Stochastic integration of predictable processes (Continued)

L_2 collection of all predictable processes X

st $\|X\|^2_{\mu_M, T} < \infty \quad \forall T < \infty.$

$$\|X\|^2_{\mu_M,T} = \int_{[0,T]\times\Omega} |X|^2 \, d\mu_M$$

M : L^2 martingale. goal: $\int_0^t X \, dM$

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simple predictable process,

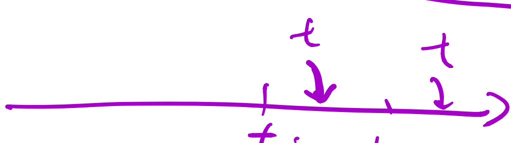
$$X_t(\omega) = \xi_0(\omega)\,\mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{n-1} \xi_i(\omega)\,\mathbb{1}_{(t_i\ t_{i+1}]}(t)$$

$S_2 \subset L_2$. left-continuous. for convenience set $t_1 = t_0 = 0$ (*)

$\Rightarrow$ X is predictable.

$$\int_0^t X_s \, dM_s = \sum_{i=1}^{n-1} \xi_i(\omega)\left(M_{t_{i+1}\wedge t}(\omega) - M_{t_i\wedge t}(\omega)\right)$$

$\left(\int_{t_i}^{t_{i+1}} dM = M_{t_{i+1}} - M_{t_i}\right)$



$t_i \quad t_{i+1}$

S_2 is the subspace of L_2.

s.t. X has the form $(*)$

and ξ_i is bounded r.v. $\xi_i \in \mathcal{F}_{t_i}$

(ξ_i is $\mathcal{F}_{t_i}$ meas)

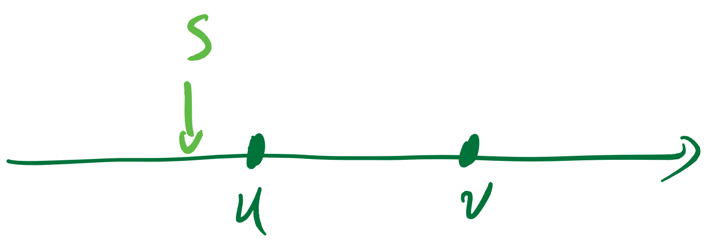
Lemma: Let $X \in S_2$.

Then $\int_0^t X\,dM$ is L^2-cadlag martingale. (Cont if M cont)

and

$$\mathbb{E}\left[\left(\int_0^t X\,dM\right)^2\right] = \int_{[0,t]\times\Omega} X^2\,d\mu_M$$

Pf:



$$Z_t \overset{\text{def}}{=} \xi\left(M_{t\wedge v} - M_{t\wedge u}\right) \quad \text{is mart?}$$

where ξ is bounded, $\mathcal{F}_u$-measurable.

$$\mathbb{E}[Z_t \mid \mathcal{F}_s] \overset{?}{=} Z_s \qquad \forall s<t$$

$s<u$.

$$\mathbb{E}[Z_t \mid \mathcal{F}_s] = \mathbb{E}[\xi(M \ldots - M \ldots) \mid \mathcal{F} \ldots]$$

$$\mathbb{E}[Z_t \mid \mathcal{F}_s] = \mathbb{E}[\xi(M_{t\wedge v} - M_{t\wedge u}) \mid \mathcal{F}_s]$$

$$= \mathbb{E}\Big[\xi\, \underbrace{\mathbb{E}[M_{t\wedge v} - M_{t\wedge u} \mid \mathcal{F}_u]}_{\overset{?}{=} 0} \,\Big|\, \mathcal{F}_s\Big]$$

$$= 0 \quad = \quad Z_s = \xi(M_{s\wedge v} - M_{s\wedge u}) = 0$$

1) $s < t \le u$.

$$\mathbb{E}[M_t - M_t \mid \mathcal{F}_u] = 0$$

2) $t > u$.

$$\mathbb{E}[M_{t\wedge v} - M_{t\wedge u} \mid \mathcal{F}_u]$$

$$= \mathbb{E}[M_{t\wedge v} \mid \mathcal{F}_u] - M_u = 0$$

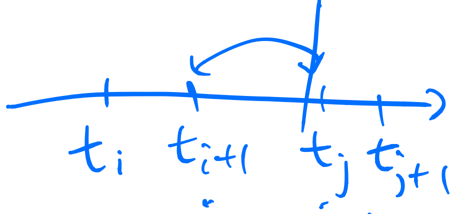
$s \geq u$, $\longrightarrow$ exercise

$\Rightarrow \int_0^t X\,dM$ is mart.

