If $X_n \Rightarrow X$, $Y_n \xrightarrow{P} c \in \mathbb{R}$

then $X_n + Y_n \Rightarrow X + c$.

Also: $X_n \cdot Y_n \Rightarrow c \cdot X$.

If $Z_n \xrightarrow{P} a \in \mathbb{R}$

$$X_n \cdot Y_n + Z_n \Rightarrow c \cdot X + a.$$

Berry-Essen thm

$$\sup_{x \in \mathbb{R}} | F_n(x) - \Phi(x) | \leq \frac{3 E|X|^3}{\sigma^3 \sqrt{n}}$$

F_n: CDF $\frac{S_n - n\mu}{\sqrt{n}\,\sigma}$

$X_1, X_2, \ldots$ iid $P(X_i = +1) = P(X_i = -1) = \frac{1}{2}$

shows that $\frac{1}{\sqrt{n}}$ is sharp.

Local limit theorems

$X_1, X_2, \ldots$ iid $E[X_i] = 0$, $E[X_i^2] = 1$

$X_i \in \mathbb{Z}$ CLT: $\frac{S_n}{\sqrt{n}} \Rightarrow N(0,1)$

$$p_n(x) := P\left(\frac{S_n}{\sqrt{n}} = x\right) \qquad x \in \frac{1}{\sqrt{n}}\mathbb{Z}$$

Local CLT: $$\sup_{x \in \frac{1}{\sqrt{n}}\mathbb{Z}} \left| \sqrt{n}\, p_n(x) - \varphi(x) \right| \xrightarrow[n\to\infty]{} 0$$

PDF of $N(0,1)$

If the support of X_i is not a subset of $b + h \cdot \mathbb{Z}$

then $$\sqrt{n}\, P\left(S_n \in (x_n + a,\, x_n + b) \right) \to (b-a)\,\varphi(x)$$

$$\frac{x_n}{\sqrt{n}} \xrightarrow[n\to\infty]{} x$$

Lindeberg replacement trick

Convergence in distribution can be proved by considering expectations of certain test functions.

Q: suppose that for $X_1, X_2, \ldots$

all moments converge:

$$E[X_n^k] \xrightarrow[n\to\infty]{} m_k \in \mathbb{R}$$

Does it follow that $X_n \Rightarrow X$ with $E[X^k] = m_k$.

If $E[X_n^2] \to m_2$ then $X_1, X_2, \ldots$ is tight.

Moment problem:

Q: is it true that the sequence of moments identifies the distribution?

i.e.: if X, Y are random variables with $E[X^k] = E[Y^k] \in \mathbb{R}$ for all k, is it true that $X \overset{d}{=} Y$?

The answer is no!

Counterexample: X is abs cont with PDF

$$f_0(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-\frac{(\log x)^2}{2}} \mathbb{1}_{x>0}$$

This is the lognormal distribution

$$X = e^{Y} \quad \text{with } Y \sim N(0,1).$$

$$E[X^k] = E[e^{k \cdot Y}] = e^{\frac{k^2}{2}}$$

Claim: if $a \in [-1,1]$ then

$$f_a(x) = f_0(x)\left(1 + a \sin(2\pi \log x)\right) \cdot \mathbb{1}_{(x>0)}$$

is a PDF with the same sequence of moments as f_0.

Proof: we need to show that for any nonnegative integer r

$$0 = \int_0^\infty x^r \cdot f_0(x) \cdot \cancel{a} \cdot \sin(2\pi \log x)\, dx$$

Change of variables: $x = e^{r+s}$ . . . .

The sequence of moments for the lognormal distribution grows too quickly!

E.g. $\limsup_{k \to \infty} (m_{2k})^{\frac{1}{2k}} = r < \infty$

$$\varphi_X(t) = E[e^{itX}] = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} E[X^k]$$

If we have this then this series converges in a neighborhood of 0.

Using the uniqueness thm from complex analysis this shows that we cannot have another distribution with the same moments

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Thm (Carleman)

Necessary and sufficient condition to identify the distribution from the moments: $\sum_{k=1}^{\infty} (m_{2k})^{-\frac{1}{2k}} = \infty$

Poisson approximation

$X \sim \text{Poisson}(\lambda) \qquad P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$

$X_n \sim \text{Binom}(n, \frac{\lambda}{n}) \qquad E[X_n] = \lambda$

$X_n \Rightarrow \text{Poisson}(\lambda)$

"counting rare events"

Thm: $X_{n,m} \sim \text{Bernoulli}(p_{n,m})$

$X_{n,1}, \ldots X_{n,k_n}$ independent.

