$$
\begin{aligned}
M_{t_1} &= E[M_{t_2}|\mathcal{F}_{t_1}] = E[M_0] + E\Big[\int_0^{t_2} f^{(t_2)}(s,\omega)dB_s(\omega)|\mathcal{F}_{t_1}\Big] \\
&= E[M_0] + \int_0^{t_1} f^{(t_2)}(s,\omega)dB_s(\omega)\ . \qquad (4.3.7)
\end{aligned}
$$

But we also have

$$
M_{t_1} = E[M_0] + \int_0^{t_1} f^{(t_1)}(s,\omega)dB_s(\omega)\ . \qquad (4.3.8)
$$

Hence, comparing (4.3.7) and (4.3.8) we get that

$$
0 = E\Big[\Big(\int_0^{t_1}(f^{(t_2)} - f^{(t_1)})dB\Big)^2\Big] = \int_0^{t_1} E[(f^{(t_2)} - f^{(t_1)})^2]ds
$$

and therefore

$$
f^{(t_1)}(s,\omega) = f^{(t_2)}(s,\omega) \qquad \text{for a.a. } (s,\omega) \in [0,t_1]\times\Omega\ .
$$

So we can define $f(s,\omega)$ for a.a. $s \in [0,\infty)\times\Omega$ by setting

$$
f(s,\omega) = f^{(N)}(s,\omega) \qquad \text{if } s \in [0,N]
$$

and then we get

$$
M_t = E[M_0] + \int_0^t f^{(t)}(s,\omega)dB_s(\omega) = E[M_0] + \int_0^t f(s,\omega)dB_s(\omega) \quad \text{for all } t \geq 0\ .
$$

□

Exercises

4.1. Use Itô's formula to write the following stochastic processes X_t on the standard form

$$
dX_t = u(t,\omega)dt + v(t,\omega)dB_t
$$

for suitable choices of $u \in \mathbf{R}^n$, $v \in \mathbf{R}^{n\times m}$ and dimensions n, m:

a) $X_t = B_t^2$, where B_t is 1-dimensional
b) $X_t = 2 + t + e^{B_t}$ (B_t is 1-dimensional)
c) $X_t = B_1^2(t) + B_2^2(t)$ where (B_1, B_2) is 2-dimensional
d) $X_t = (t_0 + t, B_t)$ (B_t is 1-dimensional)
e) $X_t = (B_1(t)+B_2(t)+B_3(t), B_2^2(t)-B_1(t)B_3(t))$, where (B_1, B_2, B_3) is 3-dimensional.

4.2. Use Itô's formula to prove that

$$\int_0^t B_s^2 dB_s = \tfrac{1}{3}B_t^3 - \int_0^t B_s ds\ .$$

4.3. Let X_t, Y_t be Itô processes in $\mathbf{R}$. Prove that

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + dX_t \cdot dY_t\ .$$

Deduce the following general *integration by parts formula*

$$\int_0^t X_s dY_s = X_tY_t - X_0Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s \cdot dY_s\ .$$

4.4. (Exponential martingales)
Suppose $\theta(t,\omega) = (\theta_1(t,\omega), \ldots, \theta_n(t,\omega)) \in \mathbf{R}^n$ with $\theta_k(t,\omega) \in \mathcal{V}[0,T]$ for $k = 1, \ldots, n$, where $T \le \infty$. Define

$$Z_t = \exp\left\{ \int_0^t \theta(s,\omega)dB(s) - \tfrac{1}{2}\int_0^t \theta^2(s,\omega)ds \right\}; \qquad 0 \le t \le T$$

where $B(s) \in \mathbf{R}^n$ and $\theta^2 = \theta \cdot \theta$ (dot product).

a) Use Itô's formula to prove that

$$dZ_t = Z_t\theta(t,\omega)dB(t)\ .$$

b) Deduce that Z_t is a martingale for $t \le T$, provided that

$$Z_t\theta_k(t,\omega) \in \mathcal{V}[0,T] \qquad \text{for } 1 \le k \le n\ .$$

Remark. A sufficient condition that Z_t be a martingale is the *Kazamaki condition*

$$E\left[\exp\left(\tfrac{1}{2}\int_0^t \theta(s,\omega)dB(s)\right)\right] < \infty \qquad \text{for all } t \le T\ . \tag{4.3.9}$$

This is implied by the following (stronger) *Novikov condition*

$$E\left[\exp\left(\tfrac{1}{2}\int_0^T \theta^2(s,\omega)ds\right)\right] < \infty\ . \tag{4.3.10}$$

See e.g. Ikeda & Watanabe (1989), Section III.5, and the references therein.

