Martingale inequalities

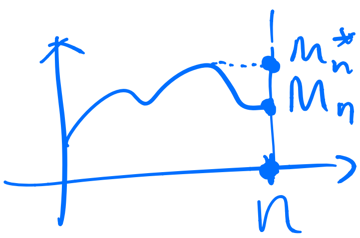
Doob's maximal inequality

Markov inequality:

For $X \geq 0$,

$\forall \lambda > 0$. $\quad \lambda P(\underset{|X|}{X} \geq \lambda) \leq E(\underset{|X|}{X})$

$\{M_n\}_{n \in \mathbb{Z}_+}$ process $\qquad M_n^* = \sup_{0 \leq m \leq n} M_m$



maximum process

Thm (max ineq in discrete-time)

Suppose M_n nonnegative submart.

$\lambda > 0$

$$\lambda P(M_n^* \geq \lambda) \leq E\left[M_n \mathbb{1}_{M_n^* \geq \lambda}\right]$$

trivial.

$$\leq E[M_n]$$

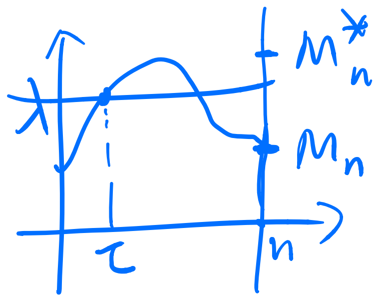
Rmk 1 If we don't have "non negative"

then $M_n^* = \sup_{0 \leq m \leq n} |M_m|$

rmk2. M_n is non-decreasing process

$\Rightarrow M_n^* = M_n$ (Markov ineq)

pf. $\tau = \min\{m : M_m \geq \lambda\}$



$M_n^* \geq \lambda \iff \tau \leq n$

If $\tau \leq n$ then

$$\lambda \mathbb{1}_{\tau \leq n} \leq M_\tau \mathbb{1}_{\tau \leq n} = \sum_{0 \leq m \leq n} M_m \mathbb{1}_{\tau = m}$$

$$\mathbb{E}\left(\downarrow \right) \qquad \left(\quad \right) \leq \mathbb{E}\left[\downarrow \right]$$

$$\mathbb{E}[M_m \mathbb{1}_{\tau = m}] \leq \mathbb{E}[M_n \mathbb{1}_{\tau = m}]$$

$$\mathbb{E}[M_m \mathbb{1}_A] \leq \mathbb{E}[M_n \mathbb{1}_A]$$

$$\forall A \in \mathcal{F}_m \qquad m \leq n$$

since, $M_m \mathbb{1}_A \leq \mathbb{E}[M_n \mid \mathcal{F}_m] \mathbb{1}_A$

$$\Rightarrow \mathbb{E}[\lambda \mathbb{1}_{\tau \leq n}] \leq \sum_{0 \leq m \leq n} \mathbb{E}[M_n \mathbb{1}_{\tau = m}]$$

$$= \mathbb{E}[M_n \mathbb{1}_{\underline{\tau \le n}}] \qquad \blacksquare$$

$$\Leftrightarrow M_n^* \ge \lambda$$

<u>Rmk</u>:

$$\boxed{\lambda^p \, \underbrace{\mathbb{P}(M_n^* \ge \lambda)}_{= \, \mathbb{P}((M_n^*)^p \ge \lambda^p)} \le \mathbb{E}[M_n^p]} \qquad \forall p \ge 1$$

<u>Thm</u> (Doob L^p ineq)

If $\{M_n\}$ is nonnegative submart.

then $\forall p > 1$,

$$\|M_n^*\|_p \le \frac{p}{p-1} \|M_n\|_p$$

(recall. $\|X\|_p = (\mathbb{E}|X|^p)^{1/p}$

Hölder: $\|XY\|_1 \le \|X\|_p \|Y\|_q$.

$\frac{1}{p} + \frac{1}{q} = 1$)

<u>pf</u>: suffice to show

$$\lambda \mathbb{P}(X \ge \lambda) \le \mathbb{E}[\overset{M}{Y} \mathbb{1}_{\overset{M^*}{X} \ge \lambda}]$$

$$\Rightarrow \|X\|_p \le \frac{p}{p-1} \|Y\|_p$$

$\forall z \geq 0.$

$$z^p = \underline{p \int_0^z x^{p-1}\,dx} = p \int_0^\infty x^{p-1} \mathbb{1}_{z \geq x}\,dx$$

$$z \mapsto X$$

$\|X\|_p^p$

$$\mathbb{E}[X^p] = p \int_0^\infty x^{p-1}\, \mathbb{P}(X \geq x)\,dx$$

$$\leq p \int_0^\infty x^{p-2}\, \mathbb{E}[Y \mathbb{1}_{X \geq x}]\,dx$$

$$= p\, \mathbb{E}\Big[Y \underbrace{\int_0^\infty x^{p-2} \mathbb{1}_{X \geq x}\,dx}\Big]$$

$$= \frac{p}{p-1} \mathbb{E}[Y \cdot X^{p-1}]$$

$$\leq \frac{p}{p-1} \|Y\|_p \|X\|_p^{p-1}$$

(Holder with $\begin{cases} p \\ q = \frac{p}{p-1} \end{cases}$

$\frac{1}{p} + \frac{p-1}{p} = 1$)

$\blacksquare$

$\boxed{\text{Cont-time}}$?

Thm: (Max ineq in cont_time)

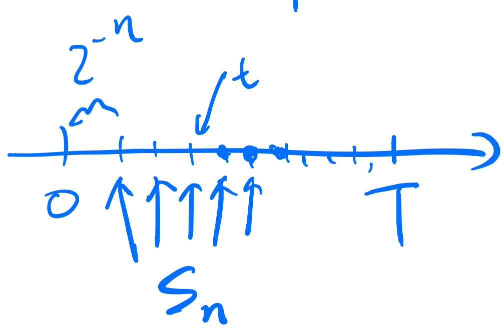
Suppose M_t is right-continuous nonnegative submartingale. $\forall \lambda > 0$

$$\lambda^p \, \mathbb{P}(\underbrace{M_T^*}_{= \sup_{0 \le t \le T} M_t} > \lambda) \le \mathbb{E}(M_T^p) \quad \forall p \ge 1$$

Moreover, if $M_T \in L^p(\Omega)$ for some $p > 1$

then, $\| M_T^* \|_p \le \frac{p}{p-1} \| M_T \|_p$

Pf (Sketch)



(use right-continuity)

$$\lim_{n \to \infty} \sup_{t \in S_n} M_t = \sup_{0 \le t \le T} M_t = M_T^*$$

$$\lambda^p \, \mathbb{P}(\sup_{t \in S_n} M_t \ge \lambda) \le \mathbb{E}(M_T^p)$$

take $n \to \infty$ (Fatou lemma)