COMPSCI726: Nonlinear Optimization I
Notebook
Zijie Zhang

2020/09/04/Friday === Create Notebook

Overview

Basic Information

Meetings: Mon and Wed 2.30-3.45pm on BBC Ultra
Instructor: Jelena (pronounced as Yelena) Diakonikolas
Email: jelena@cs.wisc.edu
Office hours: Mon 9-10am and 4-5pm
TA: Eric Lin clin353@wisc.edu; OH: Wed 9-10am and 4-5pm

Textbook

Nocedal & Wright (Numerical Optimization, 2nd ed, 2006)

Workload and Assessment

- Homework (5-6):
  - 30% of the grade
  - a combination of math problems and coding assignments
  - no collaboration allowed; any discussion must be verbal-only
- Midterm:
  - 30% of the grade
  - to be scheduled in mid-October
  - typically 4 multi-part questions of a similar format/difficulty as homework questions
- Final:
  - 40% of the grade
  - scheduled for 12/16/2020
  - similar format to midterm

Topics Covered in Class

- Introduction: optimization background; convex sets; convex functions; convergence rates.

- Background on smooth unconstrained optimization: Taylor theorem and optimality conditions.
- First-order methods:
  - gradient descent for convex and nonconvex optimization, line search methods
  - projections, gradient mapping, and their use in projected gradient descent
  - Bregman divergence and mirror descent
  - Nesterov acceleration for convex optimization
  - conjugate gradients; lower bound for smooth convex minimization
  - conditional gradients (Frank-Wolfe methods)
  - stochastic gradient descent

- Second-order methods
  - Newton method
  - trust-region Newton
  - inexact Newton methods and Newton-CG
  - cubic regularization
  - quasi-Newton methods (DFP, BFGS, SR-1, general Broyden class)
  - limited-memory quasi-Newton (L-BFGS)

Lecture 1

Our standard optimization problem:

$$\min_{x\in X} f(x) \quad (P)$$

where x is **vector**, X is **feasible set**, $f(x)$ is **objective function**.

$$max_{x\in X} f(x) \Leftrightarrow \min_{x\in X} -f(x)$$

the value of (P): val(P) = $\inf_{x\in X}\ f(x)$.
To give (P) a meaning, we need to specify:

- vector space, feasible set, objective function
- what is means to "solve" (P)

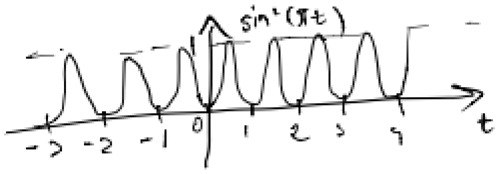
Q: Can we even hope to solve an aribitrary opt. problem?
Ex: Can you come up with an example of positive integers x, y, z s.t. (Pythagorean triples)

$$x^2 + y^2 = z^2$$

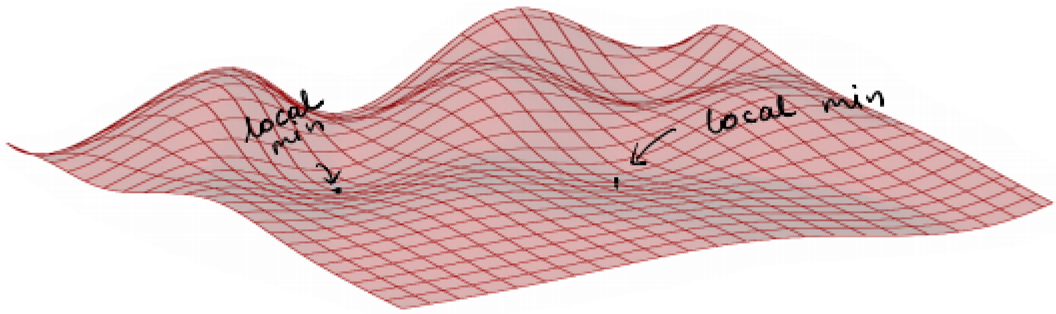
(3,4,5); (5,12,13); (8,15,17) ...
How about $x^3 + y^3 = z^3$?

