Thm: $f:[0,1]\to\mathbb{R}$, continuous.

Then there are polynomials f_n such that $f_n \to f$ uniformly on $[0,1]$.

Proof: $$f_n(x) \stackrel{def}{=} \sum_{i=0}^{n} \underbrace{\binom{n}{i} x^i (1-x)^{n-i}} f\left(\frac{i}{n}\right)$$

Fix $p\in[0,1]$ and let $X_1,\dots$ i.i.d Bernoulli(p)

$S_n = X_1+\dots+X_n$ has Binomial(n,p) distribution

We have $E[S_n]=np$ $\quad \operatorname{Var} S_n = np(1-p)$

$$E\left[f\left(\frac{S_n}{n}\right)\right] = f_n(p)$$

$$\left|f_n(p)-f(p)\right| = \left|E\left[f\left(\frac{S_n}{n}\right)\right]-f(p)\right|$$

$$= \left|E\left[f\left(\frac{S_n}{n}\right)-f(p)\right]\right| \le \underbrace{E\left[\left|f\left(\frac{S_n}{n}\right)-f(p)\right|\right]} =$$

$\delta>0$ fixed

$$A_n = \left\{\left|\frac{S_n}{n}-p\right|\le\delta\right\}$$

$$= E\Big[\underbrace{\left|f\left(\frac{S_n}{n}\right)-f(p)\right|}_{\text{small value}}\mathbb{1}_{A_n}\Big] + \underbrace{E\left[\left|f\left(\frac{S_n}{n}\right)-f(p)\right|\mathbb{1}_{A_n^c}\right]}_{\text{small probability}}$$

$$P\left(\left|\frac{S_n}{n} - p\right| > \delta \right) \le \frac{\mathrm{Var}\left(\frac{S_n}{n}\right)}{\delta^2} = \frac{\mathrm{Var}(X_1)}{n\delta^2}$$

$$= \frac{p(1-p)}{n\delta^2} \le \frac{1}{n\delta^2}$$

Let $M = \max_{x\in[0,1]} |f(x)| < \infty$ because f is cont on $[0,1]$.

$$E\left[\underbrace{\left| f\left(\frac{S_n}{n}\right) - f(p) \right|}_{\le 2M} 1_{A_n^c} \right] \le E\left[2M\, 1_{A_n^c} \right]$$

$$= 2M\, E\left[1_{A_n^c} \right] = 2M\, P(A_n^c)$$

$$\le \frac{2M}{n\delta^2}$$

$$E\left[\left| f\left(\frac{S_n}{n}\right) - f(p) \right| 1_{\left\{\left|\frac{S_n}{n} - p\right| \le \delta\right\}} \right]$$

f is uniform continuous. For a given $\varepsilon > 0$ we can choose $\delta > 0$ so that $|x-y| \le \delta$ implies $|f(x) - f(y)| \le \varepsilon$. $x, y \in [0,1]$

$$E\left[\left| f\left(\frac{S_n}{n}\right) - f(p) \right| 1_{A_n} \right] \le E\left[\varepsilon\, 1_{A_n} \right] = \varepsilon P(A_n) \le \varepsilon$$

$$E\left[\left|f\left(\tfrac{S_n}{n}\right)-f(p)\right|\right] \le \frac{2M}{n\cdot\delta^2}+\varepsilon$$

$$\limsup_{n\to\infty}\sup_{p\in[0,1]} E\left|f\left(\tfrac{S_n}{n}\right)-f(p)\right| \le \varepsilon$$

for any $\varepsilon>0$

This implies that $f_n \to f$ uniformly. ∎

Thm (WLLN) $X_1, X_2, \dots$ i.i.d with EX_1 finite then $\dfrac{\sum_{i=1}^n X_i}{n} \xrightarrow{P} E[X_1]$.

We have seen that if $E[|X|^p]<\infty$ then we can bound $P(|X|\ge y)$ using Markov's inequality.

