Martingale convergence thm

(discrete-time)

Thm. $\{M_n\}$ mart. $\mathbb{E}[|M_n|] \le B < \infty \quad \forall n$

Then $\exists$ r.v. M_∞ with $\mathbb{E}[|M_\infty|] \le B$

s.t. $\mathbb{P}\left(\lim_{n\to\infty} M_n = M_\infty\right) = 1$

$M_n(w) \to M_\infty(w)$ a.e. w

Pf: $\forall\, a < b$ want $\mathbb{P}(A_{ab}) = 0$

$$A_{ab} = \{w : \liminf_{n\to\infty} M_n(w) \le a < b \le \limsup_{n\to\infty} M_n(w)\}$$

Lemma If M_n is a nonnegative submart.

and $\tilde{M}_n = M_0 + A_1(M_1 - M_0) + A_2(M_2 - M_1) + \cdots$

$+ A_n(M_n - M_{n-1})$

where A_n nonanticipating and value in $\{0,1\}$

Then. $\mathbb{E}(\tilde{M}_n) \le \mathbb{E}(M_n)$

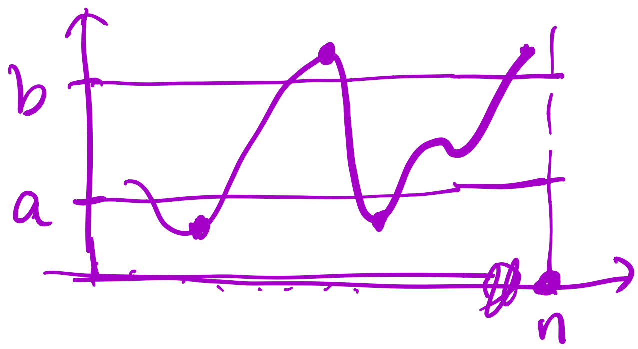
$\boxed{\text{upcrossing inequality}}$: If $\{M_n\}$ is submart.

then $\forall a<b$

$$(b-a)\,E[N_n(a,b)] \le E[(M_n-a)_+]$$

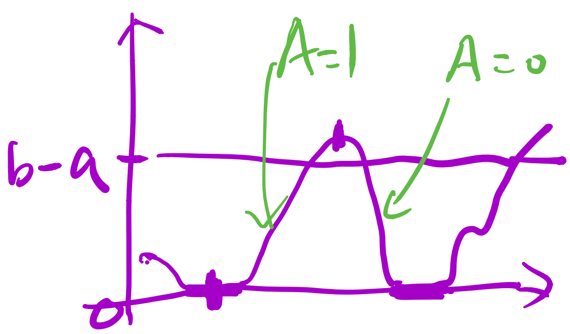
$N_n(a,b)$: upcrossing number.

$x_+ \stackrel{def}{=} \max(x,0)$



$N_n(a,b) = 2$

<u>proof</u>: $N_n(a,b)$ = # of upcrossings of $[0,b-a]$ by $\{(M_0-a)_+, (M_1-a)_+, \cdots, (M_n-a)_+\}$



$(M_n-a)_+$ a submart.

$\tilde{M}_n$ is mart. transform of $(M_n-a)_+$ by $A_n\in\{0,1\}$

$$E(\tilde{M}_n) \le E[(M_n-a)_+]$$

win at least $(b-a)N_n(a,b)$

$\blacksquare$

Back to pf: $\mathbb{P}(A_{ab}) = 0$.

$$A_{ab} \subset \{w : N_\infty(a,b) = \infty\}$$

So it suffice to prove $\mathbb{E}[N_\infty(a,b)] < \infty$

$$\mathbb{E}[N_n(a,b)] \le \frac{1}{b-a} \mathbb{E}[(M_n - a)_+]$$

$$\le \frac{1}{b-a}(|a| + B)$$

$n \to \infty$ $\quad \mathbb{E}[N_\infty(a,b)] < \infty$. $\square$

((conti-time))

Thm $\{M_t\}$ continuous mart.

(1) $\mathbb{E}(|M_t|^p) \le B < \infty$ for some $p > 1$ $\forall t$.

then $\exists$ r.v. M_∞ with $\mathbb{E}[|M_\infty|^p] \le B$

$$\mathbb{P}(\lim_{t\to\infty} M_t = M_\infty) = 1$$

$$\lim_{t\to\infty} \|M_t - M_\infty\|_p = 0$$

(2) $\mathbb{E}(|M_t|) \le B < \infty$ $\quad \forall t$

$$\mathbb{E}(|M_t|) \le B < \infty \quad \forall t$$

Then $\exists$ r.v. M_∞ $\quad \mathbb{E}|M_\infty| \le B$

$$\mathbb{P}\left(\lim_{t\to\infty} M_t = M_\infty\right) = 1$$

Pf part (1). $M_0, M_1, \dots \xrightarrow{n\to\infty} M_\infty$ a.s

$$|M_t - M_\infty| \le |M_m - M_\infty| + |M_t - M_m|$$

~~m t~~ Send $m \to \infty$ $\quad$ $|M_m - M_\infty| \xrightarrow{a.s} 0$ $\quad$ $\sup_{\{t : t \ge m\}} |M_t - M_m|$

①
$$\limsup_{t\to\infty} |M_t - M_\infty| \le \lim_{m\to\infty} \sup_{\{t:\, t\ge m\}} |M_t - M_m|$$

$\mathbb{P}(\quad - \quad - \quad . \quad)$

By Doob max ineq.

$$\mathbb{P}\left(\sup_{t\in[m,n]} |M_t - M_m| > \lambda\right)$$

$$\le \lambda^{-p}\, \mathbb{E}\left[|M_n - M_m|^p\right]$$

$n \to \infty$ $\quad M_n \to M_\infty$

Then $m \to \infty$.

then M ...

$$\mathbb{P}\left(\lim_{m\to\infty} \sup_{t\geq m} |M_t - M_m| > \lambda\right) \quad (2)$$

$$= 0$$

$$\leq \cancel{\lambda^{-p} E[|M_\infty - M_r}$$

(1) (2) $\Rightarrow$ $M_t \to M_\infty$ a.s. as $t\to\infty$

L^p - convergence:

$$\limsup_{t\to\infty} \|M_t - M_\infty\|_p \leq \|M_t - M_m\|_p + \|M_m - M_\infty\|_p$$

$$\leq \sup_{\{n:\,n\geq m\}} \|M_n - M_m\|_p$$

Send $m\to\infty$

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part (2). $\tau_n = \inf\{t: |M_t| \geq n\}$

Stopping theorem. $M_{t\wedge\tau_n}$ is mart $\forall n$.

by part (1) converge as $t\to\infty$

(If $\tau_n = \infty$. then $M_t = M_{t\wedge\tau_n}$)

So, for $w \in \underline{\{ w : \tau_n(w) = \infty \}}$.
then $\underline{M_t(w) \text{ converge}}$.

Doob max ineq.

$$\mathbb{P}\Big(\sup_{t \in [0,T]} |M_t| \geq \underbrace{\lambda}_{n} \Big) \leq \frac{\mathbb{E}(|M_T|)}{\lambda} \leq \frac{B}{\lambda}$$

Send $T \to \infty$

$$\mathbb{P}(\tau_n = \infty) \geq \underline{1 - \frac{B}{n}}$$

$$\mathbb{P}\Big(\bigcup_{n=1}^{\infty} \underline{\{\tau_n = \infty\}} \Big) = 1$$

$\Rightarrow$ a.e. $\underline{w} \in \Omega$ $M_t(w)$ converge

$\blacksquare$