MATH 735 - Fall 2020

Homework 2

Due : 11/04, 2020

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Problem 1

X, Y are two independent Brownian motions, compute $[X, Y]$.

Proof. By (2.13) in *Timo's notes*.

$$[X, Y]_t = \lim_{|\pi| \to 0} \sum_i (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})$$

We need prove $\mathbb{E}\left[\sum_i (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})\right] \to 0$ which is

$$\sum_i \mathbb{E}\left[X_{t_{i+1}} Y_{t_{i+1}}\right] + \sum_i \mathbb{E}\left[X_{t_i} Y_{t_i}\right] - \sum_i \mathbb{E}\left[X_{t_i} Y_{t_{i+1}}\right] - \sum_i \mathbb{E}\left[X_{t_{i+1}} Y_{t_i}\right] \to 0$$

By the independence of X and Y, all the expectations above are 0. So

$$[X, Y] = 0 \text{ if } X, Y \text{ are two independent Brownian motions}$$

□

Problem 2

Compute the quadratic variations [N] and [M] where N is Poisson process and M is compensated Poisson process.

1.

$$[N]_t = \sum_{0 \leqslant s \leqslant t} (\Delta N_s)^2 = N_t$$
$$[N] = N$$

2.

$$M = N - \lambda t$$

By Lemma A.10 and Lemma A.11, we know that $[f](T) = 0$ if f is continuous. So we have

$$(\Delta(N_s - \lambda s))^2 = (\Delta N_s)^2$$

Thus,

$$[M] = N$$

Problem 3

Suppose M is a right-continuous square-integrable martingale with stationary independent increments: for all $s,t \geqslant 0, M_{s+t} - M_s$ is independent of $\mathcal{F}_s$ and has the same distribution as $M_t - M_0$. Then $\langle M \rangle_t = t \cdot E[M_1^2 - M_0^2]$

Proof. The deterministic, continuous function $t \to t \cdot E[M_1^2 - M_0^2]$ is predictable. For any $t > 0$ and integer k

$$E[M_{kt}^2 - M_0^2] = \sum_{j=0}^{k-1} E[M_{(j+1)t}^2 - M_{jt}^2] = \sum_{j=0}^{k-1} E[(M_{(j+1)t} - M_{jt})^2] = kE[(M_t - M_0)^2] = kE[M_t^2 - M_0^2]$$

Using this twice, for any rational k/n,

$$E[M_{k/n}^2 - M_0^2] = kE[M_{1/n}^2 - M_0^2] = (k/n)E[M_1^2 - M_0^2]$$

Given an irrational $t > 0$, pick rationals $q_n \to t$. Fix $T \geqslant q_m$. By right-continuity of paths, $M_{q_m} \to M_t$ almost surely. Uniformly integrability of $\{M_{q_m}^2\}$ follows by the submartingale property

$$0 \leqslant M_{q_m}^2 \leqslant E[M_T^2 | \mathcal{F}_{q_m}]$$

and Lemma B.16. Uniformly integrability gives convergence of expectations $E[M_{q_m}^2] \to E[M_t^2]$. Applying this above gives

$$E[M_t^2 - M_0] = tE[M_1^2 - M_0^2]$$

Now we can check the martingale property.

$$\begin{aligned}
E[M_t^2 | \mathcal{F}_s] &= M_s^2 + E[M_t^2 - M_s^2 | \mathcal{F}_s] \\
&= M_s^2 + E[(M_t - M_s)^2 | \mathcal{F}_s] \\
&= M_s^2 + E[(M_{t-s} - M_0)^2] \\
&= M_s^2 + E[M_{t-s}^2 - M_0^2] \\
&= M_s^2 + (t-s)E[M_1^2 - M_0^2]
\end{aligned}$$

□

Problem 4

$$C_n \leqslant \sum_{i=1}^{n} (B_{t_i} - B_{t_{i-1}})^2 h(B_{t_{i-1}}, B_{t_i})$$

We need prove $(B_{t_i} - B_{t_{i-1}})^2$ converges to 0.

$$\begin{aligned}
E[(B_{t_i} - B_{t_{i-1}})^2] &= \Delta t \to 0 \\
E[(B_{t_i} - B_{t_{i-1}})^4] &= 3(\Delta t)^2 \to 0
\end{aligned}$$

So, $(B_{t_i} - B_{t_{i-1}})^2$ converges to 0. Thus $C_n \to 0$.