4.5. Let $B_t \in \mathbf{R}$, $B_0 = 0$. Define

$$\beta_k(t) = E[B_t^k]\ ; \qquad k = 0, 1, 2, \ldots;\ \ t \geq 0\ .$$

Use Itô's formula to prove that

$$\beta_k(t) = \tfrac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)ds\ ; \qquad k \geq 2\ .$$

Deduce that

$$E[B_t^4] = 3t^2 \qquad \text{(see (2.2.14))}$$

and find

$$E[B_t^6]\ .$$

4.6. a) For c, α constants, $B_t \in \mathbf{R}$ define

$$X_t = e^{ct + \alpha B_t}\ .$$

Prove that

$$dX_t = (c + \tfrac{1}{2}\alpha^2)X_t dt + \alpha X_t dB_t\ .$$

b) For $c, \alpha_1, \ldots, \alpha_n$ constants, $B_t = (B_1(t), \ldots, B_n(t)) \in \mathbf{R}^n$ define

$$X_t = \exp\left(ct + \sum_{j=1}^n \alpha_j B_j(t)\right)\ .$$

Prove that

$$dX_t = \left(c + \tfrac{1}{2}\sum_{j=1}^n \alpha_j^2\right)X_t dt + X_t\left(\sum_{j=1}^n \alpha_j dB_j\right)\ .$$

4.7. Let X_t be an Itô integral

$$dX_t = v(t,\omega)dB_t(\omega) \quad \text{where } v \in \mathbf{R}^n,\ v \in \mathcal{V}(0,T),\ B_t \in \mathbf{R}^n,\ 0 \leq t \leq T\ .$$

a) Give an example to show that X_t^2 is not in general a martingale.
b) Prove that if v is bounded then

$$M_t := X_t^2 - \int_0^t |v_s|^2 ds \quad \text{is a martingale}\ .$$

The process $\langle X, X\rangle_t := \int_0^t |v_s|^2 ds$ is called the *quadratic variation process* of the martingale X_t. For general processes X_t it is defined by

$$\langle X,X\rangle_t = \lim_{\Delta t_k\to 0}\sum_{t_k\le t}|X_{t_{k+1}}-X_{t_k}|^2 \quad \text{(limit in probability)} \quad (4.3.11)$$

where $0=t_1<t_2\cdots<t_n=t$ and $\Delta t_k=t_{k+1}-t_k$. The limit can be shown to exist for continuous square integrable martingales X_t. See e.g. Karatzas and Shreve (1991).

4.8. a) Let B_t denote n-dimensional Brownian motion and let $f:\mathbf{R}^n\to\mathbf{R}$ be C^2. Use Itô's formula to prove that

$$f(B_t)=f(B_0)+\int_0^t \nabla f(B_s)dB_s+\tfrac{1}{2}\int_0^t \Delta f(B_s)ds\ ,$$

where $\Delta=\sum\limits_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator.

b) Assume that $g:\mathbf{R}\to\mathbf{R}$ is C^1 everywhere and C^2 outside finitely many points $z_1,\dots,z_N$ with $|g''(x)|\le M$ for $x\notin\{z_1,\dots,z_N\}$. Let B_t be 1-dimensional Brownian motion. Prove that the 1-dimensional version of a) still holds, i.e.

$$g(B_t)=g(B_0)+\int_0^t g'(B_s)dB_s+\frac{1}{2}\int_0^t g''(B_s)ds\ .$$

(Hint: Choose $f_k\in C^2(\mathbf{R})$ s.t. $f_k\to g$ uniformly, $f_k'\to g'$ uniformly and $|f_k''|\le M$, $f_k''\to g''$ outside $z_1,\dots,z_N$. Apply a) to f_k and let $k\to\infty$).

