Math 733 – Fall 2020

Homework 5

Due: 10PM, November 22, 2020. Please upload your work in Canvas. Late homework will not be accepted.

1. Suppose that $X_1, X_2, \ldots$ are i.i.d. with $E[X_k] = 0$, $E[X_k^2] = 1$. Let $S_n = X_1 + \cdots + X_n$. Show that $\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} = \infty$ a.s.
   Hint: use the Kolmogorov 0-1 law and the CLT.
   (Note that we have seen already that for any $\alpha > 0$ we have $\frac{S_n}{n^{1/2+\alpha}} \xrightarrow{P} 0$.)

2. Let $X_1, X_2, \ldots$ be i.i.d. random variables with $E[X_k] = \mathrm{Var}[X_k] = 1$, and set $S_n = X_1 + \cdots + X_n$. Show that $\sqrt{S_n} - \sqrt{n}$ converges in distribution, and find the limit.

3. Let $X_1, X_2, \ldots$ be independent random variables, and set $S_n = X_1 + \cdots + X_n$. Assume that almost surely $|X_i| \le M$ for all $i \ge 1$ with a given constant $M < \infty$. Show that if $\mathrm{Var}\, S_n \to \infty$ then $\frac{S_n - ES_n}{\sqrt{\mathrm{Var}\, S_n}} \Rightarrow N(0,1)$.

4. Let $X_1, X_2, \ldots$ be i.i.d. random variables with distribution function F and a *continuous* PDF f. We have proved that $F_n(x) := \frac{1}{n}\sum_{k=1}^n 1(X_k \le x)$ converges uniformly to $F(x)$ with probability one. Now we will look at the empirical distribution on a finer scale. Let $c \in \mathbb{R}$ be a number with $f(c) > 0$ and consider $N_n(a,b) = \sum_{k=1}^n 1\big(X_k \in (c + \frac{a}{n}, c + \frac{b}{n})\big)$. Show that $N_n(a,b)$ converges in distribution for any $a < b$ and find the limit.

5. Let $X_1, X_2, \ldots$ be random variables. Assume that for every $k \ge 1$ we have $E[X_n^k] \to m_k \in \mathbb{R}$ for some $m_1, m_2, \ldots$.

   (a) Show that for any subsequential limit in distribution $X_{n_m} \Rightarrow Y$ we must have $E[Y^k] = m_k$ for all $k \ge 1$.

   (b) Assume further that the random variables satisfy $|X_n| \le C$ with a deterministic $C > 0$. Show that $X_1, X_2, \ldots$ converges in distribution.

You can use the Piazza page to ask for clarifications about a specific problem, but please don't discuss explicit solutions before the deadline. Handing in plagiarized work, whether copied from a fellow student or off the web, is not acceptable and will lead to sanctions.

Bonus problem.
Find independent random variables X, Y and Z so that Y and Z do not have the same distribution, but $X + Y \stackrel{d}{=} X + Z$.

Bonus problems are not graded, and you don't need to submit them. They are provided as an extra challenge for those who are interested.