

What is probability theory?

Goal: model uncertain events
quantifying likelihood

Simple examples with built-in symmetry

1) Flip a fair coin.
What's the probability of tails?

Natural answer: $\frac{1}{2}$

2) We flip a fair coin 10 times.
What's the probability of getting
10 tails? $\frac{1}{2^{10}}$

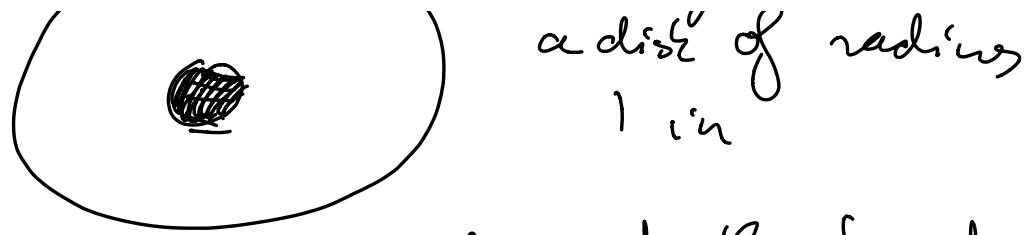
2^{10} outcomes

each outcome: a sequence of H/T

3) Dartboard with radius 10 in.



Bull's eye:

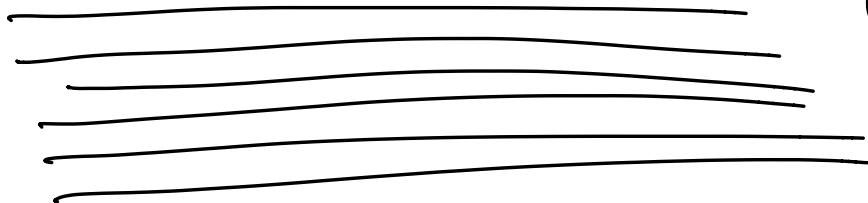


We throw a dart at the board.
What's the prob of hitting
the bull's eye?

$$\frac{\text{area of } \bullet}{\text{area of board}} = \frac{1}{100}$$

Even in this simple setup you
can come up with non-trivial
problems.

- 1) We flip a fair coin 100 times.
What is the probability that
the # of heads \geq # of tails
at any given time.
- 2) Buffon's needle problem
needle with length $l < 1$.
Infinite grid of parallel lines with
distance l from each other.



We drop
the needle.

What's the probability that the needle will not intersect any of the lines?

Natural questions:

- 1) What do we do if the outcomes are not equally likely?
- 2) What if we have infinitely many outcomes?
- 3) How to treat all those cases together?
- 4) Often we are interested in a random number.
How to describe this?
(Random variables)

5, What if we have lots
of random variables?

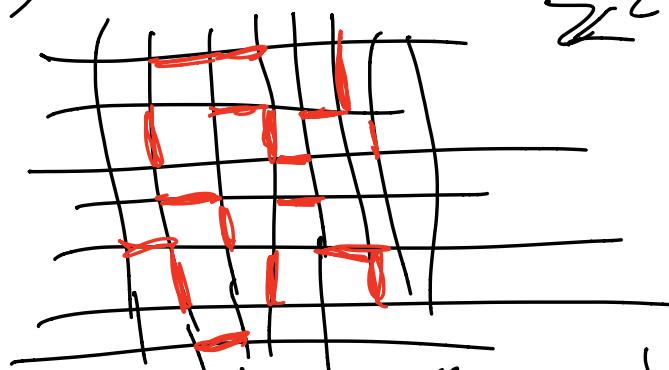
(What can we say about the
behavior of certain quantities?)

Frequency of heads when we flip
a coin over and over.

(Law of Large Numbers, Central Limit
Theorem)

Couple of problems from modern
probability theory

↳ (Percolation)



$$\mathbb{Z}^2$$

For each edge
we flip a coin,
and if it is heads
then color is red.

What is the probability that
from $(0,0)$ we can reach ∞
using only red edges?

