Math 714 – Fall 2020

Homework 2 – Tony Yuan

Github link for Matlab codes: https://github.com/YeTonyYuan/Math-714

A(a) Let $v = \sum_{j=1}^{n} a_j w_j$. Then

$$< v, w_k >=< \sum_{j=1}^{n} a_j w_j, w_k >=< a_k w_k, w_k >= a_k ||w_k||^2,$$

$a_k = \frac{<v,w_k>}{||w_k||^2}$, $v = \sum_{j=1}^{n} a_j w_j = \sum_{j=1}^{n} \frac{<v,w_j>}{||w_j||^2} w_j$.

A(b i) The solution can lie in a subspace of R^N, so it can take less than N operations to obtain the solution.

A(b ii) When $n = 1, p_1 = r_1 - \frac{<r_1,p_0>_A}{||p_0||_A^2} p_0$, and

$$< p_1, p_0 >_A=< r_1 - \frac{< r_1, p_0 >_A}{||p_0||_A^2} p_0, p_0 >_A=< r_1, p_0 >_A - \frac{< r_1, p_0 >_A}{||p_0||_A^2} ||p_0, p_0||_A^2 = 0.$$

Suppose the statement is true for $n = k$. Let $n = k+1$. For $0 \leq j < k+1$,

$$< p_{k+1}, p_j >_A=< r_{k+1} - \sum_{k=0}^{n} \frac{\langle r_{k+1}, p_k \rangle_A}{\|p_k\|_A^2} p_k, p_j >_A=< r_{k+1}, p_j >_A - \sum_{k=0}^{n} \frac{\langle r_{k+1}, p_k \rangle_A}{\|p_k\|_A^2} < p_k, p_j >_A .$$

By induction we know $< p_k, p_j >_A= 0$ except when $k = j$.Thus,

$$< p_{k+1}, p_j >_A=< r_{k+1}, p_j >_A - \frac{\langle r_{k+1}, p_j \rangle_A}{\|p_j\|_A^2} < p_j, p_j >_A= 0.$$

The statement is true for $n = k+1$. By induction, it is true for all $1 \leq n \leq n^* - 1$.

A(c i) $v = \sum_{j=1}^{N} \frac{<v,\phi_j>}{||\phi_j||^2} \phi_j$, $w = \sum_{k=1}^{N} \frac{<w,\phi_k>}{||\phi_k||^2} \phi_k$.

$$\begin{aligned} < Av, w > &= \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{< v, \phi_j >}{||\phi_j||^2} \frac{< w, \phi_k >}{||\phi_k||^2} < A\phi_j, \phi_k > && (1) \\ &= \sum_{n=1}^{N} \frac{< v, \phi_n >}{||\phi_n||^2} \frac{< w, \phi_n >}{||\phi_n||^2} \lambda_n < \phi_n, \phi_n > && (2) \\ &= \sum_{n=1}^{N} \lambda_n < v, \phi_n >< w, \phi_n > . && (3) \end{aligned}$$

A(a ii) For all $1 \leq j \leq N$, $< A\phi_n, \phi_n >= \lambda_n ||\phi_n||^2$. Since A is positive definite, $< A\phi_n, \phi_n >> 0$. So $\lambda_n > 0$ for all n.

A(a iii)

$$< Av, v > = \sum_{n=1}^{N} \lambda_n < v, \phi_n >< \phi_n, v > \tag{4}$$
$$\leq \lambda_N \sum_{n=1}^{N} < (< v, \phi_n > \phi_n), v > \tag{5}$$
$$= \lambda_N < \sum_{n=1}^{N} < v, \phi_n > \phi_n, v > \tag{6}$$
$$= \lambda_N ||v||^2. \tag{7}$$

Similarly we can obtain the other direction.

(a iv)

$$||Av||^2 =< Av, Av > = \sum_{n=1}^{N} \lambda_n < v, \phi_n >< \phi_n, Av > \tag{8}$$
$$\leq \lambda_N \sum_{n=1}^{N} < (< v, \phi_n > \phi_n), Av > \tag{9}$$
$$= \lambda_N < v, Av > \tag{10}$$
$$\leq \lambda_N \lambda_N ||v||^2. \tag{11}$$

$||Av|| \leq \lambda_N ||v||$.

(d)

$$p_{k+1} = r_{k+1} + \beta_k p_k \tag{12}$$
$$= r_k - \alpha_k w_k + \beta_k p_k \tag{13}$$
$$= p_k - \beta_{k-1} p_{k-1} - \alpha_k A p_k + \beta_k p_k \tag{14}$$
$$= (1 + +\beta_k) p_k - \alpha_k A p_k - \beta_{k-1} p_{k-1}. \tag{15}$$

(e) Let $det(\lambda I_N - A) = p(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$. By Cayley-Hamilton theorem, $p(A) = 0$. Then, $A^N = -c_{n-1}A^{n-1} + \cdots - c_1 A - c_0$.

(f i) $e_{n+1} = u_{n+1} - u = u_n + \alpha(f - Au_n) - u = u_n + \alpha(Au - Au_n) - u = (I - \alpha A)(u_n - u) = (I - \alpha A)e_n$.

(f ii) Eigenvalues of $I - \alpha A$ are $1 - \alpha\lambda_j$. Then by part (c iv),

$$||e_{n+1}|| = ||(I - \alpha A)e_n|| \leq \max_{1 \leq j \leq N} |1 - \alpha\lambda_j| \, ||e_n||.$$

(f iii) To minimize ρ, $\alpha > 0$. ρ is minimized when $|1 - \alpha\lambda_1| = |1 - \alpha\lambda_N|$, i.e., $\alpha = \frac{2}{\lambda_1 + \lambda_N}$.