- Fermat's conjecture Fermat's Last Theorem
  For any $n \geqslant 3, x^n + y^n = z^n$ has no solutions over positive integers.
  Proved by Andrew Wiles in 1994.
  Consider: P_F

$$\begin{cases} \min_{x,y,z,n}(x^n + y^n - z^n), \\ s.t. \quad x \geqslant 1, y \geqslant 1, z \geqslant 1, n \geqslant 3 \\ \sin^2(\pi n) + \sin^2(\pi x) + \sin^2(\pi y) + \sin^2(\pi z) = 0 \end{cases}$$



If you could certify whether $val(P_F) \neq 0$, you would have found a proof for Fermat's conjecture.



- Ex: Unconstrainted optimation, many minima: "Arbitrary optimization problems are hopeless, we always need some structive"

Specifying the optimization problem

1. **Vector space**(where the optimization variables and the feasible set "live")
   $(\mathbb{R}^d, ||\cdot||)$:normed vector space

- Most often, we will take $||x|| = ||x||_2 = \left(\sum_{i=1}^d x_i^2\right)^{\frac{1}{2}}$
- We might sometimes also consider l_p norms:

$$||x||_p = \left(\sum_{i=1}^d x_i^p\right)^{\frac{1}{p}}, p \geqslant 1$$

$$||x||_1 = \sum_i |x_i|, \quad ||x||_\infty = \max_{1 \leqslant i \leqslant d} |x_i|$$

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Lecture 2

We will use $\langle \cdot, \cdot \rangle$ to denote inner products.
Standard inner product:

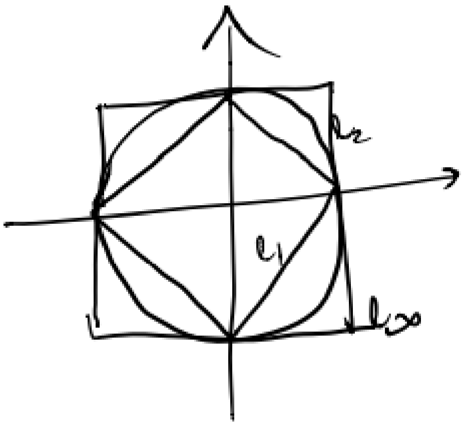
$$\langle x, y \rangle = x^T y = \sum_{i=1}^d x_i y_i$$

When we work with $(\mathbb{R}^d, ||\cdot||_p)$, view $\langle y, x \rangle$ as the value of a linear function y at x. So, if we are measuring the length x using the $||\cdot||_p$, we should measure the length of y using $||\cdot||_{p^*}$, where $\frac{1}{p} + \frac{1}{p^*} = 1$.
dual norm:$||z||_* = \sup_{||x|| \leqslant 1} \langle z, x \rangle$

$$\forall z, x : \langle z, x \rangle \leqslant ||z||_* \cdot ||x||$$

proof: Fix any two vectors x, z. Assume $x \neq 0, z \neq 0$, o.w. trivial. Define $\hat{x} = \frac{x}{||x||}$.

$$||z||_* \geqslant \langle z, \hat{x} \rangle = \frac{\langle z, x \rangle}{||x||}$$



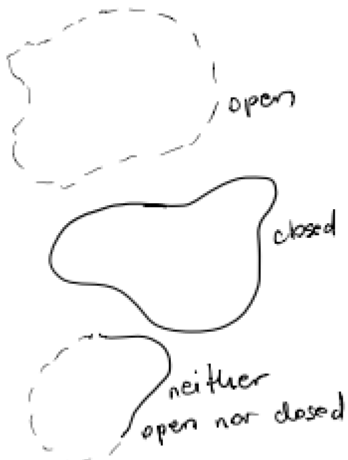
2. **Feasible set**:

- specifies what solution points we are allowed to output $X \subseteq \mathbb{R}^d$. If $x = \mathbb{R}^d$, we say that (P) is unconstrained. o.w., we say that (P) is constrained.
  X can be specified:
- as an abstract geometric body (a ball, a box, a polyhedron)
- via functional constraints: $g_i(x) \leqslant 0, i = 1, 2, \cdots, m.$, $\mu_i(x) = 0, i = 1, 2, \cdots, p$.

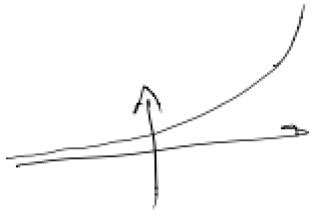
$$f_i(x) \geqslant C \Leftrightarrow g_i(x) = C - f_i(x)$$

E.g., $X = \mathcal{B}_2(0, 1)$ (Unit Euclidean ball) $X = \{x \in \mathbb{R}^d : ||x||_2 \leqslant 1\}$

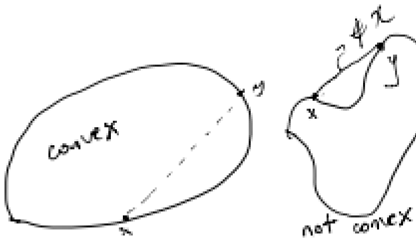
- In this class, we will always assume that X is **closed** and **convex**.



