$(\Omega, \mathcal{F}, \mathbb{P})$ probability space

(Real valued) random variable:

$X : \Omega \to \mathbb{R}$ Borel measurable

Distribution of X: prob measure on $(\mathbb{R}, \mathcal{B})$

write $\underline{Q_X(B) = P(X^{-1}(B))} = \underline{P(X \in B)}$

Cumulative distribution function of X:

$$F_X(y) = P(X \le y) = Q_X((-\infty, y]) \quad y \in \mathbb{R}$$

If $\underline{F(y) = \int_{-\infty}^{y} f(z)\,dz}$ $\left(f = \frac{dQ}{dx}\right)$ then

f is called the probability density function of X

and $\underline{P(X \in B) = \int_B f(y)\,dy}$ for B Borel.

$$f \geq 0 \quad \int_{-\infty}^{\infty} f(x)\,dx = 1$$

If X is supported on a countable (distribution of X) set (There is a countable set $S \subset \mathbb{R}$ so that $P(X \in S) = 1$.)

then Q_X can be described by the function $p_X(a) = P(X = a)$

$a \in S$

Probability mass function of X
(we say that X is a discrete r.v.)
Ex: Indicator random variable: 1_A

More general framework:
we can have $\mathbb{R}^d$-valued functions
or we can consider a measurable
space $(\mathcal{X}, S)$

$X : \Omega \to \mathcal{X}$
we need $X^{-1}(A) \in \mathcal{F}$ for $A \in S$

Q_X is a probability measure on $(\mathcal{X}, S)$

If μ is a measure on $(\mathcal{X}, S)$
and $Q_X \ll \mu$ (Q_X is abs. cont.
with respect to μ) then
R-N derivative $\frac{dQ_X}{d\mu}$

$$P(X \in A) = \int_A \frac{dQ_X}{d\mu} \cdot d\mu$$

Def: If X is a r.v on $(\Omega, \mathcal{F}, P)$ then we denote by $\sigma(X)$ the smallest σ-field with respect to which X is measurable.

$$\sigma(X) \subset \mathcal{F}$$

$\sigma(X) = \sigma$-field generated by the sets $X^{-1}(B)$ with $B \in \mathcal{B}$

Ex: $X = 1_A$

$$\sigma(X) = \{\emptyset, \Omega, A, A^c\}$$

Notation: $\boxed{X \in \mathcal{G}}$ $\leftarrow$ σ-field

if $\sigma(X) \subset \mathcal{G}$.

Recall: measurable functions are pointwise limits of "simple" functions.

simple function: finite linear combination of indicators

$$X = \sum_{i=1}^{n} c_i \mathbb{1}_{B_i} \qquad c_i \in \mathbb{R}$$

Claim (HW2)

Suppose that X and Y are random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Y is measurable wrt $\sigma(X)$ $(Y \in \sigma(X))$ if and only if $Y = f(X)$ for some measurable function f.

If X is a r.v. and $f : \mathbb{R} \to \mathbb{R}$ measurable then $f(X)$ is also r.v. and it is measble wrt $\sigma(X)$.

$f : \mathbb{R}^d \to \mathbb{R}$ measurable

$X_1, X_2, \ldots, X_n$ random variables on the same prob. space

$(X_1, X_2, \ldots, X_n)$

then $f(X_1, \ldots, X_n)$ is also a r.v.

sums, products of random variables

min, max

inf, sup, liminf, limsup of countably many random variables is also a random variable.

If $X_1, X_2, \dots$ is a sequence of r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$ then the following sets are all events:

$$\{\limsup_{n\to\infty} X_n \le c\}, \quad \{\lim_{n\to\infty} X_n \text{ exists}\}$$

$$\{\lim_{n\to\infty} X_n = c\}$$

$$\{\lim_{n\to\infty} X_n \text{ exists}\} = \{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) \text{ exists}\}$$

So we can talk about

$$P(\lim_{n\to\infty} X_n \text{ exists})$$

Def: $\{X_n\}_{n\geq 1}$ converges almost surely if $P(\lim_{n\to\infty} X_n \text{ exists}) = 1$ $\boxed{X_n \xrightarrow{a.s.} X}$

Def: Slightly weaker type of convergence: if $\boxed{\text{for every } \varepsilon > 0}$ we have

$$\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0$$

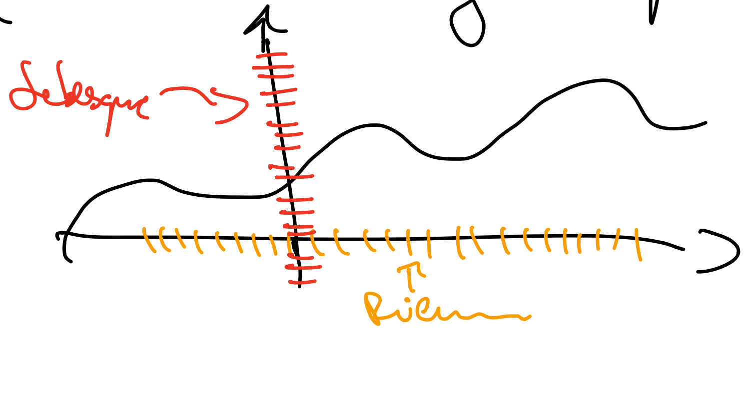
then we say that $X_n \to X$ in probability.

$$X_n \xrightarrow{P} X.$$

Q: is it possible to define a prob. space to model the choosing a positive integer uniformly?

