Itô integrals

Doob

Itô

Riemann-Stieltjes $\int f\, dg$

If f cont.
g is BV (bound variation)
Then exists

$\int_0^t f(s)\, dB_s$

price of this stock at s per unit

how much stock you hold at s

how much you gain $s \to s+ds$

→ net gain $[0, t]$

Assumption on f: $[0, T]$ $\int_0^T$ $(\Omega, \mathcal{F}_t)$

- $f(t, \omega)$ is measurable wrt $\mathcal{B} \times \mathcal{F}_T$ $\mathcal{F}_T$

  $\mathbb{R}_+$ Ω

  Borel on $\mathbb{R}_+$

- f is adapted to $\{\mathcal{F}_t\}$.

  $\forall t$ $f(t, \cdot) \in \mathcal{F}_t$

$\mathcal{H}^2$ space of all meas. adapted f s.t.

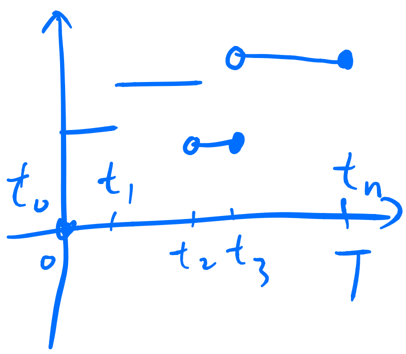
$$\mathbb{E} \int_0^T f^2(t, \omega)\, dt < \infty$$

$$\| f \|^2_{L^2([0,T] \times \Omega)}$$

$$f = \mathbb{1}_{[a,b]} \qquad [a,b] \subset [0,T]$$

$$\int_0^T f_t \, dB_t = \int_a^b dB_t = B_b - B_a$$

① $f(t,\omega) = \sum_{i=0}^{n-1} a_i(\omega) \, \mathbb{1}_{(t_i,\, t_{i+1}]}$



② $a_i \in \mathcal{F}_{t_i}$

③ $\mathbb{E}[a_i^2] < \infty \quad \forall i$

$\mathcal{H}_0^2 \subset \mathcal{H}^2$

$$\int_0^T f_t \, dB_t = \sum_{i=0}^{n-1} a_i(\omega) \left(B_{t_{i+1}} - B_{t_i}\right)$$

Lemma: Itô Isometry on $\mathcal{H}_0^2$

$$\left\| \int_0^T f \, dB \right\|_{L^2(\Omega)} = \|f\|_{L^2([0,T]\times\Omega)}$$

$$\mathbb{E}\left[\left(\int_0^T f \, dB\right)^2\right] \qquad \mathbb{E}\int_0^T f^2 \, dt$$

Pf: $f^2 = \sum_i a_i^2 \, \mathbb{1}_{(t_i,\, t_{i+1}]}$

RHS $= \mathbb{E}\int_0^T f^2 \, dt = \sum_i \mathbb{E}\, a_i^2 \, (t_{i+1} - t_i)$ ✓

LHS =

$$\sum_i \mathbb{E}\left[a_i^2 \, (B_{t_{i+1}} - B_{t_i})^2 \right]$$

(a_i is indep of $B_{t_{i+1}} - B_{t_i}$)

$$= \sum_i \mathbb{E}(a_i^2) \, \cancel{\mathbb{E}((\quad)^2)} \, (t_{i+1} - t_i)$$

✓ $\blacksquare$

Lemma: $\mathcal{H}_0^2$ is dense in $\mathcal{H}^2$.

$\forall f \in \mathcal{H}^2 \quad \exists f_n \in \mathcal{H}_0^2. \quad f_n \to f$

$\left(\|f - f_n\|_{L^2([0,T]\times\Omega)} \to 0 \right)$

$$\left\| \int_0^T f_n \, dB \right\|_{L^2(\Omega)} = \|f_n\|_{L^2([0,T]\times\Omega)}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\boxed{\int_0^T f \, dB} \qquad\qquad f$$

Thm Itô isometry hold on $\mathcal{H}^2$.

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prop: $0 \le s \le t$. $f \in \mathcal{H}^2$

$$E\left[\left(\int_s^t f(u,w)\, dB\right)^2 \middle| \mathcal{F}_s\right]$$

$$= E\left[\int_s^t f^2(u,w)\, du \middle| \mathcal{F}_s\right]$$

pf: It's enough to show $\forall A \in \mathcal{F}_s$

$$E\left[\mathbb{1}_A \left(\int_s^t f\, dB\right)^2\right] = E\left[\mathbb{1}_A \int_s^t f^2\, du\right]$$

$$\tilde{f}(u,w) = \begin{cases} \mathbb{1}_A f(u,w) & u \in (s,t] \\ 0 & u \le s \end{cases}$$

Apply Itô isometry to $\tilde{f}$ ∎

Cor: $M = \left(\int_0^t f\, dB\right)^2 - \int_0^t f^2\, du$

is martingale.

$E[M_t \mid \mathcal{F}_s] \overset{?}{=} M_s$ use the prop above.

(In case $f = 1$, $M = B^2 - t$)

$$\int_0^{\Theta} f \, dB.$$