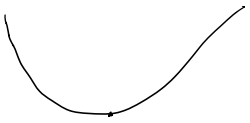
* Minima of convex functions:

(P) $\min_{x \in \mathcal{X}} f(x)$: f is convex, $\mathcal{X}$ is convex, closed, and nonempty

* **Thm 2.6** Consider (P). We have the following:

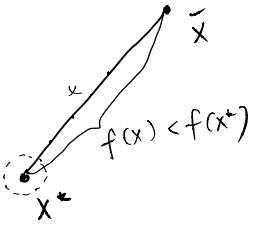


(a) Any local solution to (P) is also a global solution.

(b) The set of global solutions to (P) is convex.

Proof:

(a) Suppose f.p.o.c. that x^* is a local but not a global solution. Then $\exists\, \bar{x} \in \mathcal{X}$ s.t. $f(\bar{x}) < f(x^*)$.



As $\mathcal{X}$ is convex, $\forall\, \alpha \in (0,1)$:

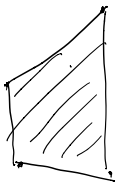
$(1-\alpha)\, x^* + \alpha\, \bar{x} \in \mathcal{X}$.

As f is convex, $\forall\, \alpha \in (0,1)$:

$$f((1-\alpha)\, x^* + \alpha\, \bar{x}) \leq (1-\alpha)\, f(x^*) + \alpha f(\bar{x}) < f(x^*).$$

$\Rightarrow$ Every neighborhood of x^* must include a point $(1-\alpha)\, x^* + \alpha\, \bar{x}$ for some $\alpha > 0$ that will have a strictly lower function value. $\Rightarrow$ x^* cannot be a local solution.

(b) Let $x^*, \bar{x} \in \mathcal{X}$ be any two global solutions.



$\mathcal{X}$ is convex $\Rightarrow \forall\, \alpha \in (0,1)$: $(1-\alpha)\, x^* + \alpha\, \bar{x} \in \mathcal{X}$.

f is convex $\Rightarrow \forall\, \alpha \in (0,1)$:

$$f((1-\alpha)\, x^* + \alpha\, \bar{x}) \leq (1-\alpha)\, f(x^*) + \alpha\, f(\bar{x})$$
$$= f(x^*) = f(\bar{x})$$

$\Rightarrow f((1-\alpha)\, x^* + \alpha\, \bar{x}) = f(x^*)$

$\Rightarrow (1-\alpha)\, x^* + \alpha\, \bar{x}$ is a global solution.

$\Rightarrow$ the set of global solutions must be convex.

■

* Thm.

(a) Let f be cont.'ly diff.'able. f is convex if and only if

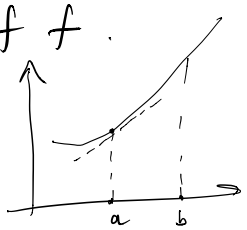
$$\forall\, x, y: \quad f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle .$$

(b) Let f be twice cont.'ly diff.'able.
f is convex if and only if $\forall x: \nabla^2 f(x) \succeq 0$.

* Thm 2.7 Let f be cont.'ly diff.'able and convex.
If $\nabla f(x^*) = 0$, then x^* is a global min of f.

Pf: Use Part (a) of the Thm above:

$\forall x:$ $f(x) \geq f(x^*) + \langle \nabla f(x^*), x - x^* \rangle$.
($\nabla f(x^*) = 0$)



(for constrained setups, we would use $\langle \nabla f(x^*), x - x^* \rangle \geq 0, \forall x$)

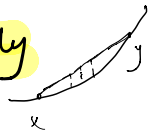
* Strongly convex functions:

* Def. Given $m > 0$, we say that $f: \mathbb{R}^d \to \bar{\mathbb{R}}$ is m-strongly convex (or strongly convex w/ modulus m), if $\forall x, y \in \mathbb{R}^d$:

$$f((1-\alpha)x + \alpha y) \leq (1-\alpha) f(x) + \alpha f(y) - \frac{m}{2}(1-\alpha)\alpha \|y - x\|^2 .$$

* Ex: 1) When f is cont.'ly diff.'able, equivalently

$$\forall x, y: \quad f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{m}{2}\|y - x\|^2 .$$



2) When f is twice cont.'ly diff.'able, equivalently:

$$\forall x: \quad \nabla^2 f(x) \succeq m I .$$

* Thm 2.8. Suppose that $f: \mathbb{R}^d \to \bar{\mathbb{R}}$ is cont.'ly diff.'able and m-strongly convex for some $m > 0$. If $\nabla f(x^*) = 0$, then x^* is the unique global min of f.

