Product measures

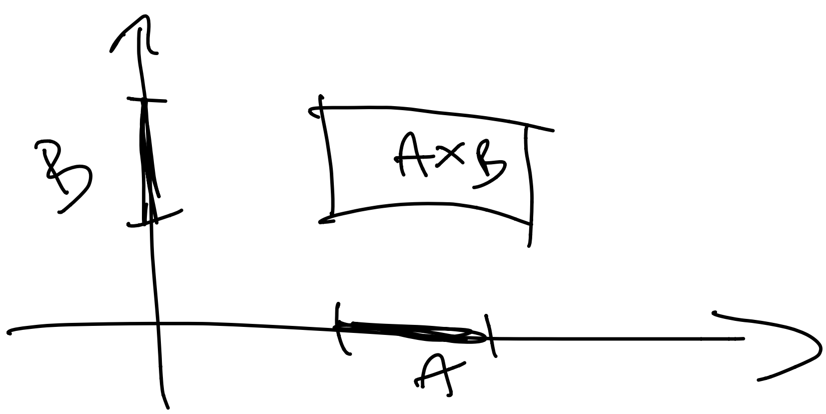
Suppose that $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$, $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ are probability spaces.

Construct a new one with

$\Omega = \Omega_1 \times \Omega_2$

$\mathcal{F}$ should contain all sets of the form $A \times B$ with $A \in \mathcal{F}_1$, $B \in \mathcal{F}_2$

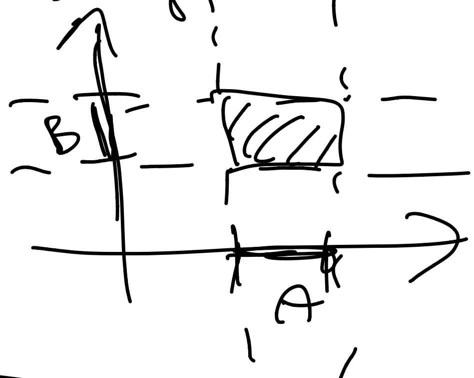
$\mathbb{P}(A \times B) = \mathbb{P}_1(A) \times \mathbb{P}_2(B)$



Thm: With $\mathcal{F} = \sigma(\mathcal{F}_1 \times \mathcal{F}_2)$ (this is the σ-field generated by sets of the form $A \times B$ with $A \in \mathcal{F}_1$, $B \in \mathcal{F}_2$)

there is a unique $\mathbb{P}$ on $(\Omega, \mathcal{F})$, $\Omega = \Omega_1 \times \Omega_2$,
with $\mathbb{P}(A \times B) = \mathbb{P}_1(A)\, \mathbb{P}_2(B)$.

Proof: The sets $A \times B$ form a semi-algebra so we can use the previous thm.



You can iterate this to define the product of finitely many probability spaces.

Soon: infinitely many probability spaces

Thm (Fubini) $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$, $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$

$f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ measurable

If $\boxed{f \geq 0 \text{ or } \int_{\Omega_1 \times \Omega_2} |f| \, d(\mathbb{P}_1 \times \mathbb{P}_2) < \infty}$

$$\int_{\Omega_1} \left(\int_{\Omega_2} f \, d\mathbb{P}_2 \right) d\mathbb{P}_1 = \int_{\Omega_2} \left(\int_{\Omega_1} f \, d\mathbb{P}_1 \right) d\mathbb{P}_2 = \int_{\Omega_1 \times \Omega_2} f \, d(\mathbb{P}_1 \times \mathbb{P}_2)$$

Independence

Def: $(\Omega, \mathcal{F}, P)$ is a prob space

1) $A, B \in \mathcal{F}$ are independent if
$$P(AB) = P(A)\,P(B)$$

2) Two random variables $X, Y : \Omega \to \mathbb{R}$ are independent if $\{X \in A\}, \{Y \in B\}$ are independent for any A, B Borel set.
$$P(X \in A, Y \in B) = P(X \in A)\,P(Y \in B)$$

3) Two σ-fields $\mathcal{A}, \mathcal{B} \subseteq \mathcal{F}$ are independent if all pairs $A \in \mathcal{A}$, $B \in \mathcal{B}$ are independent.

