HOMEWORK 1

A PREPRINT

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1 Problem A

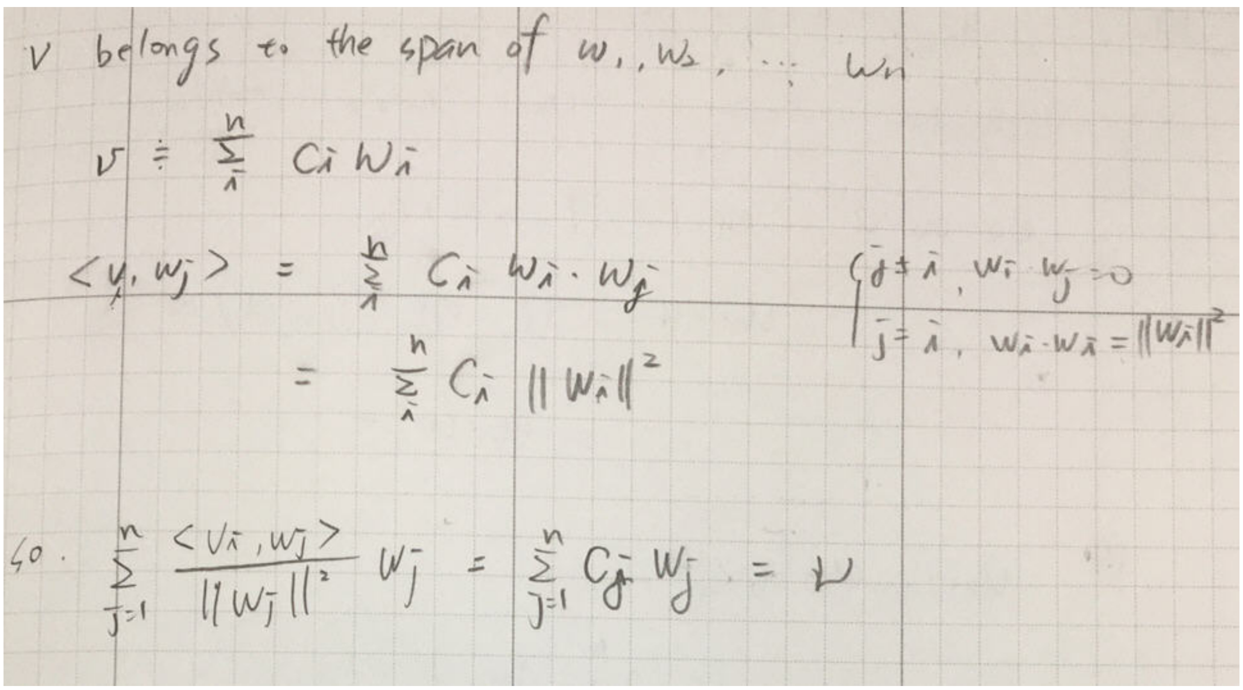

Figure 1: Problem A (a)

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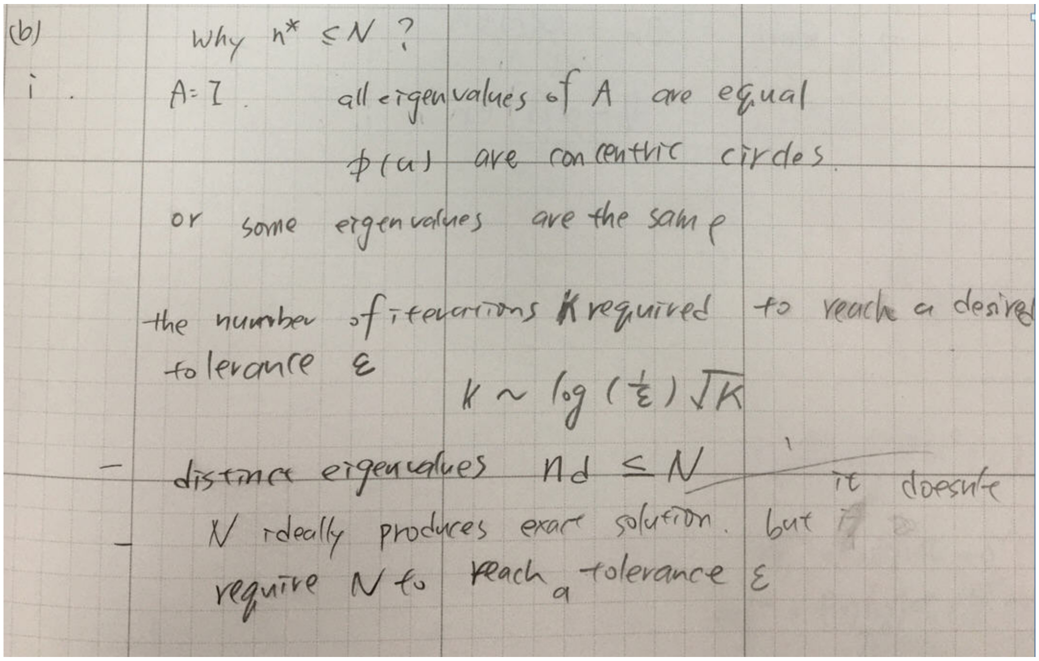


