Therefore, from this point of view it seems reasonable to use (3.3.6) (i.e. the Stratonovich interpretation) – and not the Itô interpretation

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s \tag{3.3.7}$$

as the model for the original white noise equation (3.3.2).

On the other hand, the specific feature of the Itô model of "not looking into the future" (as explained after Example 3.1.1) seems to be a reason for choosing the Itô interpretation in many cases, for example in biology (see the discussion in Turelli (1977)). The difference between the two interpretations is illustrated in Example 5.1.1. Note that (3.3.6) and (3.3.7) coincide if $\sigma(t,x)$ does not depend on x. For example, this is the situation in the linear case handled in the filtering problem in Chapter 6.

In any case, because of the explicit connection (3.3.6) between the two models (and a similar connection in higher dimensions – see (6.1.3)), it will for many purposes suffice to do the general mathematical treatment for one of the two types of integrals. In general one can say that the Stratonovich integral has the advantage of leading to ordinary chain rule formulas under a transformation (change of variable), i.e. there are no second order terms in the Stratonovich analogue of the Itô transformation formula (see Theorems 4.1.2 and 4.2.1). This property makes the Stratonovich integral natural to use for example in connection with stochastic differential equations on manifolds (see Elworthy (1982) or Ikeda and Watanabe (1989)).

However, Stratonovich integrals are not martingales, as we have seen that Itô integrals are. This gives the Itô integral an important computational advantage, even though it does not behave so nicely under transformations (as Example 3.1.9 shows). For our purposes the Itô integral will be most convenient, so we will base our discussion on that from now on.

Exercises

Unless otherwise stated B_t denotes Brownian motion in $\mathbf{R}$, $B_0 = 0$.

3.1. Prove directly from the definition of Itô integrals (Definition 3.1.6) that

$$\int_0^t s dB_s = tB_t - \int_0^t B_s ds \, .$$

(Hint: Note that

$$\sum_j \Delta(s_j B_j) = \sum_j s_j \Delta B_j + \sum_j B_{j+1} \Delta s_j \text{ .)}$$

3.2. Prove directly from the definition of Itô integrals that

$$\int_0^t B_s^2 dB_s = \tfrac{1}{3}B_t^3 - \int_0^t B_s ds \ .$$

3.3. If $X_t\colon \Omega \to \mathbf{R}^n$ is a stochastic process, let $\mathcal{H}_t = \mathcal{H}_t^{(X)}$ denote the σ-algebra generated by $\{X_s(\cdot);\ s \le t\}$ (i.e. $\{\mathcal{H}_t^{(X)}\}_{t\ge 0}$ is the *filtration of the process* $\{X_t\}_{t\ge 0}$).

a) Show that if X_t is a martingale w.r.t. *some* filtration $\{\mathcal{N}_t\}_{t\ge 0}$, then X_t is also a martingale w.r.t. its own filtration $\{\mathcal{H}_t^{(X)}\}_{t\ge 0}$.

b) Show that if X_t is a martingale w.r.t $\mathcal{H}_t^{(X)}$, then

$$E[X_t] = E[X_0] \qquad \text{for all } \ t \ge 0 \ . \tag{$*$}$$

c) Give an example of a stochastic process X_t satisfying $(*)$ and which is *not* a martingale w.r.t. its own filtration.

3.4. Check whether the following processes X_t are martingales w.r.t. $\{\mathcal{F}_t\}$:
(i) $X_t = B_t + 4t$
(ii) $X_t = B_t^2$
(iii) $X_t = t^2 B_t - 2\int_0^t sB_s ds$
(iv) $X_t = B_1(t)B_2(t)$, where $(B_1(t), B_2(t))$ is 2-dimensional Brownian motion.

3.5. Prove directly (without using Example 3.1.9) that

$$M_t = B_t^2 - t$$

is an $\mathcal{F}_t$-martingale.

3.6. Prove that $N_t = B_t^3 - 3tB_t$ is a martingale.

3.7. A famous result of Itô (1951) gives the following formula for n times *iterated Itô integrals*:

$$n! \int\limits_{0\le u_1 \le \cdots \le u_n \le t} \cdots \Big(\int \Big(\int dB_{u_1}\Big) dB_{u_2}\Big) \cdots dB_{u_n} = t^{\frac{n}{2}} h_n\left(\frac{B_t}{\sqrt{t}}\right) \tag{3.3.8}$$

where h_n is the *Hermite polynomial* of degree n, defined by

$$h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}\big(e^{-\frac{x^2}{2}}\big) \ ; \qquad n = 0, 1, 2, \ldots$$

(Thus $h_0(x) = 1$, $h_1(x) = x$, $h_2(x) = x^2 - 1$, $h_3(x) = x^3 - 3x$.)

a) Verify that in each of these n Itô integrals the integrand satisfies the requirements in Definition 3.1.4.

b) Verify formula (3.3.8) for $n = 1, 2, 3$ by combining Example 3.1.9 and Exercise 3.2.

c) Use b) to give a new proof of the statement in Exercise 3.6.

