Math 735 - Fall 2020

Homework 1

Due: 10/14

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Question 1

Proof. If Y has a single continuous path, then it means there exists a ω s.t. $Y_t(\omega) = 0$ for all t. This is equivalent to

$$\exists\ \omega \in [0,1] \text{ s.t. } t \neq \omega\ \forall t \in \mathbb{R}_+$$

This is impossible, so Y does not have a single continuous path.
X and Y are modifications of each other means

$$\mathbb{P}(X_t = Y_t) = \mathbb{P}(Y_t = 0) = \mathbb{P}(t \neq \omega) = 1$$

For any fixed ω, this is true. So, X and Y are modifications of each other. □

Question 2

Proof. We need prove $\widetilde{B_t}$ is a Gaussian process.

$$\widetilde{\mathbb{P}}(t,x) = \mathbb{P}(\lambda^2 t, \lambda x) = \frac{1}{\sqrt{2\pi\lambda^2 t}} e^{-\frac{(\lambda x)^2}{2\lambda^2 t}} = \frac{1}{\lambda} \cdot \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} = \frac{1}{\lambda}\mathbb{P}(t,x)$$

So, $\widetilde{B_t} \sim \frac{1}{\lambda} N(0,t) = N(0, \frac{t}{\lambda^2})$ □

Question 3

Proof.

$$\begin{aligned}
E[B_t^k] &= \int_{-\infty}^{\infty} x^k \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \\
&= \frac{1}{k+1} x^{k+1} \cdot \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \Big|_{-\infty}^{\infty} + \frac{1}{t(k+1)} \int_{-\infty}^{\infty} x^{k+2} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \\
&= \frac{1}{t(k+1)} E[B_t^{k+2}]
\end{aligned}$$

$$E[B_t^{k+2}] = t(k+1)E[B_t^k]$$

We know $E[B_t^0] = 1$, $E[B_t^1] = 0$, so we have
When $k = 2j$,

$$E[B_t^k] = t^j \prod_{i=1}^{j} (2i-1)$$

When $k = 2j-1$

$$E[B_t^k] = 0$$

□

Question 4

Proof. Stopping time

1. Play until there is no money or play 500 games before stopping.
2. Play N games to stop,$\{\tau = N\}$
3. Flip a coin N times, stop at the first appearance of *head, tail, head*.

Not stopping time

1. Stopping gambling when the gambler gets the maximum amount of money he can win is not a stopping time.
   Explanation: Because it needs not only the information of the present and the past, but also the information of the future.
2. Stopping when the gambler doubles his wager is not a stopping time.
   Explanation: Because there is a positive probability that he will never double his money.
3. Sold the day before the stock fell is not a stopping time.
   Explanation: Because we don't know the future information.

□

Question 5

(L^p martingale convergence for $p > 1$ in discrete time)
Show that if $\{M_n\}$ is a martingale that satisfies

$$\mathbb{E}\left[|M_n|^p\right] \leqslant B < \infty$$

for some $p > 1$ and for all $n \geqslant 0$, then there exists a random variable M_∞ with $\mathbb{E}\left[|M_\infty|^p\right] \leqslant B$ such that

$$\mathbb{P}\left(\lim_{n\to\infty} M_n = M_\infty\right) = 1 \text{ and } \lim_{n\to\infty} ||M_n - M_\infty||_p = 0$$

Proof. By Jensen's inequality,

$$f(x) = x^p \text{ is a convex function for } p > 1$$

It gives $f(\mathbb{E}[|M_n|]) \leqslant \mathbb{E}[f(|M_n|)] \Rightarrow \mathbb{E}[|M_n|]^p \leqslant \mathbb{E}[|M_n|^p] \leqslant B < \infty$.
By L^1 martingale convergence Theorem, there exists M_∞ bounded such that

$$\mathbb{P}\left(\lim_{n\to\infty} M_n = M_\infty\right) = 1 \text{ and } \lim_{n\to\infty} ||M_n - M_\infty||_p = 0$$

By fatou's Lemma,$\mathbb{E}[|M_\infty|^p]$ is bounded, because

$$\mathbb{E}[|M_\infty|^p] = \mathbb{E}[\lim_{n\to\infty} |M_n|^p] \leqslant \liminf_{n\to\infty} \mathbb{E}[|M_n|^p] \leqslant B < \infty$$

By Doob's L^p maximal inequality, it gives

$$\mathbb{E}[|M_n^*|^p]^{\frac{1}{p}} \leqslant \frac{p}{1-p}\mathbb{E}[|M_n|^p]^{\frac{1}{p}} \leqslant \frac{p}{p-1}B^{\frac{1}{p}} < \infty$$

Then $\mathbb{E}[|M_n^*|^p]^{\frac{1}{p}} \to$ a finite number. Let

$$\lim_{n\to\infty} ||M_n^*||_p = ||M_\infty^*||_p < \infty$$

Thus,

$$\lim_{n\to\infty} ||M_n - M_\infty||_p = ||\lim_{n\to\infty} M_n - M_\infty||_p = 0$$

□

Question 6

Proof. Assume that M_t is a submartingale. By Theorem 3.6, we know

$$\mathbb{E}[M_{\tau\wedge T|\mathcal{F}_s}] \geqslant M_{\tau\wedge T\wedge s}$$

We know $M_{\tau\wedge T\wedge s} = M_{\tau\wedge s}$, since $s < T$. Thus

$$\mathbb{E}[M_{\tau\wedge T|\mathcal{F}_s}] \geqslant M_{\tau\wedge s}$$

Hence, $M_{\tau\wedge s}$ is also a submartingale. In Ex3.2, we know B is a standard Brownian motion and $\tau = \inf t \geqslant 0 : B_t = 1$. By Stopping Time Theorem, we have $\mathbb{E}[B_{\tau\wedge t}] = \mathbb{E}[B_0] = 0$.
However,

$$\mathbb{E}[B_\tau] = -P(B_\tau = -1) = -1 \neq 0$$

So, Theorem3.6 cannot hold without the truncation by T. □

Question 7

Proof. Consider the *1D-symmetric random walk* with

$$\forall i \in \mathbb{N},\ P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$$

Set

$$S_n = \sum_{i=1}^{n} X_k$$

the sum of first n-th terms of X_i, it is a martingale.
Let σ be the time arrive position Y, and let $\tau = \sigma + 1$. Here σ is a stopping time.
If τ is also a stopping time,

$$\mathbb{E}[S_\tau|\mathcal{F}_\sigma] = S_\sigma$$

Then, we have

$$\mathbb{E}[S_\tau] = \mathbb{E}[S_\sigma] = \mathbb{E}[S_{\sigma-1}] = \cdots = 0$$

However,

$$\mathbb{E}[S_\tau] = Y \neq 0$$

□