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Math 714

Homework # 2

A.

a) $w_1, w_2, \dots, w_n$ orthogonal, $v \in \text{span}\{w_1, \dots, w_n\}$ then $v = \sum_{j=1}^{n} \frac{\langle v, w_j \rangle}{\|w_j\|^2} w_j$

Based on span: $v = \alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_n w_n$ ①

Taking inner product of v & w_1:

$$\langle v, w_1 \rangle = \langle \alpha_1 w_1, w_1 \rangle + \overset{0}{\cancel{\langle \alpha_2 w_2, w_1 \rangle}} + \dots + \overset{0}{\cancel{\langle \alpha_n w_n, w_1 \rangle}}$$

$$\langle v, w_1 \rangle = \alpha_1 \langle w_1, w_1 \rangle = \alpha_1 \|w_1\|^2$$

$$\Rightarrow \alpha_1 = \frac{\langle v, w_1 \rangle}{\|w_1\|^2}$$

For any $i = 1, \dots, n$ by similar analysis: $\alpha_i = \frac{\langle v, w_i \rangle}{\|w_i\|^2}$ ②

Combining ① & ②: $\boxed{v = \sum_{i=1}^{n} \frac{\langle v, w_i \rangle}{\|w_i\|^2} w_i}$

b) i) n^* may be strictly less than N because convergence occurs when the magnitude of the residual reaches a certain tolerance. This tolerance can be reached before the N^{th} iteration, allowing n^* to be strictly smaller than N.

ii) $$p_n = r_n - \sum_{j=0}^{n-1} \frac{\langle r_n, p_j \rangle_A}{\|p_j\|_A^2} p_j$$

For $n=1$: $p_1 = r_1 - \frac{\langle r_1, p_0 \rangle_A}{\|p_0\|_A^2} p_0$

$$\langle p_1, p_0 \rangle_A = \langle r_1, p_0 \rangle_A - \frac{\langle r_1, p_0 \rangle_A}{\|p_0\|_A^2} \langle p_0, p_0 \rangle_A$$

$$\langle p_1, p_0 \rangle_A = \langle r_1, p_0 \rangle_A - \langle r_1, p_0 \rangle_A = 0 \checkmark$$

Now assume result holds for p_{n-1}: $\langle p_{n-1}, p_j \rangle_A = 0$ for $0 \le j < n \le n^*-1$

Now for p_n:

$$p_n = r_n - \frac{\langle r_n, p_0 \rangle_A}{\|p_0\|_A^2} p_0 - \cdots - \frac{\langle r_n, p_{n-1} \rangle_A}{\|p_{n-1}\|_A^2} p_{n-1}$$

$$\langle p_n, p_j \rangle_A = \langle r_n, p_j \rangle_A - \frac{\langle r_n, p_0 \rangle_A}{\|p_0\|_A^2} \langle p_0, p_j \rangle_A - \cdots - \frac{\langle r_n, p_{n-1} \rangle_A}{\|p_{n-1}\|_A^2} \langle p_{n-1}, p_j \rangle$$

Since all p_j are orthogonal $\langle p_i, p_j \rangle = 0$ for $i \ne j$

$$\Rightarrow \langle p_n, p_j \rangle_A = \langle r_n, p_j \rangle_A - \frac{\langle r_n, p_j \rangle_A}{\|p_j\|_A^2} \langle p_j, p_j \rangle_A$$

$$= \langle r_n, p_j \rangle_A - \langle r_n, p_j \rangle_A$$

$$\Rightarrow \boxed{\langle p_n, p_j \rangle_A = 0 \text{ for } 0 \le j < n \le n^*-1}$$

c) $A\phi_n = \lambda_n \phi_n$ and $\langle \phi_n, \phi_j \rangle = \delta_{nj}$ $\{\phi\}$ orthonormal

