Math 733 – Fall 2020

Homework 3

Due: 10PM, October 11, 2020. Please upload your work in Canvas. Late homework will not be accepted.

1. Suppose that X and Y are independent. Find the distribution of $X+Y$ if
   (a) X has Binomial(n,p) and Y has Binomial(m,p) distribution.
   (b) X has Poisson(λ) and Y has Poisson(μ) distribution.

   (Check the posted summary on Canvas for the definition of these distributions.)

2. Suppose that X and Y are independent with CDF F_X and F_Y.
   (a) Express the CDF of XY in terms of F_X and F_Y.
   (b) Suppose that X and Y are absolutely continuous with PDF f_X and f_Y. Show that XY is absolutely continuous and find its PDF.

3. Suppose that $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ as $n \to \infty$, and all these random variables are defined on the same probability space. Show that $X_n + Y_n \xrightarrow{P} X+Y$ and $X_n Y_n \xrightarrow{P} XY$.

4. Let $f : [0,1] \to \mathbb{R}$ be a bounded, three time continuously differentiable function. Evaluate the following limit:
$$\lim_{n\to\infty} \int_0^1 \int_0^1 \dots \int_0^1 n \left(f\left(\frac{1}{n}(x_1 + x_2 + \dots + x_n)\right) - f(1/2)\right) dx_1\, dx_2 \dots dx_n$$
   Hint: express the integral as an expectation.

5. Suppose that $X_1, X_2, \dots$ are independent and identically distributed non-negative integer valued random variables with common CDF $F(x)$. Assume that $F(0) < 1$ and let $F^{(n)}$ denote the n-fold convolution of F. (This is the convolution of n copies of F.)
   (a) For each $x > 0$ find a random variable N_x for which $E[N_x] = \sum_{n=1}^{\infty} F^{(n)}(x)$.
   (b) Show that $\sum_{n=1}^{\infty} F^{(n)}(x)$ is finite for all $x \geq 0$.
   (c) Is the previous statement true if we don't assume that the X_i are integer valued?

6. Suppose that $X_1, X_2, \dots$ are i.i.d. Cauchy distributed random variables, i.e. X_1 has PDF $\frac{1}{\pi(1+x^2)}$. Let $S_n = X_1 + \dots + X_n$, and suppose that $c_n > 0$ is a sequence with $c_n/n \to \infty$ as $n \to \infty$. Show that $\frac{S_n}{c_n} \xrightarrow{P} 0$ as $n \to \infty$.
   Here are two possible approaches:
   (1) Adjust the proof of the (general) Weak Law of Large Numbers discussed in class.
   (2) Show that S_n/n has the same distribution as X_1.
   (The second approach shows that the condition $c_n/n \to \infty$ is sharp.)

You can use the Piazza page to ask for clarifications about a specific problem, but please don't discuss explicit solutions before the deadline. Handing in plagiarized work, whether copied from a fellow student or off the web, is not acceptable and will lead to sanctions.

Bonus problem.
A random walker moves on the lattice $\mathbb{Z}^2$ according to the following rule: in the first step it moves to one of its neighbors with probability $1/4 - 1/4$, and then in step $n > 1$ it moves to one of the three neighbors that it didn't visit in the step $n-1$ with equal probability (independently of its history before step $n-1$). Let T be the time when the random walker steps on a site that it already visited. Show T has a finite expectation. What is the best bound you can give on $E[T]$?

Bonus problems are not graded, and you don't need to submit them. They are provided as an extra challenge for those who are interested.