n-1}}{\|r_n\|} \right) \\
q_{n+1} \sqrt{\beta_n} = q_n - \alpha_n A q_n + \frac{\alpha_n \beta_{n-1}}{\alpha_{n-1}} \left(q_n - \frac{q_{n-1}}{\sqrt{\beta_{n-1}}} \right) \\
A q_n = -\delta_{n-1} q_{n-1} + \gamma_n q_n - \delta_n q_{n+1}
\end{gathered}
$$

iv

v

$$
Q_n^T A Q_n = Q_n^T (Q_n T_n - \delta_{n-1} q_n e_n^T) = T_n
$$