

$([0,1], \mathcal{B}, P)$

↑
Lebesgue measure

$$a \in [0,1]$$

$$P(\{a\}) = 0$$

$$\begin{aligned} a < 1 \\ \{a\} &\subset [a, a+\varepsilon] \\ P(\{a\}) &\leq \varepsilon \\ \rightsquigarrow P(\{a\}) &= 0 \end{aligned}$$

Probability measures on \mathbb{R}

P is a prob measure on $(\mathbb{R}, \mathcal{B})$

$$F(x) = P((-\infty, x])$$

non-decreasing right-cont F_2 計算

$$\lim_{x \rightarrow -\infty} F(x) = 0$$

$$\lim_{x \rightarrow \infty} F(x) = 1$$



Thm : If $F: \mathbb{R} \rightarrow \mathbb{R}$ is
nondecreasing, right-cont with
 $\lim_{x \rightarrow \infty} F(x) - \lim_{x \rightarrow -\infty} F(x) = 1$ then
 there is a unique prob measure μ
 such that $\mu([a, b]) = F(b) - F(a)$.

More generally, without this condition
 we get a measure μ on $(\mathbb{R}, \mathcal{B})$.

① STES Semi-algebra : collection of sets
 SPTES closed for intersection, complement is
 a finite disjoint union of some of ~~sets~~

② SGS S^c is ~~有限~~ algebra : closed for union and
 集之并 complement (\leadsto intersection)
 $A, B \in \mathcal{G}$ then $A \cup B \in \mathcal{G}, A^c \in \mathcal{G}$

The algebra generated by a given
 semi-algebra is given by the collection
 of finite disjoint unions.

Theorem: Let S be a semi-algebra and μ a set function $S \rightarrow \mathbb{R}_+$, which is finitely additive:

$$\text{if } A_1, A_2, \dots, A_n \in S \text{ disjoint} \Rightarrow \mu(\bigcup_{\varepsilon=1}^n A_\varepsilon) = \sum_{\varepsilon=1}^n \mu(A_\varepsilon)$$

and if it is σ -subadditive:

$$A_1, A_2, \dots \in S \text{ disjoint} \quad \mu(\bigcup_{\varepsilon} A_\varepsilon) \leq \sum_{\varepsilon} \mu(A_\varepsilon).$$

Then μ has a unique extension to the algebra generated by S , and if the extension is σ -finite then there is a unique extension to $\sigma(S)$.

The extension is a measure!

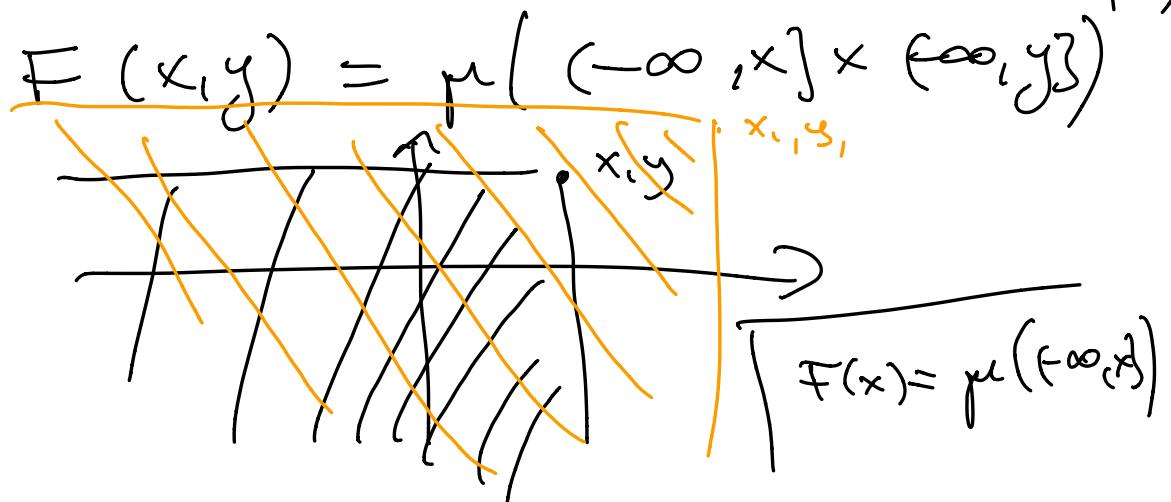
we can cover \mathbb{R} with countably many finite measure sets.