$$\boxed{\left(\int_0^t X\,dM\right)^2 = \sum_i \xi_i^2 (M_{t\wedge t_{i+1}} - M_{t\wedge t_i})^2 + 2\sum_{i<j} \xi_i \xi_j (M_{t\wedge t_{i+1}} - M_{t\wedge t_i})(M_{t\wedge t_{j+1}} - M_{t\wedge t_j})}$$

$\sum_i \xi_i(\cdots)$

$$\underline{t_{i+1} \le t_j}$$



$$\mathbb{E}\left(\mathbb{E}\left[\xi_i \xi_j \left(M_{t\wedge t_{i+1}} - M_{t\wedge t_i}\right)\left(M_{t\wedge t_{j+1}} - M_{t\wedge t_j}\right) \,\middle|\, \mathcal{F}_{t_j}\right]\right)$$

$$= 0$$

$$\mathbb{E}\left[\left(\int_0^t X\, dM\right)^2\right] = \sum_i \mathbb{E}\left[\xi_i^2 \left(M_{t\wedge t_{i+1}} - M_{t\wedge t_i}\right)^2\right]$$

$$= \sum_i \mathbb{E}\left[\xi_i^2\, \mathbb{E}\left[\left(M_{t\wedge t_{i+1}} - M_{t\wedge t_i}\right)^2 \,\middle|\, \mathcal{F}_{t_i}\right]\right]$$

$$= \sum_i \mathbb{E}\left[\xi_i^2\, \mathbb{E}\left(M^2_{t\wedge t_{i+1}} - M^2_{t\wedge t_i} \,\middle|\, \mathcal{F}_{t_i}\right)\right]$$

$$= \sum_i \mathbb{E}\left[\xi_i^2\, \mathbb{E}\left[[M]_{t\wedge t_{i+1}} - [M]_{t\wedge t_i} \,\middle|\, \mathcal{F}_{t_i}\right]\right]$$

(because $M^2 - [M]$ is mart

$u < s < t$

$$\mathbb{E}\left[M_t^2 - [M]_t \,\middle|\, \mathcal{F}_u\right] = \mathbb{E}\left[M_s^2 - [M]_s \,\middle|\, \mathcal{F}_u\right]$$

$$= \sum_i \mathbb{E}\left[\xi_i^2 \left([M]_{t\wedge t_{i+1}} - [M]_{t\wedge t_i}\right)\right]$$

$$= \sum_i \mathbb{E}\, \xi_i^2 \int_0^t \mathbb{1}_{(t_i\, t_{i+1}]}(s)\, d[M]_s$$

$$= \mathbb{E} \int_0^t \sum_i \xi_i^2\, \mathbb{1}_{(t_i\, t_{i+1}]}(s)\, d[M]_s$$

$$= E\int_0^t \Big(\sum_i \xi_i \mathbb{1}_{(t_i, t_{i+1}]}(s)\Big)^2 d[M]_s$$

$$= \int_{[0,t]\times\Omega} X^2 \, d\mu_M.$$

$$\Rightarrow \boxed{\Big\| \int_0^t X \, dM \Big\|_{L^2(\Omega)} = \| X \|_{\mu_M, t}.}$$

metric/distance on L_2.

$$\| X \|_{L_2} = \sum_{k=1}^{\infty} 2^{-k} \left(1 \wedge \| X \|_{\mu_M, k}\right)$$

metric/dist on L^2-mart.

$$\| M \|_{M_2} = \sum_{k=1}^{\infty} 2^{-k} \left(1 \wedge \| M_k \|_{L^2}\right)$$

$$\Big\| \int_0^t X \, dM \Big\|_{M_2} = \| X \|_{L_2} \quad (\text{Itô isometry})$$

$\forall X \in S_2$.

Lem. $\forall X \in L_2$. $\exists X_n \in S_2$ s.t. $\| X - X_n \|_{L_2} \to 0$

If $X \in L_2$. we can find $X_n \in S_2$.

$$\int_0^t X_n \, dM$$

$$\downarrow$$

$$X$$

Define $\int_0^t X\,dM$ to be the L^2 cadlag mart.

s.t. $\lim_{n\to\infty} \left\| \boxed{\int_0^t X_n\,dM} - \int_0^t X\,dM \right\|_{M_2} = 0.$

need to check $\exists$. and $!$ (exercise)