$\lambda_1 \le \cdots \le \lambda_N$

i) $\langle Av, w \rangle$ $\qquad V = \sum_{n=1}^{N} \frac{\langle v, \phi_n \rangle}{\cancel{\|\phi_n\|}} \phi_n = \sum_{n=1}^{N} \langle v, \phi_n \rangle \phi_n$

$$Av = \sum_{n=1}^{N} A \langle v, \phi_n \rangle \phi_n = \sum_{n=1}^{N} \langle v, \phi_n \rangle (A\phi_n)$$

$$\Rightarrow Av = \sum_{n=1}^{N} \lambda_n \langle v, \phi_n \rangle \phi_n$$

$$\langle Av, w \rangle = \left\langle \sum_{n=1}^{N} \lambda_n \langle v, \phi_n \rangle \phi_n, w \right\rangle$$

$$\Rightarrow \boxed{\langle Av, w \rangle = \sum_{n=1}^{N} \lambda_n \langle v, \phi_n \rangle \langle \phi_n, w \rangle}$$

ii) $A\phi_N = \lambda \phi_N$

$\phi_N^T A \phi_N = \lambda \phi_N^T \phi_N$

$\phi_N^T A \phi_N = \lambda \|\phi_N\|^2$

$\Rightarrow \lambda = \dfrac{\phi_N^T A \phi_N}{\|\phi_N\|^2}$ Numerator is positive since A is positive definite, and $\|\phi_N\|^2 > 0$

Therefore $\dfrac{\phi_N^T A \phi_N}{\|\phi_N\|^2} > 0$

$\Rightarrow \boxed{\lambda_n > 0 \text{ for } 1 \le n \le N}$

iii) $\lambda_1 \|v\|^2 \le \langle Av, v\rangle \le \lambda_N \|v\|^2$

$\lambda_1 v \le Av \le \lambda_N v$ since λ_1 is smallest λ_i and λ_N is largest since λ positive

$\Rightarrow \langle \lambda_1 v, v\rangle \le \langle Av, v\rangle \le \langle \lambda_N v, v\rangle$

$\boxed{\lambda_1 \|v\|^2 \le \langle Av, v\rangle \le \lambda_N \|v\|^2}$ ✓

iv) $\|Av\| \le \lambda_n \|v\|$

Since A is symmetric positive definite

$Av \le \lambda_n v$

$\Rightarrow \|Av\| \le \|\lambda_n v\|$

$\boxed{\|Av\| \le \lambda_n \|v\|}$ ✓

d) $p_{n+1} = (1+\beta_n)p_n - \alpha_n A p_n - \beta_{n-1} p_{n-1}$

$p_{n+1} = r_{n+1} + \beta_n p_n$

$r_{n+1} = r_n - \alpha_n \omega_n \ , \ r_n = p_n - \beta_{n-1} p_{n-1}$

$\omega_n = A p_n$

$\Rightarrow \quad p_{n+1} = r_n - \alpha_n A p_n + \beta_n p_n$

$p_{n+1} = p_n - \beta_{n-1} p_{n-1} - \alpha A p_n + \beta_n p_n$

$$\Rightarrow \boxed{p_{n+1} = (1+\beta_n)p_n - \alpha A p_n - \beta_{n-1} p_{n-1}}$$

e) If $A \in \mathbb{R}^{N \times N}$ non-singular, A^N can be represented as a linear combination of $I, A, A^2, \dots, A^{N-1}$

Cayley-Hamilton Thm:

$$A^N + c_{N-1}A^{N-1} + \dots + c_1 A + c_0 I = 0$$

Therefore: $\boxed{A^N = -c_{N-1}A^{N-1} - c_{N-2}A^{N-2} - \dots - c_1 A - c_0 I}$

f) $Au = f \rightarrow u = u + \alpha(f - Au)$

$u_{n+1} = u_n + \alpha(f - Au_n)$

i) $e_n = u_n - u$

$$\begin{aligned} e_{n+1} &= u_{n+1} - u \\ &= u_n + \alpha(f - Au_n) - u \\ &= u_n + \alpha(Au - Au_n) - u \\ &= (u_n - u) - \alpha A(\underbrace{u_n - u}_{e_n}) \end{aligned}$$

$$\Rightarrow \boxed{e_{n+1} = (I - \alpha A)e_n}$$

ii) $\|e_{n+1}\| \leq \rho \|e_n\|$ $\quad \rho = \max_{1 \leq j \leq N} |1 - \alpha \lambda_j|$

