Itô integrals

Doob

Itô

Riemann-Stieltjes $\int f\,dg$

If f cont.
g is BV (bound variation)
Then exists

$\int_0^t f_{(s)}\,dB_s$ ← price of this stock at s per unit

$f_{(s)}$ ↑ how much stock you hold at s

$\underbrace{\phantom{f_{(s)}dB_s}}$ how much you gain $s \to s+ds$

→ net gain $[0, t]$

Assumption on f: $[0, T]$ $\int_0^T$ $(\Omega, \mathcal{F}_t)$

- $f(t, \omega)$ is measurable wrt $\mathcal{B} \times \mathcal{F}_T$ $\quad \mathcal{F}_T$
  $\mathbb{R}_+$ Ω
  $\mathcal{B}$: Borel on $\mathbb{R}_+$
- f is adapted to $\{\mathcal{F}_t\}$.
  $\forall t \quad f(t, \cdot) \in \mathcal{F}_t$

$\mathcal{H}^2$ space of all meas. adapted f s.t.

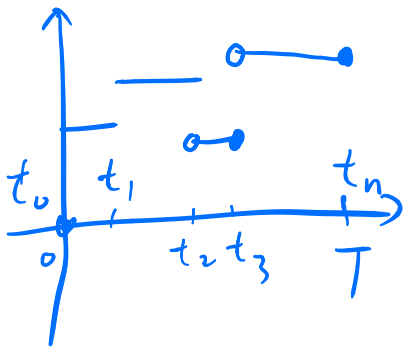
$$\mathbb{E}\int_0^T f^2(t, \omega)\,dt < \infty$$

$$\| f \|^2_{L^2([0,T]\times\Omega)}$$

$$f = 1_{[a,b]}, \quad [a,b] \subset [0,T]$$

$$\int_0^T f_t \, dB_t = \int_a^b dB_t = B_b - B_a$$

① $f(t,\omega) = \sum_{i=0}^{n-1} a_i(\omega) \, 1_{(t_i, t_{i+1}]}$



② $a_i \in \mathcal{F}_{t_i}$

③ $E[a_i^2] < \infty \quad \forall i$

$$\mathcal{H}_0^2 \subset \mathcal{H}^2$$

$$\int_0^T f_t \, dB_t = \sum_{i=0}^{n-1} a_i(\omega) \left(B_{t_{i+1}} - B_{t_i}\right)$$

Lemma: Itô Isometry on $\mathcal{H}_0^2$

$$\left\| \int_0^T f \, dB \right\|_{L^2(\Omega)} = \|f\|_{L^2([0,T]\times\Omega)}$$

$$E\left[\left(\int_0^T f \, dB\right)^2\right] \qquad E\int_0^T f^2 \, dt$$

Pf: $f^2 = \sum_i a_i^2 \, 1_{(t_i, t_{i+1}]}$

RHS $= E\int_0^T f^2 \, dt = \sum_i E a_i^2 (t_{i+1} - t_i)$.

LHS =

$$\sum_i \mathbb{E}\left[a_i^2 (B_{t_{i+1}} - B_{t_i})^2 \right]$$

(a_i is indep of $B_{t_{i+1}} - B_{t_i}$)

$$= \sum_i \mathbb{E}(a_i^2) \underbrace{\cancel{\mathbb{E}((\quad)^2)}}_{(t_{i+1}-t_i)}$$

$\blacksquare$

Lemma: $\mathcal{H}_0^2$ is dense in $\mathcal{H}^2$.

$\forall f \in \mathcal{H}^2 \quad \exists f_n \in \mathcal{H}_0^2. \quad f_n \to f$

$\left(\|f - f_n\|_{L^2([0,T]\times\Omega)} \to 0 \right)$

$$\left\| \int_0^T f_n \, dB \right\|_{L^2(\Omega)} = \|f_n\|_{L^2([0,T]\times\Omega)}$$

$$\int_0^T f_n \, dB \downarrow \int_0^T f \, dB \qquad f_n \downarrow f$$

Thm Itô isometry hold on $\mathcal{H}^2$.

prop: $0 \le s \le t$. $f \in \mathcal{H}^2$

$$\mathbb{E}\left[\left(\int_s^t f(u,\omega)\,dB\right)^2 \Big| \mathcal{F}_s\right] = \mathbb{E}\left[\int_s^t f^2(u,\omega)\,du \Big| \mathcal{F}_s\right]$$

pf: It's enough to show $\forall A \in \mathcal{F}_s$

$$\mathbb{E}\left[\mathbb{1}_A\left(\int_s^t f\,dB\right)^2\right] = \mathbb{E}\left[\mathbb{1}_A \int_s^t f^2\,du\right]$$

$$\tilde{f}(u,\omega) = \begin{cases} \mathbb{1}_A f(u,\omega) & u \in (s,t] \\ 0 & u \le s \end{cases}$$

Apply Itô isometry to $\tilde{f}$ $\square$

Cor: $M = \left(\int_0^t f\,dB\right)^2 - \int_0^t f^2\,du$

is martingale.

$\underline{\mathbb{E}[M_t | \mathcal{F}_s] \overset{?}{=} M_s}$ use the prop above.

(In case $f = 1$, $M = \underline{B^2 - t}$)

$$\int_0^{\Theta} f \, dB.$$