3.1 Prove $\displaystyle\int_0^t s\,dB_s = tB_t - \int_0^t B_s\,ds$.

So here $f(s,\omega) = s$, which is easily approximated by

$$\Phi_n = \sum_{k=1}^{N^{(n)}} s_k^{(n)} \mathbb{I}\{s_k^{(n)} \leq s < s_{k+1}^{(n)}\}.$$

So
$$\int_s^t s\,dB_s = \lim_{n\to\infty} \int_0^t \sum_{k=1}^{N^{(n)}} s_k^{(n)} \mathbb{I}\{s_k^{(n)} \leq s < s_{k+1}^{(n)}\}\,dB_s.$$

$$= \lim_{n\to\infty} \sum_{k=1}^{N^{(n)}} s_k^{(n)} \int_{s_k^{(n)}}^{s_{k+1}^{(n)}} dB_s.$$

$$= \lim_{n\to\infty} \sum_{k=1}^{N^{(n)}} s_k^{(n)}\,\Delta B_k^{(n)}.$$

Now,
$$\Delta(s_j B_j) = s_{j+1}B_{j+1} - s_j B_j$$
$$= s_j(B_{j+1} - B_j) + s_{j+1}B_{j+1} - s_j B_{j+1}$$
$$= s_j(\Delta B_j) + (\Delta s_j) B_{j+1}$$

$$\Rightarrow \quad \sum s_k^{(n)} \Delta B_k^{(n)} = \sum \Delta(s_k^{(n)} B_k^{(n)}) - \sum B_k \Delta(s_k^{(n)}).$$

$$\downarrow n\to\infty \qquad\qquad \downarrow n\to\infty \qquad\qquad \downarrow n\to\infty$$

$$\int_0^t s\,dB_s = \int_0^t d(sB_s) - \int_0^t B_s\,ds.$$

3.2 Skipping some of the rigour we used in 3.1 :

$$(\Delta B_k)^3 = (B_{k+1} - B_k)^3$$

$$= B_{k+1}^3 - B_k^3 - 3B_{k+1}B_k(B_{k+1} - B_k).$$

$$= \Delta(B_k^3) - 3(B_k + \Delta B_k)(B_k)\Delta B_k.$$

so $$(\Delta B_k)^3 = \Delta(B_k^3) - 3B_k^2\Delta B_k - 3B_k(\Delta B_k)^2. \qquad (*)$$

Now, $$\lim_{\substack{\text{mesh}\\ \Delta t_k \to 0}} \sum_k (\Delta B_k)^3 \leq \lim_{\substack{\text{mesh}\\ \Delta t_k \to 0}} \underbrace{\max_k |\Delta B_k|}_{\to 0} \cdot \underbrace{\sum_k (\Delta B_k)^2}_{\to t}$$

establishes the LHS of $*$ (summed $\sum$ as $\Delta t_k \to 0$) is 0

Thus $$0 = \int_0^t d(B_s^3) - 3\int_0^t B_s^2\, dB_s - 3\int_0^t B_s\, ds.$$

as required.

3.3

a Prove $\langle X \rangle_t$ is a martingale wrt some filtration $\{\mathcal{N}_t\}$, then it is a martingale wrt $\{\sigma(X_t)\} = \{\mathcal{H}_t\}$.

Note X_t adapted to $\mathcal{N}_t$ implies $\mathcal{H}_t \subseteq \mathcal{N}_t \;\; \forall t$.

By the Tower property,

$$\mathbb{E}[\mathbb{E}[Y \mid \mathcal{N}_t] \mid \mathcal{H}_t] = \mathbb{E}[Y \mid \mathcal{H}_t].$$

Thus for $t \geq s$, setting $X_t = Y$ above,

$$\mathbb{E}[X_t \mid \mathcal{H}_s] = \mathbb{E}[\mathbb{E}[X_t \mid \mathcal{N}_s] \mid \mathcal{H}_s]$$

