Math 733 - Fall 2020
Homework 3
Due: 10/11, 10pm

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1. (a) *Proof.*

$$X \sim B(n,p) \Rightarrow P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$
$$Y \sim B(m,p) \Rightarrow P(X=k) = \binom{m}{k} p^k (1-p)^{m-k}$$

Then

$$\begin{aligned} P(X+Y=k) &= \sum_{i=0}^{k} P(X=i, Y=k-i) \\ &= \sum_{i=0}^{k} P(X=i) \cdot P(Y=k-i) \\ &= \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} \cdot \binom{m}{k-i} p^{k-i} (1-p)^{m-k+i} \\ &= p^k (1-p)^{m+n-k} \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i} \\ &= \binom{n+m}{k} p^k (1-p)^{m+n-k} \end{aligned}$$

Thus,

$$X + Y \sim B(n+m, p)$$

□

(b) *Proof.*

$$X \sim \text{Poisson}(\lambda) \Rightarrow P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$$
$$Y \sim \text{Poisson}(\mu) \Rightarrow P(Y=k) = \frac{\mu^k}{k!} e^{-\mu}$$

Then

$$\begin{aligned}
P(X+Y=k) &= \sum_{i=0}^{k} P(X=i, Y=k-i) \\
&= \sum_{i=0}^{k} P(X=i) \cdot P(Y=k-i) \\
&= \sum_{i=0}^{k} \frac{\lambda^i}{i!} e^{-\lambda} \cdot \frac{\mu^{k-i}}{(k-i)!} e^{-\mu} \\
&= e^{-(\lambda+\mu)} \sum_{i=0}^{k} \frac{\lambda^i}{i!} \frac{\mu^{k-i}}{(k-i)!} \\
&= \frac{(\lambda+\mu)^k}{k!} e^{-(\lambda+\mu)}
\end{aligned}$$

Thus,

$$X+Y \sim \text{Poisson}(\lambda+\mu)$$

□

2. (a) *Proof.* Let $h(x,y) = \mathbb{1}_{\{xy \leqslant z\}}$, let μ, ν be the probability measures with distributions F_X and F_Y. Since for fixed $y > 0$,

$$\int h(x,y)\mu(dx) = \int \mathbb{1}_{(-\infty, z/y]}(x)\mu(dx) = F_X(\frac{z}{y})$$

If $y < 0$,

$$\int h(x,y)\mu(dx) = \int \mathbb{1}_{(z/y, \infty]}(x)\mu(dx) = 1 - F_X(\frac{z}{y})$$

So

$$\int h(x,y)\mu(dx) = \frac{1}{2} + \text{sgn}(y)\left(F_X(\frac{z}{y}) - \frac{1}{2}\right)$$

$$\begin{aligned}
F_{XY}(z) = P(XY \leqslant z) &= \iint \mathbb{1}_{\{xy \leqslant z\}}\mu(dx)\nu(dy) \\
&= \int \left(\frac{1}{2} + \text{sgn}(y)\left(F_X(\frac{z}{y}) - \frac{1}{2}\right)\right)\nu(dy) \\
&= \int \left(\frac{1}{2} + \text{sgn}(y)\left(F_X(\frac{z}{y}) - \frac{1}{2}\right)\right)dF_Y(y)
\end{aligned}$$

□

(b) *Proof.* □

3. *Proof.* □

4. *Proof.* □

5. *Proof.* □

6. *Proof.* □