Review of conditional expectation

L^p space. $\mathbb{E}|X|^p < \infty$ $\quad \|X\|_p = (\mathbb{E}|X|^p)^{1/p}$.

L^∞ space $\quad \mathbb{P}(|X| \le c) = 1$ for some const $c < \infty$

$$\|X\|_\infty = \inf\{c : \mathbb{P}(|X| \le c) = 1\}$$

properties: (1) $\|X - Y\|_p = 0 \Rightarrow X = Y$ a.s

(2) $|\mathbb{E}[XY]| \le \|X\|_p \|Y\|_q \quad \frac{1}{p} + \frac{1}{q} = 1$

(3) $\|X + Y\|_p \le \|X\|_p + \|Y\|_p$.

If $\lim_{n\to\infty} \|X_n - X\|_p = 0 \Rightarrow$ Say X_n converge to X in L^p.

L^p is metric space $\|X - Y\|_p$

Independence: X, Y are independent

$$\Longleftrightarrow P(\{X \in B_1\} \cap \{Y \in B_2\}) = P(X \in B_1)\, P(Y \in B_2)$$

"factorize"

Two σ-algebras $\mathcal{F}_1, \mathcal{F}_2$ are independent

$$\Longleftrightarrow \mathbb{P}(D_1 \cap D_2) = \mathbb{P}(D_1)\mathbb{P}(D_2) \quad \forall D_1 \in \mathcal{F}_1,\ D_2 \in \mathcal{F}_2$$

X and σ-alg $\mathcal{F}_3$ are indp

$$\Longleftrightarrow \mathbb{P}(\{X \in B\} \cap D) = P(X \in B)\, P(D) \quad \forall B \in \mathcal{B}(\mathbb{R}),\ D \in \mathcal{F}_3$$

$(\Omega, \mathcal{F}, P)$

Conditional expection: of X wrt a σ-algebra $\mathcal{D}$ $(\mathcal{D} \subset \mathcal{F})$

denoted by $\mathbb{E}[X|\mathcal{D}]$ is the unique random var Y which satisfies

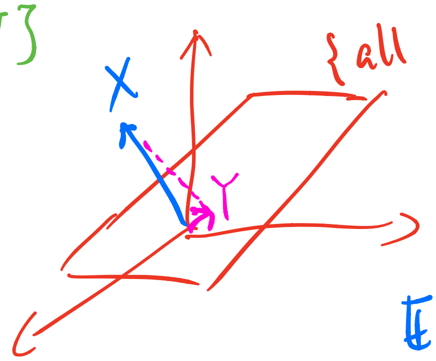
A) Y is $\mathcal{D}$-measurable

B) $\int_D X \, dP = \int_D Y \, dP \quad \forall D \in \mathcal{D}$

Ex: ① ② $\Omega = \{HH, HT, TH, TT\}$ $\mathcal{F} = \{\phi, \{HH\}, \ldots \{HH, HT\} \ldots \Omega\}$ $\in \{HT\}$

$\rightarrow \mathcal{D} = \{\phi, \{HH\}, \{HT, TH\}, \{TT\}, \{HH, TT\}, \Omega\}$

$\cancel{\in}$ $\{HT\}$



Y is $\mathcal{D}$-measuble

$\mathbb{E}[X|\mathcal{D}] = Y$.

$\forall$ $\mathcal{D}$-measurable Z $\quad \mathbb{E}[ZX] = \mathbb{E}[ZY]$

Take $Z = \mathbb{1}_D$ $\quad Z(\omega) = \mathbb{1}_D(\omega) = \begin{cases} 1 & \omega \in D \\ 0 & \omega \notin D \end{cases}$

$\mathbb{E}[ZX] = \int_D X \, dP$

$\mathbb{E}[ZY] = \int_D Y \, dP$

Radon-Nikodym theorem $\Rightarrow \exists$ of conditional expection.

Properties 1). $\mathbb{E}[\mathbb{E}[X|\mathcal{D}]] = \mathbb{E}X$

(B) $\forall D \in \mathcal{D}$ $\quad \int_D X \, dP = \int_D Y \, dP$

choose $D = \Omega$ " "

$EX \quad E[\quad]$

2) If $X \geq 0$ Then $E[X \mid \mathcal{D}] \geq 0$.

$Y = E[X \mid \mathcal{D}]$ is $\mathcal{D}$ meas.

and $\int_D Y \, dp = \int_D X \, dp \geq 0 \quad \forall D \in \mathcal{D} \quad \Rightarrow Y \geq 0$ a.s.

3). $E[aX + bY \mid \mathcal{D}] = aE[X \mid \mathcal{D}] + bE[Y \mid \mathcal{D}]$

$$\int_D aX + bY \, dp \overset{?}{=} \int_D aE[X \mid \mathcal{D}] + bE[Y \mid \mathcal{D}] \, dP$$

$$a\int_D X \, dp + b\int_D Y \, dp = a\int E[X \mid \mathcal{D}] \, dp + b\int_D E[Y \mid \mathcal{D}] \, dp$$

4) If $X \geq Y$ then $E[X \mid \mathcal{D}] \geq E[Y \mid \mathcal{D}]$

$2) + 3) \Rightarrow 4)$

5) If X is $\mathcal{D}$-measurable.
$E[X \mid \mathcal{D}] = X$

6) X, Y (Y: $\mathcal{D}$-measurable)

$$E[YX \mid \mathcal{D}] = YE[X \mid \mathcal{D}]$$

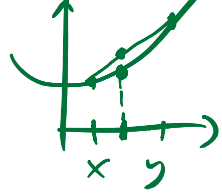
7) If X is indep of $\mathcal{D}$. then $E[X \mid \mathcal{D}] = E[X]$

$$\int_D X \, dp \overset{?}{=} \int_D E(X) \, dp$$

$$\int_\Omega X \cdot 1_D \, dp = E[X \cdot 1_D] = EX \cdot \int_\Omega 1_D \, dp$$

8). $E[E[X \mid \mathcal{D}_2] \mid \mathcal{D}_1]$

$\mathcal{D}_1 \subset \mathcal{D}_2 \qquad = \mathbb{E}[X \mid \mathcal{D}_1]$



9) Jensen ineq: recall "convex" function

$$\phi(\lambda x + (1-\lambda) y) \le \lambda \phi(x) + (1-\lambda)\phi(y)$$

$$\mathbb{E}[\phi(X) \mid \mathcal{D}] \geq \phi(\mathbb{E}[X \mid \mathcal{D}])$$

10) $$\mathbb{E}\left|\mathbb{E}[X \mid \mathcal{D}] - \mathbb{E}[Y \mid \mathcal{D}]\right|^p \le \mathbb{E}|X-Y|^p \qquad p \geq 1$$

Pf:
$$\text{LHS} = \mathbb{E}\left[\left|\mathbb{E}[X-Y \mid \mathcal{D}]\right|^p\right] \overset{\text{Jensen}}{\le} \mathbb{E}\left[\mathbb{E}[|X-Y|^p \mid \mathcal{D}]\right] = \mathbb{E}[|X-Y|^p]$$

11). $\mathbb{E}|X_n - X|^p \xrightarrow{n\to\infty} 0.$

$$\Rightarrow \lim_{n\to\infty} \mathbb{E}\left[\left|\mathbb{E}[X_n \mid \mathcal{D}] - \mathbb{E}[X \mid \mathcal{D}]\right|^p\right] = 0.$$