Characteristic function of X: $\varphi_X(t) = E[e^{itX}]$
(only depends on the distribution of X)

Last time:

1) X, Y independent $\varphi_{X+Y} = \varphi_X \varphi_Y$

2) $\varphi_X = \varphi_Y$ on $\mathbb{R}$ then $X \overset{d}{=} Y$

3) if $\int_{\mathbb{R}} |\varphi_X| dx < \infty$ then X has PDF

$$f_X(x) = \int_{-\infty}^{\infty} e^{-itx} \varphi(t)\, dt.$$

X is abs cont

If $\varphi_X(t) \geq 0$ and $\int_{-\infty}^{\infty} \varphi_X(x)\, dx < \infty$

then we can define a PDF $\frac{1}{c} \varphi_X(x)$,

The characteristic function of the new PDF is the original PDF (multiplied by a constant).

Examples

1) $X \sim N(0,1)$ $\quad \varphi_X(t) = e^{-\frac{t^2}{2}}$

$\sigma X + \mu \sim N(\mu, \sigma^2)$

$$\varphi_{\sigma X + \mu}(t) = \varphi_X(\sigma t) e^{it\mu} = e^{-\frac{\sigma^2 t^2}{2} + t\mu \cdot i}$$

2) $X \sim \text{Poisson}(\lambda)$

$$E[e^{iXt}] = \sum_{k=0}^{\infty} e^{ikt} \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= e^{\lambda e^{it} - \lambda}$$

3) $X \sim \text{Exp}(1)$

$$\varphi_X(t) = \frac{1}{1 - it}$$

4) Gamma distribution, uniform, binomial, geometric

Applications

1) $X \sim \text{Poisson}(\lambda)$ $Y \sim \text{Poisson}(\mu)$

independent

$$\varphi_{X+Y}(t) = \varphi_X(t)\,\varphi_Y(t) = e^{\lambda(e^{it}-1)} \cdot e^{\mu(e^{it}-1)}$$

$$= e^{(\lambda+\mu)(e^{it}-1)}$$

char function of Poisson$(\lambda+\mu)$

$X + Y \sim \text{Poisson}(\lambda+\mu)$

2) $X_1 \sim N(\mu_1, \sigma_1^2)$ $X_2 \sim N(\mu_2, \sigma_2^2)$
independent

$$X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

3) Sum of iid Exponential
Sum of independent Gamma
with the same λ.

4) Cauchy distribution $\frac{1}{\pi}\frac{1}{1+x^2}$
Characteristic function : $e^{-|t|}$

$X \sim Exp(1)$ $f_X(x) = e^{-x} \mathbb{1}(x \geq 0)$

$-Y \sim Exp(1)$ $f_Y(x) = e^{x} \mathbb{1}(x \leq 0)$

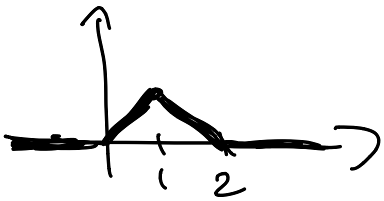
let $f_Z(z) = \frac{1}{2}(f_X(x) + f_Y(x)) = \underline{\underline{\frac{1}{2} e^{-|x|}}}$
this is a PDF

$$\varphi_Z(t) = \frac{1}{2}(\varphi_X(t) + \varphi_Y(t))$$
$$= \frac{1}{2}\left(\frac{1}{1-it} + \frac{1}{1+it}\right)$$

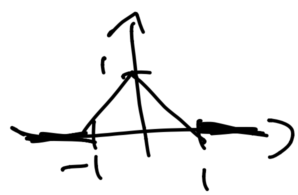
$$= \frac{1}{1+t^2}$$

If we normalize φ_2 then we get the Cauchy distribution. The char function is the appropriate constant multiple of the original PDF: $e^{-|t|}$.

5) Convolution of two Unif $[0,1]$



shifted and scaled version of $(1-|t|)^+$



$X \sim \text{Unif}\left[-\frac{1}{2}, \frac{1}{2}\right]$

$$\varphi_X(t) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{itx}\,dx = \frac{2\sin\left(\frac{t}{2}\right)}{t}$$

$$\varphi_{X_1+X_2}(t) = \frac{4\sin^2\left(\frac{t}{2}\right)}{t^2}$$

From this: PDF $\dfrac{1-\cos(x)}{\pi x^2}$

char function: $(1-|t|)^+$

How to use the char function to prove convergence in distribution?

Thm: Assume that $X_1, X_2, \ldots$ are random variables with char functions $\varphi_1, \varphi_2, \ldots$

(1) If $X_n \Rightarrow X$ then $\varphi_n \to \varphi_X$ pointwise.

