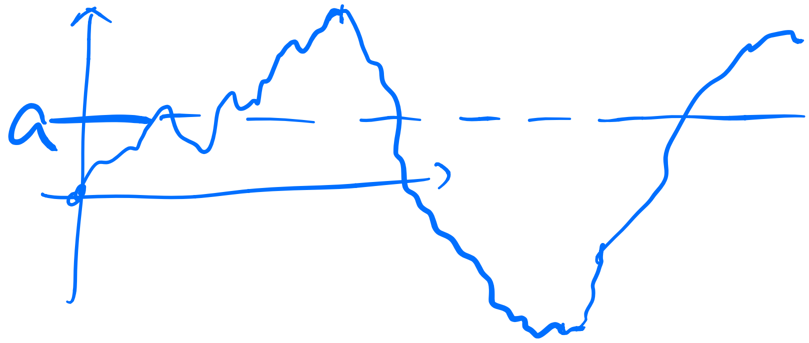
Applications of Generalized Itô formula

Ex 1

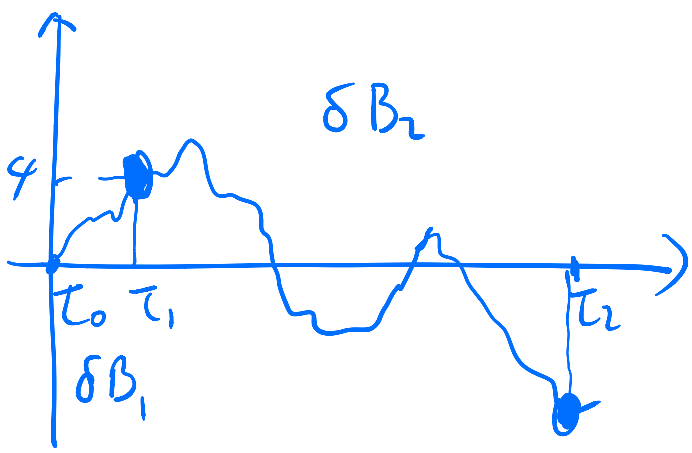


$$\overline{\lim_{t\to\infty}}\, B_t = \infty \qquad \underline{\lim_{t\to\infty}}\, B_t = -\infty \quad \text{a.s.}$$

$$\left(\text{recall } \overline{\lim_{t\to\infty}}\, f_t = \lim_{t\to\infty} \sup_{s\geq t} f_s \right)$$

In particular. B visits every value infinitely often

Pf:



$$\tau_1 = \inf\{t > \tau_0,\ |B_t - B_{\tau_0}| = 4\}$$

$$\tau_2 = \inf\{t > \tau_1,\ |B_t - B_{\tau_1}| = 16\}$$

$$\tau_{k+1} = \inf\{t > \tau_k,\ |B_t - B_{\tau_k}| = 4^{k+1}\}$$

$$\delta B_k = B_{\tau_k} - B_{\tau_{k-1}}$$

$$P(\delta B_k = 4^k) = P(\delta B_k = -4^k) = \frac{1}{2}$$

for a.e. $\omega \in \Omega$,

there are arbitrarily large j, k s.t.

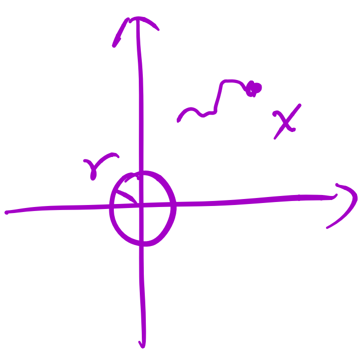
$$\delta B_j = 4^j \quad \text{and} \quad \delta B_k = -4^k$$

$$B_{\tau_j} = \delta B_1 + \delta B_2 + \cdots + \delta B_j$$

$$-4 \quad -4^2 \quad \cdots \quad +4^j > \frac{4^j}{2} > M$$

Given any $M > 0$, can take j large enough.