$$(a, b] \cup (b, c] = (a, c]$$

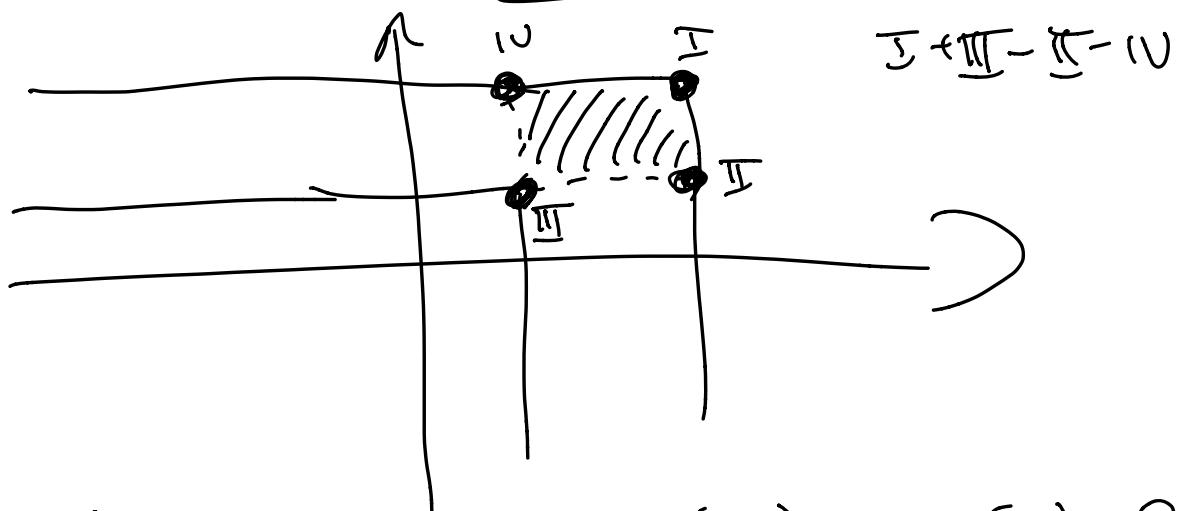
$$(a, b]^\complement = (-\infty, a] \cup [b, \infty)$$

Extension to \mathbb{R}^d

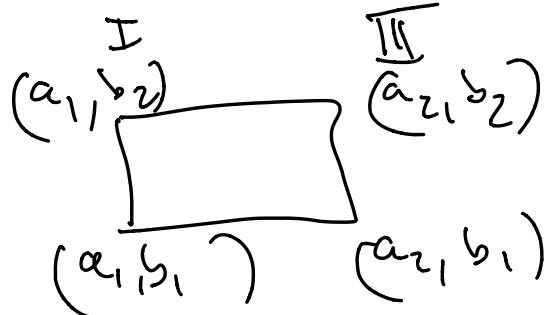
$d=2$ μ is a measure on $(\mathbb{R}^2, \mathcal{B})$



right cont | non decreasing in both variables



$$F(a_1, b_1) + F(a_2, b_2) - F(a_2, b_1) - F(a_1, b_2) \geq 0$$



$$\overline{I} \quad \overline{II}$$

$$\begin{matrix} a_1 < a_2 \\ b_1 < b_2 \end{matrix}$$

$$\overline{IV}$$

Random variables

$X: \Omega \rightarrow \mathbb{R}$ which is measurable
 $(\Omega, \mathcal{F}, \mathbb{P})$

For any Borel set B the inverse image of B : $X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\}$
is an event

$\underbrace{X^{-1}(B) \in \mathcal{F}}$ for all Borel set B .

What's the prob of $X \leq z$?

$$\underbrace{\{\omega : X(\omega) \leq z\}}_{\text{prob}} = X^{-1}((-\infty, z])$$

Claim: if \mathcal{A} generates \mathcal{B} and
 $X^{-1}(A) \in \mathcal{F}$ for $A \in \mathcal{A}$ then X is a r.v.

Enough to check $X^{-1}((-\infty, c]) \in \mathcal{F}$
for all $c \in \mathbb{R}$.

Ex: if $(\Omega, \mathcal{F}, \mathbb{P})$ is discrete then
any $X: \Omega \rightarrow \mathbb{R}$ is a r.v.

2) constant function is a r.v.

Indicator random variable

$$1_A(w) = \begin{cases} 1 & \text{if } w \in A \\ 0 & \text{if } w \in A^C \end{cases}$$

we need $\emptyset, \mathcal{S}, A, A^C$ to be measurable

$A \in \mathcal{F}$ A is an event

Distribution of a r.v.

X is a r.v. on (Ω, \mathcal{F}, P)

$$\begin{aligned}\mu(A) &= Q_X(A) = P(X \in A) \\ &= P(\tilde{X}(A))\end{aligned}$$

μ is a probability measure on $(\mathbb{R}, \mathcal{B})$

$$\mu(\mathbb{R}) = P(X \in \mathbb{R}) = P(\mathcal{S}) = 1$$

check G-additivity!

