# MATH 733 - Fall 2020

## Homework 5

**Due : 10PM, November 22, 2020**

Zijie Zhang

November 22, 2020

1. By Kolmogorov's 0-1 Law, If $X_1, X_2, \cdots$ are independent and $A \in \mathcal{T}$, then $P(A) = 0$ or $1$.
   Notice that, by Fatou's lemma
   $$P\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} > x\right) \geqslant \limsup_{n\to\infty} P\left(\frac{S_n}{\sqrt{n}} > x\right)$$
   By CLT, we know
   $$\limsup_{n\to\infty} P\left(\frac{S_n}{\sqrt{n}} > x\right) = P(\mathcal{N}(0,1) > x) > 0$$
   So, let $A_x = \{\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} > x\} \in \mathcal{T}$, $P(A_x) > 0$. Then $P(A) = 1$.
   Thus,
   $$\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} = \infty$$

2. By CLT,
   $$P\left(\frac{S_n - n}{\sqrt{n}} \leqslant \alpha\right) \to \Phi(\alpha) \text{ as } (n \to \infty)$$
   If we add $\frac{k}{\sqrt{n}}$ after $\alpha$, it still true.
   $$P\left(\frac{S_n - n}{\sqrt{n}} \leqslant \alpha + \frac{\alpha^2}{4\sqrt{n}}\right) \to \Phi(\alpha) \text{ as } (n \to \infty)$$
   $$\frac{S_n - n}{\sqrt{n}} \leqslant \alpha + \frac{\alpha^2}{4\sqrt{n}} \Leftrightarrow S_n \leqslant n + \alpha\sqrt{n} + \frac{\alpha^2}{4} = \left(\frac{\alpha}{2} + \sqrt{n}\right)^2 \Leftrightarrow \sqrt{S_n} - \sqrt{n} \leqslant \frac{\alpha}{2}$$
   So, we have
   $$P\left(\sqrt{S_n} - \sqrt{n} \leqslant \frac{\alpha}{2}\right) \to \Phi(\alpha) \text{ as } (n \to \infty)$$
   $$P\left(\frac{\sqrt{S_n} - \sqrt{n}}{\frac{1}{2}} \leqslant \alpha\right) \to \Phi(\alpha) \text{ as } (n \to \infty)$$
   $$\sqrt{S_n} - \sqrt{n} \to \mathcal{N}\left(0, \frac{1}{4}\right)$$

3. The problem satisfies Lindeberg's condition. Assume $E[X_k] = \mu_k$ and $\text{Var}[X_k] = \sigma_k^2$.
   Then, $s_n^2 = \text{Var } S_n = \sum_{k=1}^n \sigma_k^2$.
   $$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^n \left[(X_k - \mu_k)^2 \cdot \mathbb{1}_{|X_k - \mu_k| > \varepsilon s_n}\right] = 0$$
   Because, $|X_k| < M < \infty$ and $S_n \to \infty$. By CTL,
   $$Z_n = \frac{S_n - ES_n}{s_n}$$
   converge in distribution to a standard normal random variable as $n \to \infty$.

4.

$$\begin{aligned} N_n(a,b) &= \sum_{k=1}^{n} \mathbb{1}\left(X_k \in \left(c+\frac{a}{n}, c+\frac{b}{n}\right)\right) \\ &= \sum_{k=1}^{n} \left[\mathbb{1}\left(X_k \leqslant c+\frac{b}{n}\right) - \mathbb{1}\left(X_k \leqslant c+\frac{a}{n}\right)\right] \\ &= nF_n\left(c+\frac{b}{n}\right) - nF_n\left(c+\frac{a}{n}\right) \end{aligned}$$

Consider

$$\begin{aligned} \lim_{n\to\infty} \frac{N_n(a,b)}{b-a} &= \lim_{n\to\infty} \frac{nF_n\left(c+\frac{b}{n}\right) - nF_n\left(c+\frac{a}{n}\right)}{b-a} \\ &= \lim_{n\to\infty} \frac{F_n\left(c+\frac{b}{n}\right) - F_n\left(c+\frac{a}{n}\right)}{\frac{b-a}{n}} \\ &= \lim_{n\to\infty} \frac{F\left(c+\frac{b}{n}\right) - F\left(c+\frac{a}{n}\right)}{\frac{b-a}{n}} \end{aligned}$$

Notice that, this is $f(c)$. So, $N_n(a,b)$ converges in distribution for any $a < b$ and $\lim_{n\to\infty} N_n(a,b) = (b-a)f(c)$.

5. (a) Let $F_{n_m}(x)$ be the cdf. of $X_{n_m}$, $F_{n_m}(x) = P(X_{n_m} \leqslant x)$. We know $X_{n_m} \Rightarrow Y$, let $F(x)$ be the cdf. of $Y$.

   For any $k$,

$$E[Y^k] = \int_\Omega x^k dF(x) = \lim_{m\to\infty} \int_\Omega x^k dF_{n_m}(x) = \lim_{m\to\infty} E[X_{n_m}^k] = m_k$$

   (b) Consider the characteristic function of $X_m$,

$$\varphi_m(t) = E\left[e^{itX_m}\right] = \sum_{k=0}^{\infty} \frac{i^k E[X_m^k]}{k!} t^k = \sum_{k=0}^{n} \frac{i^k E[X_m^k]}{k!} t^k + o(t^{n+1})$$

$$\lim_{m\to\infty} \varphi_m(t) = \lim_{m\to\infty} \sum_{k=0}^{\infty} \frac{i^k E[X_m^k]}{k!} t^k = \lim_{m\to\infty} \sum_{k=0}^{\infty} \frac{i^k m_k}{k!} t^k = \varphi(t)$$

   Here, $\varphi(t)$ is moment-generating function with $C^\infty$ at $t=0$. Thus $X_m$ converges in distribution.