Math 733 - Fall 2020

Homework 3

Due: 10/11, 10pm

Zijie Zhang

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1. (a) *Proof.*

$$X \sim B(n,p) \Rightarrow P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$Y \sim B(m,p) \Rightarrow P(X=k) = \binom{m}{k} p^k (1-p)^{m-k}$$

Then

$$\begin{aligned} P(X+Y=k) &= \sum_{i=0}^{k} P(X=i, Y=k-i) \\ &= \sum_{i=0}^{k} P(X=i) \cdot P(Y=k-i) \\ &= \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} \cdot \binom{m}{k-i} p^{k-i} (1-p)^{m-k+i} \\ &= p^k (1-p)^{m+n-k} \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i} \\ &= \binom{n+m}{k} p^k (1-p)^{m+n-k} \end{aligned}$$

Thus,

$$X+Y \sim B(n+m, p)$$

□

(b) *Proof.*

$$X \sim \text{Poisson}(\lambda) \Rightarrow P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$Y \sim \text{Poisson}(\mu) \Rightarrow P(Y=k) = \frac{\mu^k}{k!} e^{-\mu}$$

Then

$$\begin{aligned}
P(X+Y=k) &= \sum_{i=0}^{k} P(X=i, Y=k-i) \\
&= \sum_{i=0}^{k} P(X=i) \cdot P(Y=k-i) \\
&= \sum_{i=0}^{k} \frac{\lambda^i}{i!} e^{-\lambda} \cdot \frac{\mu^{k-i}}{(k-i)!} e^{-\mu} \\
&= e^{-(\lambda+\mu)} \sum_{i=0}^{k} \frac{\lambda^i}{i!} \frac{\mu^{k-i}}{(k-i)!} \\
&= \frac{(\lambda+\mu)^k}{k!} e^{-(\lambda+\mu)}
\end{aligned}$$

Thus,

$$X+Y \sim \text{Poisson}(\lambda+\mu)$$

□

2. (a) *Proof.* Let $h(x,y) = \mathbb{1}_{\{xy \leqslant z\}}$, let μ, ν be the probability measures with distributions F_X and F_Y. Since for fixed $y > 0$,

$$\int h(x,y)\mu(dx) = \int \mathbb{1}_{(-\infty, z/y]}(x)\mu(dx) = F_X(\frac{z}{y})$$

So

$$\begin{aligned}
F_{XY}(z) = P(XY \leqslant z) &= \iint \mathbb{1}_{\{xy \leqslant z\}} \mu(dx)\nu(dy) \\
&= \int F_X(\frac{z}{y}) dF_Y(y)
\end{aligned}$$

□

(b) *Proof.* Absolutely continuous means every set of measure zeros is probability zero. Consider

$$P(XY) = P(X)P(Y)$$

$$P(XY = x_1 y_1) = P(X = x_1)P(Y = y_1) = 0$$

Then XY is absolutely continuous with p.d.f

$$f_{XY} = \int f_X\left(\frac{x}{t}\right) f_Y(t) \mathrm{d}t$$

□

3. *Proof.*

$$\begin{aligned}
&\lim_{n\to\infty} P\left(|X_n + Y_n - (X+Y)| > \varepsilon\right) \\
&\leqslant \lim_{n\to\infty} P\left(|X_n - X| > \frac{\varepsilon}{2}\right) + \lim_{n\to\infty} P\left(|Y_n - Y| > \frac{\varepsilon}{2}\right) \\
&= 0
\end{aligned}$$

Since $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$. So,

$$X_n + Y_n \xrightarrow{p} X + Y$$

$$
\begin{aligned}
&\lim_{n\to\infty} P\left(|X_nY_n - XY| > \varepsilon\right) \\
&= \lim_{n\to\infty} P\left(|(X_n - X)(Y_n - Y) + Y(X_n - X) + X(Y_n - Y)| > \varepsilon\right) \\
&\leqslant \lim_{n\to\infty} P\left(|(X_n - X)(Y_n - Y)| > \frac{\varepsilon}{3}\right) + \lim_{n\to\infty} P\left(|Y(X_n - X)| > \frac{\varepsilon}{3}\right) + \lim_{n\to\infty} P\left(|X(Y_n - Y)| > \frac{\varepsilon}{3}\right) \\
&= 0
\end{aligned}
$$

□

4. *Proof.* Consider

$$x_i \sim U[0,1] \forall i \text{ and i.i.d}$$

$$
\begin{aligned}
&\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 n\left(f\left(\frac{1}{n}(x_1 + x_2 + \cdots + x_n)\right) - f\left(\frac{1}{2}\right)\right) \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n \\
&= \lim_{n\to\infty} \mathbb{E}\left[n\left(f\left(\frac{1}{n}(x_1 + x_2 + \cdots + x_n)\right) - f\left(\frac{1}{2}\right)\right)\right] \\
&= \lim_{n\to\infty} n \cdot \mathbb{E}\left[f\left(\frac{1}{n}(x_1 + x_2 + \cdots + x_n)\right) - f\left(\frac{1}{2}\right)\right] \\
&f \text{ is continuous differentiable} \Leftrightarrow \mathbb{E}\left[f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)\right] = f\left(\mathbb{E}\left[\frac{x_1 + x_2 + \cdots + x_n}{n}\right]\right) \\
&= \lim_{n\to\infty} n \cdot \left(f\left(\mathbb{E}\left[\frac{x_1 + x_2 + \cdots + x_n}{n}\right]\right) - f\left(\frac{1}{2}\right)\right) \\
&\text{Since } \mathbb{E}\left[\frac{x_1 + x_2 + \cdots + x_n}{n}\right] = \mathbb{E}\left[x_1\right] = \int_0^1 t\mathrm{d}t = \frac{1}{2} \\
&\text{We have} \\
&= \lim_{n\to\infty} n \cdot 0 = 0
\end{aligned}
$$

□

5. (a) *Proof.* Let

$$S_n = x_1 + x_2 + \cdots + x_n$$

then

$$\mathbb{E}[\mathbb{1}_{\{S_k \leqslant x\}}] = P(S_k \leqslant x) = P(x_1 + x_2 + \cdots x_k < k) = F^{(k)}(x)$$

Let

$$N_x = \sum_{k=1}^{\infty} \mathbb{1}_{\{S_k \leqslant x\}}$$

It gives

$$\mathbb{E}[N_x] = \sum_{n=1}^{\infty} \mathbb{E}[\mathbb{1}_{\{S_n \leqslant x\}}] = \sum_{n=1}^{\infty} F^{(n)}(x)$$

□

(b) *Proof.* Let $n = \lceil x \rceil$

$$
\begin{aligned}
P(N_x = N) &= P(S_N \leqslant x < S_{N+1}) \\
&\leqslant P(S_N \leqslant n) \\
&= \binom{N}{N-n} F^{(N-n)}(0)
\end{aligned}
$$

Consider the expectation,

$$
\begin{aligned}
\mathbb{E}[N_x] &= \sum_{N=1}^{\infty} N P(N_x = N) \\
&\leqslant \sum_{N=1}^{\infty} N \binom{N}{N-n} F^{(N-n)}(0) \\
&\leqslant \sum_{N=1}^{\infty} N \binom{N}{n} F^{(N-n)}(0) \\
&< \infty
\end{aligned}
$$

□

(c) *Proof.* If $x_k < t$, let $x_k^t = x_k$ else, let $x_k^t = t$.

$$
\mathbb{E}[N_x] \leqslant \mathbb{E}[N_x^t] \leqslant \frac{x/t + 1}{P(x \geqslant t)} < \infty
$$

□

6. *Proof.* □