Helly's selection thm:

If $F_1, F_2, \ldots$ are CDFs then there is a subsequence n_k and a nondecreasing, right-cont function F so that $F_{n_k}(x) \to F(x)$ as $k \to \infty$ at all continuity points of F.

$F_{n_k} \Rightarrow_v F$ vague convergence

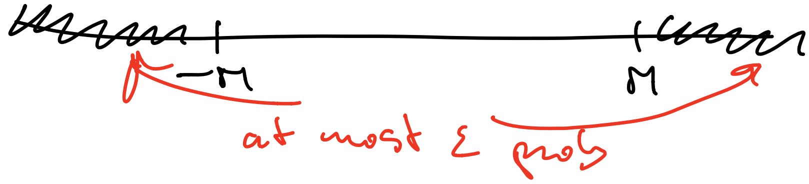
Def: We say that $F_1, F_2, \ldots$ is tight if for any $\varepsilon > 0$ we can find $M > 0$ so that
$$P(|X_n| > M) < \varepsilon$$

OR $$F_n(-M) + 1 - F_n(M) < \varepsilon$$

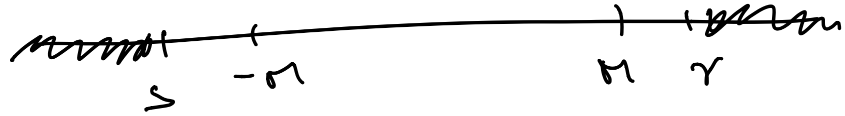
Thm: the sequence $F_1, F_2, \ldots$ is tight if and only if any subsequential vague limit is limit in distribution.

Proof: $\Longrightarrow$ suppose that $F_1, F_2, \ldots$ is tight. Suppose that F is a vague limit of a subsequence. Because $F_1, F_2, \ldots$ is tight, for $\varepsilon > 0$ we can find M so that

$$F_n(-M) + 1 - F_n(M) < \varepsilon$$



Choose $s < -M$, $r > M$ so that s and r are continuity points of F.
At these values F_{n_k} will converge to F and hence $F(s) + 1 - F(r) \le \varepsilon$.



$\rightsquigarrow$ $0 \le F(s) \le \varepsilon \qquad 1-\varepsilon \le F(r) \le 1$

We can do that for any $\varepsilon > 0$ $\rightsquigarrow$

$$\lim_{x \to -\infty} F(x) = 0 \qquad \lim_{x \to \infty} F(x) = 1 .$$

$\Leftarrow$ Try to do it on your own!

How can we check tightness?

Moment bounds + Markov

E.g. If $E|X_n| < C$ for all $n \ge 1$

$$P(|X_n| > M) \leq \frac{c}{M} \rightsquigarrow X_1, X_2, \ldots \text{ tight}$$

Convergence in distribution can be characterised using the following metrics

Levy metric:

$$\rho(F, G) = \inf\{\varepsilon > 0 : F(x-\varepsilon) - \varepsilon \leq G(x) \leq F(x+\varepsilon) + \varepsilon \text{ for all } x \in \mathbb{R}\}$$

Ky-Fan metric

$$d(F, G) = \inf\{\varepsilon > 0 : P(|X-Y| > \varepsilon) \leq \varepsilon \text{ with } X \sim F,\ Y \sim G\}$$

Characteristic Function

Def: The characteristic function of a r.v. X is the function $\varphi_X(t) = E[e^{itX}] =$ $t \in \mathbb{R}$

$$= E[\cos(t \cdot X)] + i\, E[\sin(t \cdot X)].$$

Claim: this is always well-defined.

$$\varphi_X(t) = \int_{\mathbb{R}} e^{itx}\, dQ_X(x)$$

Fourier transform of Q_X

$$M_X(t) = E\left[e^{t\cdot X}\right]$$ could be ∞

exponential moment generating function of X

Thm: φ_X identifies Q_X

$$\lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\, \varphi_X(t)\, dt =$$

$$= P\left(X\in(a,b)\right) + \frac{1}{2} P(X=a) + \frac{1}{2} P(X=b)$$

$$= \frac{1}{2}\left(P\left(X\in(a,b)\right) + P\left(X\in[a,b]\right)\right)$$

Proof: $$\left| \frac{e^{-ita} - e^{-itb}}{it}\right| = \left| \int_a^b e^{-ity}\, dy\right| \leq b-a$$

$$\frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \int_{\mathbb{R}} e^{itx}\, Q_X(dx)\, dt \overset{\text{Fubini}}{=}$$

$$\frac{1}{2\pi}\int_{\mathbb{R}}\int_{-T}^{T}\frac{e^{it(x-a)}-e^{it(x-b)}}{it}\,dt\;Q_x(dx)$$

$$=\frac{1}{2\pi}\int_{\mathbb{R}}\left(\int_{-T}^{T}\frac{\sin(t(x-a))}{t}\,dt-\int_{-T}^{T}\frac{\sin(t(x-b))}{t}\,dt\right)Q_x(dx)$$

$$R(\vartheta,t)=\int_{-T}^{T}\frac{\sin(\vartheta t)}{t}\,dt=2\int_{0}^{T\vartheta}\frac{\sin(x)}{x}\,dx$$

$$=2\cdot\operatorname{sgn}(\vartheta)\int_{0}^{T|\vartheta|}\frac{\sin x}{x}\,dx$$

