$$\varphi_X(t) = E[e^{iXt}]$$

If $E[|X|^n] < \infty$ then $\varphi^{(n)}(0) = i^n E[X^n]$

$$\varphi_X(t) = \sum_{m=0}^{n} \frac{E[X^m]\, i^m}{m!} t^m + o(t^n)$$

Lemma: $\left| e^{ix} - \sum_{m=0}^{n} \frac{(ix)^m}{m!} \right| \le \min\left(\frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!} \right)$

Proof: induction + partial integration

We apply this bound for e^{itX} and take expectations:

$$\left| \varphi(t) - \sum_{k=0}^{n} E\left[\frac{(itX)^k}{k!} \right] \right| \le E\left[\min\left(|tX|^{n+1}, 2|tX|^n \right) \right]$$

$n = 2$ (we assume $E[X^2] < \infty$)

$$\left| \varphi(t) - \left(1 + it E[X] - \frac{t^2}{2} E[X^2] \right) \right| \le$$

$$\le t^2 E\left[\min\left(t|X|^3, 2X^2 \right) \right]$$

$$\boxed{E[X^2] < \infty} \le 2t^2 \, E[X^2] \le c \cdot t^2$$

$$\overset{\shortparallel}{O}(t^2)$$

Claim:

$$\lim_{t \to 0} E\Big[\underbrace{\min\big(t|X|^3, 2X^2\big)}_{\le 2X^2}\Big] = 0$$

dominated conv thm

Back to the proof of CLT

$X_1, X_2, \dots$ iid $\quad E[X_i^2] < \infty$

$E[X_i] = \mu \quad Var(X_i) = \sigma^2$

$S_n = X_1 + \dots + X_n$

$$\frac{S_n - n\mu}{\sqrt{n} \cdot \sigma} \Rightarrow N(0,1)$$

Proof: $\quad S_n - n\mu = \sum_{i=1}^{n} (X_i - \mu)$

$\tilde{X}_i = X_i - \mu \quad Var(\tilde{X}_i) = \sigma^2$

$E[\tilde{X}_i] = 0$

We may assume that $\mu = 0$.

$$\varphi_{\frac{S_n}{\sqrt{n}\sigma}}(t) = \varphi_X\left(\frac{t}{\sqrt{n}\sigma}\right)^n = \left(1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right)^n$$

$$\Big\downarrow \text{ as } n \to \infty$$

$$\boxed{\varphi_X(t) = 1 - \frac{t^2}{2}\cdot\sigma^2 + o(t^2)} \qquad e^{-\frac{t^2}{2}}$$

$$\varphi_{\frac{S_n}{\sqrt{n}\sigma}}(t) \longrightarrow \varphi_{N(0,1)}(t) \quad \text{which}$$

$$\text{implies} \quad \frac{S_n}{\sqrt{n}\sigma} \Rightarrow N(0,1)$$

∎

Lemma: $c_n \to c \in \mathbb{C}$ then $(1+\frac{c_n}{n})^n \to e^c$.

Simple consequence: $X_n \sim \text{Poisson}(n)$

Then $\dfrac{X_n - n}{\sqrt{n}} \Rightarrow N(0,1)$.

$Y_1, Y_2, \ldots$ iid $Y_n \sim \text{Poisson}(1)$

$$S_n = Y_1 + \ldots + Y_n \sim \text{Poisson}(n)$$

$E[Y_1] = 1 \quad \text{Var}(Y_1) = 1$

$$\frac{X_n - n}{\sqrt{n}} \stackrel{d}{=} \frac{S_n - nE(Y_1)}{\sqrt{n}\cdot\sqrt{\text{Var}\, Y_1}}$$

