Review of Conditional expectation

L^p space. $\mathbb{E}|X|^p < \infty$ $\quad \|X\|_p = (\mathbb{E}|X|^p)^{1/p}$.

L^∞ space $\quad \mathbb{P}(|X| \le c) = 1$ for some const $c < \infty$

$$\|X\|_\infty = \inf\{c : \mathbb{P}(|X| \le c) = 1\}$$

properties: (1) $\|X-Y\|_p = 0 \Rightarrow X = Y$ a.s

(2) $|\mathbb{E}[XY]| \le \|X\|_p \|Y\|_q \quad \frac{1}{p} + \frac{1}{q} = 1$

(3) $\|X+Y\|_p \le \|X\|_p + \|Y\|_p$.

If $\lim_{n\to\infty} \|X_n - X\|_p = 0 \Rightarrow$ Say X_n converge to X in L^p.

L^p is metric space $\|X - Y\|_p$

Independence : 独立性 $\quad X, Y$ are independent

$\iff P(\{X \in B_1\} \cap \{Y \in B_2\}) = P(X \in B_1)\, P(Y \in B_2)$

X, Y 独立
与集 = 乘积
"factorize"

Two σ-algebras $\mathcal{F}_1, \mathcal{F}_2$ are independent

$\iff \mathbb{P}(D_1 \cap D_2) = P(D_1)P(D_2) \quad \forall D_1 \in \mathcal{F}_1,\ D_2 \in \mathcal{F}_2$

X and σ-alg $\mathcal{F}_3$ are indp

$\iff \mathbb{P}(\{X \in B\} \cap D) = P(X \in B)\, P(D)$
$\forall B \in \mathcal{B}(\mathbb{R}).\ D \in \mathcal{F}_3 \quad (\Omega, \mathcal{F}, P)$

条件期望

Conditional expection: of X wrt a σ-algebra $\mathcal{D}$ $(\mathcal{D} \subset \mathcal{F})$

denoted by $\mathbb{E}[X|\mathcal{D}]$ is the unique random var Y which satisfies

X对$\mathcal{D}$的条件期望 $\mathbb{E}[X|\mathcal{D}]$

A) Y is $\mathcal{D}$-measurable

B) $\int_D X\,dP = \int_D Y\,dP \quad \forall D \in \mathcal{D}$

对任意$D\in\mathcal{D}$ X在D上积分与其条件期望在D上积分相等.

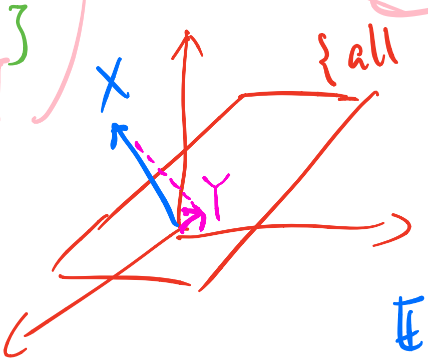
Ex: ① ② $\Omega = \{HH, HT, TH, TT\}$ $\quad \mathcal{F} = \{\phi, \{HH\}, \dots, \{HH, HT\} \dots \Omega\}$

$\in \{HT\}$

$\rightarrow \mathcal{D} = \{\phi, \{HH\}, \{HT, TH\}, \{TT\}, \{HH, TT\}, \Omega\}$

$\cancel{\{HT\}}$ 无单独的HT

$\{\text{all } L^2\text{-r.v.}\}$



Y is $\mathcal{D}$-measuble

$\mathbb{E}[X|\mathcal{D}] = Y$.

$\forall$ $\mathcal{D}$-measurable Z $\quad \mathbb{E}[ZX] = \mathbb{E}[ZY]$

Take $Z = \mathbb{1}_D$ $\quad Z(\omega) = \mathbb{1}_D(\omega) = \begin{cases} 1 & \omega \in D \\ 0 & \omega \notin D \end{cases}$

$\mathbb{E}[ZX] = \int_D X\,dP$

$\mathbb{E}[ZY] = \int_D Y\,dP$

Radon-Nikodym theorem $\Rightarrow$ $\exists$ of conditional expection.

