Math 733 – Fall 2020

Homework 6

Due: 10PM, December 6, 2020. Please upload your work in Canvas. Late homework will not be accepted.

1. Let $X_1, X_2, \dots$ be an i.i.d. sequence with $E[X_1] = \mu$, $\mathrm{Var}(X_1) = \sigma^2 < \infty$. Let N be a non-negative integer valued random variable with $E[N] = m$, $\mathrm{Var}(N) = s^2 < \infty$ that is independent of the sequence $X_1, X_2, \dots$. Let $S_n = \sum_{j=1}^n X_j$ with $S_0 = 0$, and consider the random variable S_N. (This is the element of the sequence $S_0, S_1, \dots$ corresponding to the random index N.) Find the expected value and variance of S_N.
   Hint: use conditioning.

2. Let S_n denote the position of a simple random walk after n steps. (I.e. $S_n = X_1 + \cdots + X_n$ with X_i i.i.d. $P(X_i = 1) = P(X_i = -1) = 1/2$.)

   (a) For which polynomials $g(x)$ will $g(S_n)$ be a martingale? (Here the coefficients of g cannot depend on n.)

   (b) Find a degree 3 polynomial $h(x, n)$ (with coefficients possibly depending on n) so that $h(S_n, n)$ is a martingale.

3. Let ν and τ be stopping times with respect to the same filtration $\{\mathcal{F}_n\}$. Show that $\min(\nu, \tau)$ and $\nu + \tau$ are also stopping times with respect to $\{\mathcal{F}_n\}$.

4. For a parameter $a \in (0, 1)$ let $p_a(x_1, x_2, \dots, x_n)$ denote the probability that the first n terms in the i.i.d. sequence $X_1, X_2, \dots$ of Bernoulli(a) random variables are exactly $x_1, x_2, \dots, x_n$. Let $a \neq b$ and

   $$Z_n = \frac{p_b(X_1, X_2, \dots, X_n)}{p_a(X_1, X_2, \dots, X_n)}$$

   where $X_1, X_2, \dots$ are i.i.d. Bernoulli(c) random variables. Prove that Z_n is a martingale if and only if $c = a$.

5. Suppose that $X_1, X_2, \dots$ are i.i.d. with $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$, and let $S_n = \sum_{k=1}^n X_i$. Show that the sequence $M_n = |S_n| - |\{0 \le k < n : S_k = 0\}|$ is a martingale. (M_n is the difference of $|S_n|$ and the number of visits to 0 up to $n - 1$.)

6. Let U be uniform on $[0, 1]$, and consider a sequence of random variables $X_1, X_2, \dots$ which are i.i.d. with Bernoulli(U) distribution given U. (One way to generate such a sequence is to take an i.i.d. sequence of Uniform$[0, 1]$ random variables $V_1, V_2, \dots$ that are independent of U, and set $X_k = \mathbf{1}(V_k \le U)$.)
   Let $S_n = X_1 + \cdots + X_n$, and $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$.

   (a) Find $E[X_{n+1} | \mathcal{F}_n]$.

   (b) Show that $M_n = \frac{S_n + 1}{n + 2}$ is a martingale with respect to the filtration $\mathcal{F}_n$.

   Hint: be careful, the random variables $X_1, X_2, \dots$ are not independent!

You can use the Piazza page to ask for clarifications about a specific problem, but please don't discuss explicit solutions before the deadline. Handing in plagiarized work, whether copied from a fellow student or off the web, is not acceptable and will lead to sanctions.

Bonus problem. Consider a symmetric simple random walk on the plane (i.e. each step is an independent one unit jump to one of the four directions with equal probability) and denote the distance from the origin by S_n. Let $\nu_r = \inf\{S_n > r\}$. Show that $r^{-2}E\nu_2 \to 1$ as $r \to \infty$.

Bonus problems are not graded, and you don't need to submit them. They are provided as an extra challenge for those who are interested.