$$Y_t = \int_0^t \operatorname{sign}(\widehat{B}_s) d\widehat{B}_s \ .$$

By the Tanaka formula (4.3.12) (Exercise 4.10) we have

$$Y_t = |\,\widehat{B}_t| - |\,\widehat{B}_0| - \widehat{L}_t(\omega) \ ,$$

where $\widehat{L}_t(\omega)$ is the local time for $\widehat{B}_t(\omega)$ at 0. It follows that Y_t is measurable w.r.t. the σ-algebra $\mathcal{G}_t$ generated by $|\,\widehat{B}_s(\cdot)|$; $s \le t$, which is clearly strictly contained in $\widehat{\mathcal{F}}_t$. Hence the σ-algebra $\mathcal{N}_t$ generated by $Y_s(\cdot)$; $s \le t$ is also strictly contained in $\widehat{\mathcal{F}}_t$.

Now suppose X_t is a strong solution of (5.3.1). Then by Theorem 8.4.2 it follows that X_t is a Brownian motion w.r.t. the measure P. (In case the reader is worried about the possibility of a circular argument, we point out that the proof of Theorem 8.4.2 is independent of this example!) Let $\mathcal{M}_t$ be the σ-algebra generated by $X_s(\cdot)$; $s \le t$. Since $(\operatorname{sign}(x))^2 = 1$ we can rewrite (5.3.1) as

$$dB_t = \operatorname{sign}(X_t) dX_t \ .$$

By the above argument applied to $\widehat{B}_t = X_t$, $Y_t = B_t$ we conclude that $\mathcal{F}_t$ is strictly contained in $\mathcal{M}_t$.

But this contradicts that X_t is a strong solution. Hence strong solutions of (5.3.1) do not exist.

To find a weak solution of (5.3.1) we simply choose X_t to be *any* Brownian motion $\widehat{B}_t$. Then we define $\widetilde{B}_t$ by

$$\widetilde{B}_t = \int_0^t \operatorname{sign}(\widehat{B}_s) d\widehat{B}_s = \int_0^t \operatorname{sign}(X_s) dX_s$$

i.e.

$$d\widetilde{B}_t = \operatorname{sign}(X_t) dX_t \ .$$

Then

$$dX_t = \operatorname{sign}(X_t) d\widetilde{B}_t \ ,$$

so X_t *is a weak solution*.

Finally, *weak uniqueness* follows from Theorem 8.4.2, which – as noted above – implies that any weak solution X_t must be a Brownian motion w.r.t. P.

Exercises

5.1. Verify that the given processes solve the given corresponding stochastic differential equations: (B_t denotes 1-dimensional Brownian motion)

(i) $X_t = e^{B_t}$ solves $dX_t = \frac{1}{2}X_t dt + X_t dB_t$

(ii) $X_t = \frac{B_t}{1+t}$; $B_0 = 0$ solves

$$dX_t = -\frac{1}{1+t}X_t dt + \frac{1}{1+t}dB_t\ ; \qquad X_0 = 0$$

(iii) $X_t = \sin B_t$ with $B_0 = a \in (-\frac{\pi}{2}, \frac{\pi}{2})$ solves

$$dX_t = -\tfrac{1}{2}X_t dt + \sqrt{1-X_t^2}\,dB_t \ \text{ for }\ t < \inf\big\{s > 0; B_s \notin \big[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\big]\big\}$$

(iv) $(X_1(t), X_2(t)) = (t, e^t B_t)$ solves

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^{X_1} \end{bmatrix} dB_t$$

(v) $(X_1(t), X_2(t)) = (\cosh(B_t), \sinh(B_t))$ solves

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} dB_t\ .$$

5.2. A natural candidate for what we could call *Brownian motion on the ellipse*

$$\Big\{(x,y); \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\Big\} \qquad \text{where } a > 0, b > 0$$

is the process $X_t = (X_1(t), X_2(t))$ defined by

$$X_1(t) = a\cos B_t\ , \quad X_2(t) = b\sin B_t$$

where B_t is 1-dimensional Brownian motion. Show that X_t is a solution of the stochastic differential equation

$$dX_t = -\tfrac{1}{2}X_t dt + MX_t dB_t$$

where $M = \begin{bmatrix} 0 & -\frac{a}{b} \\ \frac{b}{a} & 0 \end{bmatrix}$.

