CS714 Homework 2

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Question 1.

(a)
$\because v \in span\{\omega_1, \omega_2, \ldots, \omega_n\}$
$\therefore v = \sum_{i=1}^{n} \alpha_i \omega_i$
We know $\langle v, \omega_j \rangle = \sum_{i=1}^{n} \alpha_i \langle \omega_i, \omega_j \rangle$
$\because \langle \omega_i, \omega_j \rangle = 0$ whenever $i \neq j$
$\therefore \langle v, \omega_j \rangle = \sum_{i=1}^{n} \alpha_i \langle \omega_i, \omega_j \rangle = \alpha_i \langle \omega_j, \omega_j \rangle = \alpha_i \|\omega_i\|^2$
$\therefore \alpha_i = \frac{\langle v, \omega_j \rangle}{\|\omega_i\|^2}$
$\therefore v = \sum_{j=1}^{n} \frac{\langle v, \omega_j \rangle}{\|\omega_i\|^2} \omega_j$

(b)
i.
If rank(A)=$n^* < N$, then we only need n^* steps to converge.
Therefore, the number of iterations to converge, n^*, may be strictly smaller than N.

ii.
We know $p_0 = r_0$ and $r_n = p_n + \sum_{j=0}^{n-1} \frac{\langle r_n, p_j \rangle_A}{\|p_j\|_A^2} p_j$, so $r_n \in span\{p_0, p_1, \ldots, p_n\}$
$p_1 = r_1 - \frac{\langle r_1, p_0 \rangle_A}{\|p_0\|_A^2} p_0$ and $r_1 \in span\{p_0, p_1\} \rightarrow p_1$ and p_0 are A conjugated.

Prove by induction on n:
Assume p_i and p_j are A conjugated for $i \neq j$ and $i, j \in \{0, ..., n-1\} \rightarrow \langle p_i, p_j \rangle_A = 0$.

For $i = n$:
$\because p_n = r_n - \sum_{j=0}^{n-1}$
$\therefore \langle p_n, p_j \rangle_A = \langle r_n, p_j \rangle_A - \langle \sum_{i=0}^{n-1} \frac{\langle r_n, p_i \rangle_A}{\|p_i\|_A^2} p_i, p_j \rangle_A = \langle r_n, p_j \rangle_A - \sum_{i=0}^{n-1} \frac{\langle r_n, p_i \rangle_A}{\|p_i\|_A^2} \langle p_i, p_j \rangle_A$
If $i = j$, then $\frac{\langle r_n, p_i \rangle_A}{\|p_i\|_A^2} \langle p_i, p_j \rangle_A = \frac{\langle r_n, p_i \rangle_A}{\|p_i\|_A^2} \langle p_j, p_j \rangle_A = \langle r_n, p_j \rangle_A$
If $i \neq j$, then $\frac{\langle r_n, p_i \rangle_A}{\|p_i\|_A^2} \langle p_i, p_j \rangle_A = \frac{\langle r_n, p_i \rangle_A}{\|p_i\|_A^2} \times 0 = 0$
$\therefore \langle p_n, p_j \rangle_A = \langle r_n, p_j \rangle_A - \langle r_n, p_j \rangle_A = 0$
Therefore, $\langle p_n, p_j \rangle_A = 0$ for $0 \leq j < n \leq n^* - 1$

(c)
i.
$\because v, w \in \mathbb{R}^N$
$\therefore v = \sum_{i=1}^{N} \alpha_i \phi_i$, and $w = \sum_{j=1}^{N} \beta_j \phi_j$
$Av = A \sum_{i=1}^{N} \alpha_i \phi_i = \sum_{i=1}^{N} \lambda_i \alpha_i \phi_i$
$\langle Av, w \rangle = \langle \sum_{i=1}^{N} \lambda_i \alpha_i \phi_i, w \rangle = \langle \sum_{i=1}^{N} \lambda_i \alpha_i \phi_i, \sum_{j=1}^{N} \beta_i \phi_j \rangle$
$\because \phi_1, \ldots, \phi_n$ are orthonormal basis for $\mathbb{R}^N$
$\therefore \langle \phi_i, \phi_j \rangle = 0$ for $i \neq j$ and $\langle \phi_i, \phi_i \rangle = 1$