Simple observations

- X and Y are independent if and only if $\sigma(X)$ and $\sigma(Y)$ are independent.
- if $\mathcal{A}$ and $\mathcal{B}$ are independent and $X \in \mathcal{A}$, $Y \in \mathcal{B}$ then X and Y are independent
- A, B are independent $\iff$ 1_A and 1_B are independent

$\Updownarrow$

$\sigma(\{A\})$ and $\sigma(\{B\})$ are independent

$\sigma(\{A\}) = \{A, A^c, \phi, \Omega\}$

Ex: $(\Omega_1, \mathcal{F}_1, P_1)$, $(\Omega_2, \mathcal{F}_2, P_2)$ are prob spaces

Consider the product prob. space on $\Omega_1 \times \Omega_2$.

1, If $A \in \mathcal{F}_1$, $B \in \mathcal{F}_2$ then $A \times \Omega_2$, $\Omega_1 \times B$ are independent

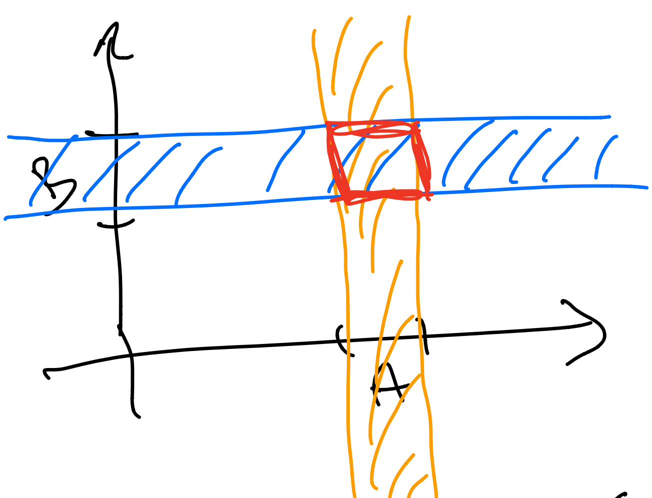
$$P(A \times \Omega_2) = P_1(A)\, P_2(\Omega_2) = P_1(A)$$

$$P(\Omega_1 \times B) = P_2(B)$$

$$P(A \times \Omega_2 \cap \Omega_1 \times B) = P(A \times B)$$

$$= P_1(A)\, P_2(B)$$

$$= P(A \times \Omega_2)\, P(\Omega_1 \times B)$$



2, The σ-fields $\{A \times \Omega_2 : A \in \mathcal{F}_1\}$,

$\{ \Omega_1 \times B : B \in \mathcal{F}_2 \}$ are independent

3) If $X, Y : \Omega_1 \times \Omega_2 \to \mathbb{R}$ are random variables so that X only depends on the first coordinate and Y only on the 2^{nd} then X and Y are independent.

Definition: We say that $\underbrace{\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n}_{\text{the } \sigma\text{-fields}} \subset \mathcal{F}$ are independent, if for every $A_i \in \mathcal{F}_i$ we have
$$P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i)$$

Random variables $X_1, X_2, \dots, X_n$ are independent if the generated σ-fields are independent.

Events $A_1, A_2, \dots, A_n$ are independent if the generated σ-fields are independent.

Claim: $A_1, A_2, \dots, A_n$ are independent if and only if

1) for any $I \subset \{1, \dots, n\}$ we have
$$P\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} P(A_i)$$
$2^n - n - 1$ equations

2) $$P\left(\bigcap_{i=1}^{n} A_i^*\right) = \prod_{i=1}^{n} P(A_i^*)$$

A_i^* : A_i or A_i^c $\qquad$ 2^n equations

Q: how can we construct a prob space with independent random variables with given distributions?