3.8. a) Let Y be a real valued random variable on $(\Omega, \mathcal{F}, P)$ such that

$$E[|Y|] < \infty \ .$$

Define

$$M_t = E[Y|\mathcal{F}_t] \ ; \qquad t \geq 0 \ .$$

Show that M_t is an $\mathcal{F}_t$-martingale.

b) Conversely, let M_t; $t \geq 0$ be a real valued $\mathcal{F}_t$-martingale such that

$$\sup_{t \geq 0} E[|M_t|^p] < \infty \qquad \text{for some } \ p > 1 \ .$$

Show that there exists $Y \in L^1(P)$ such that

$$M_t = E[Y|\mathcal{F}_t] \ .$$

(Hint: Use Corollary C.7.)

3.9. Suppose $f \in \mathcal{V}(0, T)$ and that $t \to f(t, \omega)$ is continuous for a.a. ω. Then we have shown that

$$\int_0^T f(t, \omega) dB_t(\omega) = \lim_{\Delta t_j \to 0} \sum_j f(t_j, \omega) \Delta B_j \qquad \text{in } \ L^2(P) \ .$$

Similarly we define the *Stratonovich integral* of f by

$$\int_0^T f(t, \omega) \circ dB_t(\omega) = \lim_{\Delta t_j \to 0} \sum_j f(t_j^*, \omega) \Delta B_j \ , \quad \text{where } \ t_j^* = \tfrac{1}{2}(t_j + t_{j+1}) \ ,$$

whenever the limit exists in $L^2(P)$. In general these integrals are different. For example, compute

$$\int_0^T B_t \circ dB_t$$

and compare with Example 3.1.9.

3.10. If the function f in Exercise 3.9 varies “smoothly” with t then in fact the Itô and Stratonovich integrals of f coincide. More precisely, assume that there exists $K < \infty$ and $\epsilon > 0$ such that

$$E[|f(s, \cdot) - f(t, \cdot)|^2] \leq K|s - t|^{1+\epsilon} \ ; \qquad 0 \leq s \ , \ \ t \leq T \ .$$

Prove that then we have

$$\int_0^T f(t,\omega)dB_t = \lim_{\Delta t_j \to 0} \sum_j f(t'_j,\omega)\Delta B_j \qquad (\text{limit in } L^1(P))$$

for any choice of $t'_j \in [t_j, t_{j+1}]$. In particular,

$$\int_0^T f(t,\omega)dB_t = \int_0^T f(t,\omega) \circ dB_t \,.$$

(Hint: Consider $E\big[|\sum_j f(t_j,\omega)\Delta B_j - \sum_j f(t'_j,\omega)\Delta B_j|\big]$.)

3.11. Let W_t be a stochastic process satisfying (i), (ii) and (iii) (below (3.1.2)). Prove that W_t cannot have continuous paths. (Hint: Consider $E[(W_t^{(N)} - W_s^{(N)})^2]$, where

$$W_t^{(N)} = (-N) \vee (N \wedge W_t),\ N = 1,2,3,\ldots) \,.$$

3.12. As in Exercise 3.9 we let $\circ dB_t$ denote Stratonovich differentials.

(i) Use (3.3.6) to transform the following Stratonovich differential equations into Itô differential equations:
 (a) $dX_t = \gamma X_t dt + \alpha X_t \circ dB_t$
 (b) $dX_t = \sin X_t \cos X_t dt + (t^2 + \cos X_t) \circ dB_t$

(ii) Transform the following Itô differential equations into Stratonovich differential equations:
 (a) $dX_t = rX_t dt + \alpha X_t dB_t$
 (b) $dX_t = 2e^{-X_t}dt + X_t^2 dB_t$

3.13. A stochastic process $X_t(\cdot)\colon \Omega \to \mathbf{R}$ is *continuous in mean square* if $E[X_t^2] < \infty$ for all t and

$$\lim_{s\to t} E[(X_s - X_t)^2] = 0 \qquad \text{for all } t \geq 0 \,.$$

a) Prove that Brownian motion B_t is continuous in mean square.

b) Let $f\colon \mathbf{R} \to \mathbf{R}$ be a Lipschitz continuous function, i.e. there exists $C < \infty$ such that

$$|f(x) - f(y)| \leq C|x-y| \qquad \text{for all } x,y \in \mathbf{R} \,.$$

Prove that

$$Y_t := f(B_t)$$

is continuous in mean square.

