Math 733 – Fall 2020

Midterm Exam (PM version)

1. Suppose that $X_1, X_2, \ldots$ are independent random variables with $E[X_i] = 1$ and $E[X_i^2] = 2$ for all $i \geq 1$. Show that $\sum_{i=1}^{n} \frac{1}{i} X_i \xrightarrow{P} \infty$ as $n \to \infty$.

   Note that $Y_n \xrightarrow{P} \infty$ as $n \to \infty$ if for any $K > 0$ we have $\lim_{n\to\infty} P(Y_n < K) = 0$.

2. Let Y and U be independent random variables, where U is uniform on $[0, 1]$. (We do not assume anything about the distribution of Y.) Show that the random variable $Z = \{Y + U\}$ has uniform distribution on $[0, 1]$.

   Note that $\{x\}$ denotes the fractional part of x: this is the difference between x and the largest integer that is at most as large as x. E.g. $\{5.3\} = 5.3 - 5 = 0.3$, $\{-4.6\} = -4.6 - (-5) = 0.4$.
   Hint: you could try this with a specific Y distribution first.

3. Suppose $\{p_k\}_{k\geq1}$ is a sequence in $[0, 1]$, and $X_1, X_2, \ldots$ are independent random variables with $P(X_k = 1) = p_k$, $P(X_k = 0) = 1 - p_k$.

   Show that $X_n \xrightarrow{a.s.} 0$ if and only if $\sum_{n=1}^{\infty} p_n < \infty$.