Quadratic variation II

$[B]_t = t$ $\qquad$ $B_t^2 - t$ is mart.

Thm: If M is right-cont L^2-mart.

then $M_t^2 - [M]_t$ is a mart.

(If M is ------ local L^2 ---

then -------- is local ---)

Pf: $\quad E[M_t^2 - M_s^2 \mid \mathcal{F}_s]$

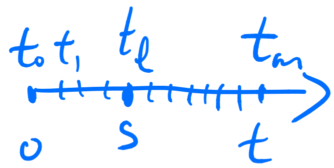
$s<t$

$$= E[(M_t - M_s)^2 \mid \mathcal{F}_s]$$

$$(M_t - M_s)^2 = M_t^2 - 2M_sM_t + M_s^2$$

$$E[M_t^2 - M_s^2 - [M]_t + [M]_s \mid \mathcal{F}_s] \overset{?}{=} 0$$

$$= E\Big[\sum_{i=l}^{m-1} (M_{t_{i+1}}^2 - M_{t_i}^2) - [M]_t + [M]_s \,\Big|\, \mathcal{F}_s\Big]$$



$s = t_l$

$$= E\Big[\sum_{i=l}^{m-1} (M_{t_{i+1}} - M_{t_i})^2 - [M]_t + [M]_s \,\Big|\, \mathcal{F}_s\Big]$$

$$= E\left[\sum_{i=0}^{m-1}(M_{t_{i+1}}-M_{t_i})^2 - [M]_t \,\middle|\, \mathcal{G}_s\right]$$

$$+ E\left[[M]_s - \sum_{i=0}^{\ell-1}(M_{t_{i+1}}-M_{t_i})^2 \,\middle|\, \mathcal{G}_s\right]$$

$$\longrightarrow 0 \quad \text{as } |\pi| \to 0 \qquad \blacksquare$$

For local mart.

τ_k is localizing seq for M.

$X := M^2 - [M]$.

$$(M^{\tau_k})^2 - \underset{\substack{\parallel \\ [M]^{\tau_k}}}{[M^{\tau_k}]} \quad \text{is mart.}$$

X^{τ_k} is mart.

$\Rightarrow$ τ_k is localizing seq for X

Thm: M, N are right-cont. L^2-mart. (local)

$\Rightarrow$ $MN - [M,N]$ is mart. (local)

Pf: $MN - [M,N]$

$$= \frac{1}{2}\left((M+N)^2 - [M+N] \right)$$

$$- \frac{1}{2}\left(M^2 - [M] \right) - \frac{1}{2}\left(N^2 - [N] \right)$$

Thm: $Y = \underline{Y_0} + M + V$

M: cadlag local mart

V: cadlag FV.

Then,

$[Y]$ exits.

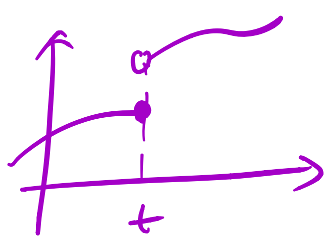
$$[Y]_t = [M]_t + \underbrace{[V]_t}_{\sum_{s\in(0,t]} (\Delta V_s)^2} + \underbrace{2[M,V]_t}_{2\sum_{s\in(0,t]} \Delta M_s \Delta V_s}$$

pf: Seppalainen's notes. (P_{103})

$[X]$ quad variation.

$\langle X \rangle$ predictable quad variation

If X is continuous



If you know X_s ~~for~~ for all $s<t$

then you can predict X_t

$X(t,\omega)$ $\qquad \mathbb{R}_+ \times \Omega$

predictable σ-algebra $\mathcal{P}$ on $\mathbb{R}_+ \times \Omega$

adapted left-cont

$\mathcal{P}$ is generated by events of the form $\{(t,\omega) : X_t(\omega) \in B\}$

Borel set in $\mathbb{R}$

A $\mathcal{P}$-measurable function

$$X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$$

is called predictable process

Thm: If M is right-cont. L^2 mart. (local)

Then $\exists$ unique predictable process $\langle M \rangle$

s.t. $M^2 - \langle M \rangle$ is mart. (local)

def of pred. quad var.

$$\boxed{\langle M, N \rangle = \frac{1}{4}\langle M+N \rangle - \frac{1}{4}\langle M-N \rangle}$$

Prop If M is continuous L^2-mart.

then $\langle M \rangle = [M]$.

Pf: M cont $\Rightarrow [M]$ cont.

Also $M^2 - \underline{[M]}$ is mart $\Rightarrow [M] = \langle M \rangle$

prop: If M right-cont. L^2 mart. with stationary independent increments

Then $\langle M\rangle_t = t\,\mathbb{E}[M_1^2 - M_0^2]$

(Pf: Seppalainen's notes P_{105})

Ex: BM B

$$\langle B\rangle_t = t\,\underline{\mathbb{E}[B_1^2 - \overset{(\mathbb{E}B_t)^2}{\overset{\|}{B_0^2}}]} = t$$

$$M_t = N_t - \alpha t \qquad M_0 = 0$$

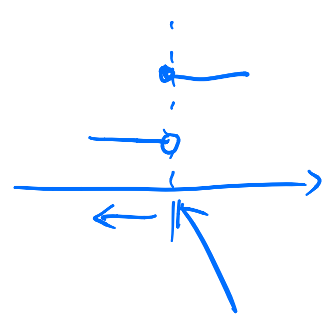
$$\langle M\rangle_t = t\,\mathbb{E}[M_1^2] = t\,\mathbb{E}[(N_1-\alpha)^2]$$

$$= \alpha t$$

predictable

$$\neq [M]_t = N_t$$

not predictable



$$\boxed{M^2 - \langle M\rangle \text{ mart.}}$$

Doob-Meyer decomposition.

Suppose X is right-cont. nonnegtive submart.

Then $\exists$ unique increasing predictable process A

s.t. $X - A$ is mart.

M^2 is nonnegtive submart,

$M^2 - \underbrace{\langle M \rangle}_{A}$ is mart.

Rmk "DL" class right-cont submart. (Seppalainen's notes P107)