As $T\to\infty$ $\displaystyle\int_0^T\frac{\sin x}{x}\,dx\to\frac{\pi}{2}$

(with complex analysis : $\operatorname{Im}\dfrac{e^{iz}-1}{z}$)

$$R(\vartheta,t)\longrightarrow\pi\operatorname{sgn}(\vartheta)$$

$$\int_{-T}^{T}\frac{\sin(t(x-a))}{t}\,dt-\int_{-T}^{T}\frac{\sin(t(x-b))}{t}\,dt$$

as $T\to\infty$ this will converge to

$$\begin{cases}2\pi & a<x<b\\ \pi & x=a \text{ or } x=b\\ 0 & x<a \text{ or } x>b\end{cases}$$

$$\sup_y \left| \int_0^y \frac{\sin(x)}{x} dx \right| < \infty$$

By the bounded convergence theorem we can take the limit inside the integral

$$\lim_{T\to\infty} \frac{1}{2\pi} \int_{\mathbb{R}} \underbrace{\left(\int_{-T}^{T} \frac{\sin(t(x-a))}{t} dt - \int_{-T}^{T} \frac{\sin(t(x-b))}{t} dt \right)}_{1_{(a,b)} + \frac{1}{2} 1_a + \frac{1}{2} 1_b} \mu_X(dx)$$

$$= P(X \in (a,b)) + \frac{1}{2}\left(P(X=a) + P(X=b)\right)$$

Thm: if $\int_{\mathbb{R}} |\varphi_X(t)| dt < \infty$ then X has a PDF

$$f_X(y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ity} \varphi_X(t) dt$$

Proof:

$$\underbrace{P(X \in (a,b)) + \frac{1}{2}\left(P(X=a) + P(X=b)\right)}= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ita} - e^{-itb}}{it} \varphi(t) dt$$

$$\leq \frac{b-a}{2\pi} \int_{\mathbb{R}} |\varphi|\, dt$$

This shows that $P(X=a)=0$ for all $a\in\mathbb{R}$.

$$P(X \in (a,b)) =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_a^b e^{-ity}\, dy\, \varphi(t)\, dt$$

$$= \int_a^b \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ity} \varphi(t)\, dt}_{\text{this is the PDF of } X}\, dy \qquad \blacksquare$$

If $\varphi_X(t) = \varphi_Y(t)$ for all $t\in\mathbb{R}$

then $X \stackrel{d}{=} Y$.

Ex: If $\varphi_X(t) \in \mathbb{R}$ for all t

then $X \stackrel{d}{=} -X$.

$$\varphi_{-X}(t) = E\left[e^{-itX}\right] = E[\cos(tX)] - i\,\cancel{E[\sin(tX)]}$$

$$\varphi_X(t) = \qquad E[\cos(tX)] + iE[\sin(tX)]$$

If X, Y are independent then

$$\boxed{\varphi_{X+Y}(t)} = E\left[e^{it(X+Y)}\right] = E\left[e^{itX}\cdot e^{itY}\right]$$

$$\overset{\text{independence}}{=} E\left[e^{itX}\right]E\left[e^{itY}\right]$$

$$= \boxed{\varphi_X(t)\,\varphi_Y(t)}$$

In the CLT we want to understand the limit in distribution of $\dfrac{S_n - n\mu}{\sqrt{n}\,\sigma}$.

$S_n = X_1 + \dots + X_n$

$$\varphi_{aY+b}(t) = E\left[e^{(aY+b)it}\right] = E\left[e^{ibt}\cdot e^{iatY}\right]$$

$$= e^{ibt}E\left[e^{iatY}\right] = e^{ibt}\varphi_Y(at)$$

$$\varphi_{S_n}(t) = \varphi_{X_1+\dots+X_n}(t) = \varphi_{X_1}(t)\cdots\varphi_{X_n}(t) = \left(\varphi_{X_1}(t)\right)^n$$

$$\varphi_{\frac{S_n - n\mu}{\sqrt{n}\sigma}}(t) = e^{-\frac{n\mu}{\sqrt{n}\sigma}} \left[\varphi_{X_1}\left(\frac{t}{\sqrt{n}\sigma}\right)\right]^n$$

If $X \sim N(0,1)$ then $\varphi_X(t) = e^{-\frac{t^2}{2}}$

$$E\left[e^{itX}\right] = \int_{\mathbb{R}} e^{itx} \frac{1}{\sqrt{2\pi}} e^{-\frac{\cancel{x}^2}{2}} dx = e^{-\frac{t^2}{2}}$$

$$\frac{1}{\sqrt{2\pi}} e^{itx - \frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{+\frac{t^2}{2} + itx - \frac{x^2}{2} - \frac{t^2}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-it)^2} \cdot e^{-\frac{t^2}{2}}$$

$$\int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-it)^2} dx = 1$$

rigorous proof: complex analysis (residue thm)