$\therefore \langle Av, w\rangle = \sum_{i=1}^{N} \lambda_i \alpha_i \beta_i$
$\because \langle v, \phi_n\rangle = \langle \sum_{i=1}^{N} \alpha_i \phi_i, \phi_n\rangle = \alpha_n$ and $\langle \phi_n, w\rangle = \langle \phi_n, \sum_{j=1}^{N} \beta_j \phi_j\rangle = \beta_n$
$\therefore \langle Av, w\rangle = \sum_{n=1}^{N} \lambda_n \langle v, \phi_n\rangle\langle \phi_n, w\rangle$

ii.
We have $A\phi_1 = \lambda_1 \phi_1$
$\because$ matrix A is symmetric positive definite matrix
$\therefore \phi_1^\top A\phi_1 > 0$ and $A^\top = A$
$\phi_1^\top A\phi_1 = (A\phi_1)^\top \phi_1 = \langle A\phi_1, \phi_1\rangle = \lambda_1 > 0$
$\because \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$
$\therefore \lambda_n > 0$ for $1 \leq n \leq N$

iii.
$\langle Av, v\rangle = \sum_{n=1}^{N} \lambda_n \langle v, \phi_n\rangle\langle \phi_n, v\rangle$
$\because v \in \mathbb{R}^N$
$\therefore v = \sum_{i=1}^{N} \alpha_i \phi_i$
$\langle Av, v\rangle = \sum_{n=1}^{N} \lambda_n \langle v, \phi_n\rangle\langle \phi_n, v\rangle = \sum_{n=1}^{N} \lambda_n \langle \sum_{i=1}^{N} \alpha_i\phi_i, \phi_n\rangle\langle \phi_n, \sum_{i=1}^{N} \alpha_i \phi_i\rangle = \sum_{n=1}^{N} \lambda_n \alpha_n^2$
$\because \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ and $\|v\|^2 = \sum_{n=1}^{N} \alpha_n^2$
$\therefore \lambda_1 \|v\|^2 \leq \langle Av, v\rangle \leq \lambda_N \|v\|^2$

iv.
$\because v \in \mathbb{R}^N$
$\therefore v = \sum_{i=1}^{N} \alpha_i \phi_i$
$\|Av\| = \left\| A(\sum_{i=1}^{N} \alpha_i \phi_i)\right\| = \left\|\sum_{i=1}^{N} \alpha_i A\phi_i\right\| = \left\|\sum_{i=1}^{N} \alpha_i \lambda_i \phi_i\right\| \leq \lambda_N \left\|\sum_{i=1}^{N} \alpha_i\phi_i\right\| = \lambda_N \|v\|$
$\therefore \|Av\| \leq \lambda_N \|v\|$

(d)
$\because r_{n+1} = r_n - \alpha_n \omega_n$ and $\omega_n = Ap_n$
$\therefore r_{n+1} = r_n - \alpha_n Ap_n$
We have $p_{n+1} = r_{n+1} + \beta_n p_n$ and $p_n = r_n + \beta_{n-1} p_{n-1}$
Then $p_{n+1} = r_{n+1} + \beta_n p_n = r_n - \alpha_n Ap_n + \beta_n p_n = p_n - \beta_{n-1}p_{n-1} - \alpha_n Ap_n + \beta_n p_n$
Therefore, $p_{n+1} = (1 + \beta_n)p_n - \alpha_n Ap_n - \beta_{n-1}p_{n-1}$ for $1 \leq n \leq n^* - 2$

(e)
Assume matrix A is non-singular, then $\det|A| \neq 0$
Using Cayley-Hamilton theorem, we have $p(\lambda) = \det(\lambda I - A)$
$p(A) = \det(AI - A) = 0 = A^N + \alpha_{N-1}A^{N-1} + \cdots + \alpha_1 A + (-1)^N \det|A| I_N$
Therefore, A^N is a linear combination of $I, A, A^2, \ldots, A^{N-1}$.

(f)
i.
$\because u_{n+1} = u_n + \alpha(f - Au) = u_n + \alpha(Au - Au_n)$
$\therefore u_{n+1} - u = u_n - u + \alpha(Au - Au_n) \rightarrow e_{n+1} = (I - \alpha A)e_n$

ii.
Assume matrix A has eigenvalue $\lambda_1, \ldots, \lambda_N$ with corresponding orthonormal eigenvector $e_i, \ldots, e_N$.
Then $\|A\| = \left\|\sum_i^N \lambda_i e_i\right\| \leq \max_{1\leq i\leq N} |\lambda_i| = \rho(A)$