(b) Why $n^* \le N$?

i. $A = I$. all eigenvalues of A are equal

$\phi(u)$ are concentric circles.

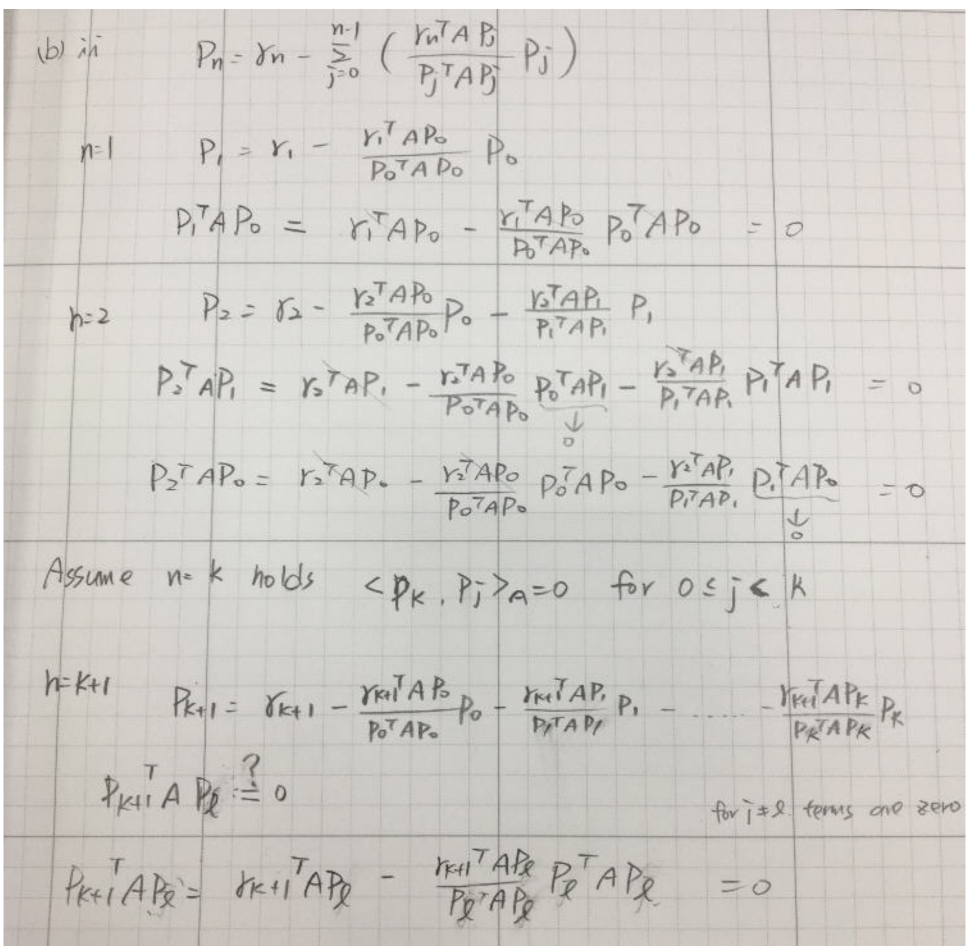
or some eigenvalues are the same

the number of iterations k required to reach a desired tolerance ε

$$k \sim \log\left(\frac{1}{\varepsilon}\right)\sqrt{K}$$

- distinct eigenvalues $n_d \le N$
- N ideally produces exact solution. but it doesn't require N to reach a tolerance ε

Figure 2: Problem A (b) i.



