CS 726: Homework #1

Posted: 09/10/2020, due: 09/21/2020 at 5pm on Canvas

Please typeset or write your solutions neatly! If we cannot read it, we cannot grade it.

Q 1. All ℓ_p norms are related via the following inequalities:

$$(\forall q > p \geq 1)(\forall \mathbf{x} \in \mathbb{R}^d): \; \|\mathbf{x}\|_q \leq \|\mathbf{x}\|_p \leq d^{\frac{1}{p}-\frac{1}{q}} \|\mathbf{x}\|_q.$$

Provide examples of non-zero vectors (vectors whose elements are not all zeros) for which these inequalities are tight (satisfied with equality).

Note: Obviously, the left and the right inequality cannot be both satisfied at the same time, so you need to come up with two separate vectors for which the left and the right inequalities are tight. [10pts]

Q 2. Let p, q be such that $p \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that you are given a differentiable function $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ and a constant L_p such that:

$$(\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d): \; \|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\|_q \leq L_p \|\mathbf{x} - \mathbf{y}\|_p.$$

What is the smallest constant L_2 (as a function of p and L_p) for which the following holds:

$$(\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d): \; \|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\|_2 \leq L_2 \|\mathbf{x} - \mathbf{y}\|_2? \qquad [7pts]$$

How large can L_2 be in the worst case (depending on the choice of p)? [3pts]

Q 3 (Jensen's Inequality)**.** Let $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ be a convex function. Prove that for any sequence of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in \mathbb{R}^d$ and any sequence of non-negative scalars $\alpha_1, \alpha_2, \ldots, \alpha_k \geq 0$ such that $\sum_{i=1}^k \alpha_i = 1$ we have:

$$f\Big(\sum_{i=1}^k \alpha_i \mathbf{x}_i\Big) \leq \sum_{i=1}^k \alpha_i f(\mathbf{x}_i). \qquad [10pts]$$

Q 4. Let $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ be an extended real valued *convex* function.

(i) Assuming that f is lower semicontinuous, prove the following: if there exists a point $\mathbf{x}$ such that $f(\mathbf{x}) = -\infty$, then f is not real-valued anywhere – it equals either $-\infty$ or $+\infty$ everywhere.
Hint: Argue first that, under these assumptions, $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, $f(\mathbf{y}) \leq \underline{\lim}_{\alpha \downarrow 0} f(\alpha \mathbf{x} + (1-\alpha)\mathbf{y})$. [10pts]

(ii) If, $\forall \mathbf{x} \in \mathbb{R}^d$, $|f(\mathbf{x})| \leq M$, for some constant $M < \infty$, then f must be a constant function (i.e., taking the same value for all $\mathbf{x} \in \mathbb{R}^d$). [10pts]

Q 5. Let $f : \mathbb{R}^d \to \bar{\mathbb{R}}$. Prove that f is convex if and only if its epigraph, defined as

$$\mathrm{epi}(f) = \{(\mathbf{x}, a) : \mathbf{x} \in \mathbb{R}^d, a \in \mathbb{R}, f(\mathbf{x}) \leq a\},$$

is convex. [15pts]

Q 6. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex continuously differentiable function. Using the definition of convexity from the class and properties of directional derivatives, prove that it must be $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d$:

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle. \qquad [10pts]$$

Q 7. Let $\mathbf{A}$ be a real symmetric $d \times d$ matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d$. Prove that, $\forall \mathbf{x} \in \mathbb{R}^d$:

(i) $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq \lambda_1\|\mathbf{x}\|_2^2$; [5pts]

(ii) $\mathbf{x}^T\mathbf{A}\mathbf{x} \leq \lambda_d\|\mathbf{x}\|_2^2$. [5pts]

Q 8. Let $\mathbf{A}$ be a $d \times d$ matrix defined by: $A_{ii} = 2$ for $1 \leq i \leq d$, $A_{i,i+1} = A_{i+1,i} = -1$, for $1 \leq i \leq d-1$ and $A_{d,1} = A_{1,d} = -1$. That is, $\mathbf{A}$ is defined as:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}.$$

Is $\mathbf{A}$ positive semidefinite (PSD)? What is its smallest eigenvalue? Justify your answers. [15pts]