From above, we have $\|e_{n+1}\| = \|(I-\alpha A)e_n\| \le \|I-\alpha A\| \, \|e_n\|$
Similarly, we have $\|I-\alpha A\| \le \max_{1\le j\le N} |1-a\lambda_j|$
Therefore, $\|e_{n+1}\| \le \rho \, \|e_n\|$, where $\rho = \max_{1\le j\le N} |1-a\lambda_j|$

iii.
We know $\lambda_1 < \lambda_2 < \cdots < \lambda_N$
$\because 1-\alpha\lambda_j$ is a linear equation
$\therefore$ the maximum value of $|1-\alpha\lambda_j|$ is either $|1-\alpha\lambda_1|$ or $|1-\alpha\lambda_N|$
To minimize $|1-\alpha\lambda_j|$, we should have $(1-\alpha\lambda_1) = -(1-\alpha\lambda_N)$
$1-\alpha\lambda_1 = -1(1-\alpha\lambda_N) \to \alpha = \frac{2}{\lambda_1+\lambda_N}$

iv.
For $\alpha = \frac{2}{c+C}$, $\rho = \max_{1\le j\le N} |1-a\lambda_j| = \max_{1\le j\le N} |1-\frac{2}{c+C}\lambda_j| = \max_{1\le j\le N} |\frac{c+C-2\lambda_j}{c+C}|$
$\because c \le \lambda_1 \le \lambda_j \le \lambda_N \le C$ and the maximum value is obtained at either λ_1 or λ_N
$\therefore \rho \le \frac{C-c}{C+c} = \frac{\kappa^{'}-1}{\kappa^{'}+1} \le 1$, where $\kappa^{'} = \frac{C}{c}$

(g)
i.
We know $r_1 = r_0 - \alpha_0\omega_0$, where $\omega_0 = Ap_0$ and $p_0 = r_0$
$\therefore r_1 = r_0 - \alpha_0 Ap_0 = r_0 - \alpha_0 Ar_0$

ii.
We know $r_{n+1} = r_n - \alpha_n\omega_n$, where $\omega_n = Ap_n$
$\therefore r_{n+1} = r_n - \alpha_n Ap_n$
$\because p_n = r_n + \beta_{n-1}p_{n-1}$
$\therefore r_{n+1} = r_n - \alpha_n A(r_n + \beta_{n-1}p_{n-1}) = r_n - \alpha_n Ar_n - \alpha_n\beta_{n-1}p_{n-1}$
$\because r_n = r_{n-1} - \alpha_{n-1}Ap_{n-1}$
$\therefore r_{n+1} = r_n - \alpha_n Ar_n + \alpha_n\beta_{n-1}\frac{r_n - r_{n-1}}{\alpha_{n-1}} = r_n - \alpha_n Ar_n + \frac{\alpha_n\beta_{n-1}}{\alpha_{n-1}}(r_n - r_{n-1})$
$\therefore r_{n+1} == r_n - \alpha_n Ar_n + \frac{\alpha_n\beta_{n-1}}{\alpha_{n-1}}(r_n - r_{n-1})$ for $1 \le n \le n^* - 1$

iii.
$\because r_1 = r_0 - \alpha_0 Ar_0$
$\therefore Ar_0 = \frac{r_0}{\alpha_0} - \frac{r_1}{\alpha_0}$
$\therefore A\frac{r_0}{\|r_0\|} = \frac{1}{\alpha_0}\frac{r_0}{\|r_0\|} - \frac{1}{\alpha_0}\frac{r_1}{\|r_0\|}$
$\because \beta_0 = \frac{\|r_1\|^2}{\|r_0\|^2}$
$\therefore A\frac{r_0}{\|r_0\|} = \frac{1}{\alpha_0}\frac{r_0}{\|r_0\|} - \frac{\sqrt{\beta_0}}{\alpha_0}\frac{r_1}{\|r_1\|}$
$\because q_0 = \frac{r_0}{\|r_0\|}$, $q_1 = \frac{r_1}{\|r_1\|}$, $\gamma_0 = \frac{1}{\alpha_0}$, and $\delta_0 = \frac{\sqrt{\beta_0}}{\alpha_0}$
$\therefore Aq_0 = \gamma_0 q_0 - \delta_0 q_1$

$\because r_{n+1} = r_n - \alpha_n Ar_n + \frac{\alpha_n\beta_{n-1}}{\alpha_{n-1}}(r_n - r_{n-1})$
$\therefore \alpha_n Ar_n = -\frac{\alpha_n\beta_{n-1}}{\alpha_{n-1}}r_{n-1} + \left(1 + \frac{\alpha_n\beta_{n-1}}{\alpha_{n-1}}\right)r_n - r_{n+1}$
$\therefore A\frac{r_n}{\|r_n\|} = -\frac{\beta_{n-1}}{\alpha_{n-1}}\frac{r_{n-1}}{\|r_n\|} + \left(\frac{1}{\alpha_n} + \frac{\beta_{n-1}}{\alpha_{n-1}}\right)\frac{r_n}{\|r_n\|} - \frac{1}{\alpha_n}\frac{r_{n+1}}{\|r_n\|}$
$\because \beta_n = \frac{\|r_{n+1}\|^2}{\|r_n\|^2}$ and $\beta_{n-1} = \frac{\|r_n\|^2}{\|r_{n-1}\|^2}$

