Martingale theory (Basic examples)

1940s ~ 1950s
Doob

$\{\text{Brownian motion}\} \subset \{\text{Martingales}\} \subset \{\text{Stochastic processes}\}$

(submartingales & supermartingales)

Def: $\{X_t\}$ adapted to $\{\mathcal{F}_t\}$

we say $\{X_t\}$ is martingale.

1) $\mathbb{E}[|X_t|] < \infty \quad \forall t.$

2) $\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s$ "fair game"

$s \le t$

$\geq$ submartingale

$\leq$ supermartingale.

Ex: BM.

Ex: $\overbrace{B_t^2 - t}^{M_t}$ is mart.

1) $\mathbb{E}|B_t^2 - t| < \infty$

$\leq$

$(\mathbb{E} B_t^2) + t \leq 2t$

2) $\mathbb{E}[B_t^2 - t \mid \mathcal{F}_s]$

$= \mathbb{E}[(B_s + (B_t - B_s))^2 - t \mid \mathcal{F}_s]$

$$= E\left[B_s^2 + 2B_s(B_t - B_s) + (B_t - B_s)^2 \mid \mathcal{F}_s \right] - t$$

$$= B_s^2 + 0 + (t-s) - t$$

$$= B_s^2 - s = M_s$$

Ex $M_t = e^{\alpha B_t - \frac{\alpha^2}{2} t}$

$$E\left[e^{\alpha B_t - \frac{\alpha^2}{2} t} \mid \mathcal{F}_s \right]$$

$$= E\left[e^{\alpha(B_s + B_t - B_s)} \mid \mathcal{F}_s \right] \cdot e^{-\frac{\alpha^2}{2} t}$$

$$= e^{\alpha B_s} e^{-\frac{\alpha^2}{2} t} E\left[e^{\alpha(B_t - B_s)} \mid \mathcal{F}_s \right]$$

$$= e^{\alpha B_s} e^{-\frac{\alpha^2}{2} t} e^{\frac{\alpha^2}{2}(t-s)}$$

$$= e^{\alpha B_s - \frac{\alpha^2}{2} s} = M_s \qquad s \le t$$

Ex. $X_t = e^{\alpha B_t}$

$$E\left[e^{\alpha B_t} \mid \mathcal{F}_s \right] = e^{\alpha B_s} \underbrace{e^{\frac{\alpha^2}{2}(t-s)}}_{\geq 1}$$

$$\geq e^{\alpha B_s} = X_s$$

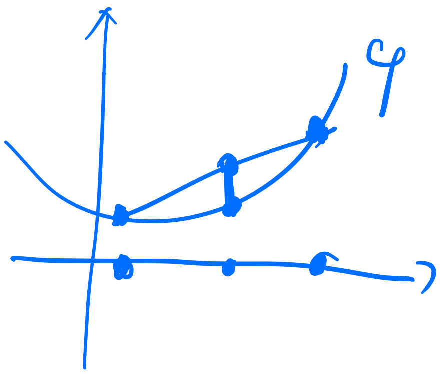
$\Rightarrow X_t$ is submart.

<u>Ex</u>. If X is mart $\Rightarrow$ $\varphi(X)$ is submart, $\forall$ convex φ.

$$\underbrace{\mathbb{E}[\varphi(X_t) \mid \mathcal{F}_s]}_{\text{Jensen ineq.}} \geq \underline{\varphi(X_s)}$$

$$\geq \varphi(\mathbb{E}[X_t \mid \mathcal{F}_s])$$

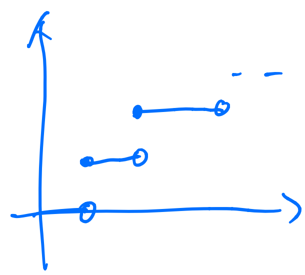
$$= \varphi(X_s)$$



$\left(\begin{array}{l} \Rightarrow B_t^2 \text{ is submart.} \\ \Rightarrow e^{\alpha B} \text{ is submart.} \end{array}\right)$

<u>Ex</u>. (Compensated Poisson)

$M_t = N_t - \underline{\lambda} t$



$$\mathbb{E}[N_t - \lambda t \mid \mathcal{F}_s]$$

$$= \mathbb{E}[N_s + \underbrace{(N_t - N_s)}_{Poi(\lambda(t-s))} \mid \mathcal{F}_s] - \lambda t$$

$$= N_s + \lambda(t-s) - \lambda t = N_s - \lambda s = M_s$$

Rmk Can Define discrete-time $\overset{(\text{sub})}{\underset{(\text{super})}{\text{martingales}}}$ in the same way.

Ex $X_1, X_2 \cdots X_n \cdots$ indep. $E(X_n)=0 \quad \forall n$

$S_0 = 0$

$S_n = X_1 + X_2 + \cdots + X_n$

$E[S_{n+1} \mid \mathcal{F}_n] = E[S_n + X_{n+1} \mid \mathcal{F}_n]$

$= S_n$

Ex: $\{X_n\}$ indep. $X_n \geq 0$ $E(X_n) = 1$

$M_n = X_1 \cdot X_2 \cdot \cdots X_n$

$E(M_{n+1} \mid \mathcal{F}_n] = E[M_n \cdot X_{n+1} \mid \mathcal{F}_n]$

$= M_n$.

iid. $\{Y_n\}$ $\phi(\lambda) = E e^{\lambda Y_n}$

$X_n = \dfrac{e^{\lambda Y_n}}{\phi(\lambda)}$ $E X_n = 1$

$\lambda \sum^{n} Y_i$

$$M_n = X_1 X_2 \cdots X_n = \frac{e^{\lambda \sum_{i=1} Y_i}}{\phi(\lambda)^n}$$