Exercise 3.1

By the definition of Itô integral. Consider a partation $\pi : 0 = t_0 < t_1 < \cdots < t_n = t$.

$$tB_t = \sum_j s_j \Delta B_j + \sum_j B_j \Delta s_j$$

$$\lim_{|\pi| \to 0} \sum_j s_j \Delta B_j = \int_0^t s dB_s$$

$$\lim_{|\pi| \to 0} \sum_j B_j \Delta s_j = \int_0^t B_s ds$$

From the definition,

$$\int_0^t s dB_s = tB_t - \int_0^t B_s ds$$

Exercise 3.3

1.

$$\begin{aligned}\mathbb{E}\left[X_t|\mathcal{H}_s\right] &= \mathbb{E}\left[\mathbb{E}[X_t|\mathcal{N}_s]|\mathcal{H}_s\right] \\ &= \mathbb{E}\left[X_s|\mathcal{H}_s\right] \\ &= X_s\end{aligned}$$

2.

$$\mathbb{E}[X_t] = \mathbb{E}\left[\mathbb{E}[X_t|\mathcal{H}_0]\right] = \mathbb{E}[X_0]$$

3. The probability of winning or losing in gambling is 1/2. If win $X_t = t$, if loss $X_t = -t$. Consider the expectation,

$$\mathbb{E}[X_t] = 0$$

$$\mathbb{E}[X_t|\mathcal{H}_s] = t, X_s \geqslant 0$$

$$\mathbb{E}[X_t|\mathcal{H}_s] = -t, X_s < 0$$

Thus, X_t is not a martingale.

Exercise 3.4

1. No. Because the expectation

$$\mathbb{E}[X_t|\mathcal{H}_s] = X_s + 4(t-s)$$

2. No. Because the expectation

$$\begin{aligned}\mathbb{E}[X_t|\mathcal{H}_s] &= \mathbb{E}[B_t^2|\mathcal{H}_s] \\ &= \mathbb{E}\left[B_s^2 + B_t^2 - B_s^2|\mathcal{H}_s\right] \\ &= B_s^2 + \mathbb{E}[B_t^2 - B_s^2|\mathcal{H}_s] \\ &= B_s^2 + \mathbb{E}\left[2B_s(B_t - B_s) + (B_t - B_s)^2|\mathcal{H}_s\right] \\ &= B_s^2 + 2B_s\mathbb{E}\left[B_t - B_s|\mathcal{H}_s\right] + \mathbb{E}\left[(B_t - B_s)^2|\mathcal{H}_s\right] \\ &= X_t + (t-s)\end{aligned}$$

3. Yes.

$$\begin{aligned}\mathbb{E}[X_t|\mathcal{H}_s] &= \mathbb{E}\left[t^2B_t - 2\int_0^t sB_s ds|\mathcal{H}_s\right] \\ &= \mathbb{E}\left[(t^2 - s^2)B_s + t^2(B_t - B_s) - 2\int_s^t uB_u du|\mathcal{H}_s\right] + X_s \\ &= X_s + (t^2 - s^2)B_s - 2\int_s^t u\left(B_s + \mathbb{E}\left[B_u - B_s|\mathcal{H}_s\right]\right)du \\ &= X_s + (t^2 - s^2)B_s - 2B_s\int_s^t udu \\ &= X_s\end{aligned}$$

4. Yes.

$$\begin{aligned}\mathbb{E}[X_t|\mathcal{H}_s] &= \mathbb{E}[B_1(t)B_2(t)|\mathcal{H}_s] \\ &= \mathbb{E}[(B_1(s) + (B_1(t) - B_1(s))(B_2(s) + (B_2(t) - B_2(s))))\,|\mathcal{H}_s] \\ &= B_1(s)B_2(s) + B_1(s)\mathbb{E}[B_2(t) - B_2(s)|\mathcal{H}_s] + B_2(s)\mathbb{E}[B_1(t) - B_1(s)|\mathcal{H}_s] + \mathbb{E}[B_1(t) - B_1(s)|\mathcal{H}_s]\mathbb{E}[B_2(t) - B_2(\\ &= B_1(s)B_2(s) \\ &= X_s\end{aligned}$$

Exercise 3.7

Proof. 1. When n = 1

$$I_1(t) = \int_0^t I_0(s)dB_s = B_t = t^{\frac{1}{2}} h_1\left(\frac{B_t}{\sqrt{t}}\right)$$

