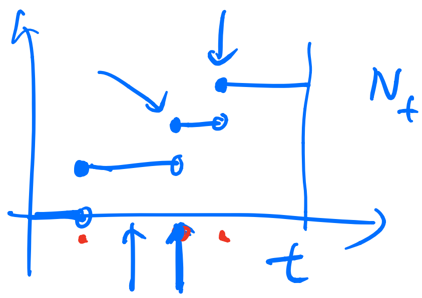
Stochastic Integration of Predictable processes

$$\int_0^t X_s \, dY_s$$

X_s: Predictable process.

Y_s: cadlag semimartingale

$$\int_0^t N_s \, dN_s$$



$$\| $$

$$1\cdot 1 + 2 \cdot 1 + 3 \cdot 1 + \cdots + N_t \cdot 1$$

$$= \frac{1}{2} N_t (N_t + 1)$$

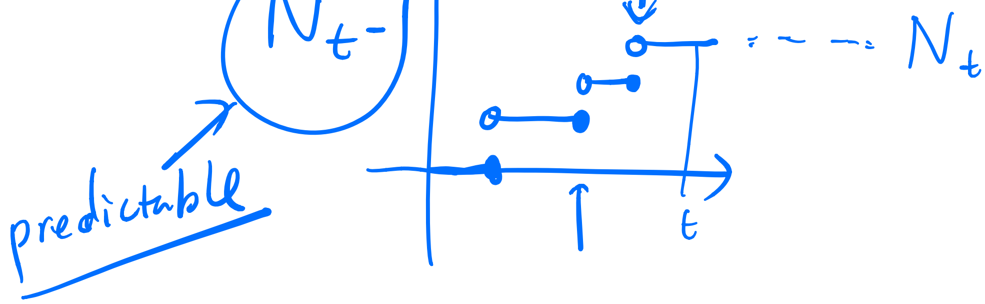
$$\int_0^t N_{s-} \, dN_s = ?$$

$$N_{s-} = \lim_{u \uparrow s} N_u$$

if N jumps at s

then $N_s \neq N_{s-}$

$N_s = N_{s-} + 1$



$$\int_0^t N_{s-}\, dN_s = 0\cdot 1 + 1\cdot 1 + \cdots + (N_t - 1)\cdot 1$$
$$= \frac{1}{2}(N_t - 1) N_t$$

"If you know what happens $< t$.

then you can predict $= t$"

N_t not.

"predictable"

$X(t, \omega)$, $t \in \mathbb{R}_+$, $\omega \in \Omega$.

$\mathbb{R}_+ \times \Omega$

$\{\mathcal{F}_t\}$

- predictable rectangles are subsets of $\mathbb{R}_+ \times \Omega$ of the type $(s, t] \times F$ where $s < t$, $F \in \mathcal{F}_s$

- $\mathcal{R}$ collection of all pred. rect's.

- The σ-field generated by $\mathcal{R}$ in $\mathbb{R}_+ \times \Omega$ (denoted $\mathcal{P}$) is called predictable σ-field.

• Any $\mathcal{P}$-measurable function $X: \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is called a predictable process.

Property. A pred process is not only adapted to $\{\mathcal{F}_t\}$ but also adapted to $\{\mathcal{F}_{t-}\}$

$$\left(\mathcal{F}_{t-} = \sigma\left(\bigcup_{s<t} \mathcal{F}_s\right) \right)$$

Lemma: The following σ-fields on $\mathbb{R}_+ \times \Omega$ are equal to $\mathcal{P}$:

(a) σ-field generated by all continuous adapted processes

(b) σ-field generated by all left-cont. adapted processes.

(c) - - - - - - all left-cont. with right-limits processes.

In particular, all cont. adapted → preditable.

left cont adapted → predictable

right-cont. ↛ left cont adapted

deterministic process

BM: $\mathbb{E}\left(\left(\int_0^t X\,dB\right)^2\right) = \mathbb{E}\int_0^t \boxed{X_s^2}\, ds$

$\boxed{\mathbb{R}_+ \times \Omega}$

$\int_0^t X\, dY$

Doléans measure:

Given a L^2 cadlag mart. M.

