Brownian motion: Properties

- BM is a Gaussian process:

$(B_{t_1}, \dots B_{t_k})$ is a joint Gaussian.

start at x

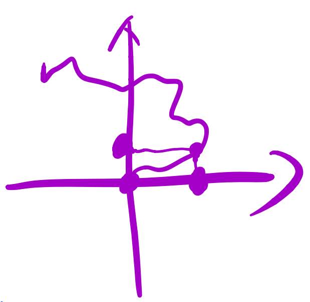
$$E^x(B_t) = x$$

$$= \int_{-\infty}^{+\infty} x_1 \underbrace{P(t, x, x_1)}_{\frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-x_1)^2}{2t}}} \, dx_1$$

$$E\left[(B_t - x)^2\right] = t$$

$$E\left[(B_s - x)(B_t - x)\right] = \min(s,t)$$

n-dim $B_t = \underbrace{(B_t^1, B_t^2, \dots B_t^n)}_{\text{are indep. 1-dim BM's}}$



$\mathbb{R}^n$

$$E^x(B_t) = x$$

$$\mathbb{E}\left[(\underbrace{B_t - x})^2\right] = \mathbb{E}\left[\underline{(B^1_t - x^1)^2} + \cdots + (B^n_t - x^n)^2\right]$$

$$= nt$$

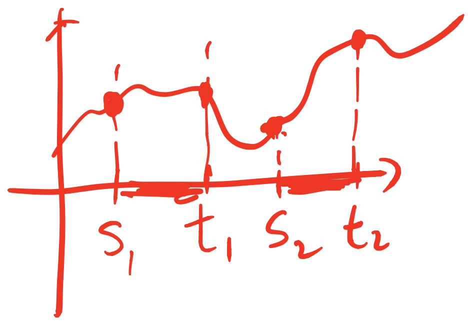
$\underline{t>s}$

$$\mathbb{E}\left[(\underbrace{B_t - B_s})^2\right] = n(t-s)$$

"increment of B from s to t"

$$\mathbb{E}^0\left[e^{iuB_t}\right] = e^{-\frac{1}{2}u^2 t} \qquad \underline{B_t \sim N(0,t)}$$

- BM has indep increments.



$\underline{s_1 < t_1 < s_2 < t_2}$

$B_{t_1} - B_{s_1}$ and $B_{t_2} - B_{s_2}$ are indep.

In general, for $0 \le t_1 < t_2 < \cdots < t_k$

B_{t_1}, $\underline{B_{t_2} - B_{t_1}}$, $\underline{B_{t_3} - B_{t_2}}$, $\cdots$, $B_{t_k} - B_{t_{k-1}}$

are indepent.

For Gaussian, <u>indep</u> if and only if <u>uncorrelated</u>

$$\mathbb{E}^{0}\left[(B_{t_2}-B_{s_2})(B_{t_1}-B_{s_1})\right]$$

$$= \mathbb{E}^{0}\left[B_{t_2}B_{t_1} - B_{t_2}B_{s_1} - B_{s_2}B_{t_1} + B_{s_2}B_{s_1}\right]$$

$$= t_1 - s_1 - t_1 + s_1 = 0$$

- $t \mapsto B_t(\omega)$ continuous for almost all ω.

Def: suppose $\{X_t\}$, $\{Y_t\}$ are two stoch processes on $(\Omega, \mathcal{F}, P)$

we say X is a "version" or "modification" of Y. if $\mathbb{P}\left(\{\omega : X_t(\omega) = Y_t(\omega)\}\right) = 1$ for all t

Kolmogorov continuity thm:

suppose $\exists\, C>0,\ \alpha, \beta>0$ $\mathbb{E}\left[|X_t - X_s|^{\alpha}\right] \le C \cdot |t-s|^{1+\beta}$ $\forall\ s, t$.

then, there is a continuous version of X.

For BM. $\mathbb{E}\left[|B_t - B_s|^2\right] = |t-s|$

$$\mathbb{E}\left[|B_t - B_s|^4\right] = 3|t-s|^2 \qquad \alpha = 4, \quad \beta = 1$$

$\Rightarrow$ B is continuous.

For n-dim BM,

$$\mathbb{E}\left[|B_t - B_s|^4\right]$$

$$= \mathbb{E}\left[\left((B_t^1 - B_s^1)^2 + \cdots + (B_t^n - B_s^n)^2\right)^2\right]$$

$$= \sum_{j=1}^{n} \mathbb{E}\left[(B_t^j - B_s^j)^4\right] + 2\sum_{i<j} \mathbb{E}\left[(B_t^i - B_s^i)^2 (B_t^j - B_s^j)^2\right]$$

$$= 3n(t-s)^2 + 2\underbrace{\sum_{i<j}}(t-s)(t-s)$$

$$= \left(3n + n(n-1)\right)(t-s)^2$$

$$= n(n+2)(t-s)^2$$

- Gaussian
- indep increments
- a.s. continuous.