Proof: From Ex 1):

$$\forall x: \quad f(x) \geq f(x^*) + \langle \nabla f(x^*), x - x^* \rangle + \frac{m}{2}\|x - x^*\|^2$$

($\nabla f(x^*) = 0$; $\frac{m}{2}\|x - x^*\|^2 > 0$ unless $x = x^*$) ∎

* Growth of sequences:

$\{a_k\}_{k\geq 1}$, $\{b_k\}_{k\geq 1}$, $\forall k: a_k, b_k \geq 0$. $\quad a_k \leq 10\, b_k \Rightarrow a_k = O(b_k)$

* "Big-Oh" notation:

$a_k = O(b_k) \iff (\exists M > 0)(\exists K < \infty)(\forall k \geq K): a_k \leq M b_k$.

(E.g., $k = O(\frac{1}{10} k^2)$, $k = O(\frac{1}{10!} k)$)

* If $a_k = O(b_k)$ and $b_k = O(a_k)$, we write $a_k = \Theta(b_k)$.

* "Little-Oh" notation:

$a_k = o(b_k) \iff \lim_{k\to\infty} \frac{a_k}{b_k} = 0$.

* Algorithmic setup:

1) first-order oracle model:

$x \longrightarrow$ [oracle] $\longrightarrow f(x), \nabla f(x)$

2) second-order oracle model:

$x \longrightarrow$ [oracle] $\longrightarrow f(x), \nabla f(x), \nabla^2 f(x)$

* All algorithms we consider in this class are iterative:
  - → start w/ some x_0, get oracle answers for x_0, choose x_1,
  - → at iteration k, get oracle answers for x_k, choose x_{k+1}

* Basic Descent Methods:

* Assumptions for this part:

(A1) f is L-smooth for some $L < \infty$ (thus also cont.'ly diff.'able)

(A2) $\mathcal{X} = \mathbb{R}^d$, i.e., the problem is unconstrained

Note: for now, and until explicitly stated otherwise, we are not assuming that f is convex.

* Def. $p \in \mathbb{R}^d$ is a descent direction for f at x if $f(x+tp) < f(x)$ for all suff. small $t > 0$.

* Prop 3.2. If f is cont.'ly diff.'able (in a neighborhood of x), then any p s.t. $\langle \nabla f(x), p \rangle < 0$ is a descent direction.

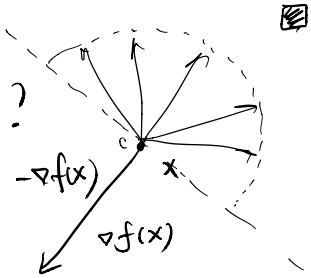
Proof: TT + continuity of ∇f: ($y = x + tp$)

$$f(x+tp) = f(x) + t \langle \nabla f(x + \gamma t p), p \rangle \text{ for some } \gamma \in (0,1).$$

We know that $\langle \nabla f(x), p \rangle < 0$. As ∇f is continuous, for all suff. small $t > 0$:

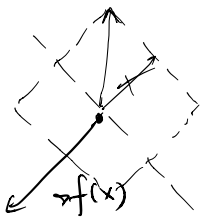
$$t \langle \nabla f(x + \gamma t p), p \rangle < 0.$$

$\Rightarrow f(x+tp) < f(x)$. ∎



* What would be a good descent direction?
  - could try to move in the direction of $-\nabla f(x)$
  - justification:
    Look at all p w/ $\|p\|_2 = 1$. Then:

$$\inf_{\|p\|_2 = 1} \langle \nabla f(x), p \rangle = -\|\nabla f(x)\|_2 \text{ attained for } p = -\frac{\nabla f(x)}{\|\nabla f(x)\|_2}$$



* "Simplest" descent algorithm:

$$x_{k+1} = x_k - \underbrace{\alpha_k}_{\text{step size}} \nabla f(x_k)$$

α_k is chosen small enough so that $f(x_{k+1}) < f(x_k)$, assuming $\nabla f(x_k) = 0$.

"gradient method", "gradient descent", "steepest descent"

* From Lemma 2.2:

$$\forall x, y: \quad f(y) \le f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2} \|y - x\|_2^2 .$$

$x = x_k$ (for $y = x_k$: $\langle \nabla f(x_k), y - x_k \rangle = 0$, $\frac{L}{2}\|y - x_k\|_2^2 = 0$)

$$\forall y: \quad f(y) \le f(x_k) + \langle \nabla f(x_k), y - x_k \rangle + \frac{L}{2} \|y - x_k\|_2^2 .$$

$$x_{k+1} = \underset{y \in \mathbb{R}^d}{\operatorname{argmin}} \left\{ \right\}$$

$$\Rightarrow \quad f(x_{k+1}) \le f(x_k) .$$

$$\nabla f(x_k) + L(x_{k+1} - x_k) = 0$$

$$\Leftrightarrow \quad x_{k+1} = x_k - \frac{1}{L} \nabla f(x_k) .$$

$$\Rightarrow \quad f(x_{k+1}) \le f(x_k) - \frac{1}{2L} \|\nabla f(x_k)\|_2^2 .$$

