* Our standard optimization problem:

$$\min_{\boxed{x} \in \boxed{\mathcal{X}}} \boxed{f(x)} \qquad (P)$$

vector — feasible set — objective function

$$\max_{x \in \mathcal{X}} f(x) \Leftarrow\Rightarrow \min_{x \in \mathcal{X}} -f(x)$$

* the value of (P): $\mathrm{val}(P) = \inf_{x \in \mathcal{X}} f(x)$.
* To give (P) a meaning, we need to specify:
  - vector space, feasible set, objective function
  - what it means to "solve" (P)

Q. Can we even hope to solve an arbitrary opt. problem?

Ex. Can you come up with an example of positive integers x, y, z s.t.

$$x^2 + y^2 = z^2 \Leftarrow (\text{Pythagorean triples})$$

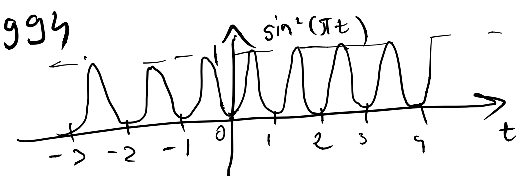
$(3, 4, 5)$; $(5, 12, 13)$; $(8, 15, 17)$ ....

How about $x^3 + y^3 = z^3$? $\Leftarrow$

* Fermat's conjecture [Fermat's Last Theorem] (1637)

For any $n \geq 3$, $x^n + y^n = z^n$ has no solutions over positive integers.

→ Proved by Andrew Wiles in 1994

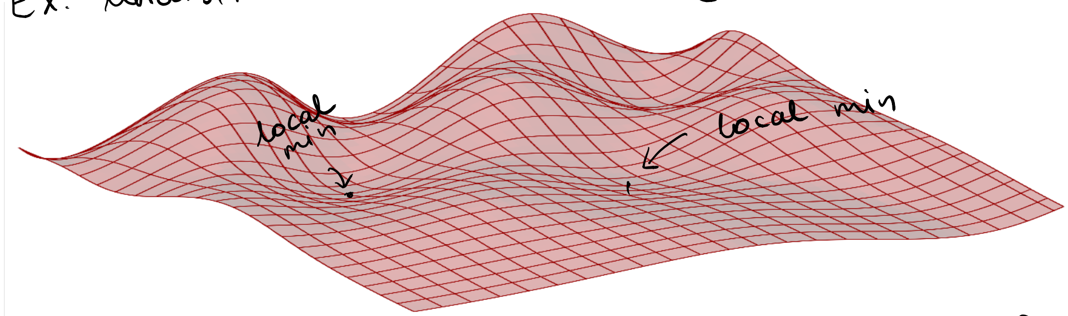


Consider:

$$(P_F) \begin{cases} \min_{x,y,z,n} (x^n + y^n - z^n)^2 \\ \text{s.t. } x \geq 1,\ y \geq 1,\ z \geq 1,\ n \geq 3 \\ \sin^2(\pi n) + \sin^2(\pi x) + \sin^2(\pi y) + \sin^2(\pi z) = 0 \end{cases}$$

If you could certify whether $\mathrm{val}(P_F) \neq 0$, you would have found a proof for Fermat's conjecture.

Ex. Unconstrained optimization, many local minima:



" Arbitrary optimization problems are hopeless, we always need some structure "

* Specifying the optimization problem:

$$\min_{x \in \mathcal{X}} f(x) \qquad (P)$$

① Vector space (where the optimization variables and the feasible set "live")

$(\underbrace{\mathbb{R}^d}_{\text{tells us that } x \text{ is a vector in } \mathbb{R}^d}, \underbrace{\|\cdot\|}_{\text{tells us how to measure distances in } \mathbb{R}^d})$: normed vector space; "primal space"

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}$$

* Most often, we will take $\|x\| = \|x\|_2 = \left(\sum_{i=1}^{d} x_i^2\right)^{1/2}$ (Euclidean norm)

* We might sometimes also consider ℓ_p norms:

$$\|x\|_p = \left(\sum_{i=1}^{d} |x_i|^p\right)^{1/p}, \quad p \geq 1$$

$$\|x\|_1 = \sum_i |x_i|, \quad \|x\|_\infty = \max_{1 \le i \le d} |x_i| .$$