(b) ii

$$p_n = r_n - \sum_{j=0}^{n-1}\left(\frac{r_n^T A p_j}{p_j^T A p_j} p_j\right)$$

$n=1$

$$p_1 = r_1 - \frac{r_1^T A p_0}{p_0^T A p_0} p_0$$

$$p_1^T A p_0 = r_1^T A p_0 - \frac{r_1^T A p_0}{p_0^T A p_0} p_0^T A p_0 = 0$$

$n=2$

$$p_2 = r_2 - \frac{r_2^T A p_0}{p_0^T A p_0} p_0 - \frac{r_2^T A p_1}{p_1^T A p_1} p_1$$

$$p_2^T A p_1 = r_2^T A p_1 - \frac{r_2^T A p_0}{p_0^T A p_0} \underbrace{p_0^T A p_1}_{0} - \frac{r_2^T A p_1}{p_1^T A p_1} p_1^T A p_1 = 0$$

$$p_2^T A p_0 = r_2^T A p_0 - \frac{r_2^T A p_0}{p_0^T A p_0} p_0^T A p_0 - \frac{r_2^T A p_1}{p_1^T A p_1} \underbrace{p_1^T A p_0}_{0} = 0$$

Assume $n = k$ holds $\langle p_k, p_j \rangle_A = 0$ for $0 \le j < k$

$n = k+1$

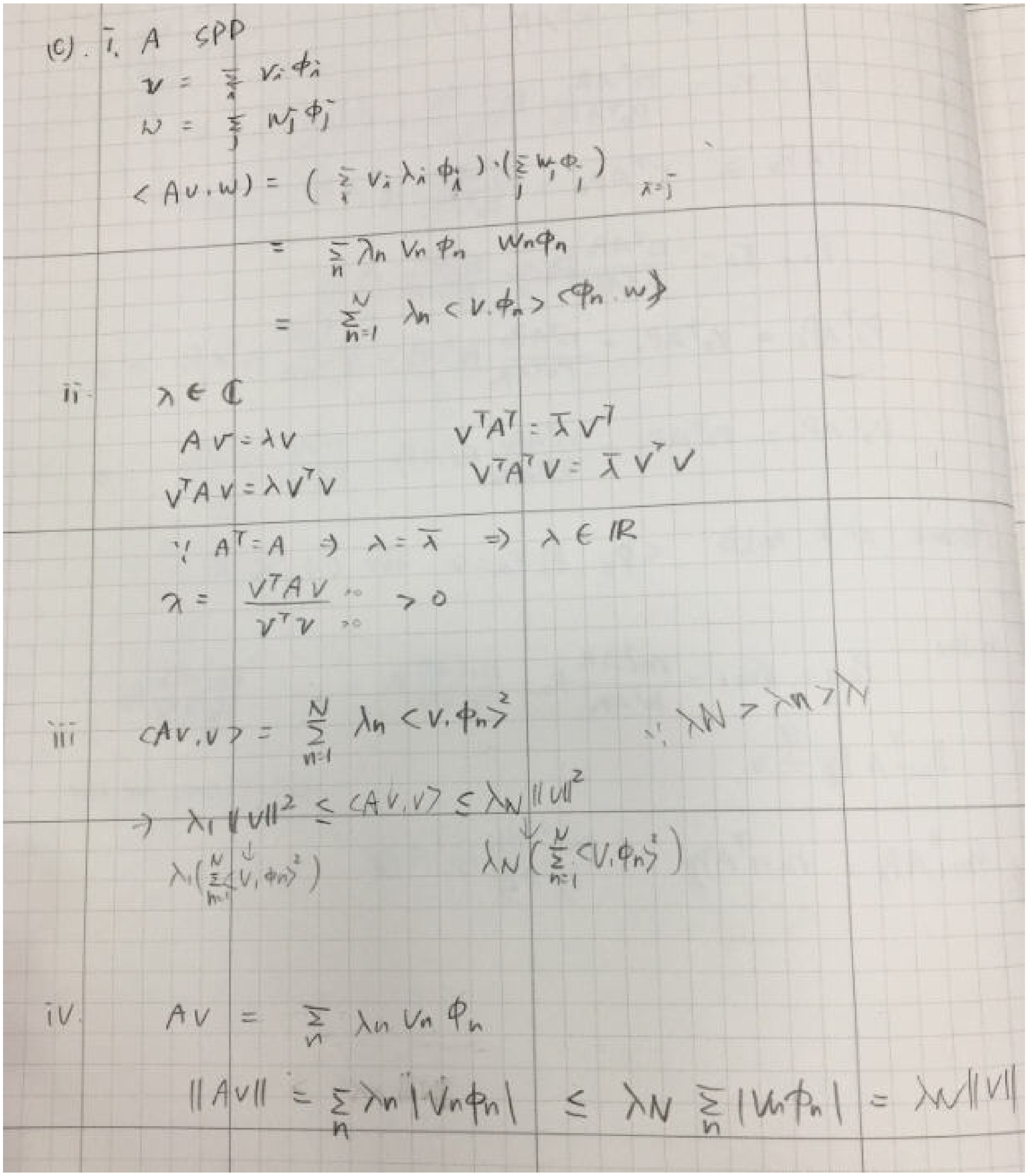
$$p_{k+1} = r_{k+1} - \frac{r_{k+1}^T A p_0}{p_0^T A p_0} p_0 - \frac{r_{k+1}^T A p_1}{p_1^T A p_1} p_1 - \cdots - \frac{r_{k+1}^T A p_k}{p_k^T A p_k} p_k$$