5.3. Let $(B_1, \ldots, B_n)$ be Brownian motion in $\mathbf{R}^n$, $\alpha_1, \ldots, \alpha_n$ constants. Solve the stochastic differential equation

$$dX_t = rX_t dt + X_t\Big(\sum_{k=1}^{n} \alpha_k dB_k(t)\Big); \qquad X_0 > 0\ .$$

(This is a model for exponential growth with several independent white noise sources in the relative growth rate).

5.4. Solve the following stochastic differential equations:

(i) $\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & X_1 \end{bmatrix} \begin{bmatrix} dB_1 \\ dB_2 \end{bmatrix}$

(ii) $dX_t = X_t dt + dB_t$
(Hint: Multiply both sides with "the integrating factor" e^{-t} and compare with $d(e^{-t}X_t)$)

(iii) $dX_t = -X_t dt + e^{-t} dB_t$.

5.5. a) Solve the *Ornstein-Uhlenbeck equation* (or *Langevin equation*)

$$dX_t = \mu X_t dt + \sigma dB_t$$

where μ, σ are real constants, $B_t \in \mathbf{R}$.
The solution is called the *Ornstein-Uhlenbeck process*. (Hint: See Exercise 5.4 (ii).)

b) Find $E[X_t]$ and $\mathrm{Var}[X_t] := E[(X_t - E[X_t])^2]$.

5.6. Solve the stochastic differential equation

$$dY_t = r\,dt + \alpha Y_t dB_t$$

where r, α are real constants, $B_t \in \mathbf{R}$.
(Hint: Multiply the equation by the 'integrating factor'

$$F_t = \exp\big(-\alpha B_t + \tfrac{1}{2}\alpha^2 t\big)\ .\ )$$

5.7. The *mean-reverting Ornstein-Uhlenbeck process* is the solution X_t of the stochastic differential equation

$$dX_t = (m - X_t)dt + \sigma dB_t$$

where m, σ are real constants, $B_t \in \mathbf{R}$.

a) Solve this equation by proceeding as in Exercise 5.5 a).

b) Find $E[X_t]$ and $\mathrm{Var}[X_t] := E[(X_t - E[X_t])^2]$.

5.8. Solve the (2-dimensional) stochastic differential equation

$$\begin{aligned} dX_1(t) &= X_2(t)dt + \alpha dB_1(t) \\ dX_2(t) &= -X_1(t)dt + \beta dB_2(t) \end{aligned}$$

where $(B_1(t), B_2(t))$ is 2-dimensional Brownian motion and α, β are constants.
This is a model for a vibrating string subject to a stochastic force. See Example 5.1.3.

5.9. Show that there is a unique strong solution X_t of the 1-dimensional stochastic differential equation

$$dX_t = \ln(1 + X_t^2)dt + \mathcal{X}_{\{X_t>0\}} X_t dB_t\ , \qquad X_0 = a \in \mathbf{R}\ .$$

5.10. Let b, σ satisfy (5.2.1), (5.2.2) and let X_t be the unique strong solution of (5.2.3). Show that

$$E[|X_t|^2] \le K_1 \cdot \exp(K_2 t) \qquad \text{for } t \le T \tag{5.3.2}$$

where $K_1 = 3E[|Z|^2] + 6C^2T(T+1)$ and $K_2 = 6(1+T)C^2$.
(Hint: Use the argument in the proof of (5.2.10)).

Remark. With global estimates of the growth of b and σ in (5.2.1) it is possible to improve (5.3.2) to a global estimate of $E[|X_t|^2]$. See Exercise 7.5.