Pairwise independence of events:

$A_1, A_2, \dots, A_n$ are pairwise independent if for any $i \neq j$ A_i and A_j are independent.

This is weaker than the "full" independence

Example: Suppose that X_1, X_2, X_3 are independent Bernoulli$(\frac{1}{2})$ random variables (independent coin flips)

Then the events $A_{ij} = \{X_i = X_j\}$ $i \neq j$ are pairwise independent, but not independent.

A_{12}, A_{13}, A_{23}

Ex: Suppose that $A_1, A_2, \ldots, A_k$ are finite subsets of $\mathbb{R}$. Consider the probability space of choosing a uniformly chosen element of $A_1 \times A_2 \times \ldots \times A_k$, and let X_j be the jth chosen index. Then $X_1, X_2, \ldots, X_k$ are independent.

If we consider the prob space of choosing uniformly from A_j, then the product of these probability spaces is choosing uniformly from $A_1 \times \ldots \times A_k$.

Q: How can we construct a prob space with independent random variables with given distributions?

For any distribution we realize that on $([0,1], \mathcal{B}, \underset{\text{Lebesgue}}{P})$ as a random variable.

Sufficient conditions for independence

Def: Collection of sets $\mathcal{A}_1, \dots, \mathcal{A}_n \subseteq \mathcal{F}$ are independent if for any subset $I \subset \{1, \dots, n\}$ $A_i \in \mathcal{A}_i$ then $P(\bigcap_{i \in I} A_i) = \prod_{i \in I} P(A_i)$.

(if $\Omega \in \mathcal{A}_i$ for each $1 \le i \le n$ then this is equivalent to $P(\bigcap_{i=1}^n A_i) = \prod_{i=1}^n P(A_i)$.)

Thm: Suppose that a collection of sets $\mathcal{A}_1, \dots, \mathcal{A}_n$ are independent, and each one is closed for intersection. Then the generated σ-fields $\sigma(\mathcal{A}_1), \sigma(\mathcal{A}_2), \dots, \sigma(\mathcal{A}_n)$ are also independent.

Def: π-system: closed for intersection

$\mathcal{B}$ is a λ-system: $\Omega \in \mathcal{B}$, if $A \subset B \in \mathcal{B}$ then $B \setminus A \in \mathcal{B}$

if $B_n \in \mathcal{B}$ with $B_n \nearrow B$ then $B \in \mathcal{B}$.

Lemma: If $\mathcal{A}$ is a π-system and $\mathcal{A} \subset \mathcal{B}$ is a λ-system then $\sigma(\mathcal{A}) \subset \mathcal{B}$.

Outline of the proof of the theorem using the lemma: Assume that $\Omega \in \mathcal{A}_i$.

Fix $A_i \in \mathcal{A}_i$ $i \geq 2$. Let $F = \bigcap_{i=2}^{n} A_i$.

$\mathcal{B} \stackrel{def}{=} \{ A \in \mathcal{F} : P(AF) = P(A)P(F) \}$

Check: $\mathcal{B}$ is a λ-system

$\sigma(\mathcal{A}_1) \subseteq \mathcal{B}$ which means that $\sigma(\mathcal{A}_1), \mathcal{A}_2, \mathcal{A}_3, \ldots, \mathcal{A}_n$ are independent. Now iterate this to get that $\sigma(\mathcal{A}_1), \sigma(\mathcal{A}_2), \ldots \sigma(\mathcal{A}_n)$ are independent.

Lemma: $X_1, X_2, \ldots, X_n$ random variables are independent if and only if

$$P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) = \prod_{i=1}^{n} P(X_i \leq x_i)$$

for all choices of $x_i \in (-\infty, \infty]$

joint CDF — product of the individual CDFs