$\therefore A\frac{r_n}{\|r_n\|} = -\frac{\sqrt{\beta_{n-1}}}{\alpha_{n-1}}\frac{r_{n-1}}{\|r_{n-1}\|} + \left(\frac{1}{\alpha_n} + \frac{\beta_{n-1}}{\alpha_{n-1}}\right)\frac{r_n}{\|r_n\|} - \frac{\sqrt{\beta_n}}{\alpha_n}\frac{r_{n+1}}{\|r_{n+1}\|}$
$\because q_n = \frac{r_n}{\|r_n\|}$, $\gamma_n = \frac{1}{\alpha_n} + \frac{\beta_{n-1}}{\alpha_{n-1}}$, and $\delta_n = \frac{\sqrt{\beta_n}}{\alpha_n}$
$\therefore Aq_n = -\delta_{n-1}q_{n-1} + \gamma_n q_n - \delta_n q_{n+1}$ for $1 \leq n \leq n^* - 1$

iv.
Assume $\delta_{-1} = 0$
Reorganize above two equations, we have

$$\begin{bmatrix} Aq_0 \\ Aq_1 \\ \dots \\ Aq_{n-1} \end{bmatrix}^\top = \begin{bmatrix} \gamma_0 q_0 - \delta_0 q_1 \\ -\delta_0 q_0 + \gamma_1 q_1 - \delta_2 q_2 \\ \dots \\ -\delta_{n-1}q_{n-2} + \gamma_{n-1}q_{n-1} \end{bmatrix}^\top + \begin{bmatrix} 0 \\ 0 \\ \dots \\ -\delta_{n-1}q_n \end{bmatrix}^\top$$

Therefore, we ahve $AQ_n = Q_nT_n - \delta_{n-1}q_ne_n^\top$, where $Q_n = \begin{bmatrix} q_0 & q_1 & \dots & q_{n-1} \end{bmatrix}$,

$$T_n = \begin{bmatrix} \gamma_0 & -\delta_0 & & & \\ -\delta_0 & \gamma_1 & -\delta_1 & & \\ & \ddots & \ddots & \ddots & \\ & & -\delta_{n-3} & \gamma_{n-2} & -\delta_{n-2} \\ & & & -\delta_{n-2} & \gamma_{n-1} \end{bmatrix}, \text{ and } e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

v.
$\because \{q_0, \dots, q_{n-1}\}$ is an orthonormal basis
$\therefore q_iq_j = 0$ for $0 \leq i, j \leq n-1$, and $i \neq j$.
$\because Q_n = \begin{bmatrix} q_0 & q_1 & \dots & q_{n-1} \end{bmatrix} \in \mathbb{R}^{N\times n}$
$\therefore Q^\top Q_n = I_n$ and $Q_n^\top q_n =$
$Q_n^\top AQ_n = Q_n^\top(Q_nT_n - \delta_{n-1}q_ne_n^\top) = T_n - \delta_{n-1}Q_n^\top q_ne_n^\top$
$\because$ in CG method $r_n^\top r_j = 0$ for $j = 0, 1, \dots, n-1$
$\therefore q_nq_j = 0$ for $j = 0, 1, \dots, n-1$

$$\therefore Q_n^\top q_n = \begin{bmatrix} q_0q_n \\ q_1q_n \\ \dots \\ q_{n-1}q_n \end{bmatrix} = 0$$

$\therefore Q_n^\top AQ_n = T_n$

Question 2.

The code is uploaded to GitHub: https://github.com/623586953/CS714/tree/master/HW2
Using MATLAB code, we can also find the minimum $N = 100$;

Taylor expansion: $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \mathcal{O}(h^3)$
$\therefore \frac{f(x+h)-f(x)}{h} = f'(x) + \frac{h}{2}f''(x) + \mathcal{O}(h^2)$
For $N+1$ interpolation, the point between x_j and x_{j+1} is $\frac{j+1}{N+1}$
By interpolation, $g\left(\frac{j+1}{N+1}\right) = f(x_j) + \frac{f(x_j+h)-f(x_j)}{h}\Delta x_j$, where $\Delta x_j = \frac{j+1}{N+1} - jh = \frac{jh+h}{1+h} - jh = \frac{h-jh^2}{1+h}$
The actual value is $f(x + \Delta x_j) = f(x_j) + (\Delta x_j)f'(x) + \frac{(\Delta x_j)^2}{2}f''(x_j) + \mathcal{O}((\Delta x_j)^3)$
$e(x_j) = f(x_j + \Delta x_j) - g\left(\frac{j+1}{N+1}\right)$
$e(x_j) = f(x_j) + \Delta x_jf'(x) + \frac{(\Delta x_j)^2}{2}f''(x_j) + \mathcal{O}((\Delta x_j)^3) - \left(f(x_j) + \Delta x_jf'(x) + \Delta x_j\frac{h}{2}f''(x) + \Delta x_j\mathcal{O}(h^2)\right)$
$e(x_j) = \frac{(\Delta x_j)^2 - \Delta x_jh}{2}f''(x_j) + \mathcal{O}((\Delta x_j)^3) - \Delta x_j\mathcal{O}(h^2)$
We have $f(x) = e^{-400(x-0.5)^2}$

