# An explicit calculation

$$\underbrace{\int_0^t B_s\,dB_s} \quad \left(= \frac{1}{2} B_t^2 - \frac{1}{2} t\right)$$

$$\mathrm{Var}\left\{\int_0^t B_s\,dB_s\right\} = \mathbb{E}\left[\left(\int_0^t B\,dB\right)^2\right]$$

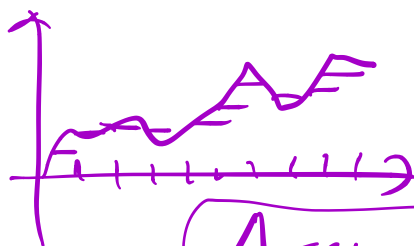
$$= \mathbb{E} \int_0^t B_s^2\,ds = \int_0^t s\,ds = \frac{1}{2} t^2$$

$$\mathrm{Var}\left(\frac{1}{2} B_t^2 - \frac{1}{2} t\right) = \frac{1}{4} \mathbb{E}\left[(B_t^2 - t)^2\right]$$

$$= \frac{1}{4}\left[\underbrace{\mathbb{E} B_t^4}_{3t^2} - 2t\,\underbrace{\mathbb{E}(B_t^2)}_{t} + t^2\right] = \frac{1}{2} t^2$$

$B \in \mathcal{H}$

$$\mathcal{H}_0^2 \ni f_n \overset{?}{\longrightarrow} B$$



Assume $B_0 = 0$

$$f_n(t,\omega) = \sum_i B_{t_i}\, 1_{(t_i, t_{i+1}]}(t)$$

$$\int_0^t f_n\,dB = \sum_i \underbrace{B_{t_i}\left(B_{t_{i+1}} - B_{t_i}\right)}$$

$$\frac{1}{2}\left(B_{t_{i+1}}^2 - B_{t_i}^2\right) - \frac{1}{2}\left(B_{t_{i+1}} - B_{t_i}\right)^2$$

$$= \frac{1}{2} B_t^2 - \frac{1}{2} \underbrace{\sum_i \left(B_{t_{i+1}} - B_{t_i}\right)^2}$$

$$\downarrow \qquad \qquad$$

$$t \quad \stackrel{def}{=} Y_n$$

$$\boxed{Y_n \xrightarrow{n\to\infty} t}$$

$$E\, Y_n = \sum_i (t_{i+1} - t_i) = t$$

$$\cancel{E(Y_n^2) = E\sum_{i,j} (B_{t_{i+1}} - B_{t_i})^2 (B_{t_{j+1}} - B_{t_j})^2}$$

$$\cancel{\text{if } i=j \quad 3(t_{i+1}-t_i)^2}$$

$$Var(Y_n) = \underbrace{\sum_i}_{n} \underbrace{Var\left((B_{t_{i+1}} - B_{t_i})^2\right)}_{\sim (t_{i+1}-t_i)^2 \sim \frac{1}{n^2}} \sim \frac{1}{n} \xrightarrow{n\to\infty} 0$$

$$\frac{1}{2} B_t^2 - \frac{1}{2} t.$$

$$\int_0^t \overbrace{B\, dB}^{\frac{1}{2} d(B^2)} = \frac{1}{2} B_t^2$$

in classical analysis