Lindeberg - Feller theorem

Triangular array $\{X_{n,j}\}_{1 \le j \le k_n}$

$X_{n,1}, \dots, X_{n,k_n}$ are independent

$S_n = \sum_{j=1}^{k_n} X_{n,j}$ $\quad E[X_{n,j}] = 0$

$E[X_{n,j}^2] = \sigma_{n,j}^2 < \infty$

$E[S_n] = 0$

$Var[S_n] = \sum_{j=1}^{k_n} \sigma_{n,j}^2$

Assume ~~that~~ $\sum_{j=1}^{k_n} \sigma_{n,j}^2 \to 1.$

Then the following two statements are equivalent:

1. $S_n \Rightarrow N(0,1)$ and $\max_{1 \le j \le k_n} \sigma_{n,j}^2 \to 0$

2. (Lindeberg condition) For any $\varepsilon > 0$ we have $\lim_{n\to\infty} \sum_{j}^{k_n} E[X_{n,j}^2 \mathbb{1}(|X_{n,j}| > \varepsilon)] = 0.$

Proof of $2 \Rightarrow 1$:

$$\varphi_{n,m}(t) = E\left[e^{itX_{n,m}}\right]$$

$$\left|\varphi_{n,m}(t) - \left(1 - \frac{t^2}{2}\sigma_{n,m}^2\right)\right| \le E\min\left(|tX_{n,m}|^3, 2|tX_{n,m}|^2\right)$$

$$X_{n,m}^2 = X_{n,m}^2 1(|\ |>\varepsilon) + X_{n,m}^2 1(|\ |\le\varepsilon)$$

$$\le E\left[1(|X_{n,m}|\le\varepsilon)|tX_{n,m}|^3\right]$$

$$+ E\left[1(|X_{n,m}|>\varepsilon)2t^2X_{n,m}^2\right]$$

$$\le \varepsilon|t|^3 E\left[X_{n,m}^2 \cancel{1(|X_{n,m}|\le\varepsilon)}\right]$$

$$+ 2t^2 E\left[X_{n,m}^2 1(|X_{n,m}|>\varepsilon)\right]$$

$$\sum_{m=1}^{k_n}\left|\varphi_{n,m}(t) - \left(1-\frac{t^2}{2}\sigma_{n,m}^2\right)\right| \le$$

$$\le \varepsilon|t|^3 \underbrace{\sum_{m=1}^{n}\sigma_{n,m}^2}_{\downarrow\ 1} + 2t^2 \underbrace{\sum_{n=1}^{m} E\left[X_{n,m}^2 1(|\ |>\varepsilon)\right]}_{\downarrow \text{ as } n\to\infty\ \ 0}$$

$$\limsup_{n\to\infty} \sum_{m=1}^{k_n}\left|\varphi_{n,m} - \left(1-\frac{t^2}{2}\sigma_{n,m}^2\right)\right| \le \varepsilon|t|^3$$

$$\varphi_{S_n}(t) = \prod_{m=1}^{k_n}\varphi_{n,m}(t)$$

if $|z_i|, |w_i| \le 1$ then $\left|\prod z_i - \prod w_i\right| \le$

$$\le \sum |z_2 - w_2|$$

$$|z_1 z_2 z_3 - w_1 w_2 w_3| \le |z_1 z_2 z_3 - z_1 z_2 w_3| + |z_1 z_2 w_3 - z_1 w_2 w_2| + |z_1 w_2 w_3 - w_1 w_2 w_3|$$

$$\le |z_3 - w_3| + |z_2 - w_2| + |z_1 - w_1|$$

$$\Big| \underbrace{\prod \varphi_{n,m}(t)}_{\varphi_{S_n}(t)} - \underbrace{\prod \left(1 - \frac{t^2 \sigma_{n,m}^2}{2}\right)}_{\downarrow ?\ e^{-\frac{t^2}{2}}} \Big| \to 0$$

