$X_n \Rightarrow X$ $\qquad \lim_{n\to\infty} P(X_n \le y) = P(X \le y)$ for all $y \in \mathbb{R}$ where F_X is cont.

Last time:

Thm: if $X_n \Rightarrow X$ then we can find $Y, Y_1, Y_2, \ldots$ on the same prob space with $Y_n \xrightarrow{a.s.} Y$ and $Y \overset{d}{=} X$, $Y_n \overset{d}{=} X_n$.

Thm: $X_n \Rightarrow X \iff E[g(X_n)] \to E[g(X)]$ for $\underline{\text{all}}$ bounded cont. g

$$\int g \, dQ_{X_n} \to \int g \, dQ_X$$

Proof: $\Rightarrow$ Choose an a.s. representation of this limit: $X_n \xrightarrow{a.s.} X$ then $g(X_n) \xrightarrow{a.s.} g(X)$

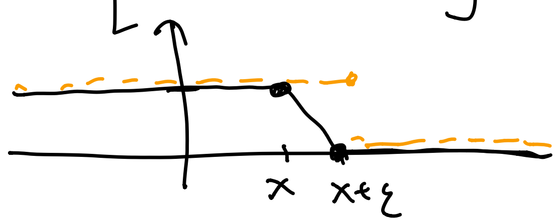
By DCT $\lim_{n\to\infty} E[g(X_n)] = E[g(X)]$

$\Leftarrow$

Goal: $\lim_{n\to\infty} P(X_n \le y) = P(X \le y)$ if F_X is cont at y.

$$P(X_n \le y) = E[\mathbb{1}(X_n \in (-\infty, y])]$$

$g_{x,\varepsilon}(y)$:



$$= \begin{cases} 1 & \text{if } y \le x \\ \text{linear} & y \in [x, x+\varepsilon] \\ 0 & \text{if } y \ge x+\varepsilon \end{cases}$$

This is continuous and bounded.

We know that $E\left[g_{x,\varepsilon}(X_n)\right] \to E\left[g_{x,\varepsilon}(X)\right]$.

Suppose that y is a cont. point of F_X.

$$P(X_n \le y) \le E\, g_{y,\varepsilon}(X_n)$$

$$\underbrace{\limsup_{n\to\infty} P(X_n \le y)} \le \lim_{n\to\infty} E\, g_{y,\varepsilon}(X_n) = E\left[g_{y,\varepsilon}(X)\right]$$

$$E\left[g_{y,\varepsilon}(X)\right] \le \underbrace{P(X \le y+\varepsilon)}$$

$$P(X_n \le y) \ge E\left[g_{y-\varepsilon,\varepsilon}(X_n)\right]$$

$$\liminf_{n\to\infty} P(X_n \le y) \ge \lim E\left(g_{y-\varepsilon,\varepsilon}(X_n)\right) = E\left[g_{y-\varepsilon,\varepsilon}(X)\right]$$

$$= P(X \le y-\varepsilon)$$

$$P(X \le y-\varepsilon) \le \liminf P(\quad) \le \limsup P(\quad)$$

$$\le P(X \le y+\varepsilon)$$

When $\varepsilon \to 0$ the upper and lower bounds both go $P(X \le y)$. ∎

Ex: Suppose that $X_n \Rightarrow X$ and g is cont.
Then $g(X_n) \Rightarrow g(X)$.

Proof: Take an a.s representation: $X_n \xrightarrow{a.s.} X$

Since g is cont, we have $g(X_n) \xrightarrow{a.s.} g(X)$.

This implies $E h(g(X_n)) \to E h(g(X))$

for any bounded cont h, $g(X_n) \Rightarrow g(X)$.

If $Y_n \xrightarrow{a.s.} Y$ then $Y_n \Rightarrow Y$.

Theorem (Continuous Mapping thm)

Suppose that $X_n \Rightarrow X$, and that g is Borel-measurable.

$D_g = \{x \in \mathbb{R} : g \text{ is discontinuous at } x\}$

If $P(X \in D_g) = 0$ then $g(X_n) \Rightarrow g(X)$.

If g is also bounded then $E g(X_n) \to E g(X)$.