Often we work with random variables that can take the value $+\infty$ or $-\infty$. We can make this rigorous by extending $\mathbb{R}$ with $\{-\infty, \infty\}$ and change the topology accordingly.

Quick overview of integration with respect to P



0. Indicator r.v.

$$\int_\Omega \mathbb{1}_B \, d\mathbb{P} = \mathbb{P}(B)$$

1. Simple functions

$$X = \sum_{i=1}^{n} c_i \mathbb{1}_{B_i}$$

X在Ω上积分 $= \int_\Omega \sum_{i=1}^{n} c_i \mathbb{1}_{B_i} \, d\mathbb{P}$

$$\int_\Omega X \, d\mathbb{P} = \int_\Omega \sum_{i=1}^{n} c_i \mathbb{1}_{B_i} \, d\mathbb{P} = \underline{\sum_{i=1}^{n} c_i \mathbb{P}(B_i)}$$

Properties: $X \geq 0 \rightsquigarrow \int_\Omega X \, d\mathbb{P} \geq 0$

$X \geq Y \rightsquigarrow \int_\Omega X \, d\mathbb{P} \geq \int_\Omega Y \, d\mathbb{P}$

same if $\mathbb{P}(X \geq Y) = 1$

$$\int_\Omega (X+Y) \, d\mathbb{P} = \int_\Omega X \, d\mathbb{P} + \int_\Omega Y \, d\mathbb{P}$$

$$\int_\Omega c \cdot X \, d\mathbb{P} = c \int_\Omega X \, d\mathbb{P}$$

$$\left| \int_\Omega X \, d\mathbb{P} \right| \leq \int_\Omega |X| \, d\mathbb{P}$$

2. Bounded functions

If X is bounded $(|X| \leq C)$

then $$\int_\Omega X \, dP = \sup_{\substack{Y \leq X \\ Y \text{ simple}}} \int_\Omega Y \, dP$$

The same properties that we had before for simple r.v.'s will extend to bounded r.v.'s.

3. Nonnegative functions

If $X \geq 0$ then

$$\int_\Omega X \, dP = \sup_{\substack{0 \leq Y \leq X \\ \text{bounded}}} \int Y \, dP$$

(could be infinite)

Same properties hold as before

4. general measurable functions

$$X = \underset{\substack{\uparrow \\ \text{positive part}}}{X_+} - \underset{\substack{\uparrow \\ \text{negative part}}}{X_-}$$

$$\int_\Omega X \, d\mathbb{P} = \int_\Omega X_+ \, d\mathbb{P} - \int_\Omega X_- \, d\mathbb{P}$$

unless both of these are ∞
(in that case $\int_\Omega X dP$ is not defined)

The same properties hold.

$\int_\Omega X \, dP$ is well-defined if

$\int_\Omega X_+ dP$ or $\int_\Omega X_- dP$ is finite.

(could be finite, ∞, $-\infty$)

Def: The expected value of a r.v. X on $(\Omega, \mathcal{F}, \mathbb{P})$ is defined as

$$E[X] = \int_\Omega X \, d\mathbb{P}$$

The expectation exists if $E[X^+]$ or $E[X^-]$ is finite.

Ex: $X = \mathbb{1}_A$ $\quad E[X] = \mathbb{P}(A)$.

EX: If X is discrete with values from $A = \{a_1, a_2, \ldots\}$ then

$$E[X] = \sum_i a_i P(X = a_i)$$

if this sum is well-defined

$$X = \sum_i \mathbb{1}_{\{a_i\}} \cdot X(a_i)$$

EX: If X is abs cont with PDF f then

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

if this integral is well-defined

Properties: If $0 \leq X$ then $E[X] \geq 0$.

If $X \leq Y$ then $E[X] \leq E[Y]$

(if these are well-defined)

$$E[X+Y] = E[X] + E[Y]$$

$$E[cX] = c\, E[X]$$

$$|E[X]| \leq E[|X|]$$

Ex: We pick 5 cards randomly from a deck of 52. What's the expected number of aces among the chosen cards?

X: # of aces

This is discrete $P(X \in \{0,1,2,3,4\}) = 1$

One can compute $E[X] = \sum_{i=0}^{4} i\, P(X=i)$

Easier way: write X as the sum of indicators.