Applications of Itô formula

$$f(t,B_t) = f(0,\overset{0}{B_0}) + \int_0^t \frac{\partial f}{\partial x}(s,B_s)\,dB_s + \int_0^t \frac{\partial f}{\partial t}(s,B_s)\,ds + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,B_s)\,ds$$

Ex: $\int_0^t B_s^k \, dB_s$ $\qquad B_0 = 0$

$$f(x) = \frac{1}{k+1} x^{k+1}$$

$$\frac{1}{k+1} B_t^{k+1} = \int_0^t B_s^k \, dB_s + \frac{1}{2}\int_0^t k \cdot B_s^{k-1}\, ds$$

Construct martingales.

$B_t^2 - t$ $\qquad e^{\alpha B_t - \frac{1}{2}\alpha^2 t}$

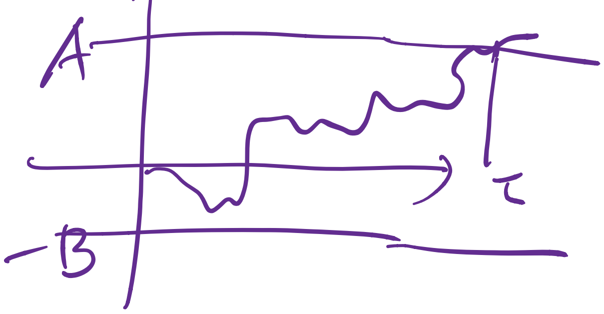
$f(t, B_t)$ $\qquad \int_0^{t} f \, dB$

Ex. suppose f satisfies $\boxed{\frac{\partial f}{\partial t} = -\frac{1}{2}\frac{\partial^2 f}{\partial x^2}}$

and $E\left[\int_0^T \left(\frac{\partial f}{\partial x}(t,B)\right)^2 dt\right] < \infty$

Then $f(t, B_t)$ is a mart.

Ex.



$$\mathbb{P}(B_\tau = A) = \frac{B}{A+B} \qquad \mathbb{P}(B_\tau = -B) = \frac{A}{A+B}$$

$$\mathrm{Cov}(\tau, B_\tau) = ?$$

$$\parallel$$

$$\mathbb{E}\left[(\tau - \mathbb{E}\tau)(B_\tau - \mathbb{E}B_\tau)\right]$$

$$= \mathbb{E}(\tau B_\tau) - \cancel{\mathbb{E}\tau \, \underbrace{\mathbb{E}B_\tau}_{= \mathbb{E}B_0 = 0}}$$

$$tB_t - (\ ?\ )$$

$$f(t,x) = tx - \left(\frac{x^3}{3}\right)$$

$$\partial_t f = -\frac{1}{2}\partial_x^2 f$$

$tB_t - \dfrac{B_t^3}{3}$ is mart.

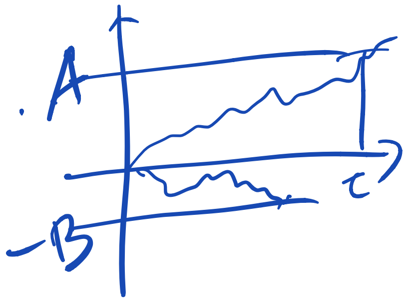
$$\mathbb{E}\left[\tau B_\tau - \frac{B_\tau^3}{3}\right] = 0$$

$$\mathbb{E}[\tau B_\tau] = \frac{1}{3}\mathbb{E}[B_\tau^3]$$

$$= \frac{1}{3}\left(\underline{P(B_\tau = A)}\cdot A + \underline{P(B_\tau = -B)}(-B)\right)$$

$$= \frac{1}{3} AB(A-B) = \mathrm{Cov}(\tau, B_\tau)$$

Ex.



$$X_t = \mu t + \sigma B_t$$

$$\mu \neq 0.$$

$$P(X_\tau = A) = ?$$

- $M_t = h(X_t)$ is mart.
- $h(A) = 1 \qquad h(-B) = 0.$

$$E(M_\tau) = E(M_0)$$

$$E(M_\tau) = P(X_\tau = A)\cdot \underbrace{h(A)}_{=1} + \cancel{P(X_\tau = -B)\cdot \underbrace{h(-B)}_{=0}}$$

$$P(X_\tau = A) = E(M_0) = E(h(X_0))$$

$$\boxed{h = ?}$$

Want $\quad h(X_t) = h(\mu t + \sigma B_t) = f(t, B_t)$

If $\quad \partial_t f = -\frac{1}{2}\partial_x^2 f$

$$\begin{cases} \sigma^2 h''(x) = -2\mu h'(x) \\ h(A) = 1 \qquad h(-B) = 0. \end{cases}$$

$$h(x) = \underline{c_1}\, e^{-\frac{2\mu}{\sigma^2}\cdot x} + \underline{c_2}$$

$$= \frac{e^{-\frac{2\mu}{\sigma^2}x} - e^{2\mu B/\sigma^2}}{e^{-2\mu A/\sigma^2} - e^{2\mu B/\sigma^2}}$$

$$\mathbb{P}(X_\tau = A) = \mathbb{E}(h(\overset{\overset{0}{\shortparallel}}{X_0}))$$

$$= \frac{e^{\frac{2\mu B}{\sigma^2}} - 1}{e^{\frac{2\mu B}{\sigma^2}} - e^{-\frac{2\mu A}{\sigma^2}}}$$

$A \to \infty \qquad B \to \infty$