* Ex. If $x_{k+1} = x_k - \alpha \nabla f(x_k)$, $\alpha \in (0, \frac{1}{L}]$, then:

$$f(x_{k+1}) \le f(x_k) - \frac{\alpha}{2} \|\nabla f(x_k)\|_2^2 .$$ "Descent Lemma"

Using Descent Lemma, if $x_{k+1} = x_k - \alpha \nabla f(x_k), \ \forall k$, (GD)

$$\begin{aligned} f(x_{k+1}) &\le f(x_k) - \frac{\alpha}{2} \|\nabla f(x_k)\|_2^2 \\ &\le f(x_{k-1}) - \frac{\alpha}{2} \|\nabla f(x_{k-1})\|_2^2 - \frac{\alpha}{2} \|\nabla f(x_k)\|_2^2 \\ &\vdots \\ &\le f(x_0) - \frac{\alpha}{2} \sum_{i=0}^{k} \|\nabla f(x_i)\|_2^2 . \end{aligned}$$

$$\frac{\alpha}{2} \sum_{i=0}^{k} \|\nabla f(x_i)\|_2^2 \le f(x_0) - f(x_{k+1})$$

Let's assume $f(x) \ge f_* > -\infty$, $\forall x$. Then:

$$(*) \qquad \frac{\alpha}{2} \sum_{i=0}^{k} \|\nabla f(x_i)\|_2^2 \le f(x_0) - f_*$$

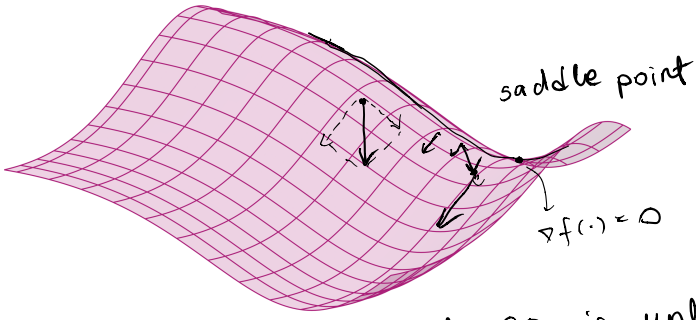
$$(**) \quad \frac{\alpha}{2} \sum_{i=0}^{k} \|\nabla f(x_i)\|_2^2 \geq \frac{\alpha}{2}(k+1) \cdot \min_{0 \leq i \leq k} \|\nabla f(x_i)\|_2^2$$

$$(*) + (**) \Rightarrow \min_{0 \leq i \leq k} \|\nabla f(x_i)\|_2^2 \leq \frac{2(f(x_0) - f_*)}{\alpha(k+1)} .$$

$$\min_{0 \leq i \leq k} \|\nabla f(x_i)\|_2 \leq \sqrt{\frac{2(f(x_0) - f_*)}{\alpha(k+1)}} \leq \epsilon$$

For any target error $\epsilon > 0$, GD satisfies

$$\min_{0 \leq i \leq k} \|\nabla f(x_i)\|_2 \leq \epsilon \quad \text{for} \quad k+1 \geq \frac{2(f(x_0) - f_*)}{\alpha \epsilon^2} .$$



"randomly initialized GD is unlikely to converge to a saddle point"

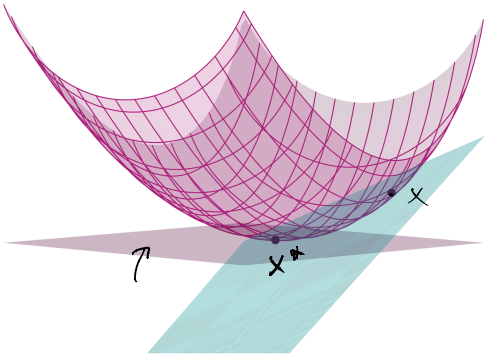
* The convex case:

How does the convexity help?

Let $x^* \in \underset{x \in \mathbb{R}^d}{\operatorname{argmin}} f(x)$

↑ might not be unique; assume all minimizers are from $\mathbb{R}^d$.

$\forall x$: $f(x^*) \geq f(x) + \langle \nabla f(x), x^* - x \rangle$



Want: bound $\underbrace{f(x) - f(x^*)}_{\text{optimality gap}}$.

GD: $x_{k+1} = x_k - \alpha \nabla f(x_k)$, $\alpha \in (0, \frac{1}{L}]$

Know from before: $f(x_{k+1}) \le f(x_k) - \frac{\alpha}{2} \|\nabla f(x_k)\|_2^2$.