Properties 1). $\mathbb{E}[\mathbb{E}[X|\mathcal{D}]] = \mathbb{E}X$

(B) $\forall D \in \mathcal{D}$ $\quad \int_D X\,dP = \int_D Y\,dP$

choose $D = \Omega$

$\mathbb{E}X \quad \mathbb{E}[\quad]$

2) If $X \geq 0$ Then $\mathbb{E}[X \mid \mathcal{D}] \geq 0.$

$Y = \mathbb{E}[X \mid \mathcal{D}]$ is $\mathcal{D}$ meas.

and $\int_D Y \, dp = \int_D X \, dp \geq 0 \quad \forall D \in \mathcal{D} \quad \Rightarrow Y \geq 0$ a.s.

3). $\mathbb{E}[aX + bY \mid \mathcal{D}] = a\mathbb{E}[X \mid \mathcal{D}] + b\mathbb{E}[Y \mid \mathcal{D}]$

$$\int_D aX + bY \, dp \overset{?}{=} \int_D a\mathbb{E}[X \mid \mathcal{D}] + b\mathbb{E}[Y \mid \mathcal{D}] \, dP$$

$$\overset{\shortparallel}{a\int_D X \, dp + b\int_D Y dp} = a\int \mathbb{E}(X \mid \mathcal{D}) \, dp + b\int_D \mathbb{E}[Y \mid \mathcal{D}] dp$$

4) If $X \geq Y$ then $\mathbb{E}[X \mid \mathcal{D}] \geq \mathbb{E}[Y \mid \mathcal{D}]$

$2) + 3) \Rightarrow 4)$

5) If X is $\mathcal{D}$-measurable. $\mathbb{E}[X \mid \mathcal{D}] = X$

6) X, Y

$\mathcal{D}$-measurable

$$\mathbb{E}[YX \mid \mathcal{D}] = Y\mathbb{E}[X \mid \mathcal{D}]$$

$\mathcal{D}$ measurable 可以提出来

7) If X is indep of $\mathcal{D}$. then $\mathbb{E}[X \mid \mathcal{D}] = \mathbb{E}[X]$

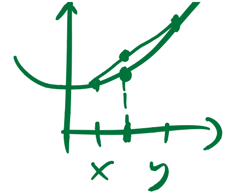
$$\int_D X \, dp \overset{?}{=} \int_D \mathbb{E}(X) \, dp$$

$$\overset{\shortparallel}{\int_\Omega X \cdot \mathbb{1}_D \, dp} = \mathbb{E}[X \cdot \mathbb{1}_D] = \mathbb{E}X \cdot \int_\Omega \mathbb{1}_D \, dp$$

8). $\mathbb{E}[\mathbb{E}[X \mid \mathcal{D}_2] \mid \mathcal{D}_1]$ 等价于在小的σ上

$\mathcal{D} \quad \mathcal{D} \quad = \mathbb{E}[X \mid \mathcal{D}]$

$\mathcal{D}_1 \subset \mathcal{D}_2$ $\qquad = \mathbb{E}[X \mid \mathcal{D}_1]$



9) Jensen ineq: recall "convex" function

$$\phi(\lambda x + (1-\lambda) y) \le \lambda \phi(x) + (1-\lambda)\phi(y)$$

ϕ - convex

$$\mathbb{E}[\phi(X) \mid \mathcal{D}] \ge \phi(\mathbb{E}[X \mid \mathcal{D}])$$

10) $$\mathbb{E}\left|\mathbb{E}[X \mid \mathcal{D}] - \mathbb{E}[Y \mid \mathcal{D}]\right|^p \qquad p \ge 1$$

$$\le \mathbb{E}|X-Y|^p$$

证: $$\text{LHS} = \mathbb{E}\left[\left|\mathbb{E}[X-Y \mid \mathcal{D}]\right|^p\right]$$

$$\overset{\text{Jensen}}{\le} \mathbb{E}\left[\mathbb{E}[|X-Y|^p \mid \mathcal{D}]\right]$$

$$= \mathbb{E}[|X-Y|^p]$$

11). $\mathbb{E}|X_n - X|^p \xrightarrow{n\to\infty} 0$

$$\Rightarrow \lim_{n\to\infty} \mathbb{E}\left[\left|\mathbb{E}[X_n \mid \mathcal{D}] - \mathbb{E}[X \mid \mathcal{D}]\right|^p\right] = 0.$$

$$\mathbb{E}\left[\left|\mathbb{E}[X-Y \mid \mathcal{D}]\right|^p\right] \qquad \phi = |\cdot|^p.$$

$$\le \mathbb{E}\left[\mathbb{E}[|X-Y|^p \mid \mathcal{D}]\right]$$

$$= \mathbb{E}[|X-Y|^p]$$