

CS 726 - Fall 2020

Homework #1

Due : 09/21/2020, 5pm

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Question 1

Proof. When $\mathbf{x} = (1, 0, 0, \dots, 0)$,

$$\|\mathbf{x}\|_q = \left(\sum_{i=1}^n |x_i|^q \right)^{\frac{1}{q}} = 1$$

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} = 1$$

$$\|\mathbf{x}\|_q = \|\mathbf{x}\|_p$$

When $\mathbf{x} = (1, 1, 1, \dots, 1)$,

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} = d^{\frac{1}{p}}$$

$$d^{\frac{1}{p}-\frac{1}{q}} \|\mathbf{x}\|_q = d^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{i=1}^n |x_i|^q \right)^{\frac{1}{q}} = d^{\frac{1}{p}}$$

$$\|\mathbf{x}\|_p = d^{\frac{1}{p}-\frac{1}{q}} \|\mathbf{x}\|_q$$

□

Question 2

Proof. We know

$$L_2 \geq \frac{\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\|_2}{\|\mathbf{x} - \mathbf{y}\|_2}$$

so, the smallest L_2 should be $\max \left(\frac{\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\|_2}{\|\mathbf{x} - \mathbf{y}\|_2} \right), \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, which is $\max \|\nabla^2 f(\mathbf{x})\|_2, \forall \mathbf{x} \in \mathbb{R}^d$.

□

Question 3

Proof. If α_1 and α_2 are two arbitrary nonnegative real numbers such that $\alpha_1 + \alpha_2 = 1$ then the convexity of f implies

$$\forall x_1, x_2, f(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

This can be easily generalized: if $\alpha_1, \dots, \alpha_n$ are nonnegative real numbers such that $\alpha_1 + \dots + \alpha_n = 1$, then

$$f(\alpha_1 x_1 + \dots + \alpha_n x_n) \leq \alpha_1 f(x_1) + \dots + \alpha_n f(x_n)$$

for any x_1, \dots, x_n .

By induction:

The statement is true for $n = 2$. Suppose it is also for some n , one needs to prove it for $n + 1$. At least one of the α_i is strictly positive, say α_1 : therefore by convexity inequality:

$$\begin{aligned} f\left(\sum_{i=1}^{n+1} \alpha_i x_i\right) &= f\left(\alpha_1 x_1 + (1 - \alpha_1) \sum_{i=2}^{n+1} \frac{\alpha_i}{1 - \alpha_1} x_i\right) \\ &\leq \alpha_1 f(x_1) + (1 - \alpha_1) f\left(\alpha_1 x_1 + (1 - \alpha_1) \sum_{i=2}^{n+1} \frac{\alpha_i}{1 - \alpha_1} x_i\right) \end{aligned}$$

Since

$$\sum_{i=2}^{n+1} \frac{\alpha_i}{1 - \alpha_1} = 1$$

one can apply the induction hypotheses to the last term in the previous formula to obtain the result, namely the finite form of the Jensen's inequality. The finite form can be rewritten as:

$$f\left(\sum_{i=1}^k \alpha_i x_i\right) \leq \sum_{i=1}^k \alpha_i f(x_i)$$

□

Question 4

Proof.

□