Math 714 - Fall 2020

Homework 2

A

(a) If $v \in \text{span}\{w_1, \cdots, w_n\}$, then there exists $\{\alpha_j\}_{j=1}^n such that$

$$v = \sum_{j=1}^n \alpha_j w_j$$

$$\frac{\langle v, w_j \rangle}{\|w_j\|^2} = \frac{\alpha_j \langle w_j, w_j \rangle}{\|w_j\|^2} = \alpha_j$$

$$v = \sum_{j=1}^n \alpha_j w_j = \sum_{j=1}^n \frac{\langle v, w_j \rangle}{\|w_j\|^2} w_j$$

(b) ii. For $n = 1$, $p_1 = r_1 - \frac{\langle r_1, p_0 \rangle}{\|p_0\|^2} p_0$.

$$\begin{aligned}\langle p_1, p_0 \rangle &= \langle r_1 - \frac{\langle r_1, p_0 \rangle}{\|p_0\|^2} p_0, p_0 \rangle \\ &= \langle r_1, p_0 \rangle - \frac{\langle r_1, p_0 \rangle}{\|p_0\|^2} \|p_0\|^2 \\ &= 0\end{aligned}$$

Suppose $n = k$ is true. When $n = k + 1$, for $0 \leqslant j \leqslant k$,

$$\begin{aligned}\langle p_{k+1}, p_j \rangle &= \langle r_{k+1}, p_j \rangle - \frac{\langle r_{k+1}, p_j \rangle}{\|p_j\|^2} \langle p_j, p_j \rangle \\ &= 0\end{aligned}$$

By induction, it is true for all $0 \leqslant j < n \leqslant n^* - 1$.

(c) i

$$\begin{aligned}\langle Av, w \rangle &= \sum_{n=1}^N \langle v, \phi_n \rangle \langle A\phi_n, w \rangle \\ &= \sum_{n=1}^N \langle v, \phi_n \rangle \langle \lambda_n \phi_n, w \rangle \\ &= \sum_{n=1}^N \lambda_n \langle v, \phi_n \rangle \langle \phi_n, w \rangle\end{aligned}$$

ii By the definition of positive definite matrix.

iii $v = \sum_{n=1}^N \alpha_n \phi_n$, then $\langle Av, v \rangle = \sum_{n=1}^N \lambda_n \alpha_n^2$, $\|v\|^2 = \sum_{n=1}^N \alpha_n^2$.
By $\lambda_1 \leqslant \cdots \leqslant \lambda_N$, we know

$$\sum_{n=1}^N \lambda_1 \alpha_n^2 \leqslant \sum_{n=1}^N \lambda_n \alpha_n^2 \leqslant \sum_{n=1}^N \lambda_N \alpha_n^2$$

$$\lambda_1 \|v\|^2 \leqslant \langle Av, v \rangle \leqslant \lambda_N \|v\|^2$$

iv

$$\|Av\|^2 = \langle Av, Av \rangle = \sum_{n=1}^{N} \alpha_n^2 \lambda_n^2 \leqslant \sum_{n=1}^{N} \alpha_n^2 \lambda_N^2 = \lambda_N^2 \|v\|^2$$
$$\|Av\| \leqslant \lambda_N \|v\|$$

(d)

$$\begin{aligned} p_{n+1} &= r_{n+1} + \beta_n p_n \\ &= r_n - \alpha_n \omega_n + \beta_n p_n \\ &= r_n - \alpha_n A p_n + \beta_n p_n \\ &= p_n - \beta_{n-1} p_{n-1} - \alpha_n A p_n + \beta_n p_n \end{aligned}$$

Thus,

$$p_{n+1} = (1 + \beta_n) p_n - \alpha_n A p_n - \beta_{n-1} p_{n-1}$$

(e) By Cayley-Hamilton theorem, $p(\lambda) = \det(\lambda I - A)$

$$p(\lambda) = A^N + \alpha_{N-1} A^{N-1} + \cdot + \alpha_1 A + (-1)^N \det |A| I_N = 0$$

Thus A^N is a linear combination of $I, A, A^2, \cdot, A^{N-1}$.

(f) i 1

ii 2

iii 3

iv 4

(g) i 1

ii 2

iii 3

iv 4

v 5

B