$$\|e_{n+1}\| = \|(I - \alpha A) e_n\| \leq \rho \|e_n\|$$

$$\|e_n - \alpha A e_n\| \leq \|e_n - \alpha \lambda_j e_n\|$$

Since λ_j is $\min_j |\lambda_j|$ of A, $\quad A e_n \geq \lambda_j e_n$

As a result $\|e_n - \alpha A e_n\| \leq \|e_n - \alpha \lambda_j e_n\|$ ✓

$$\Rightarrow \boxed{\|e_{n+1}\| \leq \rho \|e_n\| \checkmark}$$ Return to original

iii) ρ is minimized by $\alpha = \frac{2}{\lambda_1 + \lambda_N}$

$$\Rightarrow \rho = \frac{\lambda_N - \lambda_1}{\lambda_N + \lambda_1} = \frac{K - 1}{K + 1} < 1 \quad \text{where } K = \frac{\lambda_N}{\lambda_1}$$

$$\rho = \max_{1 \leq j \leq N} |1 - \alpha \lambda_j|$$

For any α, $\max \rightarrow \lambda_j = \lambda_1$

$$\rho = \left|1 - \frac{2\lambda_1}{\lambda_1 + \lambda_N}\right| = \left|\frac{\lambda_1 + \lambda_N}{\lambda_1 + \lambda_N} - \frac{2\lambda_1}{\lambda_1 + \lambda_N}\right|$$

$$\boxed{\rho = \frac{\lambda_N - \lambda_1}{\lambda_N + \lambda_1} = \frac{K-1}{K+1} < 1} \checkmark$$

iv) $0 < c \leq \lambda_1 \leq \lambda_N \leq C < \infty$ $\quad \alpha = \frac{2}{c + C}$

$$\rho = \max_{1 \leq j \leq N} |1 - \alpha \lambda_j| = |1 - \alpha \lambda_1| \leq |1 - \alpha c|$$

$$\rho \leq \left|1 - \frac{2c}{c + C}\right| = \frac{C - c}{C + c}$$

$$\Rightarrow \boxed{\rho \leq \frac{C - c}{C + c} = \frac{k' - 1}{k' + 1} < 1 \quad k' = \frac{C}{c}}$$

g) i) $r_1 = r_0 - \alpha_0 w_0$

$w_0 = A\rho_0$

$\rho_0 = r_0$

$$\Rightarrow \boxed{r_1 = r_0 - \alpha_0 A r_0}$$ ✓

ii) $r_{n+1} = r_n - \alpha_n w_n$

$w_n = A\rho_n$

$\rho_n = r_n + \beta_{n-1}\rho_{n-1}$

$$\Rightarrow r_{n+1} = r_n - \alpha_n A(r_n + \beta_{n-1}\rho_{n-1})$$

$\rho_{n-1} = \frac{w_{n-1}}{A}$

$w_{n-1} = -\frac{(r_n - r_{n-1})}{\alpha_{n-1}}$

$$\Rightarrow \boxed{r_{n+1} = r_n - \alpha_n A r_n + \frac{\alpha_n \beta_{n-1}}{\alpha_{n-1}}(r_n - r_{n-1})}$$

iii) $\gamma_0 = \frac{1}{\alpha_0}$ $\gamma_n = \frac{1}{\alpha_n} + \frac{\beta_{n-1}}{\alpha_{n-1}}$ $\delta_n = \frac{\sqrt{\beta_n}}{\alpha_n}$

from i) $r_1 = r_0 - \alpha_0 A r_0$

$A = \frac{1 - r_1 r_0^{-1}}{\alpha} = \frac{1}{\alpha_0} - \frac{r_1 r_0^{-1}}{\alpha}$

$$\Rightarrow Aq_0 = \gamma_0 q_0 - \frac{r_1 r_0^{-1}}{\alpha} \cdot \frac{r_0}{\|r_0\|}$$