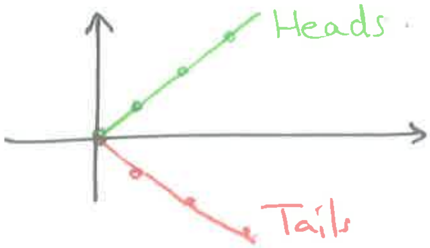
$$= \mathbb{E}[X_s \mid \mathcal{H}_s]$$

$$= X_s, \quad \text{since } X_s \in m\mathcal{H}_s.$$

b

$$\mathbb{E}[X_t] = \mathbb{E}[\mathbb{E}[X_t \mid \mathcal{H}_0]] = \mathbb{E}[X_0].$$

c Flip a coin at time 0. If heads, let $X_t = t \;\; \forall t$, if tails, let $X_t = -t \;\; \forall t$



$\mathbb{E}[X_t] = 0 \;\; \forall t$, but

$$\mathbb{E}[X_t \mid \mathcal{H}_s] = t \quad \text{if } X_s \geq 0$$

$$\mathbb{E}[X_t \mid \mathcal{H}_s] = -t \quad \text{if } X_s < 0.$$

so X_t not a martingale.

ØKSENDAL $\boxed{13.4}$

3.4 i No, $\mathbb{E}[X_t \mid \mathcal{H}_s] = X_s + 4(t-s)$.

ii No, $\mathbb{E}[X_t \mid \mathcal{H}_s] = \mathbb{E}[B_t^2 \mid \mathcal{H}_s]$

$$= \mathbb{E}\left[B_s^2 + (B_s + (B_t - B_s))^2 - B_s^2 \mid \mathcal{H}_s\right]$$

$$= B_s^2 + \mathbb{E}\left[2B_s(B_t - B_s) + (B_t - B_s)^2 \mid \mathcal{H}_s\right]$$

$$= B_s^2 + 2B_s \mathbb{E}_s[B_t - B_s] + \mathbb{E}_s[(B_t - B_s)^2]$$

$$= X_t + (t - s)$$

iii Yes,

$$\mathbb{E}_s[X_t] = \mathbb{E}_s\left[t^2 B_t - 2\int_0^t u B_u \, du\right]$$

$$= X_s + \mathbb{E}_s\left[(t^2 - s^2)B_s + t^2(B_t - B_s) - 2\int_s^t u B_u \, du\right]$$

$$= X_s + (t^2 - s^2)B_s + t^2 \mathbb{E}_s[B_t - B_s] - 2\mathbb{E}_s\left[\int_s^t u(B_s + (B_u - B_s))ds\right]$$

$$= X_s + (t^2 - s^2)B_s - 2\int_s^t u\,(B_s + \underbrace{\mathbb{E}_s(B_u - B_s)}_{=0})\, du$$

$$= X_s + (t^2 - s^2)B_s - 2B_s \int_s^t u \, du$$

$$= X_s.$$

iv Yes,

$$\mathbb{E}_s[B_1(t)B_2(t)] = \mathbb{E}_s\left[(B_1(s) + (B_1(t) - B_1(s))) \cdot (B_2(s) + (B_2(t) - B_2(s)))\right]$$

$$= \mathbb{E}[\;B_1(s)B_2(s) + B_1(s)\underbrace{\mathbb{E}_s[B_2(t) - B_2(s)]}_{=0}$$

$$+ B_2(s)\underbrace{\mathbb{E}_s[B_1(t) - B_1(s)]}_{=0}$$

By independence $\longrightarrow$ $+ \underbrace{\mathbb{E}_s[B_1(t) - B_1(s)]}_{=0}\underbrace{\mathbb{E}_s[B_2(t) - B_2(s)]}_{=0}$

$$= B_1(s)B_2(s).$$

ØKSENDAL $\boxed{3.5}$, $\boxed{3.6}$

3.5

$$\mathbb{E}_s[M_t] = \mathbb{E}_s[B_t^2 - t]$$

$$= \mathbb{E}_s\left[(B_s + (B_t - B_s))^2 - t\right]$$

$$= B_s^2 + 2B_s \underbrace{\mathbb{E}_2[(B_t - B_s)]}_{=0} + \underbrace{\mathbb{E}_s[(B_t - B_s)^2]}_{=t-s.} - t.$$