$f^{'}(x) = -800(x-0.5)e^{-400(x-0.5)^2}$, $f^{''}(x) = -800\left(-800(x-0.5)^2+1\right)e^{-400(x-0.5)^2}$
The uniform norm of error is obtained when $f^{''}$ has the largest value (largest slope change).
$\therefore x_j = 0.5 \rightarrow j = \frac{N}{2} = \frac{1}{2h} \rightarrow \Delta x_j = \frac{h-jh^2}{1+h} = \frac{h}{2(1+h)}$.
$e(x_j = 0.5) \approx \frac{(\Delta x_j)^2 - \Delta x_j h}{2} f^{''}(x_j) = \frac{-2h^3-h^2}{8(1+h)^2} \times (-800) = 100 \times \frac{2h^3+h^2}{(1+h)^2} = 0.01$
Solve above equation, we have $h = 0.01 \rightarrow N = 100$.

Therefore, the smallest value of N is 100.

Question 3.

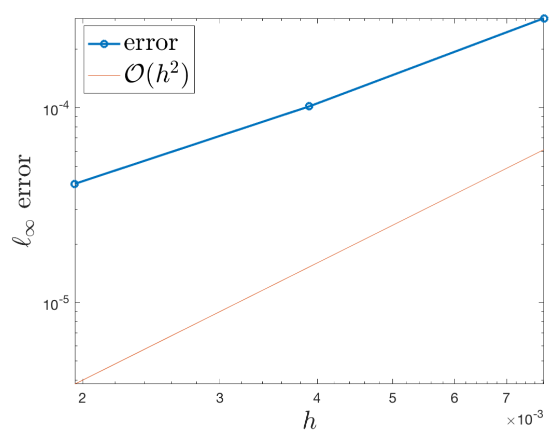
(a)
Use the Euler method for this 2D wave equation problem: $u_{tt} = \Delta u$

$$\frac{u(x,y,t+1) - 2u(x,y,t) + u(x,y,t-1)}{(\Delta t)^2} = \frac{u(x+\Delta x,y,t) + u(x-\Delta x,y,t) + u(x,y+\Delta x,t) + u(x,y-\Delta x,t) - 4u(x,y,t)}{(\Delta x)^2}$$

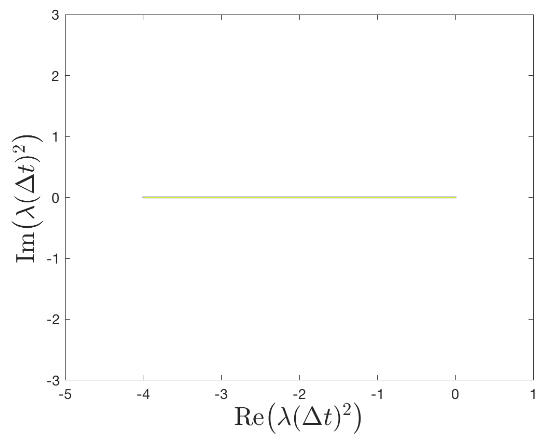
$u(x,y,t+1) = 2u(x,y,t) - u(x,y,t-1) + \left(\frac{\Delta t}{\Delta x}\right)^2 \left(u(x+\Delta x,y,t) + u(x-\Delta x,y,t) + u(x,y+\Delta x,t) + u(x,y-,t) - 4u(x,y,t)\right)$
For boundary condition $u_t(x,y,0) = f(x)f(y)$, we have $\frac{u(x,y,\Delta t) - u(x,y,-\Delta t)}{\Delta t} = f(x)f(y)$
$\therefore u(x,y,-\Delta t) = -\Delta t f(x)f(y)$

Schema setup:
1. List$(N) = [2^{10}, 2^9, 2^8, 2^7]$, all of them are greater"""" than critical N, 100, in problem B.
2. The exact solution is approximated on very fine grid, $N = 2^{10}$.
3. The timestep, $dt = a\frac{1}{N^2}$, where a = 0.5 for stability.

