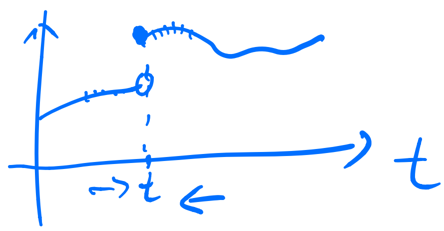
Continuous time stoch. processes (A deeper look)

{ continuous stoch. process
{ continuous-time stoch. processes. $X_t \quad t \in \mathbb{R}_+$

↳ ($X(\cdot, w)$ is a function on $\mathbb{R}_+$
fix w,

"Cadlag" : right-continuous & has left limit.



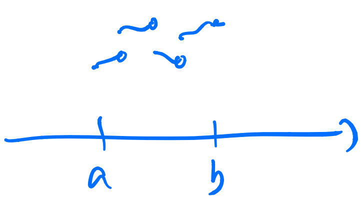
$\forall t \quad \lim_{s \to t^+} X_s = X_t$

$\lim_{s \to t^-} X_s$ exists.

$C[0,\infty)$: space of all conti-functions

$D[0,\infty)$: - - - - - - - cadlag functions.

Rmks 1) in any compact time interval
for each $\varepsilon > 0$
a cadlag function has at most finitely many dis-cont. of size $\geq \varepsilon$.



(If not true, you would have right limit point or left limit point)

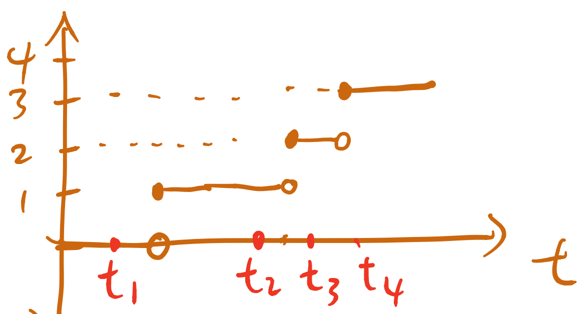
2) a cadlag function can have at most countably many discontinuities.

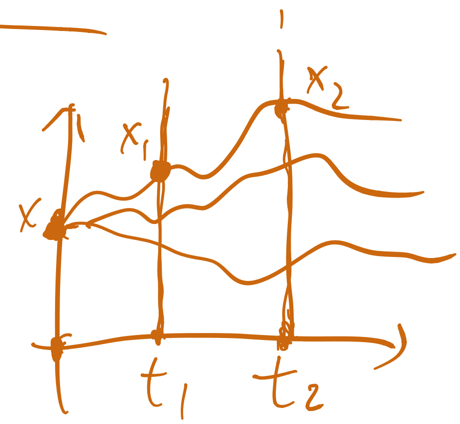
(take $\varepsilon = \frac{1}{n}$)

- Examples of cadlag stoch processes

1) Brownian motion

2) Poisson process.





Poisson(μ)

$$h(\mu, k) = e^{-\mu} \frac{\mu^k}{k!}$$

$\underline{p(t_1, x, x_1)}$
$\underline{p(t_2 - t_1, x_1, x_2)}$

$$\mathbb{P}(X_{t_1} = k_1, \cdots, X_{t_n} = k_n)$$

$$= \begin{cases} h(\lambda t_1, k_1)\, h(\lambda(t_2 - t_1), k_2 - k_1) \\ \quad \cdots h(\lambda(t_n - t_{n-1}), k_n - k_{n-1}) \\ \quad \text{if } k_1 \le k_2 \le \cdots \le k_n \end{cases}$$

if $k_1 \le k_2 \le \cdots \le k_n$

0 otherwise

called Poisson process w. parameter λ

- **Filtration**: a collection of σ-algebras $\{\mathcal{F}_t\}$, satisfying

$$\underbrace{\mathcal{F}_s \subseteq \mathcal{F}_t}_{\forall s \le t} \subseteq \mathcal{F}.$$

$(\Omega, \mathcal{F}, \mathbb{P})$

- A stochastic process X is adapted to a filtration $\{\mathcal{F}_t\}$ is X_t is $\mathcal{F}_t$-measurable for all $t \geq 0$.

- Given $\{X_t\}$ generate a filtration.

$$\mathcal{F}_t = \underbrace{\sigma(X_s : s \le t)}_{\text{the smallest } \sigma\text{-alg}}$$

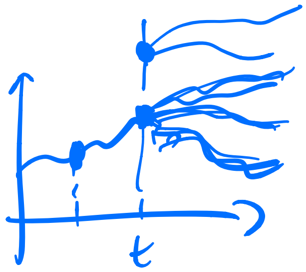
s.t. X_s is $\mathcal{F}_t$-measurable for all $s \le t$

- Markov process:

A stoch process X adapted to $\{\mathcal{F}_t\}$ is a Markov process wrt $\{\mathcal{F}_t\}$ if

$$\mathbb{E}\left[f(X_{t+s}) \mid \mathcal{F}_t\right] = \mathbb{E}\left[f(X_{t+s}) \mid X_t\right]$$

history before t



- <u>martingale</u>

X adapted to $\{\mathcal{F}_t\}$ is "martingale" w.r.t. $\{\mathcal{F}_t\}$ if

$$\mathbb{E}\left[\underline{X(t+s)} \mid \mathcal{F}_t\right] = X(t) \quad \forall\, t, s \geq 0.$$

"fair game"

Lem: BM is Markov & Martingale.

Pf: $\mathcal{F}_t = \sigma(B_s \ s \leq t)$

$\mathbb{E}(B_{t+s} \mid \mathcal{F}_t)$

$$= \mathbb{E}\left(B_{t+s} - B_t + B_t \mid \mathcal{F}_t\right)$$

$$= \mathbb{E}\left(\underline{B_{t+s} - B_t} \mid \mathcal{F}_t\right) + \mathbb{E}\left(B_t \mid \mathcal{F}_t\right)$$

$$= \mathbb{E}\left(\underline{B_{t+s} - B_t}\right) + \mathbb{E}\left(B_t \mid \mathcal{F}_t\right)$$

$$= 0 + B_t = B_t \Rightarrow \text{martingale}$$

ref: Timo Seppalainen's notes.
Prop 2.31 b)

Def (stopping time)

A random variable τ with values in $[0,\infty]$ is an $\{\mathcal{F}_t\}$-stopping time if

$$\underline{\{\tau \le t\}} \in \mathcal{F}_t \quad \forall t \geq 0$$

properties: $\tau_1 \vee \tau_2$, $\tau_1 \wedge \tau_2$ are stopping times.