Math 733 – Fall 2020

Homework 2

Due: 10PM, September 27, 2020. Please upload your work in Canvas. Late homework will not be accepted.

1. Suppose that X and Y are random variables on the same probability space. Show that Y is measurable with respect to $\sigma(X)$ if and only if $Y = f(X)$ where $f : \mathbb{R} \to \mathbb{R}$ is measurable.

2. Suppose that $0 \le X \le 1$. Find the limit of $E[X^p]$ as $p \to \infty$.

3. Let π_n be a uniformly chosen random permutation of $\{1, 2, \ldots, n\}$. Let X_n denote the number of fixed points of π_n.

   (a) Find $P(X_n = 0)$, and evaluate its limit as $n \to \infty$.
   Hint: the inclusion-exclusion formula could help.

   (b) Find $E[X_n]$.

   A permutation of the set A is a one-to-one function $A \to A$. x is a fixed point of the permutation ν if $\nu(x) = x$.
   Hint: use indicator random variables.

4. Let Y be a non-negative random variable with $E[Y^2] < \infty$. Show that

$$P(Y > 0) \ge \frac{(E[Y])^2}{E[Y^2]}.$$

5. Suppose that $X_1, \ldots, X_n$ are random variables taking values from a countable set S. Show that $X_1, \ldots, X_n$ are independent if and only if there exists non-negative functions $g_j : S \to \mathbb{R}$, $1 \le j \le n$ so that

$$P(X_1 = x_1, \ldots, X_n = x_n) = \prod_{j=1}^{n} g_j(x_j), \qquad \text{for all } x_j \in S,\ 1 \le j \le n.$$

6. Consider the probability space $([0,1), \mathcal{B}, P)$ where $\mathcal{B}$ is the set of Borel sets and P is the Lebesgue measure. Define the random variable X_n as

$$X_n(\omega) = \begin{cases} 1, & \text{if } \lfloor 2^n \omega \rfloor \text{ is odd} \\ 0, & \text{otherwise.} \end{cases}$$

   Show that $X_1, X_2, \ldots$ are independent random variables with the same Bernoulli(1/2) distribution. (This means that $P(X_n = 1) = P(X_n = 0) = 1/2$.)
   This gives an explicit construction for infinitely many independent fair coin flips.

You can use the Piazza page to ask for clarifications about a specific problem, but please don't discuss explicit solutions before the deadline. Handing in plagiarized work, whether copied from a fellow student or off the web, is not acceptable and will lead to sanctions.

Bonus problem.

In a restaurant there are infinitely many round tables. When the restaurant opens it is completely empty. As customers arrive, they can either sit at a new table or join someone else's table. If there are k customers in the restaurant then the next person may choose to sit either to the left of a customer already present or at a new table, each with probability $\frac{1}{k+1}$. The seating arrangement in the restaurant at any given time corresponds to a permutation of the customers present: the people sitting at a given table correspond to a cycle of the appropriate permutation (e.g. use clockwise orientation).

(a) Prove that if there are k guests present then the random permutation corresponding to their seating arrangement is a uniformly chosen permutation.

(b) We choose a uniform random permutation of the first n numbers. Denote by X_n the number of cycles of the permutation. Find $E[X_n]$ and $\text{Var}X_n$.

Bonus problems are not graded, and you don't need to submit them. They are provided as an extra challenge for those who are interested.