$$= B_s^2 - s.$$

3.6

$$\mathbb{E}_s[N_t] = \mathbb{E}_s\left[B_t^3 - 3tB_t\right]$$

$$= \mathbb{E}_s\left[\left(B_s + (B_t - B_s)\right)^3 - 3t\left(B_s + (B_t - B_s)\right)\right]$$

$$= B_s^3 + 3B_s^2 \underbrace{\mathbb{E}_s[(B_t - B_s)]}_{=0} + 3B_s \underbrace{\mathbb{E}_s[(B_t - B_s)^2]}_{=t-s} + \underbrace{\mathbb{E}_s[(B_t - B_s)^3]}_{=0}.$$

$$- 3tB_s - 3t \underbrace{\mathbb{E}_s[(B_t - B_s)]}_{=0}.$$

$$= B_s^3 + 3B_s(t-s) - 3tB_s.$$

$$= B_s^3 - 3sB_s$$

3.7 a

Let us prove the following :

$$\left[\text{If } \Phi \in \mathcal{V}(S_0, T_0), \text{ then } \int_{S_0}^{T_0} \Phi(s,\omega)\,dB_s \in \mathcal{V}(S_0,T_0) \right].$$

Proof.

If $\Phi \in \mathcal{V}(S_0, T_0)$ then $\exists$ Φ_n simple :

$$\mathbb{E}\left[\int_{S_0}^{T_0} (\Phi_n - \Phi)^2 ds \right] \longrightarrow 0 .$$

and by the Itô isometry.

$$\mathbb{E}\left[\left(\int_{S_0}^{T_0} \Phi_n \, dB_s - \int_{S_0}^{T_0} \Phi \, dB_s \right)^2 \right] \longrightarrow 0$$

Now, since the Φ_n are simple,

$$\int_S^T \Phi_n \, dB_s = \sum_{j=1}^{N^{(n)}} e_j(\omega) \left[B_{t_{j+1}}(\omega) - B_{t_j}(\omega) \right] .$$

The function $\Xi_n(t, \omega) = \int_{S_0}^{t} \Phi_n dB_s$ is clearly both $\mathcal{F}_t$ adapted and $(\mathcal{F} \times \mathcal{B})$ measurable.

It follows (since the lim sup of measurable functions is also measurable), that

$$\Xi(t,\omega) = \int_{S_0}^{t} \Phi \, dB_s = \limsup_n \int_{S_0} \Phi_n \, dB_s = \limsup_n \Xi_n(t,\omega) ,$$

is also both $\mathcal{F}_t$ adapted and $(\mathcal{F} \times \mathcal{B})$ measurable.

3.7 (cont.)

So we have established that 3.1.4(i) and 3.1.4(ii) hold for $\int_{S_0}^{T_0} \phi(s,\omega)\,dB_s$.

Left to show is that

$$\mathbb{E}\left[\int_{S_0}^{T_0}\left(\int_{S_0}^{t}\phi(s,\omega)\,dB_s\right)^2 dt\right] < \infty.$$

Well,

$$\mathbb{E}\left[\int_{S_0}^{T_0}\left(\int_{S_0}^{t}\phi(s,\omega)\,dB_s\right)^2 dt\right]$$

$$= \int_{S_0}^{T_0} \mathbb{E}\left[\left(\int_{S_0}^{t}\phi(s,\omega)\,dB_s\right)^2\right] dt$$

$$= \int_{S_0}^{T_0} \mathbb{E}\left[\int_{S_0}^{t}\phi(s,\omega)^2\,ds\right] dt$$

$$\leqslant \int_{S_0}^{T_0} \mathbb{E}\left[\int_{S_0}^{T_0}\phi(s,\omega)^2\,ds\right] dt$$

$$= (T_0 - S_0)\,\underbrace{\mathbb{E}\left[\int_{S_0}^{T_0}\phi(s,\omega)^2\,ds\right]}_{<\infty} < \infty.$$

3.7 a (cont.)

