Itô formula in high dimensions

Itô in 1d

$$f(t, B_t) = f(0, B_0) + \int_0^t \frac{\partial f}{\partial t}(s, B_s)\, ds + \int_0^t \frac{\partial f}{\partial x}(s, B_s)\, dB_s$$
$$+ \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s)\, ds.$$

$$df(t, B_t) = \frac{\partial}{\partial t} f(t, B_t)\, dt + \frac{\partial}{\partial x} f(t, B_t)\, dB_t$$
$$+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, B_t)$$

d-dimension

$$\vec{B}_t = \underbrace{(B_t^1, \cdots, B_t^d)}_{d \text{ indep 1d BMs.}}$$

$df(t, \vec{B}_t)$

$$= \frac{\partial}{\partial t} f(t, \vec{B}_t)\, dt + \nabla f(t, \vec{B}_t) \cdot d\vec{B}_t + \frac{1}{2}\Delta f(t, \vec{B}_t)\, dt$$

$\int_0^t \nabla f(s, \vec{B}_s) \cdot d\vec{B}_s$ is mart.

$$\nabla f(t, \vec{x}) = \left(\frac{\partial f}{\partial x_1}(t, \vec{x}), \cdots, \frac{\partial f}{\partial x_d}(t, \vec{x})\right)$$

$\vec{x} = (x_1 \cdots x_d)$

$$\Delta f(t, \vec{x}) = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_d^2}$$

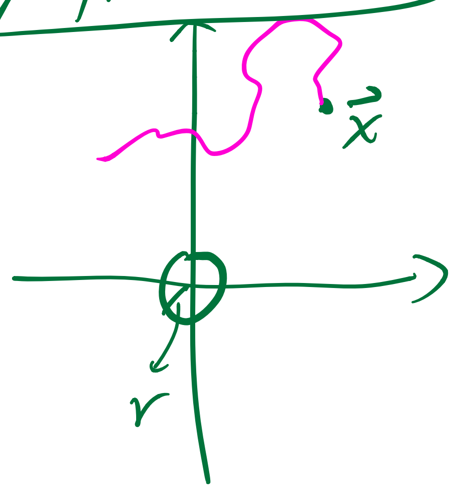
prop: $M_t = f(t, \vec{B}_t)$ is mart

if $\frac{\partial f}{\partial t} = -\frac{1}{2}\Delta f$

(special case: $f(\vec{x})$ $\quad \Delta f = 0$

$\Rightarrow f$ is harmonic function)

Application



In $\mathbb{R}^2$ $\quad \mathbb{P}_{\vec{x}}(\underline{\tau_r < \infty}) = 1$

$\boxed{\text{recurrent}}$

In $\mathbb{R}^d$ $d \geq 3$ $\quad \mathbb{P}_{\vec{x}}(\tau_r < \infty) < 1$

$\boxed{\text{Transient}}$

$\tau_r = \inf\{t : |\vec{B}_t| = r\}$

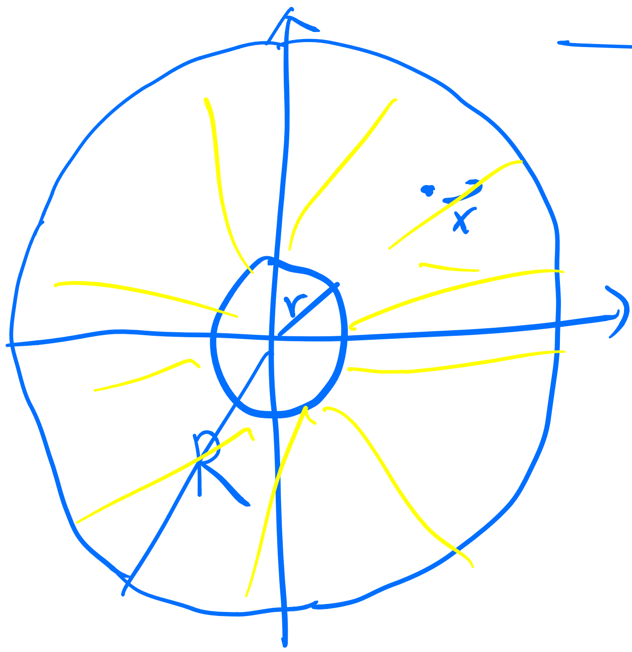
$d=2$. $\quad f(\vec{x}) = \log|\vec{x}| = \log\sqrt{x_1^2 + x_2^2}$

$\underline{\Delta f} = 0. \quad \forall \vec{x} \neq 0.$

$d \geq 3$

$f(\vec{x}) = |\vec{x}|^{2-d}$

$\Delta f = 0 \quad \forall\, x \neq 0.$



$$\begin{cases}\Delta h = 0 \\ h(\vec{x}) = 1 & |\vec{x}| = r \\ h(\vec{x}) = 0 & |\vec{x}| = R\end{cases}$$

$h(\vec{B}_t)$ is mart.

$$E[h(\vec{B}_\tau)] = E[h(\vec{B}_0)] = E[h(\vec{x})]$$
$$= h(\vec{x})$$

$$\tau = \min(\tau_r, \tau_R)$$

$$E[h(\vec{B}_\tau)] = P(|\vec{B}_\tau| = r)\cdot 1$$
$$+ P(|\vec{B}_\tau| = R)\cdot 0$$

$$= P(|\vec{B}_\tau| = r) = P(\tau_r < \tau_R)$$

$$\boxed{h(\vec{x})} = P(\tau_r < \tau_R)$$

send $R \to \infty$

<u>2D</u>

$$\underline{f(\vec{x}) = \log|\vec{x}|} \qquad \Delta f = 0.$$

$$h(\vec{x}) = \frac{\log R - \log|\vec{x}|}{\log R - \log r} \Rightarrow \Delta h = 0$$

$$\mathbb{P}(\tau_r < \infty) = \lim_{R\to\infty} \frac{\log R - \log|\vec{x}|}{\log R - \log r}$$

$$= 1$$

~~3D~~ $d \geq 3$.

$$f(\vec{x}) = |\vec{x}|^{2-d} \qquad \Delta f = 0.$$

$$h(\vec{x}) = \frac{R^{2-d} - |\vec{x}|^{2-d}}{R^{2-d} - r^{2-d}} \qquad \Delta h = 0$$

$$\mathbb{P}(\tau_r < \infty) = \lim_{R\to\infty} \frac{R^{2-d} - |\vec{x}|^{2-d}}{R^{2-d} - r^{2-d}}$$

$$= \left(\frac{r}{|\vec{x}|}\right)^{d-2} < 1$$

since $|\vec{x}| > r$

2d $\quad \mathbb{P}(\tau_r < \infty) = 1$

drunk man can alway find home

$d \geq 3 \quad \mathbb{P}(\tau_r < \infty) < 1.$

drunk bird can possibly get lost.