David Zikel, Nov. 1, 2020

GitHub: https://github.com/dzikel/dzikel-714

(Aa) Suppose that v is in the span of $w_1, w_2, \cdots, w_n$ — i.e., that $v = c_1w_1 + c_2w_2 + \cdots + c_nw_n$ for $c_1, c_2, \cdots, c_n \in \mathbf{R}$. Then, for any w_i, $w_i \cdot v = c_1(w_i \cdot w_1) + \cdots + c_i(w_i \cdot w_i) + \cdots + c_n(w_i \cdot w_n) = 0 + \cdots + c_i|w_i|^2 + \cdots + 0 = c_i|w_i|^2$. Dividing by $|w_i|^2$ yields simply c_i, so the w_i component of v is simply $\frac{w_i \cdot v}{|w_i|^2}w_i$. Summing for $i = 1 \cdots n$ gives the formula shown for v.

(Ab) i. If, for example, the solution lies directly on the 1st, 2nd, etc. gradient line, then the algorithm will converge early. In general, if step $n^* - 1$ does not change the estimate, then the algorithm will converge with less than N iterations. ii. Note that, if $n = 1$ (so $j = 0$), $\langle p_n, p_j \rangle_A = \langle p_0, r_1 \rangle_A - \frac{\langle p_0, p_0 \rangle_A}{\langle p_0, p_0 \rangle_A} \langle p_0, r_1 \rangle_A = 0$. Suppose that, for $n = 1, \cdots, k$ we have that $\langle p_n, p_j \rangle_A = 0$ for $j < n$. Then, for $j < n + 1$, $\langle p_{n+1}, p_j \rangle_A = \langle p_j, r_{n+1} \rangle_A - 0 - 0 - \cdots - \frac{\langle p_j, p_j \rangle_A}{\langle p_j, p_j \rangle_A} \langle p_j, r_{n+1} \rangle_A - \cdots - 0 = 0$. As such, $\langle p_j p_n \rangle_A = 0$ for all $0 \le j < n \le n^* - 1$.

(Ac) i. We see from the proof of (Aa) that the component of v in the direction of ϕ_n is $x_n = \frac{v \cdot \phi_n}{1}\phi_n = (v \cdot \phi_n)\phi_n$, and $Ax_n \cdot w = \lambda_n(v \cdot \phi_n)(\phi_n \cdot w)$. Since both multiplication by A and the dot product are linear, we can find $Av \cdot w$ by summing $Ax_n \cdot w$ for n from 1 to N, which is the formula shown. ii. Suppose that one of the eigenvalues $\lambda_n \le 0$. Then this eigenvalue must be associated with a nonzero eigenvector ϕ_n, such that $\langle \phi_n, \phi_n \rangle_A = \phi_n \cdot \lambda \phi_n < 0$ — but, as A is positive definite, this is a contradiction. iii. Decomposing v as $c_1\phi_1 + \cdots + c_N\phi_N$, we see that $\lambda_1|v|^2 = \lambda_1{c_1}^2 + \cdots + \lambda_1{c_N}^2$, $\lambda_N|v|^2 = \lambda_N{c_1}^2 + \cdots + \lambda_N{c_N}^2$, and, by (Aci), $Av \cdot v = \lambda_1{c_1}^2 + \lambda_2{c_2}^2 + \cdots + \lambda_N{c_N}^2$. Since each ${c_n}^2$ is nonnegative and the λ_n are nondecreasing, we have that $\lambda_1|v|^2$ is a lower bound and $\lambda_N|v|^2$ is an upper bound for $Av \cdot v$. iv. Since both sides are nonnegative (the right side by (Acii)), the equation remains unchanged when both sides are squared: $|Av|^2 \le {\lambda_N}^2|v|^2$. Again decomposing v as $c_1\phi_1 + \cdots + c_N\phi_N$, we see that $Av = \lambda_1c_1\phi_1 + \cdots + \lambda_Nc_N\phi_N$, so $|Av|^2 = {\lambda_1}^2{c_1}^2 + \cdots + {\lambda_N}^2{c_N}^2$. Since the λ_n are nondecreasing, this is bounded from above by ${\lambda_N}^2{c_1}^2 + \cdots + {\lambda_N}^2{c_N}^2 = {\lambda_N}^2|v|^2$.

