Math 733 – Fall 2020

# Homework 1

**Due: 10PM, September 13, 2020**. Please upload your work in Canvas. Late homework will not be accepted.

1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Show that for every $A, B, C \in \mathcal{F}$

   $$\mathbf{P}(A \circ B) \leq \mathbf{P}(B \circ C) + \mathbf{P}(A \circ C).$$

   $A \circ B$ denotes the symmetric difference: $(A^c \cap B) \cup (A \cap B^c)$.

2. Let $\Omega = \mathbb{R}$, and $\mathcal{F}$ be the collection of all subsets $A$ of $\mathbb{R}$ so that $A$ or $A^c$ is countable. Set $\mathbf{P}(A) = 0$ if $A$ is countable, $\mathbf{P}(A) = 1$ if $A^c$ is countable. Show that $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space.

3. Suppose that $A_1, \ldots, A_n \subset \Omega$. Show that there exists disjoint sets $B_1, \ldots, B_k \subset \Omega$ so that the $\sigma$-field generated by $\{A_1, \ldots, A_n\}$ is exactly the set of finite unions of the sets $B_i$.
   Hint: try it for $n = 2$ first.

4. Let $(S, \mathcal{S})$ be a measurable space, and $\mathcal{F}_j, j \in J$ be a (possibly uncountable) collection of sigma-fields on this space. Show that $\cap_{j \in J} \mathcal{F}_j$ is also a sigma-field.

5. Construct a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a random variable $X$ with the following properties:

   (a) $P(X \text{ is irrational}) = 1$.

   (b) For any irrational number $q$ we have $P(X = q) = 0$.

6. Suppose that we flip a fair coin infinitely many times. Show that with positive probability there will be no integer $n$ so that the coin flips $n, n+1, ..., 2n$ are all heads.
   Note: You can assume that for any fixed finite collection of coin flips each outcome is equally likely.
   Hint: you don't need the exact probability of the event, just a positive lower bound.

*You can use the Piazza page to ask for clarifications about a specific problem, but please don't discuss explicit solutions before the deadline. Handing in plagiarized work, whether copied from a fellow student or off the web, is not acceptable and will lead to sanctions.*

**Bonus problem.**

The following statement shows that if you want to prove an identity or inequality relating probabilities of certain events – like in Problem 1, or in the inclusion-exclusion formula – then it is enough to check it on the trivial probability space (i.e. where $\mathcal{F} = \{\emptyset, \Omega\}$).

Suppose that $B_1, B_2, \ldots, B_k$ are all expressed from the sets $A_1, \ldots, A_n$ using the usual set operations (in a given way). Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that

$$\sum_{j=1}^{k} c_j \mathbf{P}(B_j) \geq 0$$

on the *trivial probability space*, for any choice of events $A_1, \ldots, A_n$. (Note that on the trivial probability space any event is $\emptyset$ or $\Omega$.) Prove that in that case the inequality holds on *any* probability space, with any choice of events $A_1, \ldots, A_n$.
(The same statement holds with $=$ instead of $\geq$.)

*You don't need to submit the bonus problems, they are provided as an extra challenge for those who are interested.*