Theorem: The following statements are equivalent

1) $X_n \Rightarrow X_\infty$
2) For any open set $G \subseteq \mathbb{R}$ $\quad \liminf_{n\to\infty} P(X_n \in G) \geq P(X_\infty \in G)$
3) For any closed set $F \subseteq \mathbb{R}$ $\quad \limsup_{n\to\infty} P(X_n \in F) \leq P(X_\infty \in F)$
4) If $P(X_\infty \in \partial A) = 0$ then $\lim_{n\to\infty} P(X_n \in A) = P(X_\infty \in A)$.
   A Borel

∂A: boundary of the set $A \subset \mathbb{R}$ closure of $A \setminus$ interior A

Proof: $2 \Leftrightarrow 3$ just by taking complements

$2 + 3 \Rightarrow 4$ Use 2. for interior of A
3. for closure of A

$P(Y \in \text{interior} A) \leq P(Y \in A) \leq P(Y \in \text{closure} A)$

If $P(X_\infty \in \partial A) = 0$ then
$P(X_\infty \in \text{interior} A) = P(X_\infty \in \text{closure} A)$

$4 \Rightarrow 1$ Suppose that y is a cont point of F_∞. $P(X_\infty = y) = 0$

Set $A = (-\infty, y]$ $\partial A = \{y\}$

By 4 $\lim_{n\to\infty} P(X_n \leq y) = P(X_\infty \leq y)$.

$X_n \Rightarrow X$

Last step: $1 \Rightarrow 2$

$X_n \Rightarrow X_\infty$ we can choose an a.s. representation: $X_n \xrightarrow{a.s.} X_\infty$. G is open

$\liminf_{n\to\infty} P(X_n \in G) = \liminf_{n\to\infty} E[\mathbb{1}(X_n \in G)]$

Fatou's Lemma

$$\liminf E\left[1(X_n\in G)\right] \geq E\left[\liminf 1(X_n\in G)\right]$$
$$\geq E\left[1(X_\infty\in G)\right] = P(X_\infty\in G)$$

Since $X_n \xrightarrow{a.s.} X_\infty$, if $X_\infty \in G$ then $X_n\in G$ for n large enough hence

$$\liminf_{n\to\infty} 1(X_n\in G) \geq 1(X_\infty\in G)$$

$\blacksquare$

Compactness for convergence in distribution?

Thm (Helly's selection theorem)

If we have a sequence of CDFs $F_1, F_2, \ldots$ then we can find a subsequence which converges to a nondecreasing, right-cont function F at all continuity points of F.

Remarks:

1) the limiting F might not be a CDF because $F(\infty)=\lim_{x\to\infty} F(x)$, $F(-\infty)=\lim_{x\to-\infty} F(x)$ might not be 1 and 0. $0\leq F(-\infty)\leq F(\infty)\leq 1$

This F corresponds to a measure on $\mathbb{R}$ with total mass ≤ 1.

2) This type of convergence is called <u>vague</u> convergence.

3) $P(X_n = -n) = P(X_n = n) = \frac{1}{2}$

$$F_{X_n}(x) = \begin{cases} 0 & \text{if } x < -n \\ \frac{1}{2} & \text{if } -n \le x < n \\ 1 & \text{if } n \le x \end{cases}$$

These functions converge pointwise to $F(x) = \frac{1}{2}$.

These CDFs converge in the vague sense, but not in distribution.

Proof: For any $q \in \mathbb{Q}$ (rational) we can find a converging subsequence of $F_n(q)$.

Using a diagonal argument we can find a subsequence F_n so that $F_n(q)$ converges for <u>all</u> $q \in \mathbb{Q}$.

Denote the limit at q: $G(q)$.

This is a nondecreasing function on $\mathbb{Q}$.

Define $F(x) = \lim_{\substack{q_n \searrow x \\ q_n \in \mathbb{Q}}} G(q_n)$

Claim: F is nondecreasing, right cont and $F_n(x) \to F(x)$ for all x where F is cont.

Def: We say that a sequence of CDFs $F_1, F_2, \ldots$ is $\boxed{\text{tight}}$ if for any ε there is an $M > 0$ so that

$$\limsup_{n\to\infty} \underbrace{1 - F_n(M) + F_n(-M)}_{P(|X_n| \geq M)} \leq \varepsilon.$$

Equivalent definition: for any $\varepsilon > 0$ there is an $M > 0$ so that

$$P(|X_n| \geq M) < \varepsilon \text{ for all } n$$

Thm: $F_1, F_2, \ldots$ is tight if and only if every subsequential vague limit is a CDF.

If a sequence of CDFs is not tight then there is a subsequence along which "mass can escape to $\pm\infty$".