Martingale inequalities

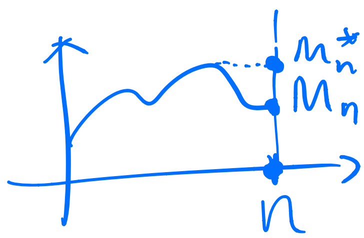
Doob's maximal inequality

Markov inequality:

For $X \geq 0$, $\forall \lambda > 0$. $\lambda P(\underset{|X|}{X} \geq \lambda) \leq E(\underset{|X|}{X})$

$\{M_n\}_{n \in \mathbb{Z}_+}$ process $\qquad M_n^* = \sup_{0 \leq m \leq n} M_m$ Mn的上界

maximum process



Thm (max ineq in discrete-time)

Suppose M_n nonnegative submart.
非负 下鞅

$\lambda > 0$

$$\lambda P(M_n^* \geq \lambda) \leq E[M_n \mathbb{1}_{M_n^* \geq \lambda}] \overset{\text{trivial.}}{\leq} E[M_n]$$

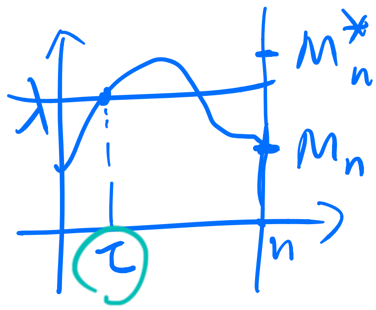
rmk1 If we don't have "non negative"

then $\boxed{M_n^* = \sup_{0 \leq m \leq n} |M_m|}$

rmk2. M_n is non-decreasing process 非降

$\Rightarrow M_n^* = M_n$ 引理 (Markov ineq)

pf. $\tau = \min\{m : M_m \geq \lambda\}$ 第一次到λ的时间



τ在n之前⇒M_n大于λ

$M_n^* \geq \lambda \iff \tau \leq n$

If $\tau \leq n$ then

$$\lambda \mathbb{1}_{\tau \leq n} \leq M_\tau \mathbb{1}_{\tau \leq n} = \sum_{0 \leq m \leq n} M_m \mathbb{1}_{\tau = m}$$

$$\mathbb{E}(\quad) \quad (\quad) \leq \mathbb{E}[\quad]$$

$$\mathbb{E}[M_m \mathbb{1}_{\tau=m}] \leq \mathbb{E}[M_n \mathbb{1}_{\tau=m}]$$

$$\mathbb{E}[M_m \mathbb{1}_A] \leq \mathbb{E}[M_n \mathbb{1}_A]$$

下鞅 $\forall A \in \mathcal{F}_m$ $\quad m \leq n$

since, $M_m \mathbb{1}_A \leq \mathbb{E}[M_n \mid \mathcal{F}_m] \mathbb{1}_A$

$$\Rightarrow \mathbb{E}[\lambda \mathbb{1}_{\tau \leq n}] \leq \sum_{0 \leq m \leq n} \mathbb{E}[M_n \mathbb{1}_{\tau = m}]$$

$$= \mathbb{E}[M_n \mathbb{1}_{\tau \le n}] \qquad \blacksquare$$

$\Leftrightarrow M_n^* \ge \lambda$

Rmk: $\lambda^p \mathbb{P}(M_n^* \ge \lambda) \le \mathbb{E}[M_n^p]$ $\forall p \ge 1$

$\mathbb{P}(M_n^* \ge \lambda) = \mathbb{P}((M_n^*)^p \ge \lambda^p)$

Thm (Doob L^p ineq) 非负下鞅

If $\{M_n\}$ is nonnegative submart.

then $\forall p > 1$,

$$\|M_n^*\|_p \le \frac{p}{p-1} \|M_n\|_p$$

$\frac{p}{p-1} \to q$, $\frac{1}{q} + \frac{1}{p} = 1$

(recall. $\|X\|_p = (\mathbb{E}|X|^p)^{\frac{1}{p}}$

Hölder: $\|XY\|_1 \le \|X\|_p \|Y\|_q$.

$\frac{1}{p} + \frac{1}{q} = 1$)

Pf: suffice to show

$$\lambda \mathbb{P}(X \ge \lambda) \le \mathbb{E}[Y \mathbb{1}_{X \ge \lambda}]$$

$Y \leftarrow M$, $X \leftarrow M^*$

$$\Rightarrow \|X\|_p \le \frac{p}{p-1} \|Y\|_p$$

$\forall z \geq 0.$

$$z^p = p\int_0^z x^{p-1}\,dx = p\int_0^\infty x^{p-1}\,\mathbb{1}_{z\geq x}\,dx$$

$z \mapsto X$

$$\|X\|_p^p \quad \mathbb{E}[X^p] = p\int_0^\infty x^{p-1}\,\mathbb{P}(X\geq x)\,dx$$

$$\leq p\int_0^\infty x^{p-2}\,\mathbb{E}[Y\,\mathbb{1}_{X\geq x}]\,dx$$

$$= p\,\mathbb{E}\Big[Y \underbrace{\int_0^\infty x^{p-2}\,\mathbb{1}_{X\geq x}\,dx}\Big]$$

$$= \frac{p}{p-1}\,\mathbb{E}[Y\cdot X^{p-1}]$$

$$\leq \frac{p}{p-1}\,\|Y\|_p\,\|X\|_p^{p-1}$$

(Holder with $\begin{cases} p \\ q = \frac{p}{p-1}\end{cases}$) $\qquad \left(\frac{1}{p} + \frac{p-1}{p} = 1\right)$

$\blacksquare$

$\boxed{\text{Cont-time}}$?

Thm: (Max ineq in cont_time)

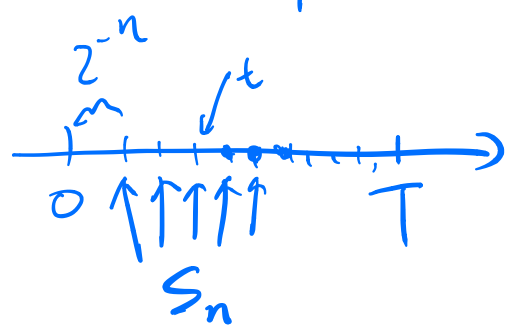
Suppose M_t is right-continuous nonnegative submartingale. $\forall \lambda > 0$

$$\lambda^p \, \mathbb{P}(\underbrace{M_T^*}_{= \sup_{0 \le t \le T} M_t} > \lambda) \le \mathbb{E}(M_T^p) \quad \forall p \ge 1$$

Moreover, if $M_T \in L^p(\Omega)$ for some $p > 1$

then, $$\|M_T^*\|_p \le \frac{p}{p-1} \|M_T\|_p$$

Pf (Sketch).



(use right-continuity)

$$\lim_{n \to \infty} \underbrace{\sup_{t \in S_n} M_t} = \sup_{0 \le t \le T} M_t = M_T^*$$

$$\lambda^p \, \mathbb{P}(\underbrace{\sup_{t \in S_n} M_t \ge \lambda}) \le \mathbb{E}(M_T^p)$$

take $n \to \infty$ (Fatou lemma)