# A generalization of Itô formula

Caution: (about notation)

$$d\left(\underbrace{\tfrac{1}{2}B_t^2}_{f(x)=\frac{1}{2}x^2}\right) = B_t\, dB_t + \frac{1}{2}\, dt.$$

is a shorthand notation for

$$\boxed{\frac{1}{2}B_t^2 = \int_0^t B_s\, dB_s + \frac{1}{2}t}$$

$$d\left(\int_0^t B_s^3\, dB_s\right) = B_t^3\, dB_t$$

$\neq f(B)$

not Itô formula.

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$$dX = a(w,t)\,dt + b(w,t)\,dB_t$$

~~(Standard form)~~.

$$\left(X = \int a(w,s)\,ds + \int b(w,s)\,dB_s\right)$$

$$f(t, X_t)$$

$$df(t, X_t) = ?$$

Special case:

$a = 0, \; b = 1$

$X = B$

<u>formula</u>:

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t)\,dt + \frac{\partial f}{\partial x}(t, X_t)\,dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t) \cdot b(\omega, t)^2\,dt$$

Ito formula. $dB \cdot dB \to dt$

$dt \cdot dt = (dt)^2$

$$\cancel{\frac{\partial^2 f}{\partial t^2}(dt)^2} \qquad \boxed{\frac{\partial^2 f}{\partial x^2}(dB)^2\,dt} \qquad \frac{\partial^2 f}{\partial t \partial x}\underbrace{dt\,dB}_{(dt)^{3/2}} \to 0$$

| $\cdot$ | $dt$ | $dB$ |
|---|---|---|
| $dt$ | 0 | 0 |
| $dB$ | 0 | $dt$. |

$$\underbrace{\frac{\partial^2 f}{\partial x^2} \cdot (dX)^2}$$

$\downarrow$

$$\frac{\partial^2 f}{\partial x^2} \cdot b^2\,dt$$

$dX = a\,dt + b\,dB$

$(dX)^2 = \cancel{a^2(dt)^2}$

$+ \cancel{2ab\,dt\,dB}$

$+ b^2(dB)^2$

$\to b^2\,dt$

Geometric BM. $\underline{X_t = e^{\alpha t + \sigma B_t}}$ (x)

$Y = f(t, X_t)$

$dY = ?$

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$$dX = \left(\alpha + \tfrac{1}{2}\sigma^2\right) X_t \, dt + \boxed{\sigma X_t} \, dB_t \quad = \text{"} a\,dt + b\,dB \text{"}$$

$$dY = \frac{\partial f}{\partial t}(t, X_t)\,dt + \frac{\partial f}{\partial x}(t, X_t)\,\boxed{dX_t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)\,\sigma^2 X_t^2\,dt.$$

$$= \frac{\partial f}{\partial t}(t, X_t)\,dt + \frac{\partial f}{\partial x}(\cdots)\left(\left(\alpha + \tfrac{1}{2}\sigma^2\right)X\,dt + \sigma X\,dB\right) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2 X_t^2\,dt$$

$$= \left(\frac{\partial f}{\partial t} + \alpha X \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 X \frac{\partial f}{\partial x} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2 X^2\right) dt + \sigma X \frac{\partial f}{\partial x}\,dB \quad (\star)$$

$\alpha t + \sigma B$ $\qquad dX = a\,dt + b\,dB$

2nd way: $X = e^{\alpha t + \sigma B}$

$$Y = f(t, X) = f(t, e^{\alpha t + \sigma B})$$

$$= \boxed{g(t, B)}$$

$$g(t, x) \stackrel{def}{=} f(t, e^{\alpha t + \sigma x})$$

$\frac{\partial g}{\partial x}$ $\frac{\partial g}{\partial t}$. chain rule.

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Another formula.

$$dX = a\,dt + b\,dB$$

$$dY = \alpha\,dt + \beta\,dB$$

$(f(t, X, Y))$

$$df(X, Y) = \frac{\partial f}{\partial x} dX + \frac{\partial f}{\partial y} dY + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} b^2\,dt + \frac{1}{2} \frac{\partial^2 f}{\partial y^2} \beta^2\,dt + \frac{1}{2} \frac{\partial^2 f}{\partial x \partial y} b \cdot \beta\,dt.$$

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Special case: $f(x, y) = xy$

$$d(XY) = Y\,dX + X\,dY$$

Product

$$+ \frac{1}{2} b \cdot \beta \, dt$$

rule

| | $dt$ | $dB^1$ | $dB^2$ |
|---|---|---|---|
| $dt$ | 0 | 0 | 0 |
| $dB^1$ | 0 | $dt$ | 0 |
| $dB^2$ | 0 | 0 | $dt$ |