# Itô formula.

$$f(t, \underset{\sim}{\overset{x}{\overset{\downarrow}{B_t}}})$$

pretend B was smooth.

$$\underline{d(\quad)} = \underline{\frac{\partial f}{\partial x}(t,B)\,dB} + \underline{\frac{\partial f}{\partial t}(t,B)\,dt}.$$

Itô formula. (assume $B_0 = 0$)

$$f(t,B_t) = f(0,\overset{B_0}{\overset{\shortparallel}{0}}) + \int_0^t \frac{\partial f}{\partial x}(s,B_s)\,dB_s$$

$$+ \underbrace{\int_0^t \frac{\partial f}{\partial s}(s,B_s)\,ds}_{\overset{\shortparallel}{0}} + \boxed{\frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,B_s)\,ds}$$

Special: $f(x)$

Remark:

$$d f(t,B) = \underline{\frac{\partial f}{\partial x}\,dB} + \frac{\partial f}{\partial t}\,dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\,dt$$

Ex: $f(x) = \underline{\frac{1}{2}x^2}$. "$\frac{1}{2}d(B^2)$"

$$\frac{1}{2}B_t^2 = \underline{\int_0^t \overbrace{B_s\,dB_s}} + \underbrace{\frac{1}{2}\int_0^t 1\,ds}_{\frac{1}{2}t}$$

Ex. $\boxed{\int_0^t B_s^2\,dB_s = ?}$

$x^2$

$\underline{f(x)} = \frac{1}{3}x^3$

$$\int_0^t \frac{\partial f}{\partial x}(B_s)\,dB_s = \frac{1}{3}B_t^3 - \frac{1}{2}\int_0^t 2B_s\,ds.$$

$$= \frac{1}{3}\underline{B_t^3} - \underline{\int_0^t B_s\,ds}.$$

Ex: $$\int_0^t \overbrace{\sin B_s}^{\frac{\partial f}{\partial x}(B)}\,dB_s = -\cos(B_t) + \cos(B_0)$$

$\left(f(x) = -\cos(x)\right)$ $$- \frac{1}{2}\int_0^t \cos B_s\,ds$$

$$= 1 - \cos B_t - \frac{1}{2}\int_0^t \underline{\cos B_s}\,ds$$

Ex. $$\underbrace{\int_0^t \overset{\frac{\partial f}{\partial x}(s,B)}{\overset{''}{s}}\,dB_s} = tB_t - \int_0^t B_s\,ds - \underline{\frac{1}{2}\int_0^t 0\,ds}$$

$\underline{f(t,x)} = tx$ $$= tB_t - \int_0^t B_s\,ds$$

$\frac{\partial f}{\partial t} = x$

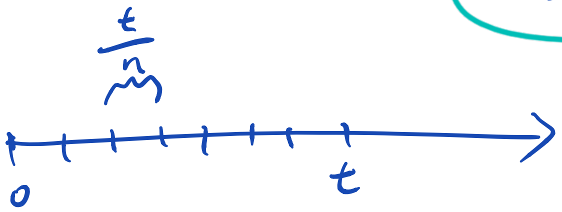
$C^\alpha\ \forall \alpha < \frac{1}{2}$

$\int \underline{B}\,d\underline{B}$

(Remark: Young thm: $\underline{\int f^\alpha\, dg^\beta}.$ $\boxed{\alpha+\beta>1}$ )