We define its Doléan measure

μ_M on $\mathcal{P}$ by

$\forall A \in \mathcal{P}$

$$\mu_M(A) = \mathbb{E}\int_0^\infty \mathbb{1}_A(t,\omega)\, d[M]_t(\omega)$$

$\boxed{\text{Ex}}$ $\mu_M(\underbrace{[0,T]\times\Omega}_{\in\mathcal{P}}) = \mathbb{E}\int_0^T d[M]_t = \mathbb{E}([M]_T)$

$= \mathbb{E}(M_T^2 - M_0^2) < \infty$

$\mathbb{E}[M] \overset{\text{def}}{=} \lim \sum_{(s,t)} \mathbb{E}(M_t - M_s)^2$

$\mathbb{E}\left[(M_t - M_s)^2\right] = \mathbb{E}M_t^2 - 2\,\mathbb{E}M_t M_s + \mathbb{E}M_s^2$

$\mathbb{E}M_t M_s = \mathbb{E}\,\mathbb{E}[M_t M_s \mid \mathcal{F}_s]$

$\|$

$\mathbb{E} M_t^2 - \mathbb{E} M_s^2$

$= \mathbb{E}[M_s M_s]$

$\boxed{\text{Ex}}$ If $M = B$. $[B]_t = t$ $\qquad (\Omega, \mathcal{F}, \mathbb{P})$

$$\mu_B(A) = \mathbb{E}\int_0^\infty \mathbb{1}_A(t,\omega)\,dt = \underbrace{(m \otimes \mathbb{P})}_{\text{on } \mathbb{R}_+ \times \Omega}(A)$$

$\boxed{\text{Ex}}$ If M = Compensated Poisson.

$M = N - \alpha t$. $\quad N$ is Poisson with rate α.

$$\mu_M = \alpha\, m \otimes \mathbb{P}$$

$$[M] = N.$$

For pred rect $A = (s,t] \times F$ $\quad \boxed{F \in \mathcal{F}_s}$

$$\mu_M(A) = \mathbb{E}\int_0^\infty \mathbb{1}_A(u,\omega)\, d[M]_u(\omega)$$

$$= \mathbb{E}\int_0^\infty \mathbb{1}_F(\omega)\, \mathbb{1}_{(s,t]}(u)\, dN_u(\omega)$$

$$= \mathbb{E}\left[\mathbb{1}_F(\omega)\left(N_t(\omega) - N_s(\omega)\right)\right]$$

$$= \mathbb{E}(\mathbb{1}_F)\, \mathbb{E}(N_t - N_s)$$

$$= \mathbb{P}(F) \cdot \alpha(t-s) = (\alpha\, m \otimes \mathbb{P})(A)$$

Fact: $\mu_M = \alpha m \otimes \mathbb{P}$ on $\mathcal{R}$ (Timo's notes Lem B.5)

then. ---- on $\mathcal{P}$.

For pred process X. define L^2 norm over $[0,T] \times \Omega$ under meas μ_M. by

$$\|X\|_{\mu_M, T} = \left(\int_{[0,T]\times\Omega} |X|^2 \, d\mu_M \right)^{1/2}$$

$$= \left(E \int_0^T |X(t,\omega)|^2 \, d[M]_t(\omega) \right)^{1/2}$$

Let $\underline{L_2}$ denote the collection of all pred processes X s.t. $\|X\|_{\mu_M,T} < \infty$ $\forall T < \infty$

<u>rmk</u>. metric

$$\|X\|_{L_2} = \sum_{k=1}^{\infty} 2^{-k} \left(1 \wedge \|X\|_{\mu_M, k} \right)$$

$\|X - Y\|_{L_2}$.

$\boxed{Ex}$ BM. $X \in L_2$ if and only if

$$E \int_0^T X(s,\omega)^2 ds < \infty \quad \forall T < \infty$$