The log-log plot of the maximum norm of the error vs. h, the grid spacing, is shown below. And we can see that this Euler method is second-order accurate.



(b)
$y^{''}(t) = \lambda y \rightarrow \frac{y^{n+1}-2y^n+y^{n-1}}{(\Delta t)^2} = \lambda y^n$
$y^{n+1} = \left(2+\lambda(\Delta t)^2\right) y^n - y^{n-1} \rightarrow \rho^2 = \left(2+\lambda(\Delta t)^2\right)\rho - 1$
$\rho = \frac{\left(2+\lambda(\Delta t)^2\right) \pm \sqrt{(2+\lambda(\Delta t)^2)^2-4}}{2}$
$|\rho| \leq 1 \rightarrow \mathrm{Re}(\lambda(\Delta t)^2) \in [-4, 0]$ and $\mathrm{Im}(\lambda(\Delta t)^2) = 0$
The stability region is shown in the following figure.



(c)
$u_{tt} = \frac{u(x+\Delta x,y,t)+u(x-\Delta x,y,t)+u(x,y+\Delta x,t)+u(x,y-\Delta x,t)-4u(x,y,t)}{(\Delta x)^2}$
Therefore, this system can be wriiten as $U^{''}(t) = AU(t)$, where A is a 5 point stencil Laplacian matrix.
The eigenvalue for matrix A is $\lambda_{k_1,k_2} = \frac{2}{(\Delta x)^2}\left((\cos(k_1\pi\Delta x) - 1) + (\cos(k_2\pi\Delta x) - 1)\right) \leq 0$
$\because -4 \leq ((\cos(k_1\pi\Delta x) - 1) + (\cos(k_2\pi\Delta x) - 1) \leq 0$
$\therefore -\frac{8}{(\Delta x)^2} \leq \lambda_{k_1,k_2} \leq 0$
Based on result on part(b), we must require $-4 \leq \lambda(\Delta t)^2 \leq 0$ for all eigenvalues.
$\because -8\left(\frac{\Delta t}{\Delta x}\right)^2 \leq \lambda_{k_1,k_2}(\Delta t)^2 \leq 0$
$\therefore$ CFL condition is $\left(\frac{\Delta t}{\Delta x}\right)^2 \leq \frac{1}{2}$

(d)
$U_j^n = e^{ik_1j_1\Delta x}e^{ik_2j_2\Delta x}$, $U_j^{n+1} = g(k_1,k_2)e^{ik_1j_1\Delta x}e^{ik_2j_2\Delta x}$, and $U_j^{n-1} = \frac{1}{g(k_1,k_2)}e^{ik_1j_1\Delta x}e^{ik_2j_2\Delta x}$
Insert these expression into equation in part(a), we have
$$g(k_1,k_2)e^{ik_1j_1\Delta x}e^{ik_2j_2\Delta x} = 2e^{ik_1j_1\Delta x}e^{ik_2j_2\Delta x} - \frac{1}{g(k_1,k_2)}e^{ik_1j_1\Delta x}e^{ik_2j_2\Delta x} + \left(\frac{\Delta t}{\Delta x}\right)^2$$
$$\left[e^{ik_1(j_1+1)\Delta x}e^{ik_2j_2\Delta x} + e^{ik_1(j_1-1)\Delta x}e^{ik_2j_2\Delta x} + e^{ik_1j_1\Delta x}e^{ik_2(j_2+1)\Delta x} + e^{ik_1j_1\Delta x}e^{ik_2(j_2-1)\Delta x} - 4e^{ik_1j_1\Delta x}e^{ik_2j_2\Delta x}\right]$$

$$g(k_1,k_2)e^{ik_1j_1\Delta x}e^{ik_2j_2\Delta x} = 2e^{ik_1j_1\Delta x}e^{ik_2j_2\Delta x} - \frac{1}{g(k_1,k_2)}e^{ik_1j_1\Delta x}e^{ik_2j_2\Delta x} + \left(\frac{\Delta t}{\Delta x}\right)^2 e^{ik_1j_1\Delta x}e^{ik_2j_2\Delta x}$$
$$\left[e^{ik_1\Delta x} + e^{-ik_1\Delta x} + e^{ik_2\Delta x} + e^{-ik_2\Delta x} - 4\right]$$

