

Math 735 - Fall 2020

Homework 1

Due: 10/14

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Question 1

Proof. If Y has a single continuous path, then it means there exists a ω s.t. $Y_t(\omega) = 0$ for all t . This is equivalent to

$$\exists \omega \in [0, 1] \text{ s.t. } t \neq \omega \forall t \in \mathbb{R}_+$$

This is impossible, so Y does not have a single continuous path.

X and Y are modifications of each other means

$$\mathbb{P}(X_t = Y_t) = \mathbb{P}(Y_t = 0) = \mathbb{P}(t \neq \omega) = 1$$

For any fixed ω , this is true. So, X and Y are modifications of each other. □

Question 2

Proof. We need prove \widetilde{B}_t is a Gaussian process.

$$\widetilde{\mathbb{P}}(t, x) = \mathbb{P}(\lambda^2 t, \lambda x) = \frac{1}{\sqrt{2\pi\lambda^2 t}} e^{-\frac{(\lambda x)^2}{2\lambda^2 t}} = \frac{1}{\lambda} \cdot \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} = \frac{1}{\lambda} \mathbb{P}(t, x)$$

So, $\widetilde{B}_t \sim \frac{1}{\lambda} N(0, t) = N(0, \frac{t}{\lambda^2})$ □

Question 3

Proof.

$$\begin{aligned} E[B_t^k] &= \int_{-\infty}^{\infty} x^k \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \\ &= \frac{1}{k+1} x^{k+1} \cdot \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \Big|_{-\infty}^{\infty} + \frac{1}{t(k+1)} \int_{-\infty}^{\infty} x^{k+2} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \\ &= \frac{1}{t(k+1)} E[B_t^{k+2}] \end{aligned}$$

$$E[B_t^{k+2}] = t(k+1)E[B_t^k]$$

We know $E[B_t^0] = 1$, $E[B_t^1] = 0$, so we have

When $k = 2j$,

$$E[B_t^k] = t^j \prod_{i=1}^j (2i-1)$$

When $k = 2j-1$

$$E[B_t^k] = 0$$

□

Question 4

Proof. Stopping time

1. Play until there is no money or play 500 games before stopping.
2. Play N games to stop, $\{\tau = N\}$
3. Flip a coin N times, stop at the first appearance of *head, tail, head*.

Not stopping time

1. Stopping gambling when the gambler gets the maximum amount of money he can win is not a stopping time.
Explanation: Because it needs not only the information of the present and the past, but also the information of the future.
2. Stopping when the gambler doubles his wager is not a stopping time.
Explanation: Because there is a positive probability that he will never double his money.
3. Sold the day before the stock fell is not a stopping time.
Explanation: Because we don't know the future information.

□

Question 5

(L^p martingale convergence for $p > 1$ in discrete time)

Show that if $\{M_n\}$ is a martingale that satisfies

$$\mathbb{E}[|M_n|^p] \leq B < \infty$$

for some $p > 1$ and for all $n \geq 0$, then there exists a random variable M_∞ with $\mathbb{E}[|M_\infty|^p] \leq B$ such that

$$\mathbb{P}\left(\lim_{n \rightarrow \infty} M_n = M_\infty\right) = 1 \text{ and } \lim_{n \rightarrow \infty} \|M_n - M_\infty\|_p = 0$$

Proof. By Jensen's inequality,

$$f(x) = x^p \text{ is a convex function for } p > 1$$

It gives $f(\mathbb{E}[|M_n|]) \leq \mathbb{E}[f(|M_n|)] \Rightarrow \mathbb{E}[|M_n|]^p \leq \mathbb{E}[|M_n|^p] \leq B < \infty$.

We need prove

$$\mathbb{P}\left(\liminf_{n \rightarrow \infty} M_n \leq a < b \leq \limsup_{n \rightarrow \infty} M_n\right) = 0, \text{ for any } a < b$$

By Doob's maximal inequality, we know

$$\mathbb{P}\left(\liminf_{n \rightarrow \infty} M_n \leq a < b \leq \limsup_{n \rightarrow \infty} M_n\right) \leq \mathbb{P}\left(\sup_{m \leq k < \infty} |M_k - M_m| \geq \varepsilon\right) \leq \mathbb{E}[M_\infty - M_n]$$

This set will converge to 0.

$$\mathbb{E}[|M_n|^p] \leq B < \infty$$

shows that

$$\|M_\infty - M_n\|_2 \rightarrow 0$$

□

Question 6

Proof. Assume that M_t is a submartingale. By Theorem 3.6, we know

$$\mathbb{E}[M_{\tau \wedge T} | \mathcal{F}_s] \geq M_{\tau \wedge T \wedge s}$$

We know $M_{\tau \wedge T \wedge s} = M_{\tau \wedge s}$, since $s < T$. Thus

$$\mathbb{E}[M_{\tau \wedge T} | \mathcal{F}_s] \geq M_{\tau \wedge s}$$

Hence, $M_{\tau \wedge s}$ is also a submartingale. In Ex3.2, we know B is a standard Brownian motion and $\tau = \inf t \geq 0 : B_t = 1$. By Stopping Time Theorem, we have $\mathbb{E}[B_{\tau \wedge t}] = \mathbb{E}[B_0] = 0$.

However,

$$\mathbb{E}[B_\tau] = -P(B_\tau = -1) = -1 \neq 0$$

So, Theorem3.6 cannot hold without the truncation by T . □

Question 7

Proof. Consider the 1D-symmetric random walk with

$$\forall i \in \mathbb{N}, P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$$

Set

$$S_n = \sum_{i=1}^n X_i$$

the sum of first n -th terms of X_i , it is a martingale.

Let σ be the time arrive position Y , and let $\tau = \sigma + 1$. Here σ is a stopping time.

If τ is also a stopping time,

$$\mathbb{E}[S_\tau | \mathcal{F}_\sigma] = S_\sigma$$

Then, we have

$$\mathbb{E}[S_\tau] = \mathbb{E}[S_\sigma] = \mathbb{E}[S_{\sigma-1}] = \cdots = 0$$

However,

$$\mathbb{E}[S_\tau] = Y \neq 0$$

□