A statement in the other direction:

Claim: $X\ge 0$. Then

$$\underline{E[X^2]} = \int_{-\infty}^{\infty} x^2\, dQ_X = \int_0^\infty x^2\, dQ_X = \int_0^\infty \int_0^x 2y\, dy\, dQ_X$$

$0<y<x$

$$= \int_0^\infty \underbrace{\int_y^\infty 2y\, dQ_X}\, dy = \underline{\int_0^\infty 2y\, P(X\ge y)\, dy}$$

Similar formula for $E[X^p]$, $E[H(X)]$
$H' > 0$

Proof of WLLN

$$X_{n,M} = X_n \mathbb{1}(|X_n| \le M) = \begin{cases} X_n & \text{if } |X_n| \le M \\ 0 & \text{otherwise} \end{cases}$$

Goal: $\frac{S_n}{n} \xrightarrow{P} E[X_1]$ $\quad S_n = \sum_{i=1}^{n} X_i$

$$\tilde{S}_n = \sum_{i=1}^{n} X_{i,n} \qquad \mu_n = E[X_{1,n}] = E[X_1 \mathbb{1}(|X_1| \le n)]$$

Then $\mu_n \to E[X_1]$ as $n \to \infty$.
(Dominated convergence for the sequence $X_1 \cdot \mathbb{1}(|X_1| \le n)$.)

We will show that $\frac{S_n}{n} - \mu_n \xrightarrow{P} 0$

Since $\mu_n - E[X_1] \to 0$ this implies
$\frac{S_n}{n} - E[X_1] \xrightarrow{P} 0$ and the theorem.

Fix $\varepsilon > 0$

$$\boxed{P\left(\left|\frac{S_n}{n} - \mu_n\right| > \varepsilon\right)} \le \underline{P(S_n \ne \tilde{S}_n)} + P\left(\left|\frac{\tilde{S}_n}{n} - \mu_n\right| > \varepsilon\right)$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{A}$$

$$P\left(\left|\frac{\tilde S_n}{n} - \mu_n\right| > \varepsilon\right)$$
$$\vee$$

$$B = \{ S_n \neq \tilde S_n \}$$

$$\underset{P(AB) + P(AB^c)}{\overset{\shortparallel}{P(A)}} \le \underset{P(S_n \neq \tilde S_n)}{\overset{\wedge}{P(B)}} + \underset{P\left(\left|\frac{\tilde S_n}{n} - \mu_n\right| > \varepsilon,\ S_n = \tilde S_n\right)}{\overset{\shortparallel}{P(AB^c)}}$$

$$P(S_n \neq \tilde S_n) \le n P(|X_1| > n) \xrightarrow[\text{as } n\to\infty]{} 0$$

$$S_n = \sum_{i=1}^n X_i \qquad \tilde S_n = \sum_{i=1}^n X_{i,n}$$

$$\{S_n \neq \tilde S_n\} \subseteq \bigcup_{i=1}^n \underset{\{|X_i| > n\}}{\overset{\shortparallel}{\{X_i \neq X_{i,n}\}}}$$

$$E[n \mathbb{1}_{|X_1|>n}]$$
$$\shortparallel$$
$$n P(|X_1| > n) \le E\left[|X_1| \mathbb{1}_{|X_1|>n}\right] \longrightarrow 0$$

dominated convergence

$$\underbrace{P\left(\left|\frac{\tilde S_n}{n} - \mu_n\right| > \varepsilon\right)}_{\mu_n = E[X_{1,n}]} \le \frac{E\left(\frac{\tilde S_n}{n} - \mu_n\right)^2}{\varepsilon^2} = \frac{\mathrm{Var}\left(\frac{\tilde S_n}{n}\right)}{\varepsilon^2}$$