Now we have proved that

$$\phi \in \mathcal{V}(0,t) \Longrightarrow \int_0^t \phi(s,\omega)\,dB_s \in \mathcal{V}(0,t),$$

we simply apply the previous statement a maximum of n times to achieve the desired result*.

b

$$1! \int_0^t dB_{u_1} = B_t = t^{1/2} \cdot \frac{B_t}{t^{1/2}} = t^{1/2} h_1\left(\frac{B_t}{\sqrt{t}}\right) \checkmark$$

$$2! \int_0^t \int_0^{u_2} dB_{u_1}\, dB_{u_2} = 2! \int_0^t B_{u_2}\, dB_{u_2}$$

$$= 2!\left[\tfrac{1}{2}B_t^2 - \tfrac{1}{2}t\right]$$

$$= t\left[\left(\frac{B_t}{\sqrt{t}}\right)^2 - 1\right]$$

$$= t^{2/2} h_2\left(\frac{B_t}{\sqrt{t}}\right) \checkmark$$

* so initially, with $\phi \equiv 1$, $\int_{S_0}^t \phi(s,\omega)\,dB_s = \int dB_s = B_t \in \mathcal{V}(S,T)$.
then $\int_{S_0}^t B_s\,dB_s \in \mathcal{V}(S,T)$, and ....

3.7b (cont.)

$$3! \int_0^t \int_0^{u_3} \int_0^{u_2} dB_{u_1}\, dB_{u_2}\, dB_{u_3} = 3\int_0^t B_{u_3}^2 - u_3 \, dB_{u_3}.$$

$$= 3\int_0^t B_s^2 \, dB_s - 3\int_0^t s \, dB_s .$$

$$= B_t^3 - 3\int_0^t B_s \, ds - 3\int_0^t s \, dB_s .$$

$$= B_t^3 - 3t B_t .$$

$$= t^{3/2} h_3\left(\frac{B_t}{\sqrt{t}}\right) .$$

$\underline{c}$ Since $B_t^3 - 3tB_t = 3\int_0^t B_s^3 - s \, dB_s$.

can be expressed as an Itô integral (without drift), it is a martingale.

ØKSENDAL $\boxed{3.8}$

3.8a $s \leq t$

$$\mathbb{E}[M_t \mid \mathcal{F}_s] = \mathbb{E}[\,\mathbb{E}[Y \mid \mathcal{F}_t] \mid \mathcal{F}_s].$$

(TOWER PROPERTY) $\Rightarrow$ $= \mathbb{E}[Y \mid \mathcal{F}_s]$

$= M_s.$

b Since M_t is a martingale, by COROLLARY 3.7,

$\exists\ M_\infty \in L^1(\mathbb{P})$:

$M_t \xrightarrow{a.s} M_\infty$ and $\mathbb{E}[|M_t - M|] \to 0.$

Left to show is that

$$\boxed{M_t = \mathbb{E}[M_\infty \mid \mathcal{F}_t]. \quad \forall t.}$$

Well (following Chapter 14 of 'Probability with Martingales'), for $\Gamma \in \mathcal{F}_t$, $r \geq t$, with

$$\mathbb{E}[M_r; \Gamma] = \mathbb{E}[M_t; \Gamma], \quad \text{since } M_t = \mathbb{E}[M_r \mid \mathcal{F}_t]$$

Now

$$|\mathbb{E}[M_r;\Gamma] - \mathbb{E}[M_\infty;\Gamma]| \leq \mathbb{E}[|M_r - M_\infty|;\Gamma]$$

$$\to 0 \text{ as } r \to \infty.$$

Thus $\mathbb{E}[M_t;\Gamma] - \mathbb{E}[M_\infty;\Gamma] = 0 \quad \forall t.$

$$\Rightarrow \mathbb{E}[M_\infty \mid \mathcal{F}_t] = M_t.$$

3.9

Compute $\displaystyle\int_0^T B_t \circ dB_t$.

