MATH714 HW2

Xin Wen

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1 Problem A

1.1 (a)

Assume $v = \sum_{i=1}^{n} a_i w_i$, so $< v, w_i >= a_i \|w_i\|^2$, which means $\frac{<v,w_i>}{\|w_i\|^2} w_i = a_i w_i$.

From the equation above, we can get $v = \sum_{i=1}^{n} \frac{<v,w_i>}{\|w_i\|^2} w_i$

1.2 (b)

First for $n = 1$, $< p_1, p_0 >=< p_1, r_0 >=< r_1, r_0 > - < \frac{<r_0,r_0>}{\|r_0\|^2} r_0, r_0 >$.

$r_1 = f - Au_1, r_0 = f - Au_0$

$r_1^T A r_0 = (f - Au_1)^T A(f - Au_0)$

So $< p_1, p_0 >= 0$.

Now when $< p_n, p_j >= 0$ for any $0 \le j < n$,

for $j < n$, $< p_{n+1}, p_j >=< r_{n+1} - \sum_{i=1}^{n} \frac{<r_{n+1},p_i>}{\|p_i\|^2} p_i, p_j >=< r_{n+1}, p_j >=< f - Au_n, p_j >= 0$.

And $< p_{n+1}, p_n >=< r_{n+1} - \frac{<r_n,p_n>}{\|p_n\|^2} p_n, p_n >$. Like $< p_1, p_0 >$, it equals to 0.

1.3 (c)

1.3.1 i

From (a) we can get $v = \sum_{j=1}^{N} \frac{<v,\phi_j>}{\|\phi_j\|^2} \phi_j$ and $w = \sum_{j=1}^{N} \frac{<w,\phi_j>}{\|\phi_j\|^2} \phi_j$.

So $< Av, w >=< \sum_{j=1}^{N} \frac{<v,\phi_j>}{\|\phi_j\|^2} A\phi_j, \sum_{j=1}^{N} \frac{<w,\phi_j>}{\|\phi_j\|^2} \phi_j >= \sum_{j=1}^{N} \lambda_j < v, \phi_j >< \phi_j, w >$

1.3.2 ii

Since A is positive definite, $0 < \phi_n^T A \phi_n = \lambda_n \phi_n^T \phi_n$. Obviously, we can get $\lambda_n > 0$ from $\phi_n^T \phi_n \ge 0$

1.3.3 iii

From i we can know $< Av, v >= \sum_{n=1}^{N} \lambda_n < v, \phi_n >^2 = \sum_{n=1}^{N} \leq \lambda^n = \sum_{n=1}^{N} \lambda_n \|v\|^2 < v*, \phi_n >^2$ where $v* = \frac{v}{\|v^2\|}$

Since $\{\phi_n\}$ are orthogonal, $\lambda_1 \leq \sum_{n=1}^{N} < v*, \phi_n >^2 \leq \lambda_N$

1.3.4 iv

$\|Av\| = \sum_{i=1}^{N} < v, \phi_i > \lambda_i \phi_i \leq \lambda_N \|v\|$ as iii.

1.4 (d)

$p_{n+1} = r_{n+1} + \beta_n p_n = r_n - \alpha_n w_n + \beta_n p_n = r_n - \alpha_n A p_n + \beta_n p_n = p_n - \beta_{n-1} p_{n-1} - \alpha_n A p_n + \beta_n p_n = (1 + \beta_n) p_n - \alpha_n A p_n - \beta_{n-1} p_{n-1}$

1.5 (e)

Now the character polynomial of A is $p(\lambda) = \lambda^n + \sum_{i=0}^{n-1} a_i \lambda^i$.

From Cayley-Hamilton theorem, $p(A) = 0$, which means $A^n = -\sum_{i=0}^{n-1} a_i A^i$, where $A^0 = I_n$.