(f iv) $\rho = max\{1 - \frac{2}{c+C}\lambda_1, \frac{2}{c+C}\lambda_N - 1\} \leq max\{1 - \frac{2c}{c+C}, \frac{2C}{c+C} - 1\} = \frac{C-c}{C+c}$.

(g i) $r_1 = r_0 - \alpha_0 w_0 = r_0 - \alpha_0 A p_0 = r_0 - \alpha_0 A r_0$.

(g ii)

$$\begin{aligned}
r_{n+1} &= r_n - \alpha_n w_n & (16)\\
&= r_n - \alpha_n A p_n & (17)\\
&= r_n - \alpha_n A(r_n + \beta_{n-1} p_{n-1} & (18)\\
&= r_n - \alpha_n A r_n - \alpha_n \beta_{n-1} w_{n-1} & (19)\\
&= r_n - \alpha_n A r_n - \alpha_n \beta_{n-1} \frac{r_{n-1} - r_n}{\alpha_{n-1}}. & (20)
\end{aligned}$$

(g iii) $Ar_0 = \gamma_0(r_0 - r_1), Aq_0 = \gamma_0 q_0 - \gamma_0 \frac{||r_1||}{||r_0||} q_1 = \gamma_0 q_0 - \delta_0 q_1$.

$\alpha_n A r_n = -r_{n+1} + r_n - \alpha_n \beta_{n-1} \frac{r_{n-1} - r_n}{\alpha_{n-1}}, Aq_n = -\delta_{n-1} q_{n-1} + \gamma_n q_n - \delta_n q_{n+1}$.

(g iv) This is just the matrix form of (iii).

(g v) $Q_n^T A Q_n = Q_n^T (Q_n T_n - \delta_{n-1} q_n e_n^T) = T_n - 0$ by orthogonality.

(B) The smallest value of N is 100.

(C(i))

$$\frac{u_{j_1,j_2}^{n+1} - 2u_{j_1,j_2}^n + u_{j_1,j_2}^{n-1}}{\Delta t^2} = \frac{u_{j_1-1,j_2}^n + u_{j_1,j_2-1}^n + u_{j_1+1,j_2}^n + u_{j_1,j_2+1}^n - 4u_{j_1,j_2}^n}{\Delta x^2}.$$

$$u_{j_1,j_2}^{n+1} = \frac{\Delta t^2}{\Delta x^2}(u_{j_1-1,j_2}^n + u_{j_1,j_2-1}^n + u_{j_1+1,j_2}^n + u_{j_1,j_2+1}^n - 4u_{j_1,j_2}^n) + 2u_{j_1,j_2}^n - u_{j_1,j_2}^{n-1}.$$

Using the initial conditions and ghost time points,

$$u_{j_1,j_2}^1 = \Delta t\, u_{j_1,j_2}^0$$

On the boundary I have $\frac{\partial^2}{\partial x^2} u_{0,j_2} = \frac{u_{2,j_2} - 2u_{1,j_2} + u_{0,j_2}}{\Delta x^2}$.

(b)

$$\frac{y^{n+1} - 2y^n + y^{n-1}}{\Delta t^2} = \lambda y^n.$$

So $\rho - 2 + 1/\rho = \lambda \Delta t^2$. The region of stability doesn't exist.

(c) $-4\frac{\Delta t^2}{\Delta x^2} + 2 \geq 0,\;\; 2\Delta t^2 \leq \Delta x^2$.

(d)

$$\frac{(g(k)^2 - 2g(k) + 1)}{\Delta t^2} = \frac{exp(-ik_1 h) + exp(ik_1 h) - 2 + exp(-ik_1 h) + exp(ik_1 h) - 2}{h^2}$$

$g(k)^2 - 2g(k) + 1 = 4\frac{\Delta t^2}{h^2}(\sin^2(k_1 h/2) + \sin^2(k_2 h/2)) \leq 2\frac{\Delta t^2}{h^2}$. Therefore, we need $2\frac{\Delta t^2}{h^2} \leq 1$, $2\Delta t^2 \leq \Delta x^2$. This is the same from (c).

(e)

$$
\begin{aligned}
&\frac{u(x,y,t+\Delta t)-2u(x,y,t)+u(x,y,t-\Delta t)}{\Delta t^2} \\
&-\frac{u(x+\Delta x,y,t)+u(x-\Delta x,y,t)+u(x,y+\Delta x,t)+u(x,y-\Delta x,t)-4u(x,y,t)}{\Delta x^2} \\
&=(u_{tt}+\frac{\Delta t^2}{12}u^{tttt}+\cdots)-(u_{xx}+\frac{\Delta x^2}{12}u^{xxxx}+(u_{yy}+\frac{\Delta x^2}{12}u^{xxxx}+\cdots) \\
&=(u_{tt}-\Delta u)+1/12\cdot(\Delta t^2(u_{xxxx}+2u_{xxyy}+u_{yyyy})-\Delta x^2(u_{xxxx}+u_{yyyy}))+cdots
\end{aligned}
$$

since $u_{tttt}=(u_{xx}+u_{yy})_{tt}=u_{xxxx}+2u_{xxyy}+u_{yyyy}$

Thus, we have $u_{xxxx}-2u_{xxyy}+u_{yyyy}=0$. Let $u=\exp(ik_1j_1\Delta x)\exp(ik_2j_2\Delta x)$. Then $k_1^4-2k_1^2k_2^2+k_2^4=0$, $k_1=k_2$. The extra terms make the waves dissipative.