$$\int_0^T B_t \circ dB_t = \lim \sum_j B_{t_j^*} \Delta B_j \quad \text{where } t_j^* = \tfrac{1}{2}(t_j + \Delta t_j)$$

$$= \lim \sum (B_{j*} - B_j + B_j)(B_{j+1} - B_j)$$

$$= \lim \sum (B_j^* - B_j)(B_{j+1} - B_j) + \lim \sum B_j (B_{j+1} - B_j).$$

$$= \lim \sum (B_j^* - B_j)\big((B_{j+1} - B_j^*) + (B_j^* - B_j)\big) + \lim \sum B_j (B_{j+1} - B_j).$$

$$= \underbrace{\lim \sum (B_j^* - B_j)(B_{j+1} - B_j^*)}_{=0} + \underbrace{\lim \sum (B_j^* - B_j)^2}_{=\frac{1}{2}t} + \underbrace{\lim \sum B_j (B_{j+1} - B_j)}_{=\int_0^t B_t \, dB_t}.$$

$$= \int_0^t B_t \, dB_t + \tfrac{1}{2}t. = \tfrac{1}{2}B_t^2$$

$$\left[\begin{array}{l} \text{Since} \quad \Delta(B_k^2) = 2B_k \Delta B_k + (\Delta B_k)^2 \\ \Longrightarrow \quad B_t^2 = 2\int_0^t B_t \, dB_t + t. \end{array}\right]$$

3.10

We will show $\mathbb{E}\left[\left| \sum f(t_j,\omega)\Delta B_j - \sum f(t_j',\omega)\Delta B_j \right| \right]$

$\to 0$ as $\Delta t_k \to 0$.

$$\mathbb{E}\left[\left| \sum_j f(t_j,\omega)\Delta B_j - \sum_j f(t_j',\omega)\Delta B_j \right| \right]$$

$$\leqslant \sum_j \mathbb{E}\left[|f(t_j,\omega) - f(t_j',\omega)|\,|\Delta B_j| \right]$$

$$\leqslant \sum_j \sqrt{\mathbb{E}\left[|f(t_j,\omega) - f(t_j',\omega)|^2 \right] \mathbb{E}\left[|\Delta B_j|^2 \right]} \quad \text{(CAUCHY SCHWARZ)}$$

$$\leqslant \sum_j \sqrt{K\,|t_j - t_j'|^{1+\epsilon}\,|\Delta t_j|}.$$

$$\leqslant \sum_j \sqrt{K\,|\Delta t_j|^{2+\epsilon}} = \sqrt{K} \sum_j |\Delta t_j|^{1+\epsilon}.$$

$$\leqslant \underbrace{\max_j |\Delta t_j|^{\epsilon}}_{\to 0 \text{ as } \Delta t_j \to 0.} \cdot \underbrace{\sum_j |\Delta t_j|}_{= t}$$

3.11 We won't use $\mathbb{E}[W_t] = 0 \ \forall\, t$ in our proof (which isn't a surprise, since the result is true for $W_t \iff$ it is true for $W_t + \alpha$, $\alpha \in \mathbb{R}$).

Prove that if $\langle W_t \rangle$ is a stochastic process:

(I) $s \neq t \implies W_s$ & W_t independent.

(II) W_t stationary.

- - - - - -

Then $\langle W_t \rangle$ can't have continuous paths.

(* unless $W_t = 0 \ \forall\, t$).

PROOF

By (I) & (II), the W_t are iid.

It follows that W_t^N are also iid.

Thus $\mathbb{E}[(W_t^N - W_s^N)^2] = 2\mathbb{E}[(W_t^N)^2] + 2\mathbb{E}[(W_t^N)]^2$.

Suppose $\mathbb{P}(W_t = 0) < 1$. Then $\exists\, N \in \mathbb{R} : \mathbb{P}(W_t \geq N) = \alpha > 0$.

Then $\mathbb{E}[(W_t^N)^2] \geq N^2\alpha =: \delta > 0$.

In particular, $\mathbb{E}[(W_t^N - W_s^N)^2] \geq 2\delta > 0$.

Thus $\mathbb{E}\left[\limsup_{s \to t}(W_s^N - W_t^N)^2\right] \geq \limsup_{s \to t} \mathbb{E}[(W_t^N - W_s^N)^2] \geq 2\delta > 0$.

Thus $\mathbb{P}(\lim_{s \to t} W_s^N = W_t^N) < 1$.

3.11 (cont.)

so $\mathbb{P}(W_t^N \text{ continuous at } t) = 1 - \epsilon < 1.$

By independence,

$$\mathbb{P}(W_t^N \text{ continuous at } t_1 \neq \cdots \neq t_k) = (1-\epsilon)^k \longrightarrow 0$$

as $k \to \infty$.