(Ad) $p_{n+1} = r_{n+1} + \beta_np_n = r_n - \alpha_nw_n + \beta_np_n = r_n - \alpha_nAp_n + \beta_np_n = p_n - \beta_{n-1}p_{n-1} - \alpha_nAp_n + \beta_np_n = (1 + \beta_n)p_n - \alpha_nAp_n - \beta_{n-1}p_{n-1}$.

(Ae) If A has characteristic equation $\lambda^N + c_{N-1}\lambda^{N-1} + \cdots + c_0$, then, by the Cayley-Hamilton theorem, $A^N + c_{N-1}A^{N-1} + \cdots + c_0I = 0$, so subtracting all of the left side but the A^N term from both sides shows that $A^N = -c_{N-1}A^{N-1} - \cdots - c_0I$, a linear combination of the desired form.

(Af) i. If $u_n = u + e_n$, then $u_{n+1} = u_n + \alpha(f - Au_n) = u + e_n + \alpha(f - Au - Ae_n) = u + (I - \alpha A)e_n$. ii. Note that the eigenvalues of $I - \alpha A)$ are $1 - \lambda_1, \cdots, 1 - \lambda_N$. By a proof similar to that for (Aciv), but using the magnitudes of the new eigenvalues as they are no longer guaranteed to be positive, we see that $|e_{n+1}| \le |1 - \alpha\lambda_j||e_n|$ for j making these bounds least strict. iii. The smallest ρ is found when $1 - \alpha\lambda_1 = -1 + \alpha\lambda_N$ — for α higher, $|1 - \alpha\lambda_N|$ will be larger, and for α lower $|1 - \alpha\lambda_1|$ will be larger. As such, we set $2 - \alpha(\lambda_1 + \lambda_N) = 0$, so $\alpha = \frac{2}{\lambda_1 + \lambda_N}$, and $\rho = \frac{\lambda_N - \lambda_1}{\lambda_N + \lambda_1} < 1$. iv. Since c and C bound the λs, we see that $\frac{c-C}{C+c} \le \frac{C+c-2\lambda_j}{C+c} \le \frac{C-c}{C+c}$ for any j, so the absolute value of $\frac{C+c-2\lambda_j}{C+c}$ is less than or equal to $\frac{C-c}{C+c}$. This holds for all j, so it must hold for the value of j maximizing the bounds.

(Ag) i. $r_1 = r_0 - \alpha_0 w_0$, w_0 is defined as Ap_0, and $r_0 = p_0$, so $r_1 = r_0 - \alpha_0 Ar_0$. ii. $r_{n+1} = r_n - \alpha_n w_n = r_n - \alpha_n Ap_n = r_n - \alpha_n A(r_n + \beta_{n-1}p_{n-1}) = r_n - \alpha_n Ar_n + \alpha_n\beta_{n-1}(-w_{n-1}) = r_n - \alpha_n Ar_n + \frac{\alpha_n\beta_{n-1}}{\alpha_{n-1}}(r_n - r_{n-1})$. iii. $q_1 = \frac{r_1}{|r_1|} = \frac{r_1/|r_0|}{|r_1|/|r_0|} = \frac{r_1/|r_0|}{\sqrt{\beta_0}}$, so $Aq_0 = \frac{Ar_0}{|r_0|} = \frac{r_0 - r_1}{\alpha_0|r_0|} = \gamma_0 q_0 - \delta_0 q_1$. For $n \ge 1$, $q_{n+1} = \frac{r_{n+1}/|r_n|}{\sqrt{\beta_n}}$ and $q_{n-1} = \sqrt{\beta_{n-1}}r_{n-1}/|r_n|$, so $Aq_n = \frac{Ar_n}{|r_n|} = \frac{1}{\alpha_n|r_n|}(r_n - r_{n+1} + \frac{\alpha_n\beta_{n-1}}{\alpha_{n-1}}(r_n - r_{n-1})) = \frac{1}{\alpha_n|r_n|}((1 + \frac{\alpha_n\beta_{n-1}}{\alpha_{n-1}})r_n - r_{n+1} - \frac{\alpha_n\beta_{n-1}}{\alpha_{n-1}}r_{n-1}) = \gamma_n q_n - \delta_n q_{n+1} - \delta_{n-1}q_{n-1}$. iv. Converting the equations from (Agiii) into matrix form (provided the matrix will be on the right) yields the coefficients of T_n, with the notable exception of the $-\delta_{n-1}q_n$ term from the equation for Aq_{n-1}. Manually subtracting this term yields the formula $AQ_n = Q_nT_n - \delta_{n-1}q_ne_n^T$, as printed. v. $Q_n^TAQ_n = Q_n^TQ_nT_n - Q_n^t\delta_{n-1}q_ne_n^T$, which, as the q_j are orthonormal, is $T_n - 0\delta_{n-1}e_n^T = T_n$.

