Independence : 2 events $P(AB) = P(A)P(B)$

2 random variables $P(X \in A, Y \in B) = P(X \in A) P(Y \in B)$ for all Borel A, B

2 σ-fields $\mathcal{A}, \mathcal{B} \subset \mathcal{F}$ $A \in \mathcal{A}, B \in \mathcal{B}$ are independent for any A, B

n sets : $\mathcal{A}_1, \dots, \mathcal{A}_n \subset \mathcal{F}$ are independent if

for any $I \subset \{1, \dots, n\}$ and $A_i \in \mathcal{A}_i, i \in I$

$$P\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} P(A_i)$$

Thm: If $\mathcal{A}_1, \dots, \mathcal{A}_n$ are independent and each one is closed for intersection then $\sigma(\mathcal{A}_1), \dots, \sigma(\mathcal{A}_n)$ are independent.

$\mathcal{A}_i \subset \mathcal{F}$

If the random variables $X_1, \dots, X_n$ are independent then

$$\underbrace{P(X_1 \le x_1, \dots, X_n \le x_n)}_{\text{Joint CDF}} = \underbrace{\prod_{i=1}^{n} P(X_i \le x_i)}_{\text{product of the individual CDFs}}$$

If this holds for all $x_1, \dots, x_n \in \mathbb{R}$ then $X_1, \dots, X_n$ are independent.

Similar statements hold for the case when $X_1, X_2, \dots, X_n$ are discrete random variables

$$P(X_1=x_1,\dots,X_n=x_n)=\prod_{i=1}^{n}P(X_i=x_i)$$

for all $x_1,\dots,x_n\in\mathbb{R}$

If $X_1,X_2,\dots,X_n$ are jointly abs continuous

$$\underset{\uparrow\ \text{joint PDF}}{f(x_1,\dots,x_n)} = \prod_{i=1}^{n}\underset{\uparrow\ \text{individual PDF}}{f_{X_i}(x_i)}$$

If $X_1,\dots,X_n$ are all abs cont, and they are independent then they are jointly abs cont.

Remark: If X_1 and X_2 are both abs cont then (X_1,X_2) might not be abs cont.

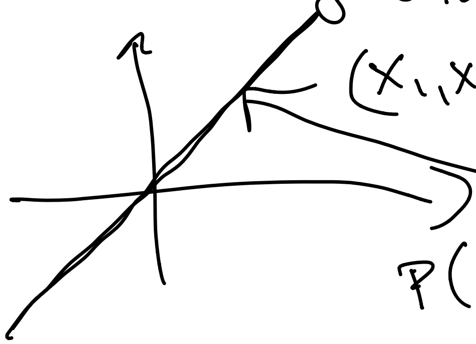
Ex: $X_1 \sim \text{Uniform}[0,1]$

Set $X_2=X_1$.

(X_1,X_2) will not be jointly abs cont!

If (Y_1,Y_2) are jointly abs cont with a PDF $f(y_1,y_2)$ then $P((Y_1,Y_2)\in B) = \int_B f(y_1,y_2)\,dy_1\,dy_2$

$B \subset \mathbb{R}^2$



(X_1,X_2) is supported on this line $y=x$

$$P((X_1,X_2)\in \text{line}) = 1$$

$$\neq \int_{\text{line}} f(x_1,x_2)\,dx_1\,dx_2 = 0$$

Why do we like independence?

If we have independent objects then we can describe the joint behavior from the individual one.

Ex: If $A_1, A_2, \dots, A_n$ are independent, $p_i = P(A_i)$ then for any $B \in \sigma(A_1, \dots, A_n)$ we can express $P(B)$ in terms of the p_i.

$$P\left(\left((A_1 \cup A_2)^c \cap A_3\right) \cup A_4\right) = ?$$

this can be written as the disjoint union of sets of the form $A_1^* A_2^* A_3^* A_4^*$

where A_i^* is A_i or A_i^c.

$$P(A_1^\circ \cdots A_n^\circ) = \prod_{i=1}^n P(A_i^\circ) \qquad P(A_i^\circ) = \begin{cases} P(A_i) & A_i^\circ = A_i \\ 1 - P(A_i) & A_i^\circ = A_i^c \end{cases}$$

Def: Events A_i, $i \in \underline{I}$ are independent if for any finite $J \subset I$ we have

$$P\Big(\bigcap_{j\in J} A_j\Big) = \prod_{j \in J} P(A_j)$$

Lemma: Suppose that $\mathcal{F}_t \subset \mathcal{F}$, $t \in T$ are independent σ-fields, and that R is a partition of T. (Elements of R are disjoint and $\cup R = T$)

Then $\sigma\big(\bigcup_{t \in S} \mathcal{F}_t\big) : S \in R$ are also independent.

E.g. $\boxed{\mathcal{F}_1, \mathcal{F}_2}, \boxed{\mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5}, \mathcal{F}_6$ (independent) $\qquad \{1,2\}, \{3,4,5\}, \{6\}$

then $\sigma(\mathcal{F}_1 \cup \mathcal{F}_2)$, $\sigma(\mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5)$, $\mathcal{F}_6$ are independent.