Thus $\mathbb{P}(W_t^N \text{ continuous}) \leq (1-\epsilon)^k \ \forall k.$

$$\boxed{\text{so } \mathbb{P}(W_t^N \text{ continuous}) = 0.}$$

Note that

$$W_t \text{ continuous} \Rightarrow W_t^N \text{ continuous}.$$

Thus $\mathbb{P}(W \text{ continuous}) = 0$.

3.12

i a : $dX_t = \gamma X_t\,dt + \alpha X_t \circ dB_t$

$$dX_t = (\gamma + \tfrac{1}{2}\alpha) X_t\,dt + \alpha X_t\,dB_t .$$

b S: $dX_t = \sin X_t \cos X_t\,dt + (t^2 + \cos X_t) \circ dB_t .$

$$\sigma(t,x) = t^2 + \cos x .$$

$$\frac{\partial\sigma}{\partial x}(t,x) = -\sin x .$$

so I: $dX_t = \left[\sin X_t \cos X_t - \tfrac{1}{2}\sin X_t (t^2 + \cos X_t)\right] dt + (t^2 + \cos X_t)\,dB_t$

$$= \tfrac{1}{2}\sin X_t (\cos X_t - t^2)\,dt + (t^2 + \cos(X_t))\,dB_t$$

ii a I: $dX_t = rX_t\,dt + \underbrace{\alpha X_t}_{\sigma} dB_t .$

S: $= (rX_t - \tfrac{1}{2}\underbrace{\alpha^2 X_t}_{\sigma_x \sigma})\,dt + \alpha X_t \circ dB_t .$

b I: $dX_t = r\cancel{X_t}\,dt + \alpha X_t\,dB_t$

S: $= (2e^{-X_t} - X_t^3)\,dt + X_t^2 \circ dB_t .$

3.13

a

$$\lim_{s\to t} \mathbb{E}\left[(B_s - B_t)^2\right] = \lim_{s\to t} (t-s) = 0 .$$

b

$$\lim_{s\to t} \mathbb{E}\left[(Y_t - Y_s)^2\right] = \lim_{s\to t} \mathbb{E}\left[(f(B_t) - f(B_s))^2\right]$$

$$\leq \lim_{s\to t} \mathbb{E}\left[c^2 |B_t - B_s|^2\right] .$$

$$= c^2 \underbrace{\lim_{s\to t} \mathbb{E}\left[(B_t - B_s)^2\right]}_{=0} = 0 .$$

c

$$\mathbb{E}\left[\left(\int_S^T X_t \, dB_t - \int_S^T \phi_n(t,\omega) \, dB_t\right)^2\right] .$$

$$= \mathbb{E}\left[\int_S^T (X_t - \phi_n(t))^2 dt\right] \qquad (\underline{\text{ITÔ ISOMETRY}})$$

$$= \sum_j \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} \mathbb{E}\left[(X_t - X_{t_j^{(n)}})^2\right] dt .$$

Now the map $\Phi : t \to L^2(\mathbb{P})$ is continuous, thus uniformly continuous on $[S,T]$. Thus given $\epsilon > 0$, $\exists\ \delta :\ \forall\ t \in [S,T]$,

$|s-t| < \delta \Rightarrow$

So provided $|t_{j+1}^{(n)} - t_j^{(n)}| < \delta\ \forall j$,

$$\leq \sum_j \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} \epsilon \, dt = \epsilon (T-S) .$$

3.14

a We may assume h is bounded, since the bounded functions $h^{(n)}(\omega) = h(\omega)\mathbb{1}\{|h(\omega)| \le n\}$ tend to h pointwise.

b claim.

$$\mathcal{F}_t = \sigma\left(\bigcup_n \mathcal{H}_n\right).$$

proof.

$\mathcal{F}_t \supseteq \sigma\left(\bigcup_n \mathcal{H}_n\right)$ Trivial, since $\mathcal{H}_n \subseteq \mathcal{F}_t \;\forall t$.

$\mathcal{F}_t \subseteq \sigma\left(\bigcup_n \mathcal{H}_n\right)$ We show $\forall s \le t$, $B_s \in m\sigma\left(\bigcup_n \mathcal{H}_n\right)$.

