CS 726 Supplementary Material: Topology, Open & Closed Sets, Continuity of Maps

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Topology is the minimum structure we need to define on a set to be able to establish continuity of maps. Topology is always defined for a given ground set M, as follows.

Definition 1.1. Let M be a set. A topology $\mathcal{O}$ is a subset of $\mathcal{P}(M)$ (the power set of M) that satisfies all of the following assumptions:

(i) $\emptyset \in \mathcal{O}$, $M \in \mathcal{O}$;

(ii) if $U \in \mathcal{O}$ and $V \in \mathcal{O}$ then also $U \cap V \in \mathcal{O}$;

(iii) if $U_\alpha \in \mathcal{O}$, for all $\alpha \in A$, where A can have uncountably many elements, then also $\cup_{\alpha \in A} U_\alpha \in \mathcal{O}$.

Some simplest examples of topologies, given any set M, are:

- chaotic topology, which consists of $\emptyset$ and M;
- discrete topology, which consists of all possible subsets of M.

(Prove that, by definition of topology, these two examples are both topologies on M.)

Some terminology is in place here. If we define a topology $\mathcal{O}_M$ on a given set M, then the pair $(M, \mathcal{O}_M)$ is called a *topological space*. The elements of $\mathcal{O}_M$, $U \in \mathcal{O}_M$, are called *open sets*. Thus, to define open sets, we need to define a topology! We say that a set is closed if it is a complement of an open set, i.e., $V \subseteq M$ is closed if $M \backslash V \in \mathcal{O}_M$.

Note that a set does not be open or closed: it can be open, closed, both open and closed, or neither open nor closed. Examples of sets that are both open and closed in topology $\mathcal{O}_M$ are $\emptyset$ and M.

Continuous Maps. Let $f : M \to N$. Whether f is continuous or not depends on what topologies are chosen on M and N. In particular:

Definition 1.2. Let $(M, \mathcal{O}_M)$, $(N, \mathcal{O}_N)$ be topological spaces. We say that a map $f : M \to N$ is continuous (w.r.t. $\mathcal{O}_M$ and $\mathcal{O}_N$) if

$$(\forall V \in \mathcal{O}_N) : \quad \mathrm{preim}_f(V) \in \mathcal{O}_M,$$

where $\mathrm{preim}_f(V) = \{m \in M : f(m) \in V\}$ is the preimage of the set V w.r.t. f.

In other words, a map $f : M \to N$ is continuous if the preimages of all open sets (in the target space) are open sets (in the domain space).

Standard Topology on $\mathbb{R}^d$. In this class, we will always work with maps $f : \mathbb{R}^d \to \bar{\mathbb{R}}$, where $d \in \mathbb{Z}_{++}$. In doing so, without explicitly specifying it, we will always assume that we adopt what is known as the standard topology on $\mathbb{R}^d$ (and $\mathbb{R}$). To define the standard topology on $\mathbb{R}^d$, we need to first define "open" or "soft" ball on $\mathbb{R}^d$. In particular, given $\mathbf{p} \in \mathbb{R}^d$, $r \in \mathbb{R}_{++}$, define

$$\mathcal{B}_r(\mathbf{p}) = \{\mathbf{q} \in \mathbb{R}^d : \|\mathbf{q} - \mathbf{p}\|_2 < r\}.$$

Now, the standard topology $\mathcal{O}_{\mathrm{standard}}$ on $\mathbb{R}^d$ will, of course, contain the empty set and all of $\mathbb{R}^d$, but, in addition, the following sets will be included:

$$U \in \mathcal{O}_{\mathrm{standard}} \quad \Leftrightarrow \quad (\forall \mathbf{p} \in U)(\exists r \in \mathbb{R}_{++}) : \mathcal{B}_r(\mathbf{p}) \subseteq U. \tag{1}$$

The standard topology on $\mathbb{R}^d$ leads to familiar definitions of open and closed sets and continuous maps that are usually introduced in undergraduate real analysis classes. Thus, with this topology, we can think of open sets as "sets that don't contain the boundary" and closed sets as those "sets that contain the boundary." Continuous maps can also be defined using different equivalent definitions, the most familiar of which is perhaps the Weierstrass-Jordan (ϵ, δ) definition. In this definition (which can be shown to be equivalent to Definition 1.2 once standard topologies on $\mathbb{R}^d$ and $\mathbb{R}$ are adopted), we say that f is continuous at a point $\mathbf{x} \in \mathbb{R}^d$ if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall \mathbf{y} \in \mathbb{R}^d): \quad \|\mathbf{y} - \mathbf{x}\| < \delta \Rightarrow |f(\mathbf{x}) - f(\mathbf{y})| < \epsilon.$$