$f(x^*) \ge f(x_k) + \langle \nabla f(x_k), x^* - x_k \rangle$ $\quad \alpha \nabla f(x_k) = x_k - x_{k+1}$

$\nabla f(x_k) = \frac{1}{\alpha}(x_k - x_{k+1})$

$= f(x_k) + \frac{1}{\alpha} \langle x_k - x_{k+1}, x^* - x_k \rangle$

$\left[(a-b)(c-a) = \frac{1}{2}(c-b)^2 - \frac{1}{2}(a-b)^2 - \frac{1}{2}(c-a)^2 \right]$

$= f(x_k) + \frac{1}{2\alpha}\|x_{k+1} - x^*\|_2^2 - \frac{1}{2\alpha}\|x_k - x^*\|_2^2 - \frac{1}{2\alpha}\|x_k - x_{k+1}\|_2^2$

$x_k - x_{k+1} = \alpha \nabla f(x_k)$

$= f(x_k) - \frac{\alpha}{2}\|\nabla f(x_k)\|_2^2 + \frac{1}{2\alpha}\|x_{k+1} - x^*\|_2^2 - \frac{1}{2\alpha}\|x_k - x^*\|_2^2$

$f(x^*) \ge f(x_{k+1}) + \frac{1}{2\alpha}\|x_{k+1} - x^*\|_2^2 - \frac{1}{2\alpha}\|x_k - x^*\|_2^2$.

1) $\|x_{k+1} - x^*\|_2^2 - \|x_k - x^*\|_2^2 \le 2\alpha \underbrace{(f(x^*) - f(x_{k+1}))}_{\le 0}$, w/ strict ineq whenever $f(x_{k+1}) \ne f(x^*)$.

$\inf_{x^* \in \operatorname{argmin}_x f(x)} \|x_k - x^*\|_2 \xrightarrow{k \to \infty} 0$.

2) $f(x_{k+1}) - f(x^*) \le \frac{1}{2\alpha}\left(\|x_k - x^*\|_2^2 - \|x_{k+1} - x^*\|_2^2\right)$

$\sum_{k=0}^{K} (f(x_{k+1}) - f(x^*)) \le \frac{1}{2\alpha}\left(\|x_0 - x^*\|_2^2 - \|x_{K+1} - x^*\|_2^2\right)$ $\quad \|x_{K+1} - x^*\|_2^2 \to 0$

$\le \frac{1}{2\alpha}\|x_0 - x^*\|_2^2$.

$\sum_{k=0}^{K} (f(x_{k+1}) - f(x^*)) \ge (K+1)(f(x_{K+1}) - f(x^*))$

$f(x_{K+1}) - f(x^*) \le \frac{\|x_0 - x^*\|_2^2}{2\alpha(K+1)}$.

$\forall \epsilon > 0$: $f(x_K) - f(x^*) \le \epsilon$ after at most

$$K = \left\lceil \frac{\|x_0 - x^*\|_2^2}{2\alpha\epsilon} \right\rceil \quad \text{iterations}.$$

* The strongly convex case:

$\forall k$:

$$f(x^*) \ge f(x_k) + \langle \underbrace{\nabla f(x_k)}_{\frac{1}{\alpha}(x_k - x_{k+1})}, x^* - x_k \rangle + \frac{m}{2}\|x^* - x_k\|_2^2$$

$$\ge f(x_{k+1}) + \frac{1}{2\alpha}\|x_{k+1} - x^*\|_2^2 - \frac{1}{2\alpha}\|x_k - x^*\|_2^2 + \frac{m}{2}\|x^* - x_k\|_2^2$$

$$= f(x_{k+1}) + \frac{1}{2\alpha}\|x_{k+1} - x^*\|_2^2 - \left(\frac{1}{2\alpha} - \frac{m}{2}\right)\|x_k - x^*\|_2^2$$

1) $$\frac{1}{2\alpha}\|x_{k+1} - x^*\|_2^2 \le \left(\frac{1}{2\alpha} - \frac{m}{2}\right)\|x_k - x^*\|_2^2 + \underbrace{f(x^*) - f(x_{k+1})}_{\le 0} \to 0$$

$$\|x_{k+1} - x^*\|_2^2 \le (1 - m\alpha)\|x_k - x^*\|_2^2.$$

Ex. $m\alpha \in (0, 1]$. ($\alpha \in (0, \frac{1}{L}]$)

$$\|x_{k+1} - x^*\|_2^2 \le (1 - m\alpha)^{k+1}\|x_0 - x^*\|_2^2$$

$$\|x_{k+1} - x^*\| \le \epsilon \quad \text{for} \quad k = \mathcal{O}\left(\frac{1}{m\alpha}\log\left(\frac{\|x_0 - x^*\|}{\epsilon}\right)\right)$$