- **Heine-Borel Thm:** If X is closed and bounded, then it is compact.
  If $X \subseteq U_{\alpha \in A} U_\alpha$ for some family of open sets $\{U_\alpha\}$ then $\exists$ a finite subfamily $\{U_{\alpha_i}\}_{i=1}^n$ s.t. $X \subseteq U_{1 \leqslant i \leqslant n} U_{\alpha_i}$
- **Weierstrass Exrtreme Value Theorem:** If X is compact and f is a function that is defined and continous on X, then f attains its extreme values on X.
- What if X is not bounded?
  Consider $f(x) = e^x$, $\inf_{x \in \mathbb{R}} f(x) = 0$.



- When we work with unconstrained problems, we will normally assume that f is bounded below.
- Convex sets:
  Def: A set $X \subseteq \mathbb{R}^d$ is convex if ($\forall x, y \in X$)($\forall \alpha \in (0,1)$): $(1-\alpha)x + \alpha y \in X$. $x + \alpha(y-x)$



3. **Object function:**

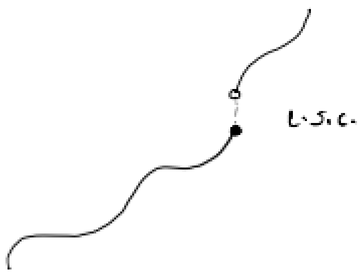
- "cost", "loss"
- Extended real valued functions:

$$f : \mathcal{D} \to \mathbb{R} \cup \{-\infty, \infty\}$$

- We will define f on all of $\mathbb{R}^d$ by assigning it value $+\infty$ at each point $X \in \mathbb{R}^d \setminus \mathcal{D}$.
- Effective domain: $dom(f) = \{x \in \mathbb{R}^d : f(x) < \infty\}$.
- "nonlinear opt" =(?) "continuous opt"
- Lower semicontinuous functions: **Def:** A function $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ if

$$\liminf_{y \to x} f(y) \geqslant f(x)$$

f is l.s.c. on $\mathbb{R}^d$ if it is l.s.c. at all $X \in \mathbb{R}^d$.



Ex. indicator of a **closed** set is l.s.c.

$$I_X(x) = \begin{cases} 0 & , x \in X \\ \infty & , x \notin X \end{cases}$$

$\min_{x \in X} f(x) \equiv \min_{x \in \mathbb{R}^d} \{f(x) + I_X(x)\}$

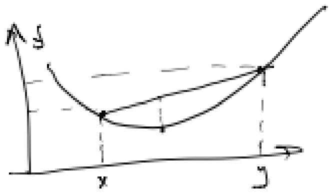
- Unless we are abstracting away constraints, the least we will assume about f is that it is continuous.
- **Def:** $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ is said to be:

- *Lipschitz-continuous* on $X \subseteq \mathbb{R}^d$ if $\exists M < \infty$
  $\forall x, y \in X : |f(x) - f(y)| \leqslant M||x - y||$.
- Smooth on $X \subseteq \mathbb{R}^d$ if f's gradients are *Lipschitz-continuous*, i.e., $\exists L < \infty$ s.t. $\forall x, y \in X$:

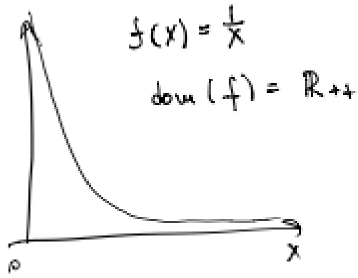
$$||\nabla f(x) - \nabla f(y)||_* \leqslant L||x - y||$$

- **Def:** $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ is convex if $\forall x, y \in \mathbb{R}^d, \forall \alpha \in (0, 1)$:

$$f((1-\alpha)x + \alpha y) \leqslant (1-\alpha)f(x) + \alpha f(y)$$



* Ex. function that is differentiable on its domain but not smooth: $f(x)=\frac{1}{x}, dom(f)=R_{++}$



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