$$p_{k+1}^T A p_\ell \overset{?}{=} 0$$

for $j \neq \ell$ terms are zero

$$p_{k+1}^T A p_\ell = r_{k+1}^T A p_\ell - \frac{r_{k+1}^T A p_\ell}{p_\ell^T A p_\ell} p_\ell^T A p_\ell = 0$$

Figure 3: Problem A (b) ii.



(c). i. A SPD

$v = \sum_i v_i \phi_i$

$w = \sum_j w_j \phi_j$

$\langle Av, w\rangle = (\sum_i v_i \lambda_i \phi_i)\cdot(\sum_j w_j \phi_j)$ $\quad i=j$

$= \sum_n \lambda_n v_n \phi_n \; w_n \phi_n$

$= \sum_{n=1}^{N} \lambda_n \langle v, \phi_n\rangle \langle \phi_n, w\rangle$

ii. $\lambda \in \mathbb{C}$

$Av = \lambda v \qquad v^T A^T = \bar{\lambda} v^T$

$v^T A v = \lambda v^T v \qquad v^T A^T v = \bar{\lambda} v^T v$

$\because A^T = A \Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}$

$\lambda = \frac{v^T A v \; {>0}}{v^T v \; {>0}} > 0$

iii. $\langle Av, v\rangle = \sum_{n=1}^{N} \lambda_n \langle v, \phi_n\rangle^2 \qquad \because \lambda_N > \lambda_n > \lambda_1$

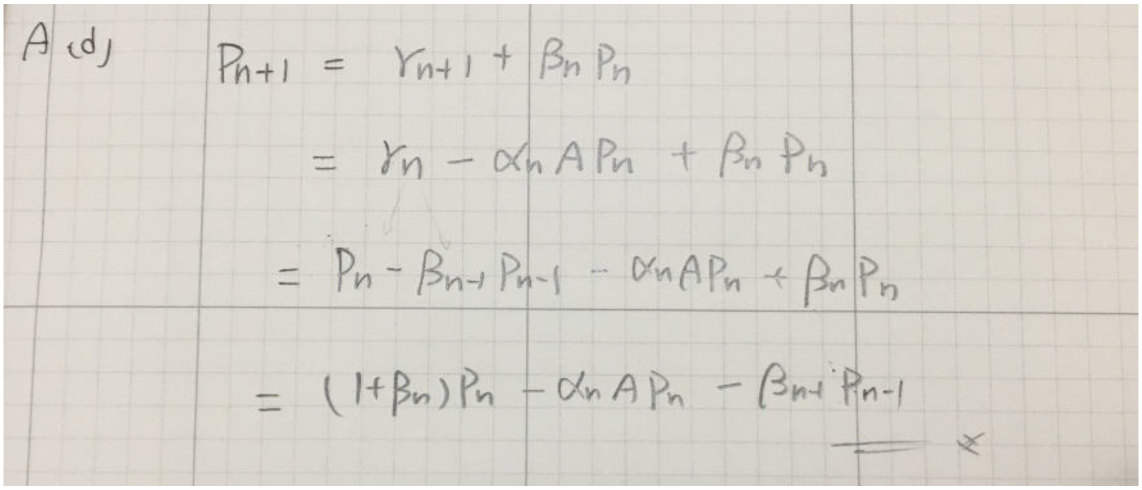
$\Rightarrow \lambda_1 \|v\|^2 \le \langle Av, v\rangle \le \lambda_N \|v\|^2$

