Probability space $(\Omega, \mathcal{F}, \mathbb{P})$

Ω: sample space (set of possible outcomes)

$\mathcal{F}$: set of events (subsets of Ω that we can "measure")
σ-field

$\mathbb{P}$: probability measure on $\mathcal{F}$

Word problem:

1) identify $(\Omega, \mathcal{F}, \mathbb{P})$
2) identify the event in question
3) Try to compute $\mathbb{P}(\text{event})$

Ex: We roll two dice.
What's the prob that the sum is even?

$\Omega = \{ (a_1, a_2) : 1 \le a_i \le 6 \}$

$\mathcal{F} = 2^{\Omega}$ $\quad \mathbb{P} : \mathcal{F} \to [0,1]$?

$$\mathbb{P}\left(\{(a_1, a_2)\}\right) = \frac{1}{36}$$

$$\# \Omega = 36$$

$$A = \{\text{the sum is even}\}$$

$$= \{(a_1, a_2) : 1 \le a_i \le 6,\ a_1 + a_2 \text{ is even}\}$$

$$\mathbb{P}(A) = \sum_{(a_1,a_2) \in A} \mathbb{P}\left(\{(a_1, a_2)\}\right) = \frac{\# A}{36}$$

$A \cup B \approx$ "A or B"

$A \cap B \approx$ "A and B"

$A^c \approx$ "not A"

Examples

1. If Ω is finite or countably infinite (nonempty). If all singletons are events then $\mathbb{P}$ can be described via the probabilities of the singletons.

$\Omega = \{\omega_1, \omega_2, \dots\}$ $\quad p_k = \mathbb{P}(\{\omega_k\}) \in [0,1]$

$$\sum_{\xi} P_\xi = 1$$

$A \subset \Omega \qquad P(A) = \sum_{\omega_\xi \in A} P_\xi$

2. A uniformly chosen number on $[0,1]$.

$\Omega = [0,1]$

$\mathcal{F} = ?$ intervals should be included

$P([a,b]) = b - a$

$0 \le a < b \le 1$

$\mathcal{F} =$ σ-field generated by the intervals
(set of Borel sets in $[0,1]$)

$\mathbb{P}$: Lebesgue measure

$$\underbrace{[0,1]}_{\Omega}, \ \underbrace{\mathcal{B}}_{\mathcal{F}}, \ \mathbb{P})$$

Lebesgue measure

More general version:

Let $\Omega \subset \mathbb{R}^n$ be a Borel set with a finite Lebesgue measure.

$$(\Omega, \underset{\substack{\uparrow \\ \text{Borel subsets of } \Omega}}{\mathcal{B}}, \mathbb{P}) \qquad \mathbb{P}(A) = \frac{|A|}{|\Omega|}$$

"Uniformly chosen point from Ω"

Two more examples

3) Product spaces

$(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$ probability spaces

$i \in I \in$ finite / infinite

$$\Omega = \times_i \Omega_i$$

$$\mathcal{F} = \sigma(\times_i \mathcal{F}_i)$$

$$\mathbb{P} = \times \mathbb{P}_i$$

4. $(\Omega, \mathcal{F}, \mathbb{P})$ is a prob space

fixed $\to B \in \mathcal{F}$, $P(B) > 0$ $\quad AB = A \cap B$

$$Q(A) = \frac{P(AB)}{P(B)} = P(A \mid B)$$

$A \in \mathcal{F}$ "Conditional prob of A given B"

$(\Omega, \mathcal{F}, Q)$ is also a prob space.

"Hat-problem"

Suppose that n guests arrive to a party (each with a hat). They take off their hats. At the end of the party everybody chooses a hat randomly.

What's the probability that nobody gets their own hat? $n \to \infty$?

Simple properties of probability measures

$\mathbb{P} : \mathcal{F} \to [0, 1]$ $\quad\quad P(\Omega) = 1$

$A_1, A_2, \ldots \in \mathcal{F}$ disjoint

$$P(\bigcup_i A) = \sum_i P(A_i)$$

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1) $P(\emptyset) = 0$

$$P(A) + P(A^c) = 1$$

$$A \cup A^c = \Omega$$

2) If $A \subset B$ then $P(A) \leq P(B)$

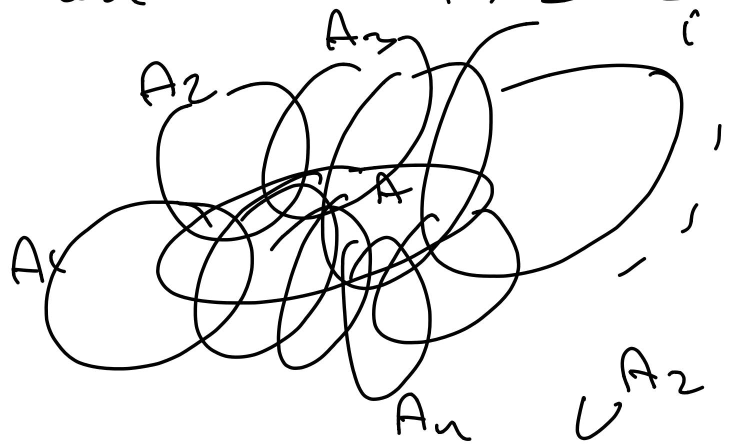
$B \setminus A = BA^c$

$B = \underbrace{A \cup BA^c}_{\text{disjoint}}$ $\quad\quad P(B) = P(A) + P(\underbrace{BA^c}_{\geq 0}) \geq P(A)$