Same holds for independent events or random variables

If $\underbrace{A_1, A_2}, \underbrace{A_3, A_4, A_5}, \underbrace{A_6}$ are independent

then $A_1 \cup A_2$, $(A_3 A_4)^c \cup A_5$, A_6^c are independent.

If X_1, X_2, X_3, X_4, X_5 are independent

then $X_1 + X_2$, $\sin\left(\frac{1}{1+X_3^2+X_4^2}\right)$, $|X_5|$ are independent.

Thm: $X_1, X_2, \ldots, X_n$ are independent if and only if the joint distribution $Q_{(X_1,\ldots,X_n)}$ is the same as $Q_{X_1} \times \ldots \times Q_{X_n}$.

Proof: $\boxed{Q_{(X_1\ldots X_n)}(A_1 \times \ldots \times A_n)} = \underline{P(X_1 \in A_1, \ldots, X_n \in A_n)}$

$\underset{\text{independence}}{=}$ $\underline{\prod_{i=1}^{n} P(X_i \in A_i)} = \underline{\prod_{i=1}^{n} Q_{X_i}(A_i)} =$

$$= \boxed{Q_{X_1} \times \dots \times Q_{X_n} (A_1 \times \dots \times A_n)}$$

The two measures agree on product sets hence they agree on the generated σ-field (Borel sets in $\mathbb{R}^n$).

Thm: X, Y are independent $h : \mathbb{R}^2 \to \mathbb{R}$

assume $h \geq 0$ OR $E[|h(X,Y)|] < \infty$

then $$E[h(X,Y)] = \int\left(\int h(x,y) Q_X(dx)\right) Q_Y(dy)$$

$$= \iint h(x,y)\, Q_X(dx) \times Q_Y(dy)$$

If $h(x,y) = f(x) g(y)$ with $f, g \geq 0$

or $E|f(X)|, E|g(Y)| < \infty$ then

$$E[f(X) g(Y)] = E[f(X)] E[g(Y)].$$

IMPORTANT: $E[X_1 + X_2] = E[X_1] + E[X_2]$

(without any additional assumption)

In general $E[X_1 X_2] \neq E[X_1] E[X_2]$

If X_1, X_2 are independent then $E[X_1 X_2] = E[X_1] E[X_2]$

If X_1, X_2 are random variables with
$$E[X_1 X_2] = E[X_1] E[X_2]$$
then we say that X_1, X_2 are uncorrelated.

independence $\not\Leftarrow$ uncorrelated

Variance of X: $\quad Var(X) = E[(X - E[X])^2]$

Covariance of X, Y:
$$Cov(X, Y) = E[(X - E[X])(Y - E[Y])]$$
$$= E[XY] - E[X] E[Y]$$

$$Cov(X, X) = Var(X)$$

Correlation of X, Y:

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}\,X}\,\sqrt{\mathrm{Var}\,Y}}$$

defined if $\mathrm{Var}\,X > 0$, $\mathrm{Var}\,Y > 0$

$$\mathrm{Corr}(X,Y) = 0 \iff E[XY] = E[X]E[Y]$$

Name: $X_1, X_2, \dots$ are independent and identically distributed ($X_1 \stackrel{d}{=} X_2 \stackrel{d}{=} X_3 \stackrel{d}{=} \dots$)

i.i.d = independent and identically distributed

How to construct independent random variables with given distributions?

If we have finitely many random variables then we can realize them on the appropriate product prob space.

$(X_1, X_2, \dots, X_n)$ can be realized on $\mathbb{R}^n$ using $Q_{X_1} \times Q_{X_2} \times \dots \times Q_{X_n}$

What if we want infinitely many independent random variables?

$X_1, X_2, \ldots$ $\qquad\qquad \mathbb{N} = \{1, 2, \ldots\}$

$$\Omega = \mathbb{R}^{\mathbb{N}}$$

$$\mathcal{F} = \sigma\left(\text{finite cylinder sets}\right)$$

$A_i \subset \mathbb{R}$

$A_1 \times \ldots \times A_n \times \mathbb{R} \times \mathbb{R} \times \ldots$

or $\quad B_n \times \mathbb{R} \times \mathbb{R} \times \ldots$

where $B_n \subset \mathbb{R}^n$

$X_1 \; X_2 \; \ldots$

$$\mathbb{P}(A_1 \times \ldots \times A_n \times \mathbb{R} \times \ldots) =$$

$$= P(X_1 \in A_1) \cdots P(X_n \in A_n)$$

$$= Q_{X_1}(A_1) Q_{X_2}(A_2) \cdots Q_{X_n}(A_n)$$

Thm: $\mathbb{P}$ can be extended uniquely to $\mathcal{F}$.

The resulting measure is called the (infinite) product measure.

Kolmogorov's extension thm,