Math 733 - Fall 2020

Homework 2

Due: 09/27, 10pm

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1. *Proof.* $\Rightarrow Y$ is measurable w.r.t. $\sigma(X)$

   - $Y = \mathbb{1}_{X^{-1}(B)}$ for some Borel set B and $f(X) = \mathbb{1}_B(X)$ then $Y(\omega) = \mathbb{1}_B(X(\omega)) = f(X(\omega))$.
   - $Y = \sum_{i=1}^n c_i \mathbb{1}_{X^{-1}(B_i)}(\omega)$ for some Borel sets B_i and $f(x) = \sum_{i=1}^n c_1 \mathbb{1}_{B_i}(X)$ then

   $$Y(\omega) = \sum_{i=1}^n c_i \mathbb{1}_{X^{-1}(B_i)}(\omega) = \sum_{i=1}^n c_i \mathbb{1}_{B_i}(X(\omega)) = f(X(\omega))$$

   - Sequential random variables Y_n, $\lim_{n\to\infty} Y_n = Y$ and f_n. Set $f(X) = \limsup f_n(X)$,

   $$f(X(\omega)) = \limsup f_n(X(\omega)) = \lim_{n\to\infty} Y_n(\omega) = Y(\omega)$$

   So, $Y = f(X)$ where $f : \mathbb{R} \to \mathbb{R}$ is measurable.
   $\Leftarrow$ Assume $Y = f(X)$ where $f : \mathbb{R} \to \mathbb{R}$ is measurable.
   Since X is a random variable on $(\Omega, \mathscr{F}, \mathbb{P})$, then $X : \Omega \to \mathbb{R}$ and $X^{-1}(B) \subseteq \mathscr{F}$ where B is any Borel set. $\sigma(X)$ is the σ-field generated by $X^{-1}(B)$. We know, $Y = f(X) : \Omega \to \mathbb{R}$ and $(f \circ X)^{-1}(B) = X^{-1} \circ f^{-1}(B)$.
   Since f is measurable, so $f^{-1}(B)$ is also a Borel set. So, $Y = f(X)$ is measurable w.r.t $\sigma(X)$.

   Assume X and Y are random variables from $(\Omega, \mathcal{F})$ to $(S, \mathcal{S})$, then

   $$\{\omega : f(X(\omega)) \in B\} = \{\omega : X(\omega) \in f^{-1}(B)\} \in \mathcal{F}$$

   If and only if $f^{-1}(B) \in \sigma(X)$, $Y = f(X)$ is measurable w.r.t $\sigma(X)$. □

2. *Proof.*

   $$\lim_{p\to\infty} E(X^p) = \lim_{p\to\infty} \lim_{n\to\infty} \sum_{i=1}^n \left(\frac{i}{n}\right)^p \mathbb{P}\left(X = \frac{i}{n}\right)$$

   We have

   $$\begin{aligned} E(X^p) &= \lim_{n\to\infty} \sum_{i=1}^n \left(\lim_{p\to\infty} \left(\frac{i}{n}\right)^p \mathbb{P}\left(X = \frac{i}{n}\right)\right) \\ &= \lim_{n\to\infty} \left(0 + \left(\frac{i}{n}\right)^p \mathbb{P}\left(X = \frac{i}{n}\right)\bigg|_{i=n} \right) \\ &= \mathbb{P}(X = 1) \end{aligned}$$

   So, $E(X^p) = \mathbb{P}(X = 1)$ as $p \to \infty$. □

3. *Proof.*

(a) Consider *Derangement formula*

$$\mathbb{P}(X_n = 0) = \frac{D(n)}{n!}$$

where

$$D(n) = n! \cdot \sum_{k=2}^{n} \frac{(-1)^k}{k!}$$

Then

$$\mathbb{P}(X_n = 0) = \sum_{k=2}^{n} \frac{(-1)^k}{k!}$$

When $n \to \infty$,

$$\mathbb{P}(X_n = 0) \to \sum_{k=2}^{\infty} \frac{(-1)^k}{k!}$$

This is one of the expression of $\frac{1}{e}$.

