Itô integral (some extensions)

$$\int X \, dY \qquad Y = B$$

$$\int_0^t B_1(s) \, dB_2(s) \qquad \int_0^t f_s \, dB_s \qquad \{\mathcal{F}_t\}$$

- More general assumption on measurability condition

Assume: there is a filtration $\{\mathcal{H}_t\}$ s.t.

- B_t is martingale with respect to $\mathcal{H}$
- and f is $\mathcal{H}$ adapted.

(Note: $\mathcal{F}_t \subset \mathcal{H}_t$)

$$(B_1, B_2, \cdots, B_n)$$

$$\mathcal{F}_t^{(n)} = \sigma\left\{ B_1(s_1), \cdots B_n(s_n) : \begin{array}{l} s_k \le t, \\ 1 \le k \le n \end{array} \right\}$$

each B_k is mart. with respect to $\mathcal{F}_t^{(n)}$

$$\left(\begin{array}{l} s < t \\ B_k(t) - B_k(s) \text{ indep of } \mathcal{F}_s^{(n)} \end{array} \right)$$

We can define $\int_0^t f(s) \, dB_k$

$$\left(\text{e.g. } \int B_2 \, dB_1 \, , \int \sin(B_1^2 + B_2^2) \, dB_2 \cdots \right)$$

Can also define

$$\int_0^t \underset{\begin{pmatrix} v_{11} & \cdots & v_{1n} \\ & \vdots & \\ v_{m1} & & v_{mn}\end{pmatrix}}{v}\; \underset{(B_1 \cdots B_n)}{dB} = \int_0^t \begin{pmatrix} v_{11} & \cdots & v_{1n} \\ & & \\ v_{m1} & \cdots & v_{mn}\end{pmatrix}\begin{pmatrix} dB_1 \\ \vdots \\ dB_n\end{pmatrix}$$

$\int_0^t f\,dB \quad f \in \mathcal{H}^2. \qquad \underline{\mathbb{E}\int_0^T f^2\,ds < \infty}.$

$$f(B) = e^{B^4} \qquad \int_0^T f(B_s)\,dB_s = \int_0^T \underline{e^{B_s^4}}\,dB_s$$

$$\mathbb{E}\int_0^T e^{2B_s^4}\,ds \;\cancel{<}\; \infty \qquad \left(\int_{-\infty}^{+\infty} e^{-x^2} e^{x^4}\,dx = \infty\right)$$

$$\mathcal{H}^2_{[0,T]} \rightsquigarrow L^2_{loc}[0,T]$$

$\|$ def

all adapted measurable functions f

s.t. $P\left(\int_0^T f^2\,dt < \infty\right) = 1$

(In fact $\forall$ continuous $g: \mathbb{R} \to \mathbb{R}$.

$f_t \stackrel{def}{=} \underline{g(B_t)} \in L^2_{loc}$

Def: (localizing sequence for $\mathcal{H}^2$)

An increasing seq of stopping times $\{\nu_n\}$ is called an $\mathcal{H}^2$ localizing seq for f

if $f_n(\omega,t) \overset{\text{def}}{=} f(\omega,t)\, \mathbb{1}_{t \le \nu_n} \in \mathcal{H}^2 \quad \forall n.$

and $$\mathbb{P}\left(\bigcup_{n=1}^{\infty} \{\omega : \nu_n = T\} \right) = 1$$

prop: $\forall f \in L^2_{loc}[0,T]$,

$$\tau_n = \inf\left\{ s : \int_0^s f^2\, dt \ge n \text{ or } s \ge T \right\}$$

is a localizing seq.

pf: $f_n^{(t)} = f^{(t)}\, \mathbb{1}_{t \le \tau_n}$

$$\int_0^T f_n^2\, ds = \int_0^{\tau_n} f^2_{(s)}\, ds \le n < \infty$$

$\Rightarrow f_n \in \mathcal{H}^2 \quad \forall n$ ✓

In general, $\displaystyle\bigcup_{n=1}^{\infty} \{w : \tau_n = T\}$

$=$

$\{\omega : \int_0 f(\omega,t)\,dt < \infty\}$

$\mathbb{P}(\ldots) = 1$ ✓

Define $\int_0^T f\,dB$ for $f \in \mathcal{L}^2_{loc}$.

1). Let $\{\nu_n\}$ be a localing seq for f.

2) $\underline{X_t^{(n)}} = \underline{\int_0^t f(\omega,s)\,\underline{1_{s \le \nu_n(\omega)}}}\,dB_s$

$\in \mathcal{H}^2$.

3). $\underline{\int_0^T f_s\,dB_s} \overset{def}{=} \lim_{n\to\infty} X_t^{(n)}$

Key Lemma: $\exists$ a continuous X

$\mathbb{P}(X_t = \lim_{n\to\infty} X_t^{(n)}) = 1$

$\forall t \in [0,T]$