Construct a probability space with random variables $X_1, X_2, \ldots$ that are independent and the CDF of X_n is a given F_n.

$$(\mathbb{R}^{\mathbb{N}}, \mathcal{F}, ?)$$

$\uparrow$ σ-field generated by the cylinder sets

$A \in \mathcal{B}_n \qquad A \times \mathbb{R} \times \mathbb{R} \times \ldots$

For each $n \geq 1$ set $\mu_n = Q_{X_1} \times Q_{X_2} \times \ldots \times Q_{X_n}$

Q_{X_k}: distribution of X_k (identified by F_n)

μ_n "lives" on $(\mathbb{R}^n, \mathcal{B}_n)$

$\mathcal{B}_n$: n-dimensional Borel sets

The sequence $\{\mu_n\}_{n \geq 1}$ is "consistent":

For any $A \in \mathcal{B}_n$

$$\mu_n(A) = \mu_{n+1}(A \times \mathbb{R})$$

Theorem (Kolmogorov's extension thm)
Suppose that μ_n is a prob measure on $(\mathbb{R}^n, \mathcal{B}_n)$ for each $n \geq 1$, and the sequence $\{\mu_n\}_{n\geq 1}$ is consistent then there is a unique extension $\mathbb{P}$ on $(\mathbb{R}^{\mathbb{N}}, \mathcal{F})$ so that

σ-field generated by the cylinder sets

for $A \in \mathcal{B}_n$ we have

$$\mathbb{P}(A \times \mathbb{R} \times \mathbb{R} \times \dots) = \mu_n(A).$$

In HW2: explicit construction of iid (independent and identically distributed) $X_1, X_2, \dots$ with Bernoulli$(\frac{1}{2})$ distribution.

$X_n : [0,1] \to \{0,1\}$
n^{th} binary digit of $\omega \in [0,1)$

Follow-up questions: How about Bernoulli(p)?
How about iid random variables with a given CDF?

Inclusion-exclusion principle

$$\boxed{P(A_1 \cup A_2 \ldots \cup A_n) = \sum_i P(A_i) - \sum_{i_1<i_2} P(A_{i_1}A_{i_2}) + - \ldots - (-1)^{j+1} \sum_{i_1<\ldots<i_j} P(A_{i_1}\ldots A_{i_j}) + \ldots}$$

$$P(B) = E[1_B]$$

$$P(B_1B_2) = E[1_{B_1B_2}] = E[1_{B_1} \cdot 1_{B_2}]$$

$$\underbrace{P(A_1 \cup \ldots \cup A_n)} = 1 - P\left(\left(\bigcup_{j=1}^{n} A_j\right)^c\right)$$

$$\underbrace{P\left(\left(\bigcup_{j=1}^{n} A_j\right)^c\right)} = P\left(\bigcap_{j=1}^{n} A_j^c\right) = E\left[1_{\bigcap_{j=1}^{n} A_j^c}\right] =$$

$$= E\left[\prod_{j=1}^{n} 1_{A_j^c}\right] = E\left[\prod_{j=1}^{n} (1 - 1_{A_j})\right]$$

$$= E\underbrace{\left[1 - \sum_{j=1}^{n} 1_{A_j} + \sum_{i_1<i_2} 1_{A_{i_1}A_{i_2}} \ldots\right]}$$

$$= 1 - \sum_{i=1}^{n} P(A_i) + \sum_{i_1<i_2} P(A_{i_1}A_{i_2}) - + \ldots$$

X: # of fixed points in a randomly (uniformly) chosen permutation of $\{1, \dots, n\}$.

$E[X] = ?$

$$X = \sum_{j=1}^{n} I_{A_j}$$ A_j: j is a fixed point of the permutation

$$E[X] = \sum_{j=1}^{n} P(A_j)$$ $P(A_j) = P(A_1) = \frac{1}{n}$

$$= n \cdot \frac{1}{n} = 1$$

Higher order moment?

k^{th} Factorial moment: $E[X(X-1)\dots(X-k+1)]$

Sum of independent random variables

Q: X, Y are independent. How can we describe the distribution of $X+Y$ in terms of Q_X, Q_Y?

Thm: Suppose that X, Y are independent with CDFs F_X and F_Y.

Then the CDF of $X+Y$ is given as

$$F_{X+Y}(z) = P(X+Y \leq z) = \int_{-\infty}^{\infty} F_X(z-y)\, d\underbrace{F_Y(y)}$$

notation for this

$$= \int_{-\infty}^{\infty} F_X(z-y)\, dQ_Y(y)$$

Notation: $\underbrace{F_X * F_Y}$ convolution

Proof: $z \in \mathbb{R}$

$$P(X+Y \leq z) = E\left[I(X+Y \leq z)\right]$$

$$= \iint I(x+y \leq z)\, d(Q_X \times Q_Y)$$

Fubini

$$= \int \underbrace{\left(\int I(x+y \leq z)\, dQ_X\right)} dQ_Y$$

$$= \int P(X+y \leq z)\, dQ_Y$$

$$= \int \underbrace{P(X \leq z-y)}\, dQ_Y = \int F_X(z-y)\, dQ_Y$$

Corollaries

1) If X has PDF f_X then $X+Y$ has a PDF given by

$$\underbrace{f_{X+Y}(z) = \int f_X(z-y)\, dQ_Y}$$

If Y also has a PDF (f_Y) then $f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy$

$f_X * f_Y$

2) If X and Y are both discrete then $X+Y$ is discrete

$$P(X+Y=a) = \sum_{\xi} P(X=\xi)\, P(Y=a-\xi)$$

convolution of PMFs.

Named distributions

- Uniform on $[a,b]$
- Bernoulli (p) $\quad p \in [0,1]$

— Binomial (n, p) "# of heads among n independent biased coin flips"

possible values: $0, 1, \ldots, n$

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k} \quad k=0,\ldots,n$$

— Geometric (p) "# of coin flips needed to see the first H"

possible values: $1, 2, 3, \ldots$ $\quad 0 \leq p \leq 1$

$$P(X=k) = (1-p)^{k-1} \cdot p$$

$$P(X=\infty) = \begin{cases} 0 & p>0 \\ 1 & p=0 \end{cases}$$

— Negative binomial (k, p) "# of coin flips needed to see the k^{th} H"

$$P(X=n) = ?$$