2. When n = 2

$$I_2(t) = 2\int_0^t I_1(s)dB_s = 2\int_0^t B_s dB_s = B_t^2 - t = t^{\frac{2}{2}} h_2\left(\frac{B_t}{\sqrt{t}}\right)$$

3. When n = 3

$$I_3(t) = 3\int_0^t I_2(s)dB_s = 3\int_0^t (B_s^2 - s)dB_s = B_t^3 - 3tB_t = t^{\frac{3}{2}} h_3\left(\frac{B_t}{\sqrt{t}}\right)$$

4. For all n, let

$$H_n(x) = \frac{t^{\frac{n}{2}}}{n!} h_n\left(\frac{x}{\sqrt{t}}\right)$$

$$\begin{aligned} H_{n+1}(x) &= \frac{t^{\frac{n+1}{2}}}{(n+1)!} h_{n+1}\left(\frac{x}{\sqrt{t}}\right) \\ &= \frac{t^{\frac{n+1}{2}}}{(n+1)!}\left[\frac{x}{\sqrt{t}} h_n\left(\frac{x}{\sqrt{t}}\right) - nh_{n-1}\left(\frac{x}{\sqrt{t}}\right)\right] \\ &= \frac{x}{n+1} H_n(x) - \frac{t}{n+1} H_{n-1}(x) \end{aligned}$$

By the properties of Hermite polynomials,

$$H_n'(x) = H_{n-1}(x) = -\frac{1}{2} H_{n-2}(x)$$

Then we have

$$dH_{n+1}(B_t) = H_n(B_t)dB_t$$

That means

$$H_{n+1}(B_t) = \int_0^t H_n(B_t)dB_t$$

$$\frac{t^{\frac{n+1}{2}}}{(n+1)!} h_{n+1}\left(\frac{B_t}{\sqrt{t}}\right) = \int_0^t \frac{t^{\frac{n}{2}}}{n!} h_n\left(\frac{B_t}{\sqrt{t}}\right) dB_t$$

□

Exercise 4.1

1. $g(t,x) = x^2$ then $dB_t^2 = 2B_t dB_t + dt$.
   $u = 1, v = 2B_t$.

2. $g(t,x) = 2 + t + e^x$ then $dX_t = dt + e^{B_t} dB_t + \frac{1}{2} e^{B_t} dt = (1 + \frac{1}{2} e^{B_t})dt + e^{B_t} dB_t$.
   $u = 1 + e^{B_t}, v = e^{B_t}$.

3. $g(t, x_1, x_2) = x_1^2 + x_2^2$ then $dX_t = 2dt + 2B_1 dB_1(t) + 2B_2 dB_2(t)$.
   $u = 2, v = \begin{bmatrix} 2B_1 \\ 2B_2 \end{bmatrix}$.

4. $dX_t = (dt, dB_t)$.

5. $dX_t = (dB_1(t) + dB_2(t) + dB_3(t), 2B_2(t)dB_t(2) + 2dt - B_1(t)dB_3(t) - B_3(t)dB_1(t))$.

Exercise 4.11

1.

$$
\begin{aligned}
dX_t &= \frac{1}{2}e^{\frac{1}{2}t}\cos{(B_t)}dt - e^{\frac{1}{2}t}\sin{(B_t)}dB_t - \frac{1}{2}e^{\frac{1}{2}t}\cos{(B_t)}dt \\
&= -e^{\frac{1}{2}t}\sin{(B_t)}dB_t
\end{aligned}
$$

So, this is martingale.

2.

$$
dX_t = e^{\frac{1}{2}t}\cos{(B_t)}
$$

So, this is martingale.

3. Let $f(x,t) = (x+t)e^{-x-\frac{1}{2}t}$

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= e^{-x-\frac{1}{2}t} - \frac{1}{2}(x+t)e^{-x-\frac{1}{2}t} \\
&= \left(1 - \frac{1}{2}x - \frac{1}{2}t\right)e^{-x-\frac{1}{2}t} \\
\frac{\partial f}{\partial x} &= \left(1-x-t\right)e^{-x-\frac{1}{2}t} \\
\frac{\partial^2 f}{\partial x^2} &= \left(x+t-2\right)e^{-x-\frac{1}{2}t}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
dX_t &= \left(1-B_t-t\right)e^{-B_t\frac{1}{2}t}dB_t + 0 \\
&= \left(1-B_t-t\right)e^{-B_t-\frac{1}{2}t}dB_t
\end{aligned}
$$

So, this is martingale.