$$Aq_0 = \gamma_0 q_0 - \frac{1}{\alpha_0}\frac{r_1}{\|r_0\|}, \quad \beta_0 = \frac{\|r_1\|^2}{\|r_0\|^2}$$

$$\Rightarrow Aq_0 = \gamma_0 q_0 - \frac{\sqrt{\beta_0}}{\alpha_0}\frac{r_1}{\|r_1\|}$$

$$\boxed{Aq_0 = \gamma_0 q_0 - \delta_0 q_1}$$

from ii) $r_{n+1} = r_n - \alpha_n A r_n + \frac{\alpha_n \beta_{n-1}}{\alpha_{n-1}}(r_n - r_{n-1})$

$$\Rightarrow A = \frac{1 - r_{n+1} r_n^{-1}}{\alpha_n} + \frac{\beta_{n-1}}{\alpha_{n-1}}(1 - r_{n-1} r_n^{-1})$$

$$A q_n = \frac{q_n}{\alpha_n} - \frac{\sqrt{\beta_n}}{\alpha_n} \frac{r_{n+1}}{\|r_{n+1}\|} + \frac{\beta_{n-1}}{\alpha_{n-1}}\left(q_n - \frac{r_{n-1}}{\|r_n\|}\right)$$

$$A q_n = \gamma_n q_n - \delta_n q_{n+1} + \frac{\beta_{n-1}}{\alpha_{n-1}} \cdot \frac{1}{\sqrt{\beta_{n-1}}} \cdot \frac{r_{n-1}}{\|r_{n-1}\|}$$

$$\Rightarrow \boxed{A q_n = \gamma_n q_n - \delta_n q_{n+1} - \delta_{n-1} q_{n-1}}$$

iv) $Q_n = [q_0 \ q_1 \ \cdots \ q_{n-1}] \in \mathbb{R}^{N \times m}$

$$T_n = \begin{bmatrix} \gamma_0 & -\delta_0 & & \\ -\delta_0 & \ddots & \ddots & \\ & \ddots & \ddots & -\delta_{n-2} \\ & & -\delta_{n-2} & \gamma_{n-1} \end{bmatrix} \in \mathbb{R}^{n \times n} \qquad e_n^T = [0 \ 0 \ \cdots \ 1]$$

$$A Q_n = Q_n T_n - \delta_{n-1} q_n e_n^T$$

$\hookrightarrow A q_k = \gamma_k q_k - \delta_k q_{k+1} - \delta_{k-1} q_{n-1}$

$$Q_n T_n - \delta_{n-1} q_n e_n^T = \begin{bmatrix} \gamma_0 q_0 - \delta_0 q_1 & \gamma_1 q_1 - \delta_1 q_2 - \delta_0 \gamma_0 & \cdots \\ | & | & \end{bmatrix} - \begin{bmatrix} 0 & \cdots & \delta_{n-1} q_n \\ | & & | \end{bmatrix}$$

$$\hookrightarrow = \begin{bmatrix} \gamma_0 q_0 - \delta_0 q_1 & \gamma_1 q_1 - \delta_1 q_2 - \delta_0 \gamma_0 & \cdots & \gamma_{n-1} q_{n-1} - \delta_{n-1} q_n - \delta_{n-2} q_{n-2} \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} A q_0 & A q_1 & \cdots & A q_{n-1} \\ | & | & & | \end{bmatrix} \Rightarrow \boxed{A Q_n = Q_n T_n - \delta_{n-1} q_n e_n^T} \checkmark$$

v) $AQ_n = Q_nT_n - \delta_{n-1} q_n e_n^T$

$Q_n^T A Q_n = Q_n^T Q_n T_n - Q_n^T \delta_{n-1} q_n e_n^T$

Since Q_n orthogonal, $Q_n^T Q_n = I$

$\Rightarrow Q_n^T A Q_n = T_n - \delta_{n-1} \underbrace{Q_n^T q_n e_n^T}$

$$Q_n^T q_n e_n^T = \begin{bmatrix} q_0^T & - \\ \vdots & \\ q_{n-1}^T & - \end{bmatrix} \begin{bmatrix} 0 & \cdots & q_n \\ | & & | \end{bmatrix}$$

Again, Q_n is orthogonal $\rightarrow q_j^T q_n = 0$ for $j = 0, 1, \ldots, n-1$

$Q_n^T A Q_n = T_n - 0$

$\Rightarrow \boxed{Q_n^T A Q_n = T_n}$

MATH 714 HOMEWORK 2

Zachary Oliver

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1 Problem B.

