$([0,1], \mathcal{B}, \mathbb{P})$

$\uparrow$ Lebesgue measure

$a \in [0,1]$

$\mathbb{P}(\{a\}) = 0$

$a<1$

$\{a\} \subset [a, a+\varepsilon]$

$\mathbb{P}(\{a\}) \leq \varepsilon$

$\rightsquigarrow \mathbb{P}(\{a\}) = 0$

Probability measures on $\mathbb{R}$

$\mathbb{P}$ is a prob measure on $(\mathbb{R}, \mathcal{B})$

$F(x) = \mathbb{P}((-\infty, x])$

non-decreasing, right-cont

$\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$

Thm: If $F: \mathbb{R} \to \mathbb{R}$ is nondecreasing, right-cont with $\lim_{x \to \infty} F(x) - \lim_{x \to -\infty} F(x) = 1$ then there is a unique prob measure μ such that $\mu((a,b]) = F(b) - F(a)$.

More generally, without this condition we get a measure μ on $(\mathbb{R}, \mathcal{B})$.

Semi algebra: collection of sets closed for intersection, complement is a finite disjoint union of some of those sets

algebra: closed for union and complement ($\rightsquigarrow$ intersection)

$A, B \in \mathcal{G}$ then $A \cup B \in \mathcal{G}$, $A^c \in \mathcal{G}$

The algebra generated by a given semialgebra is given by the collection of finite disjoint unions.

Theorem: Let S be a semi-algebra and μ a set function $S \to \mathbb{R}_+$ which is finitely additive:
if $A_1, A_2, .., A_n \in S$ disjoint $\rightsquigarrow \mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$
and it is σ-subadditive:
$A_1, A_2, .. \in S$ disjoint $\mu(\bigcup_i A_i) \leq \sum_i \mu(A_i)$.
Then μ has a unique extension to the algebra generated by S, and if the extension is σ-finite, then there is a unique extension to $\sigma(S)$.
The extension is a measure!

we can cover Ω with countably many finite measure sets.

$$(a, b] \cup (b, c] = (a, c]$$

$$(a, b]^c = (-\infty, a] \cup (b, \infty]$$

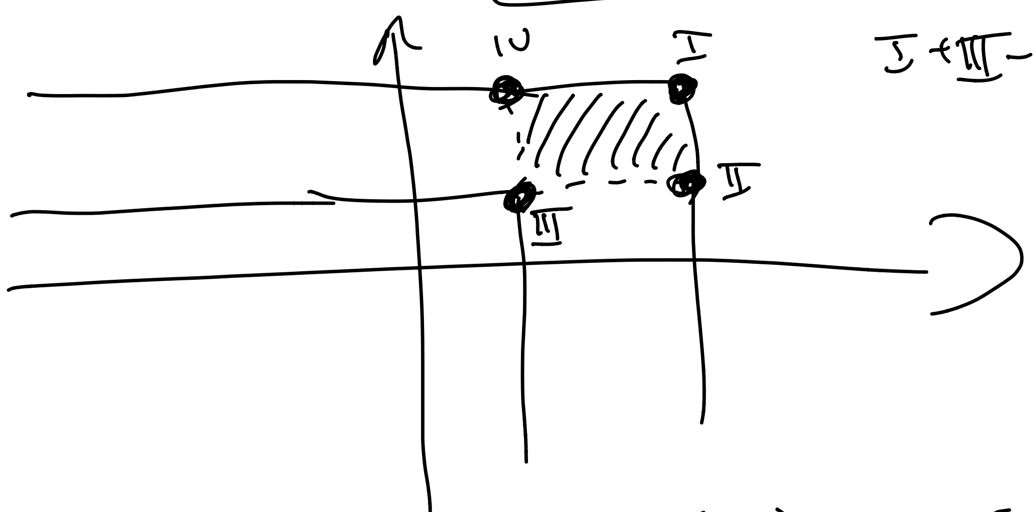
Extension to $\mathbb{R}^d$

$d = 2$ $\quad\mu$ is a measure on $(\mathbb{R}^2, \mathcal{B})$

$$F(x,y) = \mu\big((-\infty, x] \times (-\infty, y]\big)$$

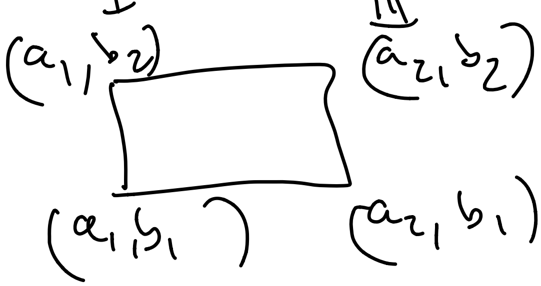
$$F(x) = \mu\big((-\infty, x]\big)$$

right cont, non decreasing in both variables



$\mathrm{I} + \mathrm{III} - \mathrm{II} - \mathrm{IV}$

$$\underset{\mathrm{I}}{F(a_1,b_1)} + \underset{\mathrm{III}}{F(a_2,b_2)} - \underset{\mathrm{II}}{F(a_2,b_1)} - \underset{\mathrm{IV}}{F(a_1,b_2)} \geq 0$$



$a_1 < a_2$

$b_1 < b_2$

Random variables

$X : \Omega \to \mathbb{R}$ which is $\overset{\text{Borel}}{\text{measurable}}$

$(\Omega, \mathcal{F}, \mathbb{P})$

For any Borel set B the inverse image of B : $X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\}$ is an event

$\underbrace{X^{-1}(B) \in \mathcal{F}}$ for all Borel set B.