Well, $\exists q_k$ of the form $j/2^n$ such that $q_k \to s$.

By continuity of B_t, $B_{q_k} \to B_s$.

$(\phi_n \in m\mathcal{G} \Rightarrow \limsup \phi_n \in m\mathcal{G})$: Thus $B_s = \limsup_k B_{q_k} \in m\sigma\left(\bigcup_n \mathcal{H}_n\right)$.

Now $\mathcal{F}_t = \sigma(B_s : s \le t)$

Thus $\mathcal{F}_t \subseteq \sigma\left(\bigcup_n \mathcal{H}_n\right)$. $\square$

Now we may apply COROLLARY C.9, so

$$h = \mathbb{E}[h \mid \mathcal{F}_t] = \lim_{n\to\infty} \mathbb{E}[h \mid \mathcal{H}_n].$$

c By DOOB-DYNKIN (LEMMA 2.1.2),

$$\mathbb{E}[h \mid \mathcal{H}_n] = g\left(B_{t_j^{(n)}} : t_j^{(n)} \le t\right).$$

3.14 (cont.)

Now there is a remarkable theorem stating that if $G : \mathbb{R}^k \to \mathbb{R}$ is measurable, then $\exists$ continuous $F_n : \mathbb{R}^k \to \mathbb{R}$ such that

$$\mathbb{E}\left[|F_n - G|\right] \to 0 \quad \text{as } n \to \infty.$$

(See RUDIN'S ("REAL AND COMPLEX ANALYSIS" CHAPTER 3)

Finally, another remarkable theorem states that if $F : \mathbb{R}^k \to \mathbb{R}$ is continuous, then $\exists$ polynomials $E_n : \mathbb{R}^k \to \mathbb{R}$ such that

$$\mathbb{E}\left[|E_n - F|\right] \to 0.$$

(This is called STONE-WEIERSTRASS THEOREM. SEE WILLIAMS "PROBABILITY WITH MARTINGALES" #7.4)

So to recap, we said:

Measurable

$\Uparrow$

Bounded + Measurable

$\Uparrow$

Defined on finitely many B_t.

$\Uparrow$

Defined on finitely many B_t, continuous.

$\Uparrow$

Polynomial in some $B_{t_1} \cdots B_{t_k}$.

ØKSENDAL $\boxed{3.14}$: (3)

However, I don't see why this approximation necessarily needs to be transitive...?

3.15 $$\mathbb{E}[(C-D)] = \mathbb{E}\left[\int_S^T (f(t,\omega)-g(t,\omega))\,dB_t\right] = 0.$$

$$\mathbb{E}[(C-D)^2] = \mathbb{E}\left[\left(\int_S^T f(t,\omega)-g(t,\omega)\,dB_t\right)^2\right].$$

ITÔ ISOMETRY $\Longrightarrow$ $$= \mathbb{E}\left[\int_S^T (f(t,\omega)-g(t,\omega))^2\,dt\right]. \quad (*)$$

Now C, D are deterministic, thus.

$$C-D = \mathbb{E}[C-D] = 0.$$

This implies $(*) = 0$ too.

Thus $f(t,\omega)-g(t,\omega) = 0$ a.e $[S,T]\times\Omega$.

3.16 By conditional Jensen's inequality, $\forall$ convex $c:\mathbb{R}\to\mathbb{R}$,

$$c(\mathbb{E}[X|\mathcal{H}]) \leq \mathbb{E}[c(X)|\mathcal{H}]$$

With $c(u) = u^2$:

$$\mathbb{E}[X|\mathcal{H}]^2 \leq \mathbb{E}[X^2|\mathcal{H}].$$

Taking expectations,

$$\mathbb{E}[\mathbb{E}[X|\mathcal{H}]^2] \leq \mathbb{E}[\mathbb{E}[X^2|\mathcal{H}]] = \mathbb{E}[X^2]$$

TOWER PROPERTY.