$\lambda_1\left(\sum_{n=1}^{N} \langle v, \phi_n\rangle^2\right) \qquad \lambda_N\left(\sum_{n=1}^{N} \langle v, \phi_n\rangle^2\right)$

iv. $Av = \sum_n \lambda_n v_n \phi_n$

$\|Av\| = \sum_n \lambda_n |v_n \phi_n| \le \lambda_N \sum_n |v_n \phi_n| = \lambda_N \|v\|$

Figure 4: Problem A (c)



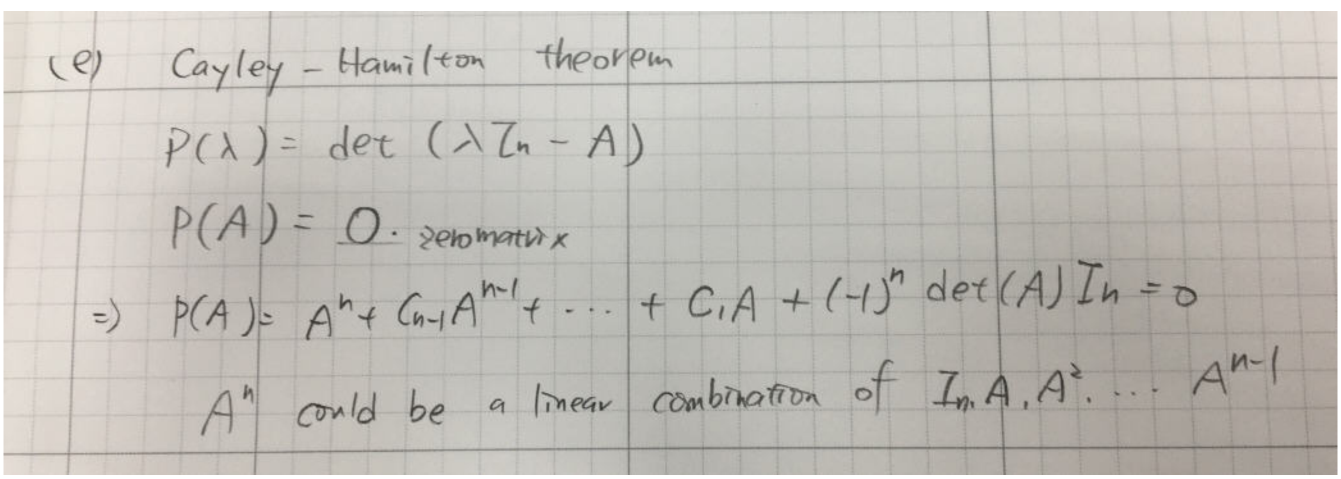
A (d) $p_{n+1} = r_{n+1} + \beta_n p_n$

$= r_n - \alpha_n A p_n + \beta_n p_n$

$= p_n - \beta_{n-1} p_{n-1} - \alpha_n A p_n + \beta_n p_n$

$= (1+\beta_n) p_n - \alpha_n A p_n - \beta_{n-1} p_{n-1}$

Figure 5: Problem A (d)



(e) Cayley – Hamilton theorem

$P(\lambda) = \det(\lambda I_n - A)$

$P(A) = \underline{O}$. zero matrix

$\Rightarrow P(A) = A^n + C_{n-1}A^{n-1} + \cdots + C_1 A + (-1)^n \det(A) I_n = 0$

A^n could be a linear combination of $I_n, A, A^2, \ldots A^{n-1}$

Figure 6: Problem A (e)

