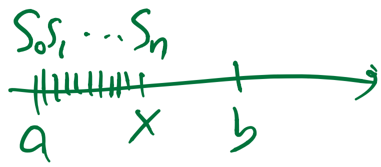
Quadratic variation.

f function on $[a,b]$



Total variation function of f

$$V_f(x) = \sup_{\pi} \left\{ \sum_{i=1}^{n} |f(s_i) - f(s_{i-1})| : a = s_0 < s_1 < \cdots < s_n = x \right\}$$

partitions of $[0, x]$

If V_f finite $\Rightarrow$ say f is BV (bounded variation)

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local martingale

We say M is a local mart. is

$\exists\ \tau_1 \le \tau_2 \le \tau_3 \le \cdots \xrightarrow{a.s} \infty$ "localizing sequence"

s.t. for each k, M^{τ_k} is a mart.

$$M^{\tau_k} = M_{t \wedge \tau_k}$$

- M is local L^2 mart. if M^{τ_k} is a L^2 mart.

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properties.

1). M is local mart, σ is stopping time

$\Rightarrow$ M^σ is a local mart.

2) If $\{\tau_k\}$ is localizing seq for M,

then $\{\tau_k\}$ is also localizing seq for M^σ.

Thm: "Fundamental thm for local marts"

Suppose M is cadlag local mart.

and $c > 0$

Then, $\exists$ cadlag local marts $\tilde{M}$ and A

s.t. • the jumps of $\tilde{M}$ are bounded by c

• A is FV

$M = \tilde{M} + A$.

Cor: If M is cadlag local mart,

then $M = \tilde{M} + A$

• $\tilde{M}$ is cadlag local L^2-mart.

• A is FV.

Def : a cadlag process Y.

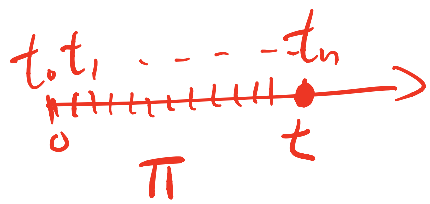
If $Y_t = \underline{Y_0} + M_t + V_t$

- M cadlag local mart.
- V cadlag FV.

$M_0 = V_0 = 0$.

then say Y semimart.

Def Y: stochastic process.



$$Q_t^{\pi} = \sum_i \left(Y_{t_{i+1}} - Y_{t_i} \right)^2$$

$$\lim_{|\pi| \to 0} Q_t^{\pi} = [Y]_t \quad \text{in probability.}$$

called quadratic variation process

- $[Y]_0 = 0$
- $t \mapsto [Y]_t$ is nondecreasing.

Def. X, Y are stoch processes.

quadratic covariation process

$$[X,Y] = \left[\tfrac{1}{2}(X+Y)\right] - \left[\tfrac{1}{2}(X-Y)\right]$$

$$\left(\tfrac{1}{4}(x+y)^2 - \tfrac{1}{4}(x-y)^2 = xy \right)$$

Alternatively: $[X,Y]_t = \lim_{|\pi|\to 0} \sum_i (X_{t_{i+1}} - X_{t_i}) \cdot (Y_{t_{i+1}} - Y_{t_i})$

(in prob)

properties: ① $[X,Y]_t = \frac{1}{2}\left([X+Y]_t - [X]_t - [Y]_t\right)$

$$= \frac{1}{2}\left([X]_t + [Y]_t - [X-Y]_t\right)$$

② suppose X, Y are cadlag.
suppose $[X,Y]$ exist.

Then $[X,Y]$ cadlag and, $\forall t$. $\Delta[X,Y]_t = (\Delta X_t)(\Delta Y_t)$

$$\left(\Delta X_t = X_t - \underset{\underset{\lim_{s\nearrow t} X_s}{\parallel}}{X_{t-}} \right)$$

Ex BM. $[B]_t = t$

$$\lim_{|\pi|\to 0} \sum_i (B_{t_{i+1}} - B_{t_i})^2 \to t \quad L^2.$$

$$\underbrace{\mathbb{E}\Big[\Big(\sum_i (B_{t_{i+1}} - B_{t_i})^2 - t\Big)^2\Big]} \to 0$$

$$\|$$

$$\sum_i \mathbb{E}\Big[\underbrace{(B_{t_{i+1}} - B_{t_i})}_{\sim N(0,\, \delta t_i)}{}^4\Big] + \sum_{i \neq j} \mathbb{E}\Big[\underbrace{(B_{t_{i+1}} - B_{t_i})^2}\,\underbrace{(B_{t_{j+1}} - B_{t_j})^2}\Big]$$

$$- 2t \sum_i \mathbb{E}\Big[(B_{t_{i+1}} - B_{t_i})^2\Big] + t^2$$

$$= 3\sum_i (\delta t_i)^2 + \sum_{i \neq j} \delta t_i \delta t_j - 2t \underbrace{\sum_i \delta t_i}_{t} + t^2$$

$$= 2\sum_i (\delta t_i)^2 + \underbrace{\sum_{i,j} \delta t_i \delta t_j}_{= (\sum_i \delta t_i)(\sum_j \delta t_j)} - t^2$$

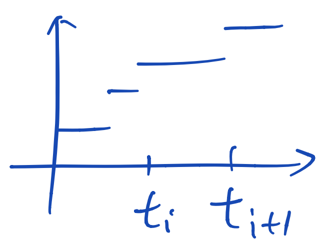
$$= 2\sum_i (\delta t_i)^2 \le 2|\pi| \underbrace{\sum_i |\delta t_i|} = 2|\pi|\, t \xrightarrow{|\pi| \to 0} 0 \qquad \blacksquare$$

Ex: $B, \tilde{B}$ indep BMs.

$$[B, \tilde{B}] = 0$$

$$\| \quad \frac{1}{2}\Big(\underbrace{[B + \tilde{B}]}_{\overset{\|}{2t}} - \underbrace{[B]}_{t} - \underbrace{[\tilde{B}]}_{t}\Big) = 0$$

Ex: Poisson process N_t.



$$[N] = N$$

$$M_t = N_t - \alpha t \longrightarrow \text{martingale}$$

$$[M] = N$$

Some results:

① If M is right-cont local mart.

Then $[M]$ exists.

and $[M]$ is right-cont.

② If M is L^2 martingale.

Then the convergence in def of $[M]$.

holds in L^1.

i.e. $\lim_{|\pi| \to 0} \mathbb{E} \left| \sum_i (M_{t_{i+1}} - M_{t_i})^2 - [M]_t \right| = 0$