MATH 735 - Fall 2020

Homework 2

Due : 11/04, 2020

Zijie Zhang

November 2, 2020

Problem 1

X, Y are two independent Brownian motions, compute $[X, Y]$.

Proof. By (2.13) in *Timo's notes*.

$$[X, Y]_t = \lim_{|\pi| \to 0} \sum_i (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})$$

We need prove $\mathbb{E}\left[\sum_i (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})\right] \to 0$ which is

$$\sum_i \mathbb{E}\left[X_{t_{i+1}} Y_{t_{i+1}}\right] + \sum_i \mathbb{E}\left[X_{t_i} Y_{t_i}\right] - \sum_i \mathbb{E}\left[X_{t_i} Y_{t_{i+1}}\right] - \sum_i \mathbb{E}\left[X_{t_{i+1}} Y_{t_i}\right] \to 0$$

By the independence of X and Y, all the expectations above are 0. So

$$[X, Y] = 0 \text{ if } X, Y \text{ are two independent Brownian motions}$$

□

Problem 2

Compute the quadratic variations [N] and [M] where N is Poisson process and M is compensated Poisson process.

1.
$$[N]_t = \sum_{0 \leqslant s \leqslant t} (\nabla N_s)^2 = N_t$$
$$[N] = N$$

2.
$$M = N - \lambda t$$

By Lemma A.10 and Lemma A.11, we know that $[f](T) = 0$ if f is continuous. So we have

$$(\nabla(N_s - \lambda s))^2 = (\nabla N_s)^2$$

Thus,

$$[M] = N$$

Problem 3

Suppose M is a right-continuous square-integrable martingale with stationary independent increments: for all $s, t \geqslant 0, M_{s+t} - M_s$ is independent of $\mathcal{F}_s$ and has the same distribution as $M_t - M_0$. Then $\langle M \rangle_t = t \cdot E[M_1^2 - M_0^2]$

Proof. The deterministic, continuous function $t \to t \cdot E[M_1^2 - M_0^2]$ is predictable. For any $t > 0$ and integer k

$$E[M_{kt}^2 - M_0^2] = \sum_{j=0}^{k-1} E[M_{(j+1)t}^2 - M_{jt}^2] = \sum_{j=0}^{k-1} E[(M_{(j+1)t} - M_{jt})^2] = kE[(M_t - M_0)^2] = kE[M_t^2 - M_0^2]$$

Using this twice, for any rational k/n,

$$E[M_{k/n}^2 - M_0^2] = kE[M_{1/n}^2 - M_0^2] = (k/n)E[M_1^2 - M_0^2]$$

Given an irrational $t > 0$, pick rationals $q_n \to t$. Fix $T \geqslant q_m$. By right-continuity of paths, $M_{q_m} \to M_t$ almost surely. Uniformly integrability of $\{M_{q_m}^2\}$ follows by the submartingale property

$$0 \leqslant M_{q_m}^2 \leqslant E[M_T^2 | \mathcal{F}_{q_m}]$$

and Lemma B.16. Uniformly integrability gives convergence of expectations $E[M_{q_m}^2] \to E[M_t^2]$. Applying this above gives

$$E[M_t^2 - M_0] = tE[M_1^2 - M_0^2]$$

Now we can check the martingale property.

$$\begin{aligned}
E[M_t^2 | \mathcal{F}_s] &= M_s^2 + E[M_t^2 - M_s^2 | \mathcal{F}_s] \\
&= M_s^2 + E[(M_t - M_s)^2 | \mathcal{F}_s] \\
&= M_s^2 + E[(M_{t-s} - M_0)^2] \\
&= M_s^2 + E[M_{t-s}^2 - M_0^2] \\
&= M_s^2 + (t-s)E[M_1^2 - M_0^2]
\end{aligned}$$

□

Problem 4