Assume:

1) $\sum_{m=1}^{k_n} p_{n,m} \xrightarrow[n\to\infty]{} \lambda \in (0,\infty)$

2) $\max_{1\le m\le k_n} p_{n,m} \xrightarrow[n\to\infty]{} 0$.

$S_n = X_{n,1} + \ldots + X_{n,k_n}$

$E[S_n] = \sum_{m=1}^{k_n} p_{n,m}$

In this case $S_n \Rightarrow \text{Poisson}(\lambda)$.

Proof #1:

$$\varphi_{S_n}(t) = \prod_{m=1}^{k_n} \varphi_{X_{n,m}}(t) = \prod_{m=1}^{k_n}\left(1+p_{n,m}(e^{it}-1)\right)$$

$Y \sim \text{Poisson}(\lambda)$

$$\varphi_Y(t) = E[e^{itY}] = \sum_{k=0}^{\infty} e^{itk}\frac{\lambda^k}{k!}e^{-\lambda} = e^{\lambda(e^{it}-1)}$$

$$\left| \underbrace{e^{\sum_{m=1}^{k_n} p_{n,m}(e^{it}-1)}}_{\prod_{m=1}^{k_n} e^{p_{n,m}(e^{it}-1)}} - \prod_{m=1}^{k_n}\left(1+p_{n,m}(e^{it}-1)\right)\right| \le$$

$$\le \sum_{m=1}^{k_n}\left| e^{p_{n,m}(e^{it}-1)} - \left(1+p_{n,m}(e^{it}-1)\right)\right|$$

$$\le C\sum_{m=1}^{k_n} p_{n,m}^2 \le C \underbrace{\max p_{n,m}}_{\to 0} \underbrace{\sum p_{n,m}}_{\to \lambda}$$

Proof #2: Coupling

Def: Total variation distance of two probability measures μ, ν on $\mathbb{R}$

$$\|\mu-\nu\|_{TV} := 2 \sup_{A \subset \mathbb{R}} |\mu(A)-\nu(A)|$$

If μ, ν are supported on $\mathbb{Z}$ then

$$\|\mu-\nu\|_{TV} = \sum_{z} |\mu(\{z\})-\nu(\{z\})|$$

(just use $A = \{z : \mu(\{z\}) \geq \nu(\{z\})\}$)

If $\|\mu_n - \nu\|_{TV} \to 0$ then $\mu_n \Rightarrow \nu$.

Lemma:

$$\|\mu_1 \times \mu_2 - \nu_1 \times \nu_2\|_{TV} \leq$$

$$\leq \|\mu_1-\nu_1\|_{TV} + \|\mu_2-\nu_2\|_{TV}$$

Lemma:

$$\|\mu_1 * \mu_2 - \nu_1 * \nu_2\|_{TV} \leq$$

$$\leq \|\mu_1 \times \mu_2 - \nu_1 \times \nu_2\|_{TV}$$

$$\leq \|\mu_1 - \mu_2\|_{TV} + \|\nu_1 - \nu_2\|_{TV}$$

$\mu \sim$ Bernoulli(p)

$\nu \sim$ Poisson(p)

	0	1	2	3
μ	$1-p$	p	0	0
ν	e^{-p}	pe^{-p}	$\frac{p^2}{2}e^{-p}$	

$$\|\mu - \nu\|_{TV} = |1-p-e^{-p}| + |p - pe^{-p}| + \underbrace{\sum_{k=2}^{\infty} \frac{p^k}{k!} e^{-p}}_{= 1-p-pe^{-p}}$$

$$\leq 2p^2$$

Back to the Poisson approximation

$\mu_{n,m}$: distribution of $X_{n,m}$
Bernoulli$(p_{n,m})$

$\nu_{n,m}$: Poisson$(p_{n,m})$

The distribution of S_n : $\mu_{n,1} * \mu_{n,2} * \cdots * \mu_{n,k_n}$

$Y_n \sim$ Poisson$\left(\sum_{m=1}^{k_n} p_{n,m}\right)$

distribution of Y_n : $\nu_{n,1} * \nu_{n,2} * \cdots * \nu_{n,k_n}$

$$\|Q_{S_n} - Q_{Y_n}\|_{TV} = \|\mu_{n,1} * \cdots * \mu_{n,k_n} - \nu_{n,1} * \cdots * \nu_{n,k_n}\|_{TV}$$

$$\leq \sum_{m} \|\mu_{n,m} - \nu_{n,m}\|_{TV} \leq 2 \sum_{n \geq 1}^{k_n} p_{n,m}^2$$ $\blacksquare$