Itô formula.

$$f(t, \overset{x}{B_t})$$

pretend B was smooth.

$$d(\quad) = \frac{\partial f}{\partial x}(t,B)\,dB + \frac{\partial f}{\partial t}(t,B)\,dt.$$

Itô formula. (assume $B_0 = 0$)

$$f(t, B_t) = f(0; \overset{B_0}{0}) + \int_0^t \frac{\partial f}{\partial x}(s, B_s)\,dB_s$$

$$+ \underbrace{\int_0^t \frac{\partial f}{\partial s}(s, B_s)\,ds}_{=0} + \boxed{\frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,B_s)\,ds}$$

Special: $f(x)$

Remark:

$$df(t,B) = \frac{\partial f}{\partial x}\,dB + \frac{\partial f}{\partial t}\,dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\,dt$$

Ex: $f(x) = \frac{1}{2}x^2$. $\frac{1}{2}d(B^2)$"

$$\frac{1}{2}B_t^2 = \int_0^t B_s\,dB_s + \underbrace{\frac{1}{2}\int_0^t 1\,ds}_{\frac{1}{2}t}$$

Ex. $\int_0^t B_s^2\,dB_s = ?$

x^2

$\underline{f(x) = \frac{1}{3}x^3}$

$$\int_0^t \frac{\partial f}{\partial x}(B_s)\,dB_s = \frac{1}{3}B_t^3 - \frac{1}{2}\int_0^t 2B_s\,ds.$$

$$= \frac{1}{3}B_t^3 - \int_0^t B_s\,ds.$$

Ex:

$$\int_0^t \overbrace{\sin B_s}^{\frac{\partial f}{\partial x}(B)}\,dB_s = -\cos(B_t) + \cos(B_0)$$

$$\left(f(x) = -\cos(x) \right) \qquad - \frac{1}{2}\int_0^t \cos B_s\,ds$$

$$= 1 - \cos B_t - \frac{1}{2}\int_0^t \cos B_s\,ds$$

Ex.

$$\int_0^t \overset{\frac{\partial f}{\partial x}(s,B)}{s}\,dB_s = tB_t - \int_0^t B_s\,ds - \frac{1}{2}\int_0^t 0\,ds$$

$\underline{f(t,x) = tx}$

$$\frac{\partial f}{\partial t} = x$$

$$= tB_t - \int_0^t B_s\,ds$$

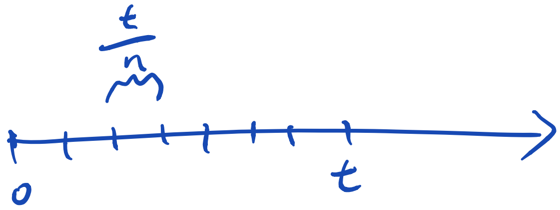
$C^\alpha \; \forall \alpha < \frac{1}{2}$

$\int B\,dB$

(Remark: Young thm: $\int f^\alpha\, dg^\beta$. $\boxed{\alpha+\beta>1}$)

R-S integral

$$f(B_t) = f(\overset{0}{\overset{\shortparallel}{B_0}}) + \int_0^t f'(B_s)\,dB_s + \frac{1}{2}\int_0^t f''(B_s)\,ds.$$



$$f(B_t) - f(B_0) = \sum_{i=1}^{n} \left\{ \underline{f(B_{t_i}) - f(B_{t_{i-1}})} \right\}$$

$$f(y) - f(x) = \underline{(y-x) f'(x)} + \underline{\tfrac{1}{2}(y-x)^2 f''(x)} + \underline{r(x,y)}$$

$$r(x,y) = \int_x^y (y-u)\left(f''(u) - f''(x)\right) du.$$

$$|r(x,y)| \le \underline{(y-x)^2} \cdot \underline{h(x,y)}$$

uniformly conti. bounded

$h(x,x) = 0.$

$$A_n = \sum_{i=1}^{n} f'(B_{t_{i-1}})\left(B_{t_i} - B_{t_{i-1}}\right)$$

$$B_n = \frac{1}{2}\sum_{i=1}^{n} f''(B_{t_{i-1}})\left(B_{t_i} - B_{t_{i-1}}\right)^2$$

$$C_n \le \sum_{i=1}^{n} \left(B_{t_i} - B_{t_{i-1}}\right)^2 h(B_{t_{i-1}}, B_{t_i})$$

$\downarrow 0$

$$A_n \longrightarrow \int_0^t f'(B_s)\,dB_s$$

$\tilde{B}_n \to 0$

$$B_n = \frac{1}{2}\sum_{i=1}^{n} f''(B_{t_{i-1}}) \Big[\left(B_{t_i} - B_{t_{i-1}}\right)^2$$

$$- (t_i - t_{i-1}) \Big]$$

$$+ \frac{1}{2} \sum_{i=1}^n f''(B_{t_{i-1}})(t_i - t_{i-1})$$

$$\downarrow d$$

$$\frac{1}{2} \int_0^t f''(B_s)\, ds$$

$$\frac{1}{2} \sum_i \mathbb{E}\Big[\mathbb{E}\big[f''(B_{t_{i-1}}) \big((B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1}) \big) \big| \mathcal{F}_{t_{i-1}} \big] \Big]$$

$$\|$$

$$\cancel{\mathbb{E}\, \tilde{B}_n \neq 0}$$

$$\mathbb{E}(\tilde{B}_n^2) \xrightarrow{?} 0 \quad \Bigg\} \Rightarrow \tilde{B}_n \xrightarrow{n\to\infty} 0 \text{ in prob}$$

$$\|$$

$$t_i - t_{i-1}$$
$$\|$$

$$\Rightarrow \frac{1}{4} \sum_{i=1}^n \mathbb{E}\Big[f''(B_{t_{i-1}})^2 \big[(B_{t_i} - B_{t_{i-1}})^2 - \delta t_i \big]^2 \Big]$$

$$+ \frac{1}{4} \sum_{i \neq j} \mathbb{E}\Big[f''(B_{i-1}) f''(B_{j-1}) \big[(B_{t_i} - B_{t_{i-1}})^2 - \delta t_i \big] \big[(B_{t_j} - B_{t_{j-1}})^2 - \delta t_j \big] \Big]$$

$(i<j)$ Condition on $\mathcal{F}_{t_i}$

$$\leq C \|f''\|_{L^\infty}^2 \sum_{i=1}^n \mathbb{E}\Big[\big((B_{t_i} - B_{t_{i-1}})^2 - \delta t_i \big)^2 \Big]$$

$$= C \|f''\|_{L^\infty}^2 \underbrace{\sum_{i=1}^n (\delta t)^2}_{n \cdot \left(\frac{t}{n}\right)^2} = \frac{C t^2}{n} \|f''\|_{L^\infty}^2 \xrightarrow{n\to\infty} 0$$

$$\left(\begin{array}{l} X \sim N(0, \delta t) \\ \mathbb{E}\big[(X^2 - \delta t)^2 \big] = \mathrm{Var}(X^2) = 2(\delta t)^2 \end{array} \right)$$

So, $B_n \xrightarrow{n\to\infty} \int_0^t f''(B_s)\, ds$