Ex2 BM in $\mathbb{R}^d$.



$$\tau_r = \inf\{t \geq 0, |B_t| \leq r\}$$

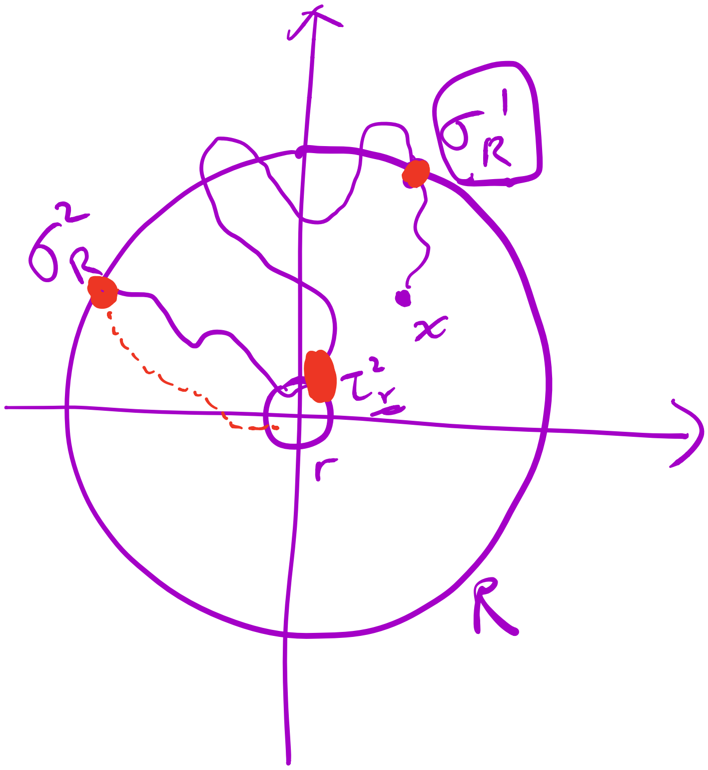
$d=2$. $\quad P(\tau_r < \infty) = 1$

$d \geq 3$. $\quad P(\tau_r < \infty) = \left(\frac{r}{|x|}\right)^{d-2}$

$\exists$ a.s. finite ~~time~~ time T. s.t.

$$|B_t| > r \quad \forall t \geq T.$$

Pf.



$$\sigma_R^1 < \tau_r^2 < \sigma_R^2 < \tau_r^3 < \cdots$$

$$\begin{cases} \tau_r^n = \inf\{t > \sigma_R^{n-1} : |B_t| \leq r\} \\ \sigma_R^n = \inf\{t > \tau_r^n : |B_t| \geq R\} \end{cases}$$

$$\mathbb{P}(\tau_r^2 < \infty) = \left(\frac{r}{R}\right)^{d-2} = \alpha$$

$$\mathbb{P}(\tau_r^3 < \infty) = \alpha \cdot \alpha = \alpha^2$$

$$\mathbb{P}(\tau_r^n < \infty) = \alpha^{n-1}$$

$$\sum_n P(\tau_r^n < \infty) < \infty .$$

By Borel-Cantelli,

$\tau_r^n < \infty$ can happen only finitely many times.

e.g. $\tau_r^1 \ \tau_r^2 \ \tau_r^3 < \infty$

$\tau_r^4 = \infty$.

Ex 3. Lévy characterization of BM.

X_t continuous local martingale

and $[X]_t = t$.

$\Rightarrow$ X must be BM

Pf.

$$Z = e^{i\theta X + \frac{1}{2}\theta^2 t}$$

$$f(t,x) = e^{i\theta x + \frac{1}{2}\theta^2 t}$$

$$\cancel{dZ = \frac{1}{2}\theta^2 Z \boxed{dt}}$$

$+ \; i\theta Z \boxed{dX}$

$+ \; \cancel{\frac{1}{2} \cdot (i\theta)^2} Z \boxed{dt}$ $\quad d[X]$

local mart.

$$Z_t = Z_0 + i\theta \underbrace{\boxed{\int_0^t Z\, dX}}_{\text{local mart}}$$

$\Rightarrow Z_t$ is local mart.

$$|Z_t| \le e^{\frac{1}{2}\theta^2 t}$$

In fact Z is mart.

$$\mathbb{E}\left[e^{i\theta X_t + \frac{1}{2}\theta^2 t} \,\middle|\, \mathcal{F}_s\right] = e^{i\theta X_s + \frac{1}{2}\theta^2 s}$$

$$\Leftrightarrow \mathbb{E}\left[e^{i\theta \underbrace{(X_t - X_s)}} \,\middle|\, \mathcal{F}_s\right] = e^{-\frac{1}{2}\theta^2 (t-s)}$$

$\Rightarrow X$ is BM.