2) Fill a large $n \times n$ matrix
using coin flips (-1 : Heads)
 $+1$: tails)

What's the probability that
the matrix is singular?

What can we say about the
eigenvalues as n gets large?

Quick problem for the break:
(Birthday problem)

Party with n guests

Let P_n denote the probability
of having a date with multiple
birthdays. $P_1 = 0$ $P_{366} = 1$

(disregard leap years)

Find the smallest n for
which $P_n > \frac{1}{2}$.

Rigorous framework

Kolmogorov's axioms (1933)

Probability space (Ω, \mathcal{F}, P)

- Ω : state space
sample space
(set of outcomes)
- \mathcal{F} : set of events
collection of subsets of Ω
 σ -field
- P : probability measure on $\bar{\mathcal{F}}$

Def of a σ -field (or σ -algebra)
(non-empty collection of sets)

1) If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$

2) If $A_i \in \mathcal{F}, i \in \mathbb{N}$ then
 $\bigcup_i A_i \in \bar{\mathcal{F}}$.

$\varnothing \in \mathcal{F}$, $\Omega \in \mathcal{F}$

If \mathcal{F}_i , $i \in I$ are σ -fields on Ω then

$\cap \mathcal{F}_i$ is also a σ -field.

(one of the problems on HW1)
non-empty

Suppose that \mathcal{A} is a set of subsets of Ω . Then there is a smallest σ -field that contains \mathcal{A} as a subset.

$\sigma(\mathcal{A})$: σ -field generated by \mathcal{A} .

Proof: Collect all σ -fields \mathcal{F} on Ω with $\mathcal{A} \subset \mathcal{F}$.

Then $\sigma(\mathcal{A}) = \cap \mathcal{F}$.

Ω is a σ -field so the intersection is non-empty \blacksquare

Probability measure:

1. $P : \mathcal{F} \rightarrow [0, 1]$

$$P(\emptyset) = 0$$

2. $A_i \in \mathcal{F}$ $i \in \mathbb{N}$ disjoint

$$P\left(\bigcup_i A_i\right) = \sum_i P(A_i)$$

P is σ -additive

3. $P(\Omega) = 1$

Events are elements of \mathcal{F} .

These are the sets that we can measure with P .

A_1, A_2, \dots, A_m are disjoint events

$$P\left(\bigcup_{i=1}^m A_i\right) = \sum_{i=1}^m P(A_i)$$

(\emptyset is disjoint from any other set)

If Ω is finite or countably infinite then usually we can

take $\mathcal{F} = 2^{\mathcal{S}}$.

Example:

1) Flipping a fair coin

$$\mathcal{S} = \{H, T\}$$

$$\mathcal{F} = \{\emptyset, \mathcal{S}, \{H\}, \{T\}\}$$

$$P(\{H\}) = \frac{1}{2} \quad P(\{T\}) = \frac{1}{2}$$

$$P(\emptyset) = 0 \quad P(\mathcal{S}) = 1$$

2) Rolling a fair die

$$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$$

$$\mathcal{F} = 2^{\mathcal{S}}$$

$$|\mathcal{F}| = 2^6$$

$$P(\{1\}) = \frac{1}{6} \quad (1 \leq 6)$$

$$A \in \mathcal{F} \quad P(A) = \frac{\#A}{6}$$

3, Random experiment with n equally likely outcomes

$$|\mathcal{S}| = n$$

$$\mathcal{S} = \{\omega_1, \dots, \omega_n\}$$

$$\mathcal{F} = 2^{\mathcal{S}}$$

$$P(\{\omega_i\}) = \frac{1}{n} \quad 1 \leq i \leq n$$

$$P(A) = \frac{|A|}{|\mathcal{S}|} \quad A \subset \mathcal{S}$$

(4), More generally:

discrete probability space

\mathcal{S} is finite or countably infinite

$$\mathcal{S} = \{\omega_1, \omega_2, \dots\}$$

$$P(\{\omega_i\}) = p_i$$

$$P(A) = \sum_{\omega_i \in A} P(\{\omega_i\})$$

$$\mathcal{F} = 2^{\mathcal{S}}$$