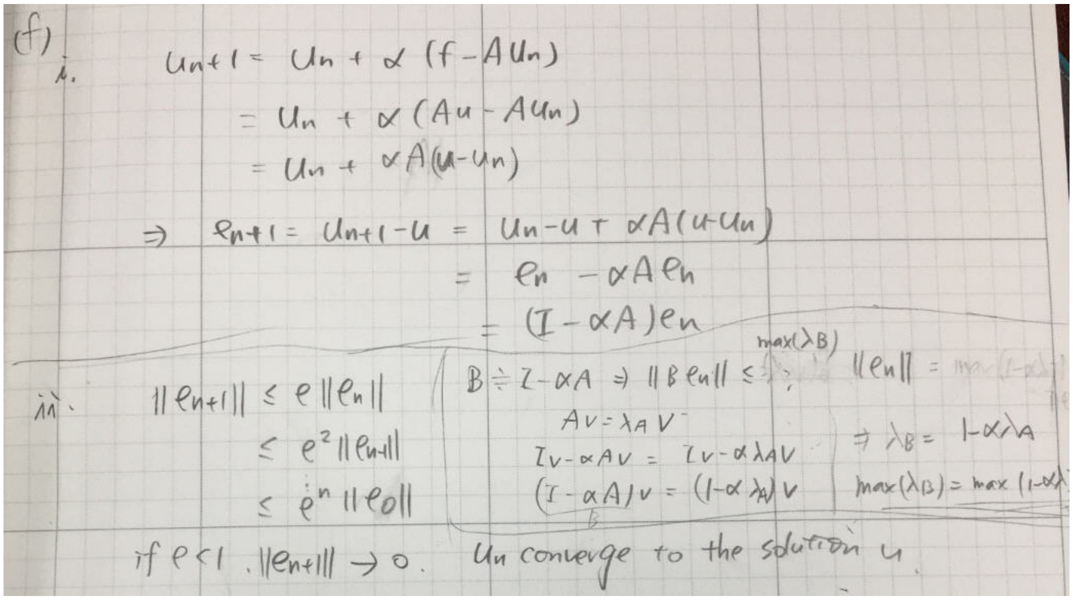


Figure 7: Problem A (f) i. and ii.

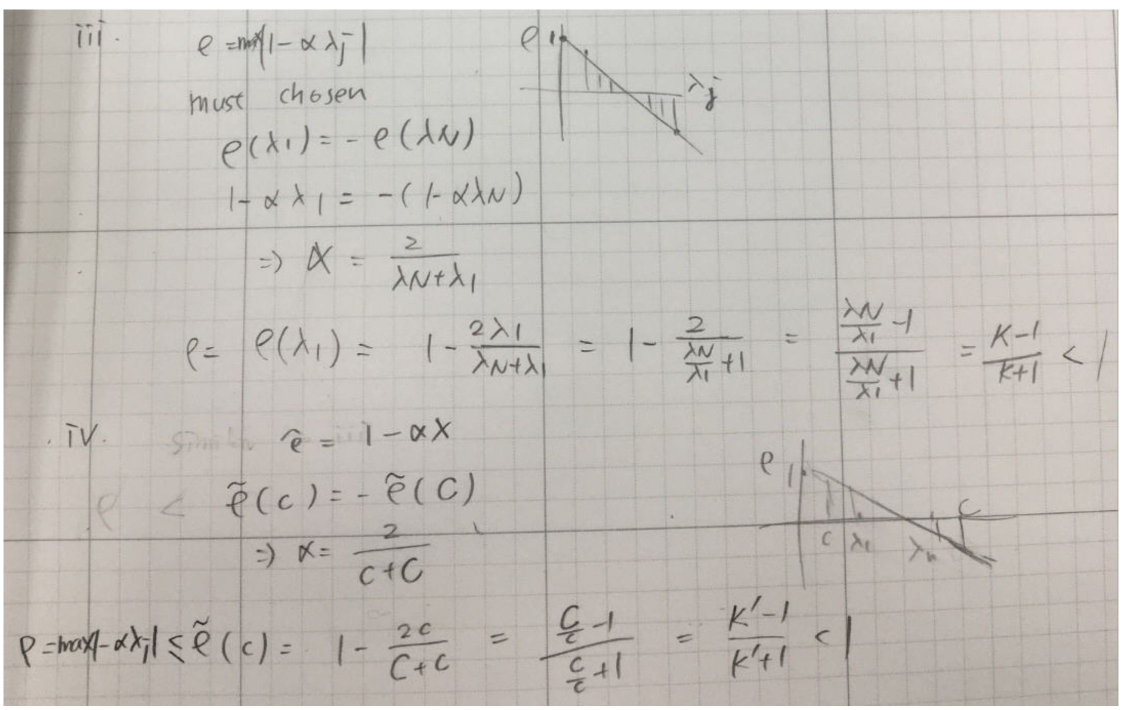


Figure 8: Problem A (f) iii. and iv.