The following function was given:

$$f(x) = e^{-400(x-0.5)^2} \tag{1}$$

The smallest value of N such that f differs from its linear interpolant by at most 10^{-2} for f computed on N+1 samples was determined numerically. It was determined that $N = 98$ was the smallest value of N. The Python code that was written to solve this problem will be provided.

2 Problem C.

2.1 Part (a)

This problem focuses on the 2D wave equation given by Equation 2 with homogeneous Dirichelet boundary conditions and the outlined initial conditions.

$$\begin{aligned} u_{tt} &= \Delta u \\ u(x, y, 0) &= 0 \\ u_t(x, y, 0) &= f(x)f(y) \end{aligned} \tag{2}$$

In the Neumann initial condition, f(x) and f(y) represent the function from Problem B. shown in Equation 1. This system was numerically solved using finite differences, with a 3-point formula for the second derivative in time and the 5-point Laplacian at time t_n. The resulting finite differences scheme was used to solve the system, where $x = jh$, $y = ih$, and $t = nk$:

$$U_{j,i}^{n+1} = 2U_{j,i}^{n} - U_{j,i}^{n-1} + \frac{k^2}{h^2}(U_{j-1,i}^{n} + U_{j,i-1}^{n} - 4U_{j,i}^{n} + U_{j+1,i}^{n} + U_{j,i+1}^{n}) \tag{3}$$

The scheme resulted in a two-step method for solving the system. The system was initialized through the use of the two given initial conditions. With the first Dirichlet initial condition, the grid points at U^0 were all set to 0. Then, using the Neumann initial condition, an expression for the grid points at U^1 was determined. A point outside of the time-domain was established, U^-1, and was used to represent both the first derivative and second derivative of u in time at $t = 0$ through the following expressions:

$$\frac{1}{k^2}(U^{-1} - 2U^0 + U^1) = \Delta u(t_0) = 0 \tag{4}$$

$$\frac{1}{2k}(U^1 - U^{-1}) = f(x)f(y) \tag{5}$$

Combining these two expressions can be completed to eliminate the point U^{-1} and the following expression for U^1 is determined. It should also be noted that $U^0 = 0$.

$$U^1 = kf(x)f(y) \tag{6}$$

This expression was used to initialize every grid point at t_1. Then, since the system was initialized, Equation 3 was completed for every proceeding time step. The code outlining this method is provided.

The numerical results were attempted to be compared to the analytical result of the 2D wave equation for the specific boundary and initial conditions, but there were errors in the code that prevented an accurate comparison. If the code had worked adequately, then max norm errors taken at multiple grid spacing values of h would have been taken and plotted on a log-log plot. This plot would likely show that the error is second-order dependent on the grid spacing.

2.2 Part (b)

The region of stability of the ODE, $y''(t) = \lambda y$ was determined. The three-point rule for the second derivative of y was used, which is shown in Equation 7.

$$y''(t) = \frac{Y^{n+1} - 2Y^n + Y^{n-1}}{(\Delta t)^2} = \lambda Y^n \tag{7}$$

Given this expression, the absolute stability can be analyzed. Equation 7 can be rearranged to the following:

$$Y^{n+1} - (2 + \lambda\Delta t^2)Y^n + Y^{n-1} = 0 \tag{8}$$

From the Difference Equation, we can insert the associated polynomial roots ρ^n as follows:

$$\rho - (2 + \lambda\Delta t^2) + \frac{1}{\rho} = 0 \tag{9}$$

Multiplying both sides by ρ yields the following result:

$$\rho^2 - (2 + \lambda\Delta t^2)\rho + 1 = 0 \tag{10}$$

The stability region is determined for when $|\rho| \leq 1$. Solving Equation 10 gives the result for the stability region.