We need to prove that $\max\limits_{1 \le j \le k_n} \sigma_{n,j}^2 \to 0$

$$\sigma_{n,m}^2 = E[X_{n,m}^2]$$

$$= E[X_{n,m}^2 \mathbb{1}(|X_{n,m}| < \varepsilon)] + E[X_{n,m}^2 \mathbb{1}(|\ | > \varepsilon)]$$

$$\le \varepsilon^2 + E[X_{n,m}^2 \mathbb{1}(|X_{n,m}| > \varepsilon)]$$

$$\max_{1 \le m \le k_n} \sigma_{n,m}^2 \le \varepsilon^2 + \underbrace{\sum_{m=1}^{k_n} E[X_{n,m}^2 \mathbb{1}(|X_{n,m}| > \varepsilon)]}_{\downarrow\ 0}$$

This shows that $\max \sigma_{n,m}^2 \to 0$.

$$\left| \prod e^{-\frac{t^2}{2}\sigma_{n,m}^2} - \prod \left(1 - \frac{t^2}{2}\sigma_{n,m}^2\right) \right| \to 0$$

$$\wedge\!\!\!\backslash$$

$$\sum \left| e^{-\frac{t^2}{2}\sigma_{n,m}^2} - \left(1 - \frac{t^2}{2}\sigma_{n,m}^2\right) \right|$$

$$\wedge\!\!\!\backslash$$

$$\sum_m C \cdot |t|^3 \sigma_{n,m}^3 \le \underbrace{\max \sigma_{n,m}}_{\downarrow\, 0} \cdot C |t|^3 \underbrace{\sum \sigma_{n,m}^2}_{\downarrow\, 1}$$

$$\prod e^{-\frac{t^2}{2}\sigma_{n,m}^2} \to e^{-\frac{t^2}{2}} \qquad \blacksquare$$

CLT follows from Lindeberg-Feller

$Y_1, Y_2, \ldots$ $\quad E[Y_i] = 0, \quad \mathrm{Var}(Y_i) = 1$

$$\frac{\sum_{i=1}^n Y_i}{\sqrt{n}} \Rightarrow N(0,1)$$

Define $X_{n,k} = \frac{Y_k}{\sqrt{n}}$ for $1 \le k \le n$

$\sigma_{n,j}^2 = \frac{1}{n} \qquad \sum_{j=1}^n \sigma_{n,j}^2 = 1$

We need to check the Lindeberg condition

$$\sum_j E[X_{n,j}^2 \mathbb{1}(|X_{n,j}|>\varepsilon)] =$$

$$= \sum_j E\left[\left(\frac{Y_j}{\sqrt{n}}\right)^2 \mathbb{1}(|Y_j|>\sqrt{n}\,\varepsilon)\right]$$

$$= E\left[Y_1^2 \mathbb{1}(|Y_1|>\sqrt{n}\,\varepsilon)\right]$$

This converges to 0 for any fixed $\varepsilon>0$ as $n\to\infty$ by the dominated convergence theorem. ∎

Lyapunov's theorem

Suppose that $X_{n,j}$ form an independent triangular array. $E[X_{n,j}]=0$ $E[X_{n,j}^2]=\sigma_{n,j}^2$

$\sum_j \sigma_{n,j}^2 = 1$. $\delta>0$

let $\Gamma_n = \left(\sum_j E|X_{n,j}|^{2+\delta}\right)^{\frac{1}{2+\delta}}$.

If $\Gamma_n \to 0$ then $\sum_j X_{n,j} \Rightarrow N(0,1)$.

Proof: We need to check the L-condition

$$\sum_j E\left[X_{n,j}^2 \, \mathbb{1}(|X_{n,j}| > \varepsilon)\right] \le$$

$$\sum_j E\left[|X_{n,j}|^{2+\delta} \varepsilon^{-\delta}\right] = \frac{\Gamma_n^{2+\delta}}{\varepsilon^{\delta}} \to 0$$

Rate of convergence

Berry-Essen: $X_1, X_2, \ldots$ iid

$$F_n(x) = P\left(\frac{S_n - n\mu}{\sqrt{n}\,\sigma} \le x\right)$$

$$\sup_x \left|F_n(x) - \Phi(x)\right| \le \frac{3\, E|X_1|^3}{\sigma^3 \sqrt{n}}$$

$\frac{1}{\sqrt{n}}$ rate of convergence is optimal in the general case