$$= \frac{\mathrm{Var}(X_{1,n})}{n\varepsilon^2}$$

$$\le \frac{E[X_{1,n}^2]}{n\varepsilon^2}$$

$$E[X_{1,n}^2] = E[X_1^2 \, \mathbb{1}(|X_1| \le n)]$$

$$= \int_0^\infty 2y \, P(|X_{1,n}| \ge y) \, dy$$

$$= \int_0^n 2y \, P(|X_{1,n}| \ge y) \, dy$$

$$\le \int_0^n 2y \, P(|X_1| \ge y) \, dy$$

$$P\left(\left| \frac{\tilde{S}_n}{n} - \mu_n \right| > \varepsilon \right) \le \frac{1}{\varepsilon^2} \cdot \frac{1}{n} \int_0^n 2y \, P(|X_1| \ge y) \, dy$$

We know that $y P(|X_1| > y) \to 0$ as $y \to \infty$. Choose $M > 0$ so that $y P(|X_1| > y) < \delta$ for $y \ge M$.

$$\frac{1}{n} \int_0^n 2y \, P(|X_1| \ge y) \, dy \le \frac{1}{n} \left(\int_0^M 2y \, P(|X_1| \ge y) \, dy + \int_M^n 2\delta \, dy \right)$$

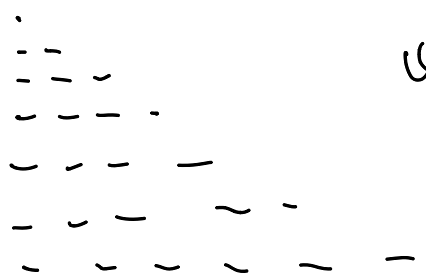
$$\limsup_{n \to \infty} \frac{1}{n} \int_0^n 2y \, P(|X_1| \ge y) \, dy \le 2\delta$$

This shows that $\frac{S_n}{n} - \mu_n \xrightarrow{P} 0$, and $\frac{S_n}{n} \xrightarrow{P} E[X_1]$.

Other questions: when will $\frac{S_n - a_n}{b_n}$ converge?

Triangular arrays

$\{X_{n,k}\}$ $\quad n \geq 1, \quad 1 \leq k \leq m_n$



We might be interested in a situation where $X_{n,1}, \dots X_{n,m_n}$ are iid, and we would want to find the limit of

$$\frac{\sum_{k=1}^{m_n} X_{n,k} - a_n}{b_n}$$

Theorem: $X_{n,k}$ is a triangular array with independent rows ($X_{n,1}, \dots, X_{n,m_n}$ are indep)

Assume that there is a sequence b_n with

- $$\underbrace{\sum_{k=1}^{m_n} P(|X_{n,k}| > b_n)}\longrightarrow 0$$ as $n \to \infty$

- $$\frac{1}{b_n^2} \sum_{k=1}^{m_n} E\left[X_{n,k}^2 \mathbb{1}(|X_{n,k}| \le b_n)\right] \to 0$$

Then
$$\frac{\sum_{k=1}^{m_n} X_{n,k} - a_n}{b_n} \xrightarrow{P} 0$$

$$a_n = \sum_{k=1}^{m_n} E\left[X_{n,k} \mathbb{1}(|X_{n,k}| \le b_n)\right].$$

Proof: Exactly the same as WLLN

$$\tilde{S}_n = \sum_{k=1}^{m_n} X_{n,k} \mathbb{1}(|X_{n,k}| \le b_n)$$

$$P\left(\left|\frac{S_n - a_n}{b_n}\right| > \varepsilon\right) \le \underset{\substack{\uparrow \\ \text{estimate using} \\ \text{the first cond}}}{P(S_n \ne \tilde{S}_n)} + \underset{\substack{\uparrow \\ \text{Chebyshev}}}{P\left(\left|\frac{\tilde{S}_n - a_n}{b_n}\right| > \varepsilon\right)}$$

∎

Examples: 1, Coupon collector

2, Cycles in uniform permutation of $\{1, \dots, n\}$

3, St. Petersburg problem

Read these in Durrett!!

(Ex 2.2.7, 2.2.8, 2.2.16