Simplify above equation,
$g(k_1,k_2) = 2 - \frac{1}{g(k_1,k_2)} + \left(\frac{\Delta t}{\Delta x}\right)^2\left[e^{ik_1\Delta x} + e^{-ik_1\Delta x} + e^{ik_2\Delta x} + e^{-ik_2\Delta x} - 4\right]$
Use g to represent $g(k_1,k_2)$, we have
$g^2 = 2g - 1 + g\left(\frac{\Delta t}{\Delta x}\right)^2[2\cos k_1\Delta x + 2\cos k_2\Delta x - 4]$
$g^2 = g\left(2 + \left(\frac{\Delta t}{\Delta x}\right)^2[2\cos k_1\Delta x + 2\cos k_2\Delta x - 4]\right) - 1$
To guarantee $|g(k_1,k_2)| \leq 1$ for all k_1, k_2, we need $-2 \leq 2 + \left(\frac{\Delta t}{\Delta x}\right)^2[2\cos k_1\Delta x + 2\cos k_2\Delta x - 4] \leq 2$

$\therefore -4 \leq \left(\frac{\Delta t}{\Delta x}\right)^2 [2\cos k_1\Delta x + 2\cos k_2\Delta x - 4] \leq 0$
$\because -2 \leq (\cos k_1\Delta x + \cos k_2\Delta x) \leq 2$
$\therefore -8 \leq (2\cos k_1\Delta x + 2\cos k_2\Delta x - 4) \leq 0$
$\therefore$ CFL condition is $\left(\frac{\Delta t}{\Delta x}\right)^2 \leq \frac{1}{2}$, which agrees with the result in part(c).

(e)
Approximation equation:

$$\frac{u(x,y,t+1) - 2u(x,y,t) + u(x,y,t-1)}{(\Delta t)^2} = \frac{u(x+\Delta x,y,t) + u(x-\Delta x,y,t) + u(x,y+\Delta x,t) + u(x,y-\Delta x,t) - 4u(x,y,t)}{(\Delta x)^2}$$

By Taylor expansion, we have
$u_{tt} + \frac{1}{12}(\Delta t)^2 u_{tttt} + \mathcal{O}(\Delta t)^4 = u_{xx} + \frac{1}{12}(\Delta x)^2 u_{xxxx} + \mathcal{O}(\Delta x)^4 + u_{yy} + \frac{1}{12}(\Delta x)^2 u_{yyyy} + \mathcal{O}(\Delta x)^4$
$u_{tt} - u_{xx} - u_{yy} = \frac{1}{12}\left(-(\Delta t)^2 u_{tttt} + (\Delta x)^2 u_{xxxx} + (\Delta x)^2 u_{yyyy}\right) - \mathcal{O}(\Delta t)^4 + \mathcal{O}(\Delta x)^4$
$\because u_{tt} \approx \Delta u = u_{xx} + u_{yy}$
$\therefore u_{tttt} = \frac{\partial^2}{\partial t^2} u_{tt} = \frac{\partial^2}{\partial t^2}(u_{xx} + u_{yy}) = \frac{\partial^2}{\partial x^2} u_{tt} + \frac{\partial^2}{\partial y^2} u_{tt} = u_{xxxx} + u_{yyyy} + 2u_{xxyy}$

Reorganized above equation, the modified equation is:
$u_{tt} - u_{xx} - u_{yy} = \frac{(\Delta x)^2}{12}\left(1 - \left(\frac{\Delta t}{\Delta x}\right)^2\right)(u_{xxxx} + u_{yyyy}) - \frac{(\Delta t)^2}{6} u_{xxyy}$
Assume $v = \frac{\Delta t}{\Delta x}$,
$u_{tt} - u_{xx} - u_{yy} = \frac{(\Delta x)^2}{12}\left(1 - v^2\right)(u_{xxxx} + u_{yyyy}) - \frac{(\Delta t)^2}{6} u_{xxyy}$
Fourier transform in space: $\hat{u} = \hat{u}(\xi_x, \xi_y, t)$
$\hat{u}_{tt} = -(\xi_x^2 + \xi_y^2)\hat{u} + \frac{(\Delta x)^2}{12}\left(1 - v^2\right)(\xi_x^4 + \xi_y^4)\hat{u} - \frac{(\Delta t)^2}{6}\xi_x^2\xi_y^2\hat{u} = \left(-(\xi_x^2 + \xi_y^2) + \frac{(\Delta x)^2}{12}\left(1 - v^2\right)(\xi_x^4 + \xi_y^4) - \frac{(\Delta t)^2}{6}\xi_x^2\xi_y^2\right)\hat{u}$
$\hat{u}(\xi_x, \xi_y, t) = e^{ct}\hat{u}_0(\xi_x, \xi_y)$, where $c = \sqrt{-(\xi_x^2 + \xi_y^2) + \frac{(\Delta x)^2}{12}(1 - v^2)(\xi_x^4 + \xi_y^4) - \frac{(\Delta t)^2}{6}\xi_x^2\xi_y^2}$
$u(x,y,t) = \frac{1}{2\pi}\int e^{i(\xi_x x + i\xi_y y)} e^{ct}\hat{u}_0(\xi_x, \xi_y) d\xi_x d\xi_y$
The extra terms lead to exponential growth.

