Itô formula.

$$f(t, \overset{x}{\downarrow}{B_t})$$

pretend B was smooth.

$$d(\quad) = \frac{\partial f}{\partial x}(t, B)\, dB + \frac{\partial f}{\partial t}(t, B)\, dt.$$

Itô formula, (assume $B_0 = 0$)

$$f(t, B_t) = f(0; \overset{B_0}{\overset{\shortparallel}{0}}) + \int_0^t \frac{\partial f}{\partial x}(s, B_s)\, dB_s$$

$$+ \underbrace{\int_0^t \frac{\partial f}{\partial s}(s, B_s)\, ds}_{\overset{\shortparallel}{0}} + \boxed{\frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s)\, ds}$$

Special: $f(x)$

Remark:

$$d f(t, B) = \frac{\partial f}{\partial x}\, dB + \frac{\partial f}{\partial t}\, dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\, dt$$

Ex: $f(x) = \frac{1}{2}x^2$. "$\frac{1}{2}d(B^2)$"

$$\frac{1}{2}B_t^2 = \int_0^t B_s\, dB_s + \underbrace{\frac{1}{2}\int_0^t 1\, ds}_{\frac{1}{2}t}$$

Ex. $\int_0^t B_s^2\, dB_s = ?$

x^2

$f(x) = \frac{1}{3}x^3$

$$\int_0^t \frac{\partial f}{\partial x}(B_s)\,dB_s = \frac{1}{3}B_t^3 - \frac{1}{2}\int_0^t 2B_s\,ds.$$

$$= \frac{1}{3}B_t^3 - \int_0^t B_s\,ds.$$

Ex: $\int_0^t \overbrace{\sin B_s}^{\frac{\partial f}{\partial x}(B)}\,dB_s = -\cos(B_t) + \cos(B_0)$

$\left(f(x) = -\cos(x) \right)$ $\quad - \frac{1}{2}\int_0^t \cos B_s\,ds$

$$= 1 - \cos B_t - \frac{1}{2}\int_0^t \cos B_s\,ds$$

Ex. $\int_0^t \overset{\frac{\partial f}{\partial x}(s,B)}{\overset{\shortparallel}{s}}\,dB_s = tB_t - \int_0^t B_s\,ds - \frac{1}{2}\int_0^t 0\,ds$

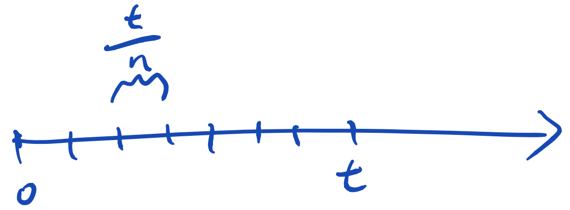
$f(t,x) = tx$

$\frac{\partial f}{\partial t} = x$

$$= tB_t - \int_0^t B_s\,ds$$

(Remark: Young thm. $\int f^{\alpha}\,dg^{\beta}$. $\boxed{\alpha+\beta>1}$ R-S integral $\int B\,dB$ $C^\alpha\ \forall \alpha<\frac{1}{2}$)

$$f(B_t) = f(\overset{0}{\overset{\shortparallel}{B_0}}) + \int_0^t f'(B_s)\,dB_s + \frac{1}{2}\int_0^t f''(B_s)\,ds.$$



$$f(B_t) - f(B_0) = \sum_{i=1}^{n} \left\{ \underline{f(B_{t_i}) - f(B_{t_{i-1}})} \right\}$$

$$f(y) - f(x) = \underline{(y-x) f'(x)} + \underline{\frac{1}{2}(y-x)^2 f''(x)} + \underline{r(x,y)}$$

$$r(x,y) = \int_x^y (y-u)\left(f''(u) - f''(x)\right) du.$$

$$|r(x,y)| \le \underline{(y-x)^2} \cdot \underline{h(x,y)}$$

uniformly conti. bounded

$h(x,x) = 0.$

$$A_n = \sum_{i=1}^{n} f'(B_{t_{i-1}})\,(\overbrace{B_{t_i} - B_{t_{i-1}}})$$

$$B_n = \frac{1}{2}\sum_{i=1}^{n} f''(B_{t_{i-1}})\,(B_{t_i} - B_{t_{i-1}})^2$$

$$C_n \le \sum_{i=1}^{n} (B_{t_i} - B_{t_{i-1}})^2\, h(B_{t_{i-1}}, B_{t_i})$$

$\downarrow 0$

$$A_n \longrightarrow \int_0^t f'(B_s)\,dB_s$$

$\tilde{B}_n \to 0$

$$B_n = \frac{1}{2}\sum_{i=1}^{n} f''(B_{t_{i-1}})\left[(B_{t_i} - B_{t_{i-1}})^2\right.$$

$$- (t_i - t_{i-1}))$$

$$+ \frac{1}{2} \sum_{i=1}^{n} f''(B_{t_{i-1}})(t_i - t_{i-1})$$

$$\downarrow d$$

$$\frac{1}{2}\int_0^t f''(B_s)\,ds$$

$$\frac{1}{2}\sum_i \mathbb{E}\Big[\mathbb{E}\Big[f''(B_{t_{i-1}})\big((B_{t_i}-B_{t_{i-1}})^2-(t_i-t_{i-1})\big)\Big|\mathcal{F}_{t_{i-1}}\Big]\Big]$$

$$\|$$

$$\cancel{\mathbb{E}\,\tilde{B}_n \neq 0}$$

$$\mathbb{E}(\tilde{B}_n^2) \overset{?}{\to} 0 \quad \Big\} \Rightarrow \tilde{B}_n \xrightarrow{n\to\infty} 0 \text{ in prob}$$

$$\|$$

$$\Rightarrow \frac{1}{4}\sum_{i=1}^{n} \mathbb{E}\Big[f''(B_{t_{i-1}})^2 \big[(B_{t_i}-B_{t_{i-1}})^2 - \underset{t_i - t_{i-1}}{\delta t_i}\big]^2\Big]$$

$$+ \frac{1}{4}\sum_{i\neq j} \mathbb{E}\Big[f''(B_{i-1})f''(B_{j-1}) \big[(B_{t_i}-B_{t_{i-1}})^2-\delta t_i\big]\big[(B_{t_j}-B_{t_{j-1}})^2-\delta t_j\big]\Big]$$

$(i<j)$ Condition on $\mathcal{F}_{t_i}$

$$\leq C\|f''\|^2_{L^\infty} \sum_{i=1}^{n} \mathbb{E}\Big[\big((B_{t_i}-B_{t_{i-1}})^2-\delta t_i\big)^2\Big]$$

$$= C\|f''\|^2_{L^\infty} \underbrace{\sum_{i=1}^{n} (\delta t)^2}_{n\cdot\left(\frac{t}{n}\right)^2} = \frac{Ct^2}{n}\|f''\|^2_{L^\infty} \xrightarrow{n\to\infty} 0$$

$$\left(\begin{array}{l} X \sim N(0,\delta t) \\ \mathbb{E}\big[(X^2-\delta t)^2\big] = \mathrm{Var}(X^2) = 2(\delta t)^2 \end{array}\right)$$

So, $B_n \xrightarrow{n\to\infty} \int_0^t f''(B_s)\,ds$