1.6 (f)

1.6.1 i

$e_{n+1} = u_{n+1} - u = u_n - u - \alpha A(u_n - u) + \alpha f - \alpha A u = e_n - \alpha A e_n = (I - \alpha A) e_n$

1.6.2 ii

$\|e_{n+1}\| = \|(I - \alpha A) e_n\| \leq \|I - \alpha A\| \|e_n\| \leq \max\{\lambda(I - \alpha A)\} \|e_n\| = \max\{1 - \alpha \lambda_j\} \|e_n\|$

1.6.3 iii

Now $\rho = |1 - \alpha \lambda_1|$ or $|1 - \alpha \lambda_N|$.

for each case, $\alpha = \frac{1}{\lambda}$ minimize ρ. But to minimize both cases, it's to minimize $(1 - \alpha \lambda_j)^2$. After calculating its Jacobi matrix, we can get $\alpha = \frac{2}{\lambda_1 + \lambda_N}$

1.6.4 iv

Now $\max |1 - \alpha \lambda_j| = \max |\frac{C + c - 2\lambda_j}{C + c}| \leq \max |\frac{C - c}{C + c}| = \frac{C - c}{C + c}$

1.7 g

1.7.1 i

$r_1 = f - A u_1 = f - A(u_0 + \alpha_0 (f - A u_0)) = f - A u_0 - \alpha A r_0 = r_0 - \alpha_0 A r_0$

1.7.2 ii

Just like i.

1.7.3 iii

$Aq_0 = A\frac{r_0}{\|r_0\|} = \frac{r_0 - r_0 + \alpha_0 A r_0}{\alpha_0 \|r_0\|} = \frac{r_0 - r_1}{\alpha_0\|r_0\|} = \frac{r_0}{\alpha_0\|r_0\|} - \frac{r_1}{\alpha_0\|r_0\|} = \frac{1}{\alpha_0}q_0 - \frac{\sqrt{\beta_0}}{\alpha_0}q_1$

$Aq_n = A\frac{r_n}{\|r_n\|} = \frac{r_{n-1} - \alpha_{n-1}Ar_{n-1} + \frac{\alpha_{n-1}\beta_{n-2}}{\alpha_{n-2}}(r_{n-1} - r_{n-2})}{\|r_n\|} = -\delta_{n-1}q_{n-1} + \gamma_n q_n - \delta_n q_{n+1}$

1.7.4 iv

From (iii), $Aq_0 = (Q_nT_n)_0 + 0$, $Aq_n = (Q_nT_n)_n - \delta_{n-1}q_n e_n^T$

1.7.5 v

$Q_n^T A Q_n = Q_n^T Q_n T_n - Q_n^T \delta_{n-1} q_n e_n^T = T_n$

2 Problem B

The result of the problem is $N = 100$, and the link to the code is:
https://github.com/Skystaryou/Study/tree/main/MATH714_MethodOfComputatnlMathI/linearInterpolant_HW2
Just run "main.m".

3 Problem C

3.1 (a)

Now the scheme is $\frac{u_{t+1}^{x,y} - 2u_t^{x,y} + u_{t-1}^{x,y}}{\Delta t^2} = \frac{u_t^{x+1,y} - 2u_t^{x,y} + u_t^{x-1,y}}{\Delta x^2} + \frac{u_t^{x,y+1} - 2u_t^{x,y} + u_t^{x,y-1}}{\Delta x^2}$
Dirichlet boundary condition: we can simply let $u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0$.
Initial condition: $u(x, y, 0) = 0$ and $\frac{u(x,y,\Delta t) - u(x,y,0)}{\Delta t} = f(x)f(y)$
The link to the code is:

https://github.com/Skystaryou/Study/tree/main/MATH714_MethodOfComputatnlMathI/2DWave_HW2
Just run main.m and the plot will show up.

3.2 (b)

Now $f'(u_{t+1})\ \frac{u_{t+2} - u_t}{2\Delta t}$, $f'(u_{t-1})\ \frac{u_t - u_{t-2}}{2\Delta t}$.
So the scheme: $\frac{u_{t+2} - 2u_t + u_{t-2}}{4\Delta t^2} = \lambda u(t)$