3) Subadditive property

A, $A_1, A_2, \ldots$ with $A \subset \bigcup_i A_i$

Then $P(A) \le \sum_i P(A_i)$



$$\bigcup A_i = \underbrace{A_1}_{B_1} \cup \underbrace{(A_2 A_1^c)}_{B_2} \cup \underbrace{(A_3 (A_1 \cup A_2)^c)}_{B_3} \cup \dots$$

$\cup A_2$ $\quad$ $\subset A_3$

disjoint

$B_1 = A_1 \qquad B_n = A_n (A_1 \cup \dots \cup A_{n-1})^c$

These are disjoint. $\bigcup A_i = \bigcup B_i$

$A_k \supset B_k$

$$\boxed{\bigcup_k B_k = \bigcup_k (B_k A \cup B_k A^c)}$$

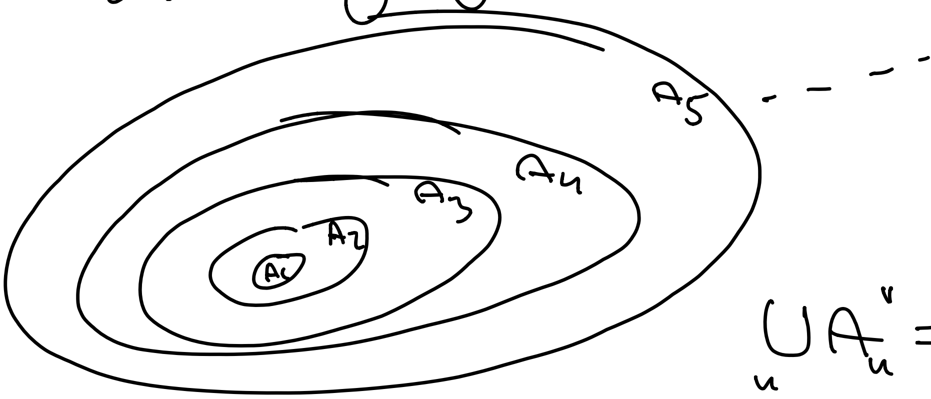
$$P(A) \le P\left(\bigcup B_k\right) = \sum_k P(B_k) \le \sum_k P(A_k)$$

$A \subset \bigcup B_k$

3, $A_1 \subset A_2 \subset A_3 \subset \dots$.

$$\lim_{n\to\infty} P(A_n) = P\Big(\bigcup_n A_n\Big)$$

"Continuity of probability measure"



$\bigcup_n A_n$ "=" $\lim_{n\to\infty} A_n$

$A_n \nearrow \bigcup_n A_n$

$B = \bigcup_n A_n \qquad \bigcup_{i=1}^{n} B_i = A_n$

$B_1 = A_1 \qquad B_i = A_i \setminus A_{i-1} = A_i A_{i-1}^c$

$B_1, B_2, \dots$ are disjoint and $B = \bigcup_i B_i$.

$$P(B) = \sum_{i=1}^{\infty} P(B_i)$$

$$\overset{\shortparallel}{P\Big(\bigcup_n A_n\Big)} = \lim_{n\to\infty} \sum_{i=1}^{n} P(B_i)$$

$$= \lim_{n\to\infty} P\Big(\bigcup_{i=1}^{n} B_i\Big) = \lim_{n\to\infty} P(A_n)$$

4) $A_1 \supset A_2 \supset A_3 \supset \dots$

$$\lim_{n\to\infty} P(A_n) = P\left(\bigcap_{n=1}^{\infty} A_n\right)$$

$$A_n \searrow \bigcap_{n=1}^{\infty} A_n$$

We can prove this from the previous case by taking complements.

Probability measures on $\mathbb{R}$ (or $\mathbb{R}^d$)

$\mathbb{P}$ is a probability measure on $\mathbb{R}$

Set $F(x) := \mathbb{P}\big((-\infty, x]\big)$

Then
- F is non decreasing
- right continuous $\quad a<b \quad (-\infty,a] \subset (-\infty,b]$
- $\lim_{x\to-\infty} F(x) = 0$
- $\lim_{x\to\infty} F(x) = 1$

$$\lim_{x\to\infty} F(x) = \lim_{\substack{n\to\infty \\ x_n\to-\infty}} F(x_n) = \lim_{n\to\infty} P\big((-\infty, x_n]\big) = 0$$

$x_1 > x_2 > x_3 > \cdots$

$(-\infty, x_1] \supset (-\infty, x_2] \supset \cdots$

$$\bigcap_n (-\infty, x_n] = \phi$$

$$\lim_{x \to a^+} F(x) = F(a)$$

$$\overset{\text{“}}{\lim_{x_n \to a}} F(x_n) = \lim_{x_n \to a} P((-\infty, x_n]) = F(a)$$

$x_1 > x_2 > x_3 > \cdots$

$x_n \gg a$

$$\bigcap_n (-\infty, x_n] = (-\infty, a]$$