$$\rho = \frac{(2 + \lambda\Delta t^2) \pm \sqrt{(-2 - \lambda\Delta t^2)^2 - 4}}{2} \tag{11}$$

$$\rho = \frac{(2 + \lambda\Delta t^2) \pm \sqrt{\lambda^2\Delta t^4 + 4\lambda\Delta t^2}}{2} \tag{12}$$

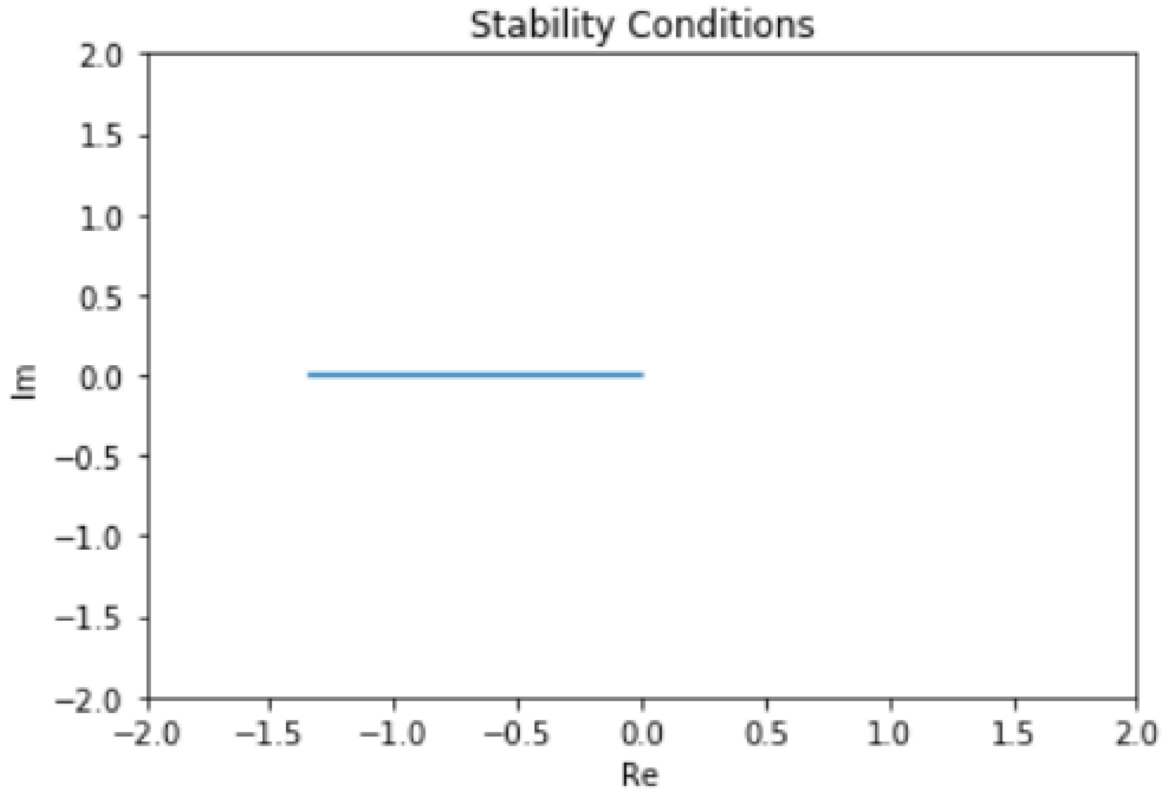
Therefore, for this expression to be less than or equal to 1, the following can be determined:

$$|(2 + \lambda\Delta t^2) \pm \sqrt{(\lambda\Delta t^2)^2 + 4\lambda\Delta t^2}| \leq 2 \tag{13}$$

For this expression to be true, the following must hold.

$$\lambda\Delta t^2 = \pm\sqrt{(\lambda\Delta t^2)^2 + 4\lambda\Delta t^2} \tag{14}$$

In order for this to be true, $-\frac{4}{3} \leq \lambda\Delta t^2 \leq 0$.


Stability Conditions
2.0
1.5
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