Math 733 – Fall 2020

Homework 1

Due: 10PM, September 13, 2020. Please upload your work in Canvas. Late homework will not be accepted.

1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Show that for every $A, B, C \in \mathcal{F}$
$$\mathbf{P}(A \circ B) \leq \mathbf{P}(B \circ C) + \mathbf{P}(A \circ C).$$
$A \circ B$ denotes the symmetric difference: $(A^c \cap B) \cup (A \cap B^c)$.

2. Let $\Omega = \mathbb{R}$, and $\mathcal{F}$ be the collection of all subsets A of $\mathbb{R}$ so that A or A^c is countable. Set $\mathbf{P}(A) = 0$ if A is countable, $\mathbf{P}(A) = 1$ if A^c is countable. Show that $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space.

3. Suppose that $A_1, \ldots, A_n \subset \Omega$. Show that there exists disjoint sets $B_1, \ldots, B_k \subset \Omega$ so that the σ-field generated by $\{A_1, \ldots, A_n\}$ is exactly the set of finite unions of the sets B_i.
Hint: try it for $n = 2$ first.

4. Let $(S, \mathcal{S})$ be a measurable space, and $\mathcal{F}_j, j \in J$ be a (possibly uncountable) collection of sigma-fields on this space. Show that $\cap_{j \in J} \mathcal{F}_j$ is also a sigma-field.

5. Construct a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a random variable X with the following properties:
   (a) $P(X \text{ is irrational}) = 1$.
   (b) For any irrational number q we have $P(X = q) = 0$.

6. Suppose that we flip a fair coin infinitely many times. Show that with positive probability
$P($ there will be no integer n so that the coin flips $n, n+1, \ldots, 2n$ are all heads $) > 0$
Note: You can assume that for any fixed finite collection of coin flips each outcome is equally likely.
Hint: you don't need the exact probability of the event, just a positive lower bound.

You can use the Piazza page to ask for clarifications about a specific problem, but please don't discuss explicit solutions before the deadline. Handing in plagiarized work, whether copied from a fellow student or off the web, is not acceptable and will lead to sanctions.

Bonus problem.
The following statement shows that if you want to prove an identity or inequality relating probabilities of certain events – like in Problem 1, or in the inclusion-exclusion formula – then it is enough to check it on the trivial probability space (i.e. where $\mathcal{F} = \{\emptyset, \Omega\}$).

Suppose that $B_1, B_2, \ldots, B_k$ are all expressed from the sets $A_1, \ldots, A_n$ using the usual set operations (in a given way). Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that
$$\sum_{j=1}^{k} c_j \mathbf{P}(B_j) \geq 0$$

on the *trivial probability space*, for any choice of events $A_1, \dots, A_n$. (Note that on the trivial probability space any event is $\emptyset$ or Ω.) Prove that in that case the inequality holds on *any* probability space, with any choice of events $A_1, \dots, A_n$.
(The same statement holds with $=$ instead of $\geq$.)

You don't need to submit the bonus problems, they are provided as an extra challenge for those who are interested.