1. $E[S_N] = \sum_{n=0}^{\infty} E[S_n] \cdot \mathbb{P}(n=N)$

$= \mu \cdot \sum_{n=0}^{\infty} n \cdot \mathbb{P}(n=N) = \mu \cdot \mathbb{E}[N]$

$= \mu m$

$Var[S_N] = E[S_N^2] - (E[S_N])^2$

$= \sum_{n=0}^{\infty} E[S_n^2] \cdot \mathbb{P}(n=N) - \mu^2 m^2$

$= \sum_{n=0}^{\infty} (n\sigma^2 + n^2\mu^2) \cdot \mathbb{P}(n=N) - \mu^2 m^2$

$= \sigma^2 m + \mu^2 (s^2 + m^2) - \mu^2 m^2$

$= m\sigma^2 + \mu^2 s^2$

2. (a) Suppose $g(x) = ax+b$.

$\mathbb{E}[g(S_{n+1}) | \mathcal{F}_n] = \mathbb{E}[g(S_n) | \mathcal{F}_n] + \mathbb{E}[aX_{n+1} | \mathcal{F}_n] = g(S_n)$

Suppose $g(x) = ax^2 + bx + c$.

$\mathbb{E}[g(S_{n+1}) | \mathcal{F}_n] = g(S_n) + \mathbb{E}[a X_{n+1}^2 | \mathcal{F}_n] \neq g(S_n)$

Noticed that the kth moment of X_n.

$\mathbb{E}[X_n^k] = (-1)^k \cdot \frac{1}{2} + 1^k \cdot \frac{1}{2} = \frac{1}{2}((-1)^k + 1)$

$\mathbb{E}[X_n^k] = 1$, if k is even.

$\mathbb{E}[X_n^k] = 0$, if k is odd.

Consider. $g(x) = \sum_{k=0}^{m} a_k x^k$

$$g(S_{n+1})-g(S_n)=\sum_{k=0}^{m} a_k\cdot(S_n+X_{n+1})^k-S_n^k$$

$$=X_{n+1}\sum_{k=0}^{m} a_k\cdot\sum_{j=0}^{k} S_n^{k-j-1}\cdot(S_n+X_{n+1})^j$$

$$=X_{n+1}\sum_{k=0}^{m}\sum_{j=0}^{k} a_i\cdot S_n^{k-j-1}\cdot\sum_{i=0}^{j}\binom{j}{i}\cdot S_n^{j-i}X_{n+1}^{i}$$

$$=\sum_{k=0}^{m}\sum_{j=0}^{k}\sum_{i=0}^{j} a_i S_n^{k-i-1}\cdot\binom{j}{i}\cdot X_{n+1}^{i+1}$$

So $g(x)=ax+b$

(b) Consider $h(x,n)=x^3-3nx$.

$$\mathbb{E}[h(S_{n+1},n+1)\mid\mathcal{F}_n]=h(S_n,n)$$
$$+\mathbb{E}[(S_n+X_{n+1})^3-3n(S_n+X_{n+1})-S_n^3+3nS_n\mid\mathcal{F}_n]$$

$$=h(S_n,n)+\mathbb{E}[3S_n^2X_{n+1}+X_{n+1}^3-3nX_{n+1}\mid\mathcal{F}_n]$$
$$=h(S_n,n)$$

So $h(S_n,n)$ is a Martingale.

3. $\{\nu=n\}\in\mathcal{F}_n$, $\{\tau=n\}\in\mathcal{F}_n$.

$\min\{\nu,\tau\}\in\{\nu,\tau\}\in\mathcal{F}_n$

$\{\nu+\tau=n\}\subset\{\nu=n\}\cup\{\tau=n\}\in\mathcal{F}_n$.

4. $Z_{n+1} = \dfrac{P_b(X_1, X_2, \cdots, X_n) \cdot P_b(X_{n+1})}{P_a(X_1, \cdots X_n) \cdot P_a(X_{n+1})} = Z_n \cdot \dfrac{P_b(X_{n+1})}{P_a(X_{n+1})}$

$$\mathbb{E}[Z_{n+1} | \mathcal{F}_n] = Z_n \cdot \mathbb{E}\left[\frac{P_b(X_{n+1})}{P_a(X_{n+1})} \Big| \mathcal{F}_n\right]$$

$$= Z_n \cdot \left[c \cdot \frac{b}{a} + (1-c) \cdot \frac{1-b}{1-a}\right]$$

Let $c \cdot \frac{b}{a} + (1-c) \cdot \frac{1-b}{1-a} = 1 \Rightarrow (a-c)(a-b) = 0$

So $c = a$. iff. Z_n is martigale.

5. $\mathbb{E}[M_{n+1} | \mathcal{F}_n] = M_n$

$+ \mathbb{E}[\,|S_n + X_{n+1}| - |S_n| \,|\, \mathcal{F}_n]$

$+ \mathbb{E}[\{0 \le k < n+1 : S_k = 0\} - \{0 \le k < n : S_k = 0\} | \mathcal{F}_n]$

If $|S_n| \neq 0$. $\mathbb{E}[M_{n+1} | \mathcal{F}_n] = 0 + 0 = 0$

If $|S_n| = 0$. $\mathbb{E}[M_{n+1} | \mathcal{F}_n] = 1 - 1 = 0$

So M_n is martingale.

6. (a) $\mathbb{E}[X_{n+1} | \mathcal{F}_n] = \dfrac{P(X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n, X_{n+1} = 1)}{P(X_1 = x_1, \cdots, X_n = x_n)}$

$$= \frac{\int_0^1 p^{S_n + 1} (1-p)^{n - S_n} \, dp}{\int_0^1 p^{S_n} \cdot (1-p)^{n - S_n} \, dp}$$

$$= \frac{B(S_n+2,\ n+1-S_n)}{B(S_n+1,\ n+1-S_n)}$$

$$= \frac{\Gamma(S_n+2)\cdot\Gamma(n+1-S_n)}{\Gamma(n+3)} \cdot \frac{\Gamma(n+2)}{\Gamma(S_n+1)\cdot\Gamma(n+1-S_n)}$$

$$= \frac{1+S_n}{n+2}$$

(b) $\mathbb{E}[M_{n+1}|\mathcal{F}_n]$

$$= \mathbb{E}\left[\frac{S_{n+1}+1}{n+3}\Big|\mathcal{F}_n\right]$$

$$= \mathbb{E}\left[\frac{S_n+1}{n+3} + \frac{X_{n+1}}{n+3}\Big|\mathcal{F}_n\right]$$

$$= \frac{1+S_n}{n+3} + \frac{1}{n+3}\cdot\frac{1+S_n}{n+2}$$

$$= \frac{1+S_n}{n+2} = M_n$$

So M_n is a Martingale.