5.11. (The Brownian bridge).
For fixed $a, b \in \mathbf{R}$ consider the following 1-dimensional equation

$$dY_t = \frac{b - Y_t}{1-t} dt + dB_t ; \qquad 0 \le t < 1 , \ Y_0 = a . \tag{5.3.3}$$

Verify that

$$Y_t = a(1-t) + bt + (1-t) \int_0^t \frac{dB_s}{1-s} ; \qquad 0 \le t < 1 \tag{5.3.4}$$

solves the equation and prove that $\lim_{t \to 1} Y_t = b$ a.s. The process Y_t is called *the Brownian bridge* (from a to b). For other characterizations of Y_t see Rogers and Williams (1987, pp. 86–89).

5.12. To describe the motion of a pendulum with small, random perturbations in its environment we try an equation of the form

$$y''(t) + (1 + \epsilon W_t) y = 0 ; \qquad y(0), y'(0) \text{ given} ,$$

where $W_t = \frac{dB_t}{dt}$ is 1-dimensional white noise, $\epsilon > 0$ is constant.

a) Discuss this equation, for example by proceeding as in Example 5.1.3.
b) Show that $y(t)$ solves a *stochastic Volterra equation* of the form

$$y(t) = y(0) + y'(0) \cdot t + \int_0^t a(t,r) y(r) dr + \int_0^t \gamma(t,r) y(r) dB_r$$

where $a(t,r) = r - t$, $\gamma(t,r) = \epsilon(r - t)$.

5.13. As a model for the horizontal slow drift motions of a moored floating platform or ship responding to incoming irregular waves John Grue (1989) introduced the equation

$$x_t'' + a_0 x_t' + w^2 x_t = (T_0 - \alpha_0 x_t') \eta W_t , \tag{5.3.5}$$

where W_t is 1-dimensional white noise, a_0, w, T_0, α_0 and η are constants.

(i) Put $X_t = \begin{bmatrix} x_t \\ x'_t \end{bmatrix}$ and rewrite the equation in the form

$$dX_t = AX_t dt + KX_t dB_t + M dB_t \ ,$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -w^2 & -a_0 \end{bmatrix}, \quad K = \alpha_0 \eta \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad M = T_0 \eta \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(ii) Show that X_t satisfies the integral equation

$$X_t = \int_0^t e^{A(t-s)} K X_s dB_s + \int_0^t e^{A(t-s)} M dB_s \qquad \text{if } X_0 = 0 \ .$$

(iii) Verify that

$$e^{At} = \frac{e^{-\lambda t}}{\xi} \{(\xi \cos \xi t + \lambda \sin \xi t) I + A \sin \xi t\}$$

where $\lambda = \frac{a_0}{2}, \xi = (w^2 - \frac{a_0^2}{4})^{\frac{1}{2}}$ and use this to prove that

$$x_t = \eta \int_0^t (T_0 - \alpha_0 y_s) g_{t-s} dB_s \tag{5.3.6}$$

and

$$y_t = \eta \int_0^t (T_0 - \alpha_0 y_s) h_{t-s} dB_s \ , \qquad \text{with } y_t := x'_t \ , \tag{5.3.7}$$

where

$$g_t = \frac{1}{\xi} \mathrm{Im}(e^{\zeta t})$$
$$h_t = \frac{1}{\xi} \mathrm{Im}(\zeta e^{\bar{\zeta} t}) \ , \qquad \zeta = -\lambda + i\xi \quad (i = \sqrt{-1}) \ .$$

So we can solve for y_t first in (5.3.7) and then substitute in (5.3.6) to find x_t.

5.14. If (B_1, B_2) denotes 2-dimensional Brownian motion we may introduce complex notation and put

$$\mathbf{B}(t) := B_1(t) + iB_2(t) \quad (i = \sqrt{-1}) \ .$$

$\mathbf{B}(t)$ is called *complex Brownian motion*.

(i) If $F(z) = u(z) + iv(z)$ is an *analytic* function i.e. F satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\,; \qquad z = x + iy$$

and we define

$$Z_t = F(\mathbf{B}(t))$$

prove that

$$dZ_t = F'(\mathbf{B}(t))d\mathbf{B}(t)\,, \tag{5.3.8}$$

where F' is the (complex) derivative of F. (Note that the usual second order terms in the (real) Itô formula are not present in (5.3.8)!)