(B) Dividing $[0, 1]$ into N subintervals — i.e., taking $N + 1$ samples — and performing brute-force calculations on a very fine grid (1000000 times finer than the value of N tested) shows that the uniform error appears to shrink to less than 0.01 at $N = 1716$.

(Ca) Estimating the function's second derivatives using the standard 3- and 5-point stencils and treating its value as 0 outside of the boundaries, in addition to initializing the function to 0 at time 0 and to $f(x)f(y)\Delta t$ after the first step, results in a solver found on GitHub (Python 3). Unfortunately, due to the large quantity of variables considered, I ran out of memory before being able to compute many high-quality numerical solutions. A log-log plot is found in the GitHub repository, with the admittedly low Δt of 0.05, showing a slope of roughly 4 — quadratic in the total number of grid points, and even more efficient in terms of the grid spacing.

(Cb) We see that $\frac{1}{\Delta t^2}(y_{i+1}-2y_i+y_{i-1}) = \lambda y_i$, so $y_{i+1} = (2+\lambda\Delta t^2)y_i - y_{i-1}$. Setting $z = \lambda\Delta t^2$, we see that the two generating solutions to this equation are consecutive powers of $\frac{1}{2}(2+z+\sqrt{4z+z^2})$ and its reciprocal. The system is stable, then, when the absolute value of this root is 1 — that is, when $z \in (-4, 0)$. A graph of this (simple) region is in the GitHub.

(Cc) The one-dimensional Laplacian with grid spacing Δx has eigenvalues ranging from 0 to $\frac{-4}{\Delta x^2}$ and the two-dimensional case has double this range (by HW1's (Cd)), so our criterion for stability is $\frac{-8\Delta t^2}{\Delta x^2} \in (0, 4)$ (i.e. $\frac{\Delta t^2}{\Delta x^2} \in (0, 0.5)$).

(Cd) Performing the discrete Laplacian on a plane wave $e^{i\Delta x(xj+yk)}$ for j, k wave parameters results in $\nabla^2 y = \frac{2}{\Delta x^2}(\cos j\Delta x + \cos k\Delta x - 2)y$, which lies between 0 and $\frac{-8y}{\Delta x^2}$. The resulting quadratic equation for the scaling factor in time is $g^2 - (2 - \frac{8\Delta t^2}{\Delta x^2})g + 1 = 0$, which results in the same range $\frac{\Delta t^2}{\Delta x^2} \in (0, 0.5)$ found in (Cc).

(Ce) A simple analysis of the component second-derivative formulas for this method yields the modified equation $v_{xx} + v_{yy} + \frac{\Delta x^2}{12}(v_{xxxx} + v_{yyyy}) = v_{tt} + \frac{\Delta t^2}{12}v_{tttt}$. For comparison, the normal heat equation implies the PDE $v_{xx} + v_{yy} + \frac{\Delta t^2}{12}(v_{xxxx} + v_{yyyy} + 2v_{xxyy}) = v_{tt} + \frac{\Delta t^2}{12}v_{tttt}$, so we see that if $\Delta x = \Delta t$ then v only solves the heat equation when $v_{xxyy} = 0$. Looking at the behavior of plane waves $v(x, y, t) = e^{i(a_x x + a_y y + a_t t)}$ gives us the criterion that $\frac{\Delta x^2}{12}(a_x^4 + a_y^4) - a_x^2 - a_y^2 = \frac{\Delta t^2}{12}a_t^4 - a_t^2$, resulting in different solution waves from the normal heat equation ($a_x^2 + a_y^2 = a_t^2$). The extra terms are fourth-order and therefore dissipative.

(D) I have not completed this problem, but I would imagine an approach could be taken which considers roots of polynomials akin to the roots required to be ≤ 1 for the stability of second-order equations. This notion of stability likely applies to second-order equations much like Lax-Richtmyer stability does for first-order ones, similarly to how eigenvalues for new equations created by the method of lines can be checked for in preexisting stability regions.