Itô integrals (continued)

$$\int_0^T f\, dB \quad \in L^2(\Omega)$$

where $f \in \mathcal{H}^2[0,T] = \mathcal{H}^2$.

$$0 \le t \le T \quad \text{fix}$$

$$\int_0^t f\, dB \overset{?}{=} \int_0^T \mathbb{1}_{[0,t]} \cdot f\, dB$$

If $X \in L^2(\Omega)$. $A \subseteq \Omega$ $\mathbb{P}(A) = 0$.

X can be equal to anything on A.

Thm: $\forall f \in \mathcal{H}^2[0,T]$.

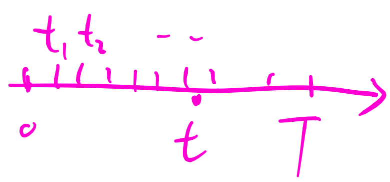
There is a stoch process $\{X_t\}_{t \in [0,T]}$.

which is a continuous martingale.

s.t. $\mathbb{P}\left(X_t = \int_0^T \mathbb{1}_{[0,t]} f\, dB\right) = 1 \quad \forall t \in [0,T]$

Pf: $\exists f_n \in \mathcal{H}_0^2[0,T]$ $\|f_n - f\|_{L^2([0,T]\times\Omega)} \to 0$

$$X_t^{(n)} = \int_0^T \mathbb{1}_{[0,t]} f_n\, dB$$



$$= a_k(\omega)(B_t - B_{t_k})$$

$$t_k < t \le t_{k+1}$$

$$+ \sum_{i=0}^{k-1} a_i(\omega)(B_{t_{i+1}} - B_{t_i})$$

$X_t^{(n)}$ is continuous martingale.

$$\mathbb{P}\left(\sup_{t\in[0,T]} \left|X_t^{(n)} - X_t^{(m)}\right| \geq \varepsilon\right)$$

max ineq

$$\leq \frac{1}{\varepsilon^2} \mathbb{E}\left[\left|X_T^{(n)} - X_T^{(m)}\right|^2\right]$$

Itô

$$\leq \frac{1}{\varepsilon^2} \| f_n - f_m \|^2_{L^2([0,T]\times\Omega)}$$

$$f_n \xrightarrow{L^2} f$$

$\exists\, n_k$, s.t $\max_{n \geq n_k} \| f_n - f_{n_k} \|^2_{L^2([0,T]\times\Omega)} \leq 2^{-3k}$

take $\varepsilon = 2^{-k}$

$$\mathbb{P}\left(\sup_{t\in[0,T]} \left|X_t^{(n_{k+1})} - X_t^{(n_k)}\right| \geq 2^{-k}\right) \leq 2^{-k}$$

Borel-Cantelli Lemma:

$\exists\, \Omega_0 \subseteq \Omega$, $\mathbb{P}(\Omega_0) = 1$, and $C(\omega) < \infty$ for $\omega \in \Omega_0$

$$\sup_{t\in[0,T]} \left|X_t^{(n_{k+1})}(\omega) - X_t^{(n_k)}(\omega)\right| \leq 2^{-k} \quad \forall k \geq C(\omega)$$

summable.

So, $\forall \omega \in \Omega_0$. $\{X_t^{n_k}(\omega)\}$ is Cauchy in uniform norm on $C[0,T]$

So, $\forall \omega \in \Omega_0$ $\exists$ continuous function $t \mapsto X_t(\omega)$

s.t. $X_t^{(n_k)}(\omega) \to X_t(\omega)$ uniformly on $[0,T]$

$$\int_0^T \mathbb{1}_{[0,t]} f \, dB$$

$$\mathbb{1}_{[0,t]} f_{n_k} \to \mathbb{1}_{[0,t]} f \quad \text{in } L^2([0,T]\times\Omega)$$

$$\Rightarrow \underbrace{\int_0^T \downarrow dB}_{X^{(n_k)}} \to \int_0^T \downarrow dB \quad \text{in } L^2(\Omega)$$

$$X^{(n_k)} \longrightarrow X_t \quad \text{in } L^2(\Omega)$$

$$\Rightarrow \left\| X_t - \int_0^T \mathbb{1}_{[0,t]} f \, dB \right\|_{L^2(\Omega)} = 0 \qquad \blacksquare$$