(ii) Solve the complex stochastic differential equation

$$dZ_t = \alpha Z_t d\mathbf{B}(t) \quad \alpha \text{ constant})\,.$$

For more information about complex stochastic calculus involving analytic functions see e.g. Ubøe (1987).

5.15. (Population growth in a stochastic, crowded environment)
The nonlinear stochastic differential equation

$$dX_t = rX_t(K - X_t)dt + \beta X_t dB_t\,; \qquad X_0 = x > 0 \tag{5.3.9}$$

is often used as a model for the growth of a population of size X_t in a stochastic, crowded environment. The constant $K > 0$ is called the *carrying capacity* of the environment, the constant $r \in \mathbf{R}$ is a measure of the quality of the environment and the constant $\beta \in \mathbf{R}$ is a measure of the size of the noise in the system.
Verify that

$$X_t = \frac{\exp\{(rK - \frac{1}{2}\beta^2)t + \beta B_t\}}{x^{-1} + r\int\limits_0^t \exp\{(rK - \frac{1}{2}\beta^2)s + \beta B_s\}ds}\,; \qquad t \geq 0 \tag{5.3.10}$$

is the unique (strong) solution of (5.3.9). (This solution can be found by performing a substitution (change of variables) which reduces (5.3.9) to a linear equation. See Gard (1988), Chapter 4 for details.)

5.16. The technique used in Exercise 5.6 can be applied to more general nonlinear stochastic differential equations of the form

$$dX_t = f(t, X_t)dt + c(t)X_t dB_t\,, \qquad X_0 = x \tag{5.3.11}$$

where $f: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $c: \mathbf{R} \to \mathbf{R}$ are given continuous (deterministic) functions. Proceed as follows:

a) Define the 'integrating factor'

$$F_t = F_t(\omega) = \exp\Big(- \int_0^t c(s)dB_s + \tfrac{1}{2}\int_0^t c^2(s)ds \Big) . \qquad (5.3.12)$$

Show that (5.3.11) can be written

$$d(F_t X_t) = F_t \cdot f(t, X_t)dt . \qquad (5.3.13)$$

b) Now define

$$Y_t(\omega) = F_t(\omega) X_t(\omega) \qquad (5.3.14)$$

so that

$$X_t = F_t^{-1} Y_t . \qquad (5.3.15)$$

Deduce that equation (5.3.13) gets the form

$$\frac{dY_t(\omega)}{dt} = F_t(\omega) \cdot f(t, F_t^{-1}(\omega)Y_t(\omega)) ; \qquad Y_0 = x . \qquad (5.3.16)$$

Note that this is just a *deterministic* differential equation in the function $t \to Y_t(\omega)$, for each $\omega \in \Omega$. *We can therefore solve (5.3.16) with ω as a parameter to find $Y_t(\omega)$ and then obtain $X_t(\omega)$ from (5.3.15).*

c) Apply this method to solve the stochastic differential equation

$$dX_t = \frac{1}{X_t}dt + \alpha X_t dB_t ; \qquad X_0 = x > 0 \qquad (5.3.17)$$

where α is constant.

d) Apply the method to study the solutions of the stochastic differential equation

$$dX_t = X_t^\gamma dt + \alpha X_t dB_t ; \qquad X_0 = x > 0 \qquad (5.3.18)$$

where α and γ are constants.
For what values of γ do we get explosion?

5.17. (The Gronwall inequality)
Let $v(t)$ be a nonnegative function such that

$$v(t) \le C + A \int_0^t v(s)ds \qquad \text{for } 0 \le t \le T$$

for some constants C, A. Prove that

$$v(t) \le C \exp(At) \qquad \text{for } 0 \le t \le T . \qquad (5.3.19)$$

(Hint: We may assume $A \neq 0$. Define $w(t) = \int\limits_0^t v(s)ds$. Then $w'(t) \leq C + Aw(t)$. Deduce that

$$w(t) \leq \frac{C}{A}(\exp(At) - 1) \tag{5.3.20}$$

by considering $f(t) := w(t)\exp(-At)$.
Use (5.3.20) to deduce (5.3.19.)