Math 733 – Fall 2020

Midterm Exam (AM version)

1. Suppose that $Y_1, Y_2, \ldots$ are random variables with finite second moment. Show that

$$\text{if } \lim_{n\to\infty} \frac{E[Y_n^2]}{(E[Y_n])^2} = 1 \quad \text{then} \quad \frac{Y_n}{E[Y_n]} \xrightarrow{P} 1, \text{ as } n \to \infty. \tag{1}$$

   There was a typo in the problem (some of the students noticed this), but all students found the path that would have led to the solution.

2. Suppose that X is a discrete random variable with support $A \subset \mathbb{R}$ and probability mass function $p_X : A \to [0,1]$. Let U be a random variable which is independent of X, and has uniform distribution on $[0,1]$. Show that $Z = X + U$ has absolutely continuous distribution, and express its probability density function in terms of p_X.
   Hint: you could try to solve the problem with a specific p_X first.

3. Suppose that $X_1, X_2, \ldots$ are independent (not necessarily identically distributed) random variables. Show that $P\left(\sup_{n\geq 1} X_n < \infty\right) = 1$ if and only if there exists $c \in \mathbb{R}$ with $\sum_{n=1}^{\infty} P(X_n > c) < \infty$.