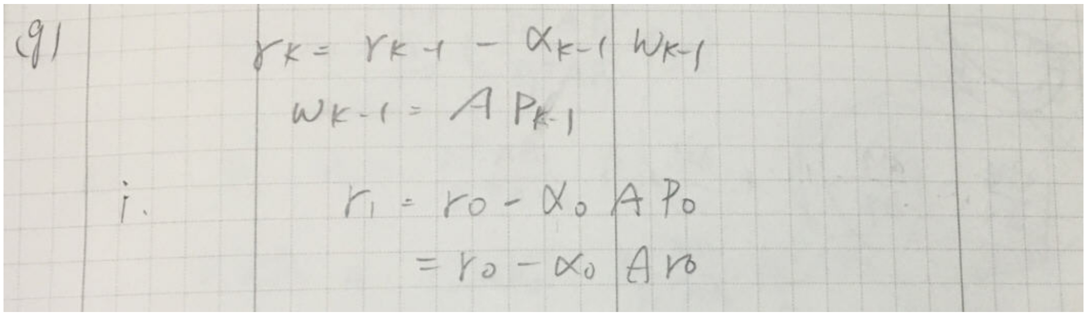


Figure 9: Problem A (g) i.

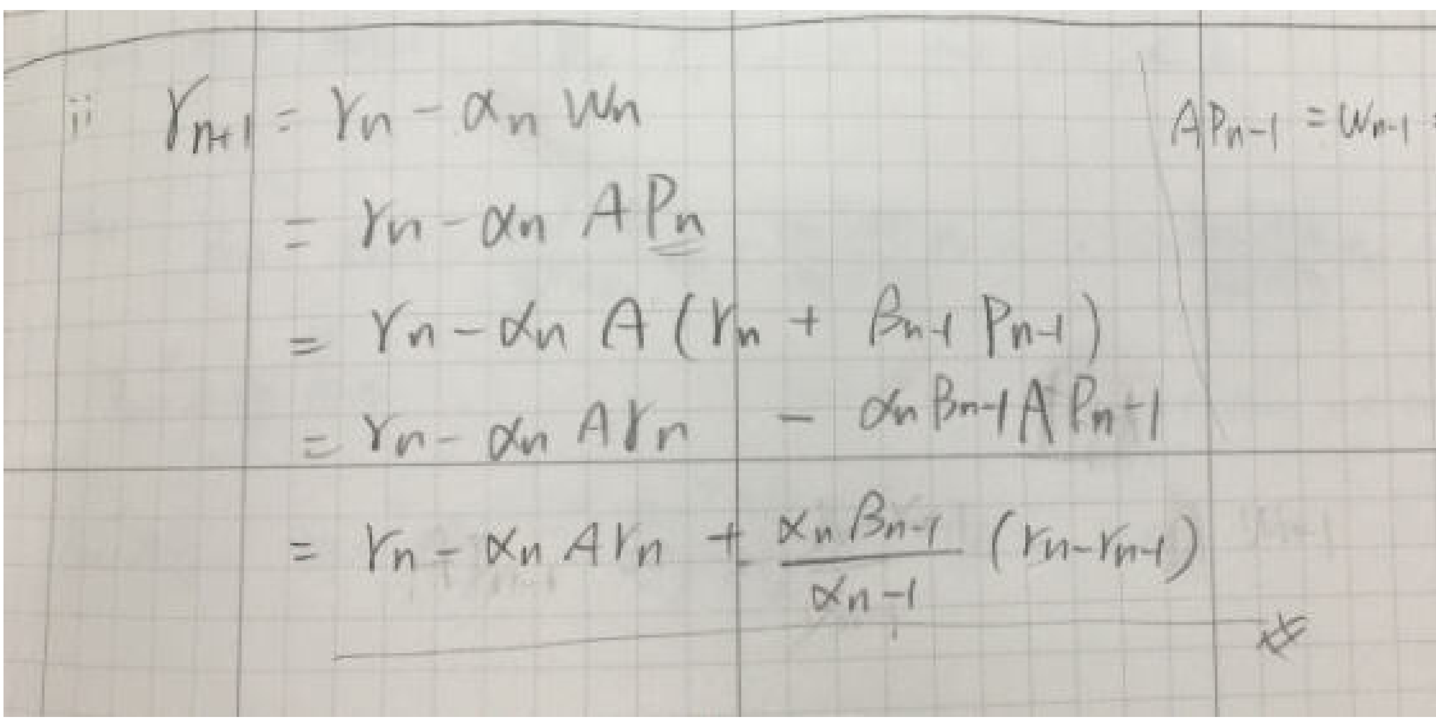