R-S integral

$$f(B_t) = f(\overset{0}{\overset{\shortparallel}{B_0}}) + \underbrace{\int_0^t f'(B_s)\,dB_s}_{A_n} + \underbrace{\frac{1}{2}\int_0^t f''(B_s)\,ds}_{B_n}$$



$$f(B_t) - f(B_0) = \sum_{i=1}^{n} \left\{ \underline{f(B_{t_i}) - f(B_{t_{i-1}})} \right\}$$

$$f(y) - f(x) = \underline{(y-x)f'(x)} + \underline{\frac{1}{2}(y-x)^2 f''(x)} + \underline{r(x,y)}$$

$$r(x,y) = \int_x^y (y-u)\left(f''(u) - f''(x)\right)du.$$

$$|r(x,y)| \leq \underline{(y-x)^2} \cdot \underline{h(x,y)}$$

uniformly conti. bounded

$h(x,x) = 0.$

$$A_n = \sum_{i=1}^{n} f'(B_{t_{i-1}})\left(B_{t_i} - B_{t_{i-1}}\right)$$

$$B_n = \frac{1}{2}\sum_{i=1}^{n} f''(B_{t_{i-1}})\left(B_{t_i} - B_{t_{i-1}}\right)^2$$

$$C_n \leq \sum_{i=1}^{n} \left(B_{t_i} - B_{t_{i-1}}\right)^2 h(B_{t_{i-1}}, B_{t_i})$$

$\downarrow 0$

$$A_n \longrightarrow \int_0^t f'(B_s)\,dB_s$$

$\tilde{B}_n \rightarrow 0$

$$B_n = \frac{1}{2}\sum_{i=1}^{n} f''(B_{t_{i-1}}) \left[ \left(B_{t_i} - B_{t_{i-1}}\right)^2 \right.$$

$$- (t_i - t_{i-1})$$

$$\underline{+ \frac{1}{2} \sum_{i=1}^{n} f''(B_{t_{i-1}})(t_i - t_{i-1})}$$

$$\downarrow d$$

$$\boxed{\frac{1}{2}\int_0^t f''(B_s)\, ds}$$

$$\frac{1}{2}\sum_i \mathbb{E}\Big[\mathbb{E}\Big[f''(B_{t_{i-1}})\big((B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})\big)\Big|\mathcal{F}_{t_{i-1}}\Big]\Big]$$

$$\|$$

$$\mathbb{E}\tilde{B}_n \neq 0$$

$$\mathbb{E}(\tilde{B}_n^2) \overset{?}{\longrightarrow} 0 .$$

$$\Bigg\} \Rightarrow \tilde{B}_n \xrightarrow{n\to\infty} 0 \text{ in prob}$$

$$\|$$

$$t_i - t_{i-1}$$
$$\|$$

$$\Rightarrow \frac{1}{4}\sum_{i=1}^{n} \mathbb{E}\Big[f''(B_{t_{i-1}})^2 \big[(B_{t_i} - B_{t_{i-1}})^2 - \delta t_i\big]^2\Big]$$

$$+ \frac{1}{4}\sum_{i\neq j} \mathbb{E}\Big[f''(B_{i-1})f''(B_{j-1}) \big[(B_{t_i} - B_{t_{i-1}})^2 - \delta t_i\big]\big[(B_{t_j} - B_{t_{j-1}})^2 - \delta t_j\big]\Big]$$

$(i<j)$ Condition on $\mathcal{F}_{t_i}$

$$\leq C\|f''\|_{L^\infty}^2 \sum_{i=1}^{n} \mathbb{E}\Big[\big((B_{t_i} - B_{t_{i-1}})^2 - \delta t_i\big)^2\Big]$$

$$= C\|f''\|_{L^\infty}^2 \underbrace{\sum_{i=1}^{n} (\delta t)^2}_{n\cdot\left(\frac{t}{n}\right)^2} = \frac{Ct^2}{n}\|f''\|_{L^\infty}^2$$

$$\downarrow n\to\infty$$
$$0$$

$$\left(\begin{array}{l} X \sim N(0, \delta t) \\ \mathbb{E}\big[(X^2 - \delta t)^2\big] = \mathrm{Var}(X^2) = 2(\delta t)^2 \end{array}\right)$$

So, $B_n \xrightarrow{n\to\infty} \underline{\underline{\int_0^t f''(B_s)\,ds}}$