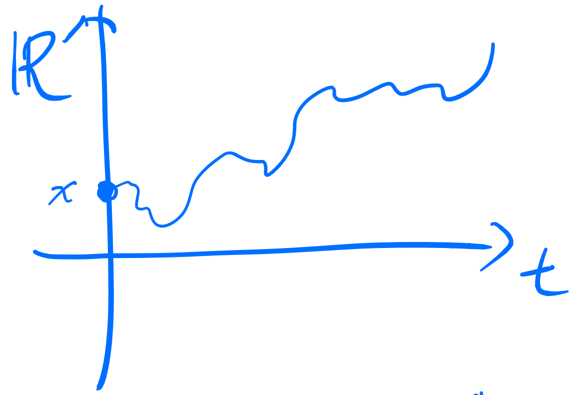
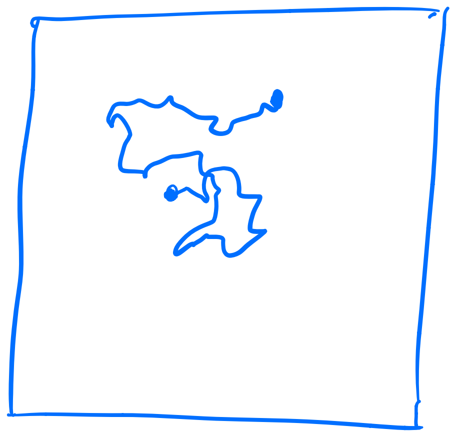
# Brownian Motion

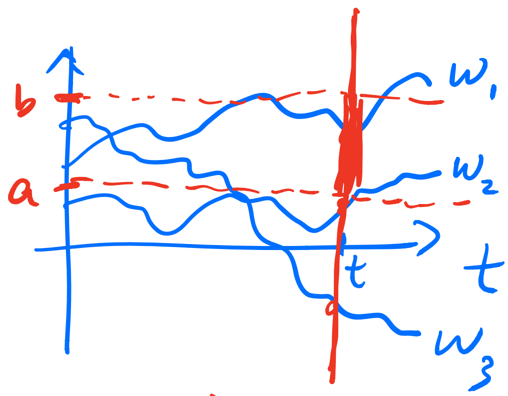
1827 Robert Brown (pollen in water)

1905 Einstein



Last time :




$X_t(w)$

$$\mu_t([a,b]) = P(X_t \in [a,b])$$

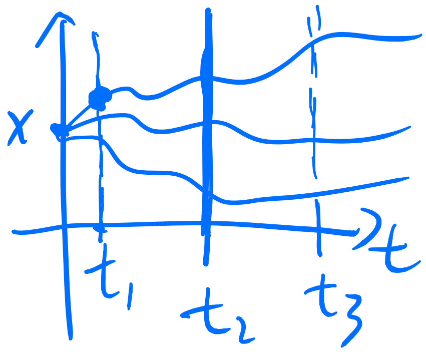
$$\mu_t(-\infty,\infty) = 1$$

$$\longrightarrow \mu_{t_1,t_2}(F_1 \times F_2)$$

Kol

$$\left( \begin{array}{l} \text{Gaussian: } \quad p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{|x|^2}{2\sigma^2}} \\ \exists (\Omega, F, \mathbb{P}), X \quad \text{s.t.} \\ \mathbb{P}(X \in F) = \int_F p(x)\,dx \end{array} \right)$$

# Brownian motion




$B_t$

$$\mu_t(F) = \int_F \underbrace{p(t, x, x_1)}_{\parallel \; \frac{1}{\sqrt{2\pi t}} e^{-\frac{|x_1 - x|^2}{2t}}} dx_1$$

$$\underline{t_1 \le t_2 \le \cdots \le t_k}$$

$$\mu_{t_1 \cdots t_k}(F_1 \times \cdots \times F_k)$$

$$\left( = \mathbb{P}(B_{t_1} \in F_1, \cdots, B_{t_k} \in F_k) \right)$$

$$\overset{def}{=} \int \cdots \int_{F_1 \times \cdots \times F_k} \underbrace{p(t_1, x, x_1)}\, p(t_2 - t_1, x_1, x_2) \cdots \underline{p(t_k - t_{k-1}, x_{k-1}, x_k)} \, dx_1 \cdots dx_k$$

how likely B goes from x to $x_1$ during time $t_1$

- $\underline{\mu_{t_1 t_2}(F_1 \times F_2) = \mu_{t_2 t_1}(F_2 \times F_1)}$

$\left( \text{if } t_1 > t_2 \text{ \underline{define} } \mu_{t_1 t_2}(F_1 \times F_2) \overset{def}{=} \mu_{t_2 t_1}(F_2 \times F_1) \right)$

- $\underline{\nu_{t_1}(F_1)} = \underline{\nu_{t_1 t_2}(F_1 \times \mathbb{R})}$

$$\| \quad \iint_{F_1 \times \mathbb{R}} p(t_1, x, x_1)\, \underline{p(t_2 - t_1, x_1, x_2)}\, dx_1 dx_2$$

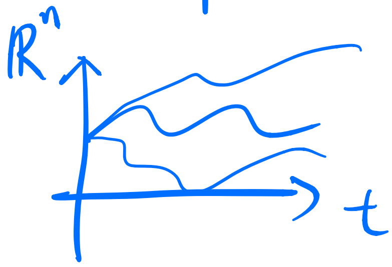
**Kol**: $\exists (\Omega, \mathcal{F}, \mathbb{P})$, and a stochastic process $\{B_t\}_{t \geq 0}$ s.t. the finite-dim distribution of $\{B_t\}_{t \geq 0}$ is given by $(\star)$

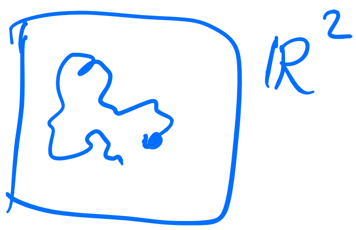
We call this stoch process (a "version" of) Brownian motion.

**n-dim BM**. (same)

$$p(t, x, y) = \frac{1}{(2\pi t)^{n/2}} e^{-\frac{|x-y|^2}{2t}}$$

$x, y \in \mathbb{R}^n$.

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"Canonical choice" of $\Omega$

$$\Omega = C([0, \infty), \mathbb{R}^n)$$

think of BM as a prob measure $P$ on $C([0,\infty), \mathbb{R}^n)$

"Polish"