(2) If $\varphi_n \to \varphi$ pointwise and φ is continuous at 0 then $X_1, X_2, \ldots$ is tight and $X_n \Rightarrow X$ where $\varphi = \varphi_X$.

Proof of (1): e^{itx} is bounded and continuous hence

$$E\left[e^{itX_n}\right] \to E\left[e^{itX}\right]$$

and hence $\varphi_n(t) \to \varphi(t)$.

Proof of (2):

Fix $u > 0$.

$$\frac{1}{u}\int_{-u}^{u}\left(1 - e^{itx}\right)dt = 2\left(1 - \frac{\sin(ux)}{ux}\right)$$

We integrate both sides with respect to the distribution of X_n.

$$\frac{1}{u}\int_{-u}^{u}\left(1-\varphi_n(t)\right)dt = 2\int\underbrace{\left(1-\frac{\sin(ux)}{ux}\right)}_{\geq 0} dQ_n(x)$$

$$\geq 2\int_{|x|\geq\frac{2}{u}}\left(1-\frac{1}{ux}\right)dQ_n(x)$$

$$\geq \int_{|x|\geq\frac{2}{u}} dQ_n(x) = P\left(|X_n|\geq\frac{2}{u}\right)$$

Since $\varphi_n\to\varphi$ and φ is cont at 0

$\varphi_n(0)=1 \Rightarrow \varphi(0)=1$.

For a given $\varepsilon>0$ we can choose $u>0$ so that $\frac{1}{u}\int_{-u}^{u}(1-\varphi(t))dt\leq\varepsilon$.

Then if n is large enough we have

$$\frac{1}{u}\int_{-u}^{u}(1-\varphi_n(t))dt\leq 2\varepsilon$$

hence $P(|X_n|\geq u)\leq 2\varepsilon$ if n is large enough. This means that $X_1, X_2, \ldots$ is tight.

Then there is a subsequence which converges in distribution. This shows that φe^{ilx} is the characteristic function of a distribution. (Using part ①.)

We also have that from any subsequence we can choose a further subsequence along which we converge in distribution to X. This implies that the original sequence has to converge in distribution to X.

De Moivre – Laplace thm in two lines

$X_1, X_2, \ldots$ iid $P(X_1 = 1) = P(X_1 = -1) = \frac{1}{2}$

Thm: $\frac{S_n}{\sqrt{n}} \Rightarrow N(0,1)$

Proof: $\varphi_{X_1}(t) = \frac{1}{2} e^{it} + \frac{1}{2} e^{-it} = \cos(t)$

$\varphi_{\frac{S_n}{\sqrt{n}}}(t) = \varphi_{\frac{X_1 + \ldots + X_n}{\sqrt{n}}}(t) = \cos\left(\frac{t}{\sqrt{n}}\right)^n$

$$= \left(1 - \frac{t^2}{2n} + O\left(\frac{t^4}{n^2}\right)\right)^n$$

$\boxed{\cos(x) = 1 - \frac{x^2}{2} + O(x^4)}$

$$\lim_{n\to\infty} \varphi_{\frac{S_n}{\sqrt{n}}}(t) = e^{-\frac{t^2}{2}}$$

This is the char function of $N(0,1)$.

Proper version of the limit:

$c_n \to 0$, $a_n \to \infty$, $c_n a_n \to \lambda$ then

$$(1+c_n)^{a_n} \to e^{\lambda}.$$

Moments and derivatives of the characteristic function

If $u>0$

$$u^{-1}\int_{-u}^{u} (1-\varphi(t))\,dt \geq P\left(|X| \geq \frac{2}{u}\right)$$

The smoother φ is at 0, the better tail bound we get.

If $E[|X|^n] < \infty$ then

we can differentiate $\varphi_X(t) = E[e^{itX}]$ inside the expectation n times to give

$$\varphi_X^{(n)}(t) = E[(iX)^n e^{itX}]$$

With $t=0$ $\varphi^{(n)}(0) = i^n E[X^n]$

[Note: for the Cauchy distribution the $\varphi_X(t) = e^{-|t|}$ which is not differentiable at $t=0$. $E|X| = \infty$]

If $E|X|^n < \infty$ then

$$\varphi_X(t) = \sum_{k=0}^{n} \frac{E[(iX)^k]}{k!} t^k + o(t^n)$$

If $E[X]=0$ and $E[X^2] = \sigma^2 < \infty$

then $\varphi_X(t) = 1 - \frac{t^2 \cdot \sigma^2}{2} + o(t^2)$

This is almost enough for the CLT, we just need a better estimate on the error term.