c) Let X_t be a stochastic process which is continuous in mean square and assume that $X_t \in \mathcal{V}(S,T)$, $T < \infty$. Show that

$$\int_S^T X_t dB_t = \lim_{n\to\infty} \int_S^T \phi_n(t,\omega) dB_t(\omega) \qquad (\text{limit in } L^2(P))$$

where

$$\phi_n(t,\omega) = \sum_j X_{t_j^{(n)}}(\omega) \mathcal{X}_{[t_j^{(n)}, t_{j+1}^{(n)})}(t)\ , \qquad T < \infty\ .$$

(Hint: Consider

$$E\Big[\int_S^T (X_t - \phi_n(t))^2 dt\Big] = E\Big[\sum_j \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} (X_t - X_{t_j^{(n)}})^2 dt\Big])\ .$$

3.14. Show that a function $h(\omega)$ is $\mathcal{F}_t$-measurable if and only if h is a pointwise limit (for a.a. ω) of sums of functions of the form

$$g_1(B_{t_1}) \cdot g_2(B_{t_2}) \cdots g_k(B_{t_k})$$

where $g_1, \dots, g_k$ are bounded continuous functions and $t_j \le t$ for $j \le k$, $k = 1, 2, \dots$

Hint: Complete the following steps:

a) We may assume that h is bounded.

b) For $n = 1, 2, \dots$ and $j = 1, 2, \dots$ put $t_j = t_j^{(n)} = j \cdot 2^{-n}$. For fixed n let $\mathcal{H}_n$ be the σ-algebra generated by $\{B_{t_j}(\cdot)\}_{t_j \le t}$. Then by Corollary C.9

$$h = E[h|\mathcal{F}_t] = \lim_{n\to\infty} E[h|\mathcal{H}_n] \qquad (\text{pointwise a.e. limit})$$

c) Define $h_n := E[h|\mathcal{H}_n]$. Then by the Doob-Dynkin lemma (Lemma 2.1.2) we have

$$h_n(\omega) = G_n(B_{t_1}(\omega), \dots, B_{t_k}(\omega))$$

for some Borel function $G_n: \mathbf{R}^k \to \mathbf{R}$, where $k = \max\{j; j \cdot 2^{-n} \le t\}$. Now use that any Borel function $G: \mathbf{R}^k \to \mathbf{R}$ can be approximated pointwise a.e. by a continuous function $F: \mathbf{R}^k \to \mathbf{R}$ and complete the proof by applying the Stone-Weierstrass theorem.

3.15. Suppose $f, g \in \mathcal{V}(S,T)$ and that there exist constants C, D such that

$$C + \int_S^T f(t,\omega) dB_t(\omega) = D + \int_S^T g(t,\omega) dB_t(\omega) \qquad \text{for a.a. } \omega \in \Omega\ .$$

Show that

$$C = D$$

and

$$f(t,\omega) = g(t,\omega) \qquad \text{for a.a. } (t,\omega) \in [S,T] \times \Omega\ .$$

3.16. Let $X\colon \Omega \to \mathbf{R}$ be a random variable such that $E[X^2] < \infty$ and let $\mathcal{H} \subset \mathcal{F}$ be a σ-algebra. Show that

$$E\big[(E[X|\mathcal{H}])^2\big] \le E[X^2]\ .$$

(See Lemma 6.1.1. See also the Jensen inequality for conditional expectation (Appendix B).)

3.17. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X\colon \Omega \to \mathbf{R}$ be a random variable with $E[|X|] < \infty$. If $\mathcal{G} \subset \mathcal{F}$ is a *finite* σ-algebra, then by Exercise 2.7 there exists a partition $\Omega = \bigcup\limits_{i=1}^{n} G_i$ such that $\mathcal{G}$ consists of $\emptyset$ and unions of some (or all) of $G_1, \ldots, G_n$.

a) Explain why $E[X|\mathcal{G}](\omega)$ is constant on each G_i. (See Exercise 2.7 c).)

b) Assume that $P[G_i] > 0$. Show that

$$E[X|\mathcal{G}](\omega) = \frac{\int_{G_i} X dP}{P(G_i)} \qquad \text{for } \omega \in G_i\ .$$

c) Suppose X assumes only finitely many values $a_1, \ldots, a_m$. Then from elementary probability theory we know that (see Exercise 2.1)

$$E[X|G_i] = \sum_{k=1}^{m} a_k P[X = a_k | G_i]\ .$$

Compare with b) and verify that

$$E[X|G_i] = E[X|\mathcal{G}](\omega) \qquad \text{for } \omega \in G_i\ .$$

Thus we may regard the conditional expectation as defined in Appendix B as a (substantial) generalization of the conditional expectation in elementary probability theory.