The distribution function (or cumulative distribution function) of

a r.v. X is

$$F(x) = P(X \leq x) = Q_x((-\infty, x])$$

Properties: {
1, non decreasing
2, right cont}

{
3, $\lim_{x \rightarrow -\infty} F(x) = 1$ $\lim_{x \rightarrow \infty} F(x) = 0$

The CDF of X identifies the distribution of X .

Thm: If F satisfies three conditions
then there is a prob space $(\Omega, \mathcal{F}, \mathbb{P})$
and a r.v. $X: \Omega \rightarrow \mathbb{R}$ so that
the CDF of X is exactly F .

Proof: $(\Omega, \mathcal{B}, \mathbb{P}, \mathbb{R})$
Lebesgue

$$X(\omega) = \sup \{ y : F(y) < \omega \}$$

Suppose that the inverse F^{-1} is well-defined

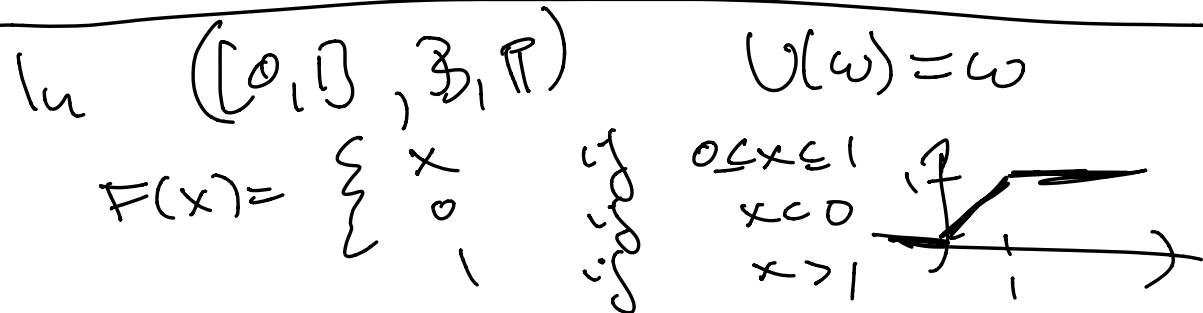
Then $X(\omega) = F^{-1}(\omega)$ works

$$P(X \leq x) = P(F^{-1}(\omega) \leq x) = P(\omega \leq F(x))$$

$$= P([0, F(x)]) = F(x)$$

this only works if F is "weakly"

Try to finish the proof by considering the general case.



The distribution of V is called the uniform distribution on $[0, 1]$.

Def: if X and Y have the same distribution if $P_1(X^{-1}(B)) = P_2(Y^{-1}(B))$

X lives on $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$

Y lives on $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$

Notation $X \stackrel{d}{=} Y$

Ex: (Ω, \mathcal{F}, P) we roll a fair die

Give an example of two random variables on (Ω, \mathcal{F}, P) with the same distribution.

$$I_{\{\xi_1\}} \stackrel{d}{=} I_{\{\xi_2\}}$$

$$\left\{ \begin{array}{l} \text{If } X = c \text{ i.e.} \\ \text{a constant} \\ Q_X(A) = \begin{cases} 1 & \text{if } c \in A \\ 0 & \text{if } c \notin A \end{cases} \end{array} \right.$$

The distribution of an indicator random variable:

$$X = I_A = I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

Q_X is supported on $\{0, 1\}$

$$Q_X(\{0\}) = P(A^c) = 1 - P(A)$$

$$Q_X(\{1\}) = P(A)$$

The distribution is identified by $P(A)$.

This distribution is called the Bernoulli distribution with parameter P .

$$P \in \{0, 1\}$$

$$P(\xi_B) = P$$

$$P(\xi_{\bar{B}}) = 1 - P$$

H.W.: try to construct a r.v. with
Bernoulli(P) distribution on $(\{0, 1\}, \mathcal{B}, P)$.

F is the CDF of X then

$$\begin{aligned} P(X \in (a, b]) &= P(a < X \leq b) \\ &= F(b) - F(a) \end{aligned}$$

$$P(X = a) = F(a) - \lim_{x \nearrow a^-} F(x)$$

If F is the CDF of the r.v. X
and there is a measurable function
 f so that $F(x) = \int_{-\infty}^x f(y) dy$
then f is called the probability
density function of X .

$$P(X \in (a, b]) = \int_a^b f(y) dy \quad \frac{dQ_X}{dx}$$

$$P(X \in B) = \int_B f(y) dy$$

↑
Borel

Lebesgue

In this case we say that the distribution of X is continuous.