Math 733 – Fall 2020

Homework 4

Due: 10PM, November 8, 2020. Please upload your work in Canvas. Late homework will not be accepted.

1. Suppose that $X, X_1, X_2, \ldots$ are integer valued random variables. Show that $X_n \Rightarrow X$ if and only if $\lim_{n\to\infty} P(X_n = k) = P(X = k)$ for all $k \in \mathbb{Z}$.

2. Show that if $X_n \xrightarrow{P} X$ then $X_n \Rightarrow X$. For the converse statement show that if $X_n \Rightarrow c$ for a $c \in \mathbb{R}$ then $X_n \xrightarrow{P} c$.

3. Suppose that $X_n \Rightarrow X$ and $Y_n \Rightarrow c$ where $c \in \mathbb{R}$. Show that $X_n + Y_n \Rightarrow X + c$.

4. Give an example for random variables $X_1, X_2, \ldots,\ Y_1, Y_2, \ldots,\ X, Y$ where $X_n \Rightarrow X$, $Y_n \Rightarrow Y$, but $X_n + Y_n$ does not converge in distribution.

5. Let $X_1, X_2, \ldots$ i.i.d with distribution function $F(x)$. Denote the maximum of the first n element by M_n. Show that if
$$\lim_{x\to\infty} x^{\alpha}(1 - F(x)) = b$$
with fixed positive constants α, b then $n^{-1/\alpha} M_n$ converges in distribution and identify the limiting distribution.

6. Use characteristic functions to prove the following identity
$$\frac{\sin(t)}{t} = \prod_{k=1}^{\infty} \cos\left(\frac{t}{2^k}\right).$$
Hint: this follows from one of the previous HW problems.

7. Let $X_1, X_2, \ldots$ independent with the following distribution:
$$\mathbf{P}(X_m = m) = \mathbf{P}(X_m = -m) = \frac{1}{2m^2}, \quad \mathbf{P}(X_m = 1) = \mathbf{P}(X_m = -1) = \frac{1}{2} - \frac{1}{2m^2}.$$
Let $S_n = X_1 + \cdots + X_n$. Show that $\frac{Var(S_n)}{n} \to 2$, but $\frac{S_n}{\sqrt{n}} \Rightarrow N(0,1)$.

You can use the Piazza page to ask for clarifications about a specific problem, but please don't discuss explicit solutions before the deadline. Handing in plagiarized work, whether copied from a fellow student or off the web, is not acceptable and will lead to sanctions.

Bonus problem.
Let $X_1, X_2, \ldots$ be i.i.d. Bernoulli(1/2) random variables. Let ν_n denote the index k when we first have at least n zeros and n ones among $X_1, \ldots, X_k$. Prove that $\frac{\nu_n - 2n}{\sqrt{n}}$ converges in distribution and find the limit.

Bonus problems are not graded, and you don't need to submit them. They are provided as an extra challenge for those who are interested.