3.17 a

- $\mathbb{E}[X \mid \mathcal{G}](\omega)$ is $\mathcal{G}$-measurable.
- By $\boxed{2.7c}$, $\mathbb{E}[X \mid \mathcal{G}] = \sum_{k=1}^{m} c_k \mathbb{1}\{\omega \in G_k\}$.

b Show $\mathbb{E}[X \mid \mathcal{G}] = \sum_{k=1}^{m} \frac{\mathbb{E}[X \mathbb{1}_{G_k}]}{\mathbb{P}(G_k)} \mathbb{1}\{\omega \in G_k\}$.

- Well need to show $\forall\, \Gamma \in m\mathcal{G}$, $\mathbb{E}[X\Gamma] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{G}]\Gamma]$.
- Now $\Gamma \in m\mathcal{G} \Rightarrow \Gamma = \sum_{j=1}^{m} a_j \mathbb{1}\{\omega \in G_j\}$.
- So $\mathbb{E}\left[\Gamma \cdot \sum_{k=1}^{m} \frac{\mathbb{E}[X \mathbb{1}_{G_k}]}{\mathbb{P}(G_k)} \mathbb{1}\{\omega \in G_k\}\right]$

$$= \mathbb{E}\left[\sum_{k=1}^{m}\sum_{j=1}^{m} \frac{\mathbb{E}[X \mathbb{1}_{G_k}]}{\mathbb{P}[G_k]} a_j \mathbb{1}\{\omega \in G_k\} \mathbb{1}\{\omega \in G_j\}\right]$$

$$= \mathbb{E}\left[\sum_{k=1}^{m} \frac{\mathbb{E}[X \mathbb{1}_{G_k}] a_k}{\mathbb{P}[G_k]} \mathbb{1}\{\omega \in G_k\}\right]$$

$$= \sum_{k=1}^{m} \frac{\mathbb{E}[X \mathbb{1}_{G_k}] a_k}{\mathbb{P}[G_k]} \mathbb{E}[\mathbb{1}\{\omega \in G_k\}]$$

$$= \sum_{k=1}^{m} \mathbb{E}[X \mathbb{1}_{G_k} a_k]$$

$$= \mathbb{E}\left[\sum_{k=1}^{m} X \mathbb{1}_{G_k} a_k\right]$$

$$= \mathbb{E}\left[X \sum_{k=1}^{m} a_k \mathbb{1}_{G_k}\right]$$

$$= \mathbb{E}[X\Gamma]. \quad \checkmark$$

3.17 c

By 3.17 b,

$$\mathbb{E}[X \mid \mathcal{G}](\omega) = \frac{\mathbb{E}[X \mathbb{1}_{G_i}]}{\mathbb{P}(G_i)} \quad \text{when } \omega \in G_i .$$

If $X = \sum_{k=1}^{m} a_k \mathbb{1}\{X = a_k\}$.

Then
$$\frac{\mathbb{E}[X \mathbb{1}_{G_i}]}{\mathbb{P}(G_i)} = \frac{1}{\mathbb{P}(G_i)} \mathbb{E}\left[\sum_{k=1}^{m} a_k \mathbb{1}\{X = a_k\} \mathbb{1}\{G_i\}\right]$$

$$= \sum_{k=1}^{m} a_k \frac{\mathbb{E}[\mathbb{1}\{X = a_k\} \mathbb{1}\{G_i\}]}{\mathbb{P}(G_i)}.$$

$$= \sum_{k=1}^{m} a_k \mathbb{P}(X = a_k \mid G_i).$$

ØKSENDAL $\boxed{3.18}$: (1)

Let $s > t$

$\underline{\underline{3.18}}$ $$\mathbb{E}[M_s \mid \mathcal{F}_t] = \mathbb{E}\left[e^{\sigma B_s - \frac{1}{2}\sigma^2 s} \mid \mathcal{F}_t\right]$$

$$= \mathbb{E}\left[e^{\sigma(B_s - B_t)}\right] e^{-\frac{1}{2}\sigma^2 s} e^{\sigma B_t}$$

By $\boxed{2.20}$

$$= e^{\frac{1}{2}\sigma^2(s-t)} e^{-\frac{1}{2}\sigma^2 s} e^{\sigma B_t}$$

$$= e^{-\frac{1}{2}\sigma^2 t} e^{\sigma B_t}$$