What's the prob of $X \leq 2$?

$\{\omega : X(\omega) \leq 2\} = X^{-1}((-\infty, 2])$

Claim : if $\mathcal{A}$ generates $\mathcal{B}$ and $X^{-1}(A) \in \mathcal{F}$ for $A \in \mathcal{A}$ then X is a r.v.

Enough to check $X^{-1}((-\infty, c]) \in \mathcal{F}$ for all $c \in \mathbb{R}$.

Ex:

1) If $(\Omega, \mathcal{F}, \mathbb{P})$ is discrete then any $X : \Omega \to \mathbb{R}$ is a r.v.

2) constant function is a r.v.

indicator random variable

$$1_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \in A^c \end{cases}$$

we need $\emptyset, \Omega, A, A^c$ to be measurable

$A \in \mathcal{F}$ A is an event

Distribution of a r.v.

X is a r.v on $(\Omega, \mathcal{F}, \mathbb{P})$

$$\mu(A) = Q_X(A) = \mathbb{P}(X \in A)$$
$$= \mathbb{P}(X^{-1}(A))$$

μ is a probability measure on $(\mathbb{R}, \mathcal{B})$

$$\mu(\mathbb{R}) = \mathbb{P}(X \in \mathbb{R}) = \mathbb{P}(\Omega) = 1$$

check σ- additivity!

The distribution function (or cumulative distribution function) of a r.v. X is

$$F(x) = \mathbb{P}(X \le x) = Q_X((-\infty, x])$$

Properties:
1. non decreasing
2. right cont
3. $\lim_{x \to \infty} F(x) = 1$ $\quad \lim_{x \to -\infty} F(x) = 0$

The CDF of X identifies the distribution of X.

Thm: If F satisfies these conditions then there is a prob space $(\Omega, \mathcal{F}, \mathbb{P})$ and a r.v. $X : \Omega \to \mathbb{R}$ so that the CDF of X is exactly F.

Proof: $([0,1], \mathcal{B}, \mathbb{P})$
Lebesgue

$$X(\omega) = \sup\{y : F(y) < \omega\}$$

Suppose that the inverse F^{-1} is well-defined

Then $X(\omega) = F^{-1}(\omega)$ works

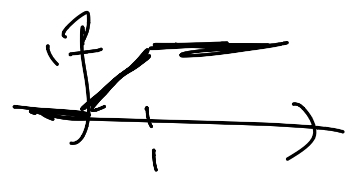
$$P(X \le x) = P(F^{-1}(\omega) \le x) = P(\omega \le F(x))$$
$$= P([0, F(x)]) = F(x)$$

this only works if F is invertible

Try to finish the proof by considering the general case.

In $([0,1], \mathcal{B}, \mathbb{P})$ $\quad U(\omega) = \omega$

$$F(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x < 0 \\ 1 & \text{if } x > 1 \end{cases}$$



The distribution of U is called the uniform distribution on $[0,1]$.

Def: if X and Y have the same distribution if $P_1(X^{-1}(B)) = P_2(Y^{-1}(B))$

X lives on $(\Omega_1, \mathcal{F}_1, P_1)$

Y lives on $(\Omega_2, \mathcal{F}_2, P_2)$

Notation $X \overset{d}{=} Y$

Ex: $(\underset{\{1,\dots,6\}}{\Omega}, \mathcal{F}, \mathbb{P})$ we roll a fair die

Give an example of two random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with the same distribution.

$$I_{\{1\}} \overset{d}{=} I_{\{2\}}$$

If $X = c$ is a constant
$$Q_X(A) = \begin{cases} 1 & \text{if } c \in A \\ 0 & \text{if } c \notin A \end{cases}$$

The distribution of an indicator random variable:

$$X = I_A = \mathbb{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

Q_X is supported on $\{0, 1\}$

$$Q_X(\{0\}) = \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$$
$$Q_X(\{1\}) = \mathbb{P}(A)$$

The distribution is identified by $\mathbb{P}(A)$.

This distribution is called the Bernoulli distribution with parameter p.

$p \in [0,1]$ $\quad\quad P(\{1\}) = p$

$\quad\quad\quad\quad\quad\quad P(\{0\}) = 1-p$

HW: try to construct a r.v. with Bernoulli(p) distribution on $([0,1], \mathcal{B}, P)$.

F is the CDF of X then

$$P(X \in (a,b]) = P(a < X \leq b) = F(b) - F(a)$$

$$P(X = a) = F(a) - \lim_{x \nearrow a^-} F(x)$$

If F is the CDF of the r.v. X and there is a measurable function f so that $F(x) = \int_{-\infty}^{x} f(y)\,dy$ then f is called the probability density function of X.

$$P(X \in (a,b]) = \int_a^b f(y)\,dy \quad\quad \frac{dQ_X}{d\lambda}$$

$$P(X \in B) = \int_B f(y)\,dy$$

Borel Lebesgue

In this case we say that the distribution of X is abs continuous.