4.9. Prove that we may assume that g and its first two derivatives are bounded in the proof of the Itô formula (Theorem 4.1.2) by proceeding as follows: For fixed $t\ge 0$ and $n=1,2,\dots$ choose g_n as in the statement such that $g_n(s,x)=g(s,x)$ for all $s\le t$ and all $|x|\le n$. Suppose we have proved that (4.1.9) holds for each g_n. Define the stochastic time

$$\tau_n=\tau_n(\omega)=\inf\{s>0;|X_s(\omega)|\ge n\}$$

(τ_n is called a *stopping* time (See Chapter 7)) and prove that

$$\left(\int_0^t v\frac{\partial g_n}{\partial x}(s,X_s)\mathcal{X}_{s\le\tau_n}dB_s:=\right)$$
$$\int_0^{t\wedge\tau_n} v\frac{\partial g_n}{\partial x}(s,X_s)dB_s=\int_0^{t\wedge\tau_n} v\frac{\partial g}{\partial x}(s,X_s)dB_s$$

for each n. This gives that

$$g(t \wedge \tau_n, X_{t \wedge \tau_n}) = g(0, X_0) + \int_0^{t \wedge \tau_n} \left(\frac{\partial g}{\partial s} + u \frac{\partial g}{\partial x} + \tfrac{1}{2} v^2 \frac{\partial^2 g}{\partial x^2} \right) ds + \int_0^{t \wedge \tau_n} v \frac{\partial g}{\partial x} dB_s$$

and since

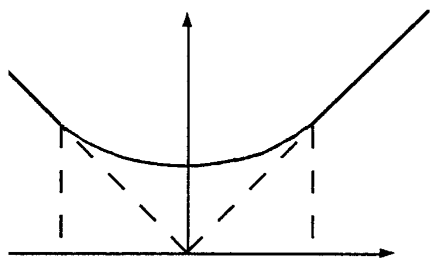
$$P[\tau_n > t] \to 1 \qquad \text{as } n \to \infty$$

we can conclude that (4.1.9) holds (a.s.) for g.

4.10. (Tanaka's formula and local time).
What happens if we try to apply the Itô formula to $g(B_t)$ when B_t is 1-dimensional and $g(x) = |x|$? In this case g is not C^2 at $x = 0$, so we modify $g(x)$ near $x = 0$ to $g_\epsilon(x)$ as follows:

$$g_\epsilon(x) = \begin{cases} |x| & \text{if} \quad |x| \geq \epsilon \\ \frac{1}{2}(\epsilon + \frac{x^2}{\epsilon}) & \text{if} \quad |x| < \epsilon \end{cases}$$

where $\epsilon > 0$.



a) Apply Exercise 4.8 b) to show that

$$g_\epsilon(B_t) = g_\epsilon(B_0) + \int_0^t g'_\epsilon(B_s) dB_s + \frac{1}{2\epsilon} \cdot |\{s \in [0, t]; B_s \in (-\epsilon, \epsilon)\}|$$

where $|F|$ denotes the Lebesgue measure of the set F.

b) Prove that

$$\int_0^t g'_\epsilon(B_s) \cdot \mathcal{X}_{B_s \in (-\epsilon, \epsilon)} dB_s = \int_0^t \frac{B_s}{\epsilon} \cdot \mathcal{X}_{B_s \in (-\epsilon, \epsilon)} dB_s \to 0$$

in $L^2(P)$ as $\epsilon \to 0$.
(Hint: Apply the Itô isometry to

$$E\left[\left(\int_0^t \frac{B_s}{\epsilon} \cdot \mathcal{X}_{B_s \in (-\epsilon, \epsilon)} dB_s \right)^2 \right].$$

c) By letting $\epsilon \to 0$ prove that

$$|B_t| = |B_0| + \int_0^t \operatorname{sign}(B_s)dB_s + L_t(\omega) , \tag{4.3.12}$$

where

$$L_t = \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \cdot |\{s \in [0,t]; B_s \in (-\epsilon,\epsilon)\}| \quad (\text{limit in } L^2(P))$$

and

$$\operatorname{sign}(x) = \begin{cases} -1 & \text{for} \quad x \le 0 \\ 1 & \text{for} \quad x > 0 \end{cases} .$$

L_t is called the *local time* for Brownian motion at 0 and (4.3.12) is the *Tanaka formula* (for Brownian motion). (See e.g. Rogers and Williams (1987)).