Question 4.

ODE problem: $y^{''}(t) = \lambda y$
$\frac{U^{n+1} - 2U^n + U^{n-1}}{(\Delta t)^2} = \lambda U^n \rightarrow U^{n+1} = (2 + \lambda(\Delta t)^2)U^n - U^{n-1}$
It can be written as $U^{n+1} = B((\Delta t)^2)U^n - U^{n-1}$, where $B((\Delta t)^2) = 2 + \lambda(\Delta t)^2$.

U_n is the approximation, and u_n is the true solution, then
$\begin{cases} U^{n+1} = BU^n - U^{n-1} \\ u^{n+1} = B((\Delta t)^2)u^n - u^{n-1} + (\Delta t)^2\tau^n \end{cases}$, where τ^n is the LTE.

$\because E^n = U^n - u^n$
$\therefore E^{n+1} = BE^n - E^{n-1} - (\Delta t)^2\tau^n$

Assume E^0 and E^1 is given as initial value,
$$\begin{cases} E^2 = BE^1 - E^0 - (\Delta t)^2\tau^1 \\ E^3 = BE^2 - E^1 - (\Delta t)^2\tau^2 \\ E^4 = BE^3 - E^2 - (\Delta t)^2\tau^3 \\ \qquad\qquad \vdots \end{cases}$$

$E^N = \alpha^N(B)E^1 - \alpha^{N-1}(B)E^0 - (\Delta t)^2 \sum_{i=2}^{N} \alpha^{N-i}(B)\tau^i$

, where $\alpha^j(B) = B \times \alpha^{j-1}(B) - \alpha^{j-2}(B)$, $\alpha^1 = 1$, $\alpha^2 = B$, $j = 2, 3, \ldots, N$.

Assume $\left\|\alpha^N(B)\right\| \leq \left\|B^{N-1}\right\|$ holds for $n = k$
When $n = 1$, $\left\|\alpha^1(B)\right\| = 1 \leq \left\|B^0\right\| = 1$
When $n = k+1$, $\alpha^N(B) = B \times \alpha^{N-1}(B) - \alpha^{N-2}(B)$
$\therefore \left\|\alpha^N(B)\right\| = \left\|\alpha^{N-1}(B)\right\| \|B\| - \left\|\alpha^{N-2}(B)\right\| \leq \left\|B^{N-2}\right\| \|B\| - \left\|B^{N-3}\right\| = \left\|B^{N-1}\right\| - \left\|B^{N-3}\right\| \leq \left\|B^{N-1}\right\|$
Therefore, $\left\|\alpha^N(B)\right\| \leq \left\|B^{N-1}\right\|$ holds for every natural number.

$\left\|\alpha^N(B)\right\| = \left\|\alpha^{N-1}(B) \times B - \alpha^{N-2}(B)\right\| \leq \left\|\alpha^{N-1}(B) \times B - \alpha^{N-2}(B)\right\| \leq \left\|B^{N-1}\right\|$

$\left\|E^N\right\| = \left\|\alpha^N(B)\right\| \left\|E^1\right\| - \left\|\alpha^{N-1}(B)\right\| \left\|E^0\right\| - (\Delta t)^2 \sum_{i=2}^{N} \left\|\alpha^{N-i}(B)\right\| \left\|\tau^i\right\|$
$\because \left\|\alpha^N(B)\right\| \leq \left\|B^{N-1}\right\|$
$\therefore \left\|E^N\right\| \leq \left\|B^{N-1}\right\| \left\|E^1\right\| - \left\|B^{N-2}\right\| \left\|E^0\right\| - (\Delta t)^2 \sum_{i=2}^{N} \left\|B^{N-i}\right\| \left\|\tau^i\right\|$
From weak stability, we know $\exists C_T$ that $\left\|B^i\right\| \leq C_T$ for $i = 2, 3, \ldots, N$
$\therefore \left\|E^N\right\| \leq C_T \left\|E^1\right\| - C_T \left\|E^0\right\| - (\Delta t)^2 (N-1) C_T \max_i \left\|\tau^i\right\|$

When the method is consistent(As $\Delta t \to 0$, $\|\tau\| \to 0$), and when we use appropriate initial data($\left\|E^0\right\| \to 0$ and $\left\|E^1\right\| \to 0$), the above method is convergent($\|E^n\| \to 0$).