Brownian Motion: more properties and calculations

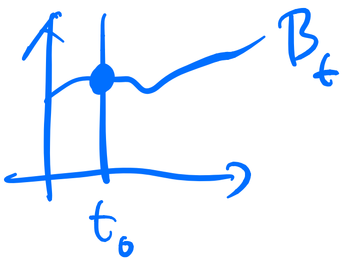
Ex 1. Fix $h>0$. $B_{t+h} - B_t$ distribution is indep of t

"Stationary"

pf: $\underline{B_{t_1+h}} - \underline{B_{t_1}} \sim N(0, \underbrace{(t_1+h)-t_1}_{h})$

$B_{t_2+h} - B_{t_2} \sim N(0, \underline{h})$

Ex 2.

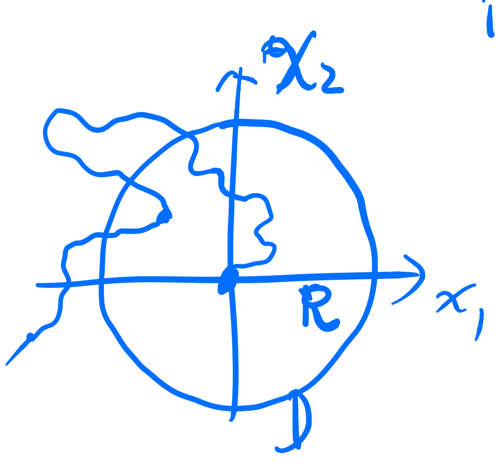


Define

$\tilde{B}_t \stackrel{\text{def}}{=} B_{t_0+t} - B_{t_0}$

is a BM.

Ex 3.



$P(B_t \in D) = ?$

$B_t^1 \sim N(0, t)$

$B_t^2 \sim$

$$P(B_t \in D) = \iint_D \frac{1}{2\pi t} \underline{e^{-\frac{|x|^2}{2t}}}\, dx$$

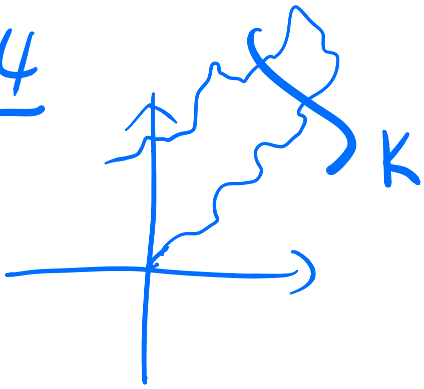
$$\begin{cases} x_1 = r\cos\theta \\ x_2 = r\sin\theta \end{cases}$$

$$= \frac{1}{2\pi t}\int_0^{2\pi}\int_0^R e^{-\frac{r^2}{2t}}\, r\,dr\,d\theta$$

$$= \frac{(2\pi)}{2\pi t} \left. -e^{-\frac{r^2}{2t}}\cdot t \right|_0^R$$

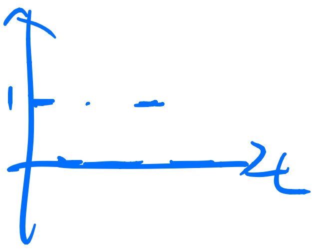
$$= 1 - e^{-R^2/2t}$$

Ex4



$$\mathbb{E}\left(\text{total length of time that } B \in K\right) = 0.$$

$$\parallel$$

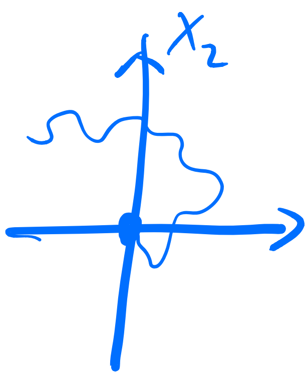


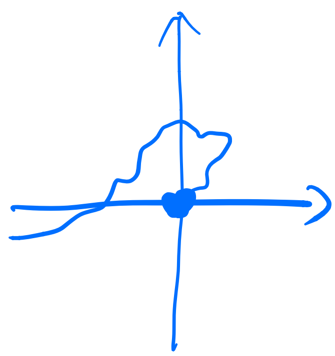
$$\int_0^\infty \mathbb{P}(B_t \in K)\,dt$$

$$= \int_0^\infty \int_K \frac{1}{(2\pi t)^{n/2}} e^{-\frac{|x|^2}{2t}}\,dx\,dt$$

$$\parallel$$
$$0$$

Ex5.





If B is n-dim BM,
U is an orthogonal matrix ($UU^T = I$)

Then $\tilde{B}_t = U B_t$

pf: $\mathbb{P}(\tilde{B}_{t_1} \in F_1, \cdots, \tilde{B}_{t_k} \in F_k)$

$\stackrel{?}{=} \mathbb{P}(B_{t_1} \in F_1, \cdots, B_{t_k} \in F_k)$

$$\text{LHS} = \mathbb{P}(B_{t_1} \in U^{-1}F_1, \cdots, B_{t_k} \in U^{-1}F_k)$$

$$= \int_{U^{-1}F_1 \times \cdots \times U^{-1}F_k} P(t_1, 0, x_1)\, P(t_2 - t_1, x_1, x_2) \cdots P(t_k - t_{k-1}, x_{k-1}, x_k)\, dx_1 \cdots dx_k$$

$y_i = U x_i$

$$= \int_{F_1 \times \cdots \times F_k} P(t_1, 0, y_1) \cdots P(t_k - t_{k-1}, y_{k-1}, y_k)\, dy_1 \cdots dy_k$$

$$= \text{RHS}.$$

$$P(\overbrace{t_k - t_{k-1}}^{t}, x_{k-1}, x_k)$$

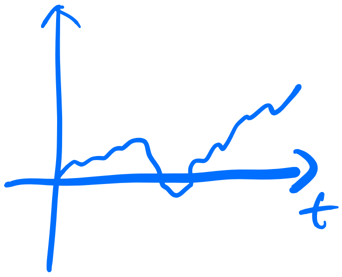
$$= \frac{1}{(2\pi t)^{n/2}} e^{-\frac{(x_k - x_{k-1})^2}{2t}}$$

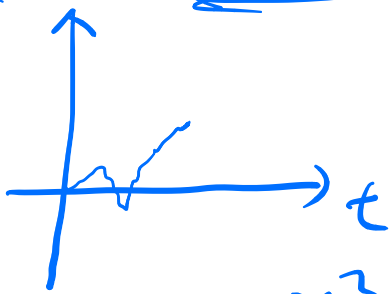
$$= \frac{1}{(2\pi t)^{n/2}} e^{-\frac{(y_k - y_{k-1})^2}{2t}}$$

Ex6: If B is Id BM

$\tilde{B}_t \stackrel{def}{=} \frac{1}{\lambda} B_{\lambda^2 t}$ is also BM.

$\lambda > 0$ const.





key: $\frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{t}} = p(t,x)$

$$p(\lambda^2 t, \lambda x) = \frac{1}{\sqrt{2\pi\lambda^2 t}} e^{-\frac{(\cancel{\lambda} x)^2}{\cancel{\lambda^2} t}}$$

$$= \frac{1}{\lambda} \, p(t, x)$$