$$\lim_{n\to\infty} \mathbb{P}(X_n = 0) = \frac{1}{e}$$

(b) Noticed that

$$\mathbb{P}(X_n = 1) = \frac{\binom{n}{1}D(n-1)}{n!} = \frac{D(n-1)}{(n-1)!} = \mathbb{P}(X_{n-1} = 0)$$

$$\mathbb{P}(X_n = 2) = \frac{\binom{n}{2}D(n-2)}{n!} = \frac{1}{2} \cdot \frac{D(n-2)}{(n-2)!} = \frac{1}{2}\mathbb{P}(X_n = 0)$$

$$\mathbb{P}(X_n = k) = \frac{\binom{n}{k}D(n-k)}{n!} = \frac{1}{k}\mathbb{P}(X_n = k-1)$$

So, we have

$$\mathbb{P}(X_n = k) = \frac{1}{k!} \cdot \mathbb{P}(X_n = 0)$$

$$\begin{aligned}
E[X_n] &= \sum_{k=0}^{n} k \cdot \mathbb{P}(X_n = k) \\
&= \sum_{k=0}^{n} k \cdot \frac{1}{k!}\mathbb{P}(X_n = 0) \\
&= \mathbb{P}(X_n = 0) \cdot \sum_{k=1}^{n} \frac{1}{(k-1)!} \\
&= \sum_{k=2}^{n} \frac{(-1)^k}{k!} \cdot \sum_{k=1}^{n} \frac{1}{(k-1)!}
\end{aligned}$$

By the way, when n is large enough, $E[X_n] \to 1$.

$\square$

4. *Proof.*

$$E[Y] = \int_{\Omega} y \cdot \mathbb{1}(Y > 0) d\mathbb{P}$$

$$E[Y^2] = \int_{\Omega} y^2 d\mathbb{P}$$

Consider the Integral form of *Cauchy–Schwarz inequality.*

$$\begin{aligned}(E[Y])^2 &= \left(\int_{\Omega} y \cdot \mathbb{1}(y > 0) d\mathbb{P}\right)^2 \\ &\leqslant \int_{\Omega} \left(y \cdot \sqrt{\mathbb{1}(y > 0)}\right)^2 d\mathbb{P} \cdot \int_{\Omega} \left(\sqrt{\mathbb{1}(y > 0)}\right)^2 d\mathbb{P} \\ &\leqslant \mathbb{P}(Y > 0) \cdot E[Y^2]\end{aligned}$$

$$\mathbb{P}(Y > 0) \geqslant \frac{(E[Y])^2}{E[Y^2]}$$

□

5. *Proof.* $\Leftarrow$ Assume that

$$g_j(x) = \sum_{i \in S} \mathbb{P}(X_j = x_j) \mathbb{1}_{\{x_j\}}(x) \geqslant 0$$

So we have

$$\mathbb{P}(X_1 = x_1, \cdots, X_n = x_n) = \prod_{j=1}^{n} g_j(x_j) = \prod_{j=1}^{n} \mathbb{P}(X_j = x_j) = \mathbb{P}(X_1 = x_1) \cdots \mathbb{P}(X_n = x_n)$$

By definition, they are independent.

$\Rightarrow$ If they are independent, we have

$$\mathbb{P}\left(X_1 = x_1, \cdots, X_n = x_n\right) = \prod_{j=1}^{n} \mathbb{P}\left(X_j = x_j\right)$$

Let $g_j(x) = \mathbb{P}\left(X_j = x\right)$ for all $1 \leqslant j \leqslant n$, they are non-negative functions from S to $\mathbb{R}$. □

6. *Proof.* Consider the expression of ω in binary. The probability of $X_n = 1$ means the probability that the nth decimal place is 1. By the definition and arbitrariness of ω, we know

$$\mathbb{P}\left(X_n = 0\right) = \mathbb{P}\left(X_n = 1\right) = \frac{1}{2}$$

That means,

$$X_n \sim \text{Bernoulli}(1/2),\ \forall n \in \mathbb{N}$$

And their independence is obvious. □