4.11. Use Itô's formula (for example in the form of Exercise 4.3) to prove that the following stochastic processes are $\{\mathcal{F}_t\}$-martingales:

a) $X_t = e^{\frac{1}{2}t}\cos B_t \qquad (B_t \in \mathbf{R})$
b) $X_t = e^{\frac{1}{2}t}\sin B_t \qquad (B_t \in \mathbf{R})$
c) $X_t = (B_t + t)\exp(-B_t - \frac{1}{2}t) \qquad (B_t \in \mathbf{R})$.

4.12. Let $dX_t = u(t,\omega)dt + v(t,\omega)dB_t$ be an Itô process in $\mathbf{R}^n$ such that

$$E\Big[\int_0^t |u(r,\omega)|dr\Big] + E\Big[\int_0^t |vv^T(r,\omega)|dr\Big] < \infty \qquad \text{for all } t \ge 0 .$$

Suppose X_t is an $\{\mathcal{F}_t^{(n)}\}$-martingale. Prove that

$$u(s,\omega) = 0 \qquad \text{for a.a. } (s,\omega) \in [0,\infty) \times \Omega . \tag{4.3.13}$$

Remarks:

0) 1) This result may be regarded as a special case of the Martingale Representation Theorem.

2) The conclusion (4.3.13) does not hold if the filtration $\mathcal{F}_t^{(n)}$ is replaced by the σ-algebras $\mathcal{M}_t$ generated by $X_s(\cdot)$; $s \le t$, i.e. if we only assume that X_t is a martingale w.r.t. its own filtration. See e.g. the Brownian motion characterization in Chapter 8.

Hint for the solution:

If X_t is an $\mathcal{F}_t^{(n)}$-martingale, then deduce that

$$E\Big[\int_t^s u(r,\omega)dr | \mathcal{F}_t^{(n)}\Big] = 0 \qquad \text{for all } s \ge t .$$

Differentiate w.r.t. s to deduce that

$$E[u(s,\omega)|\mathcal{F}_t^{(n)}] = 0 \qquad \text{a.s., for a.a. } s > t\ .$$

Then let $t \uparrow s$ and apply Corollary C.9.

4.13. Let $dX_t = u(t,\omega)dt + dB_t$ $(u \in \mathbf{R},\ B_t \in \mathbf{R})$ be an Itô process and assume for simplicity that u is bounded. Then from Exercise 4.12 we know that unless $u = 0$ the process X_t is not an $\mathcal{F}_t$-martingale. However, it turns out that we can construct an $\mathcal{F}_t$-martingale from X_t by multiplying by a suitable exponential martingale. More precisely, define

$$Y_t = X_t M_t$$

where

$$M_t = \exp\Big(- \int_0^t u(r,\omega)dB_r - \tfrac{1}{2}\int_0^t u^2(r,\omega)dr \Big)\ .$$

Use Itô's formula to prove that

$$Y_t \quad \text{is an } \mathcal{F}_t\text{-martingale}\ .$$

Remarks:

a) a) Compare with Exercise 4.11 c).

b) This result is a special case of the important *Girsanov Theorem*. It can be interpreted as follows: $\{X_t\}_{t\le T}$ is a martingale w.r.t the measure Q defined on $\mathcal{F}_T$ by

$$dQ = M_T dP \qquad (T < \infty)\ .$$

See Section 8.6.

4.14. In each of the cases below find the process $f(t,\omega) \in \mathcal{V}[0,T]$ such that (4.3.6) holds, i.e.

$$F(\omega) = E[F] + \int_0^T f(t,\omega)dB_t(\omega)\ .$$

a) $F(\omega) = B_T(\omega)$ b) $F(\omega) = \int_0^T B_t(\omega)dt$

c) $F(\omega) = B_T^2(\omega)$ d) $F(\omega) = B_T^3(\omega)$

e) $F(\omega) = e^{B_T(\omega)}$ f) $F(\omega) = \sin B_T(\omega)$

4.15. Let $x > 0$ be a constant and define

$$X_t = (x^{1/3} + \tfrac{1}{3}B_t)^3\ ; \qquad t \ge 0\ .$$

Show that

$$dX_t = \tfrac{1}{3}X_t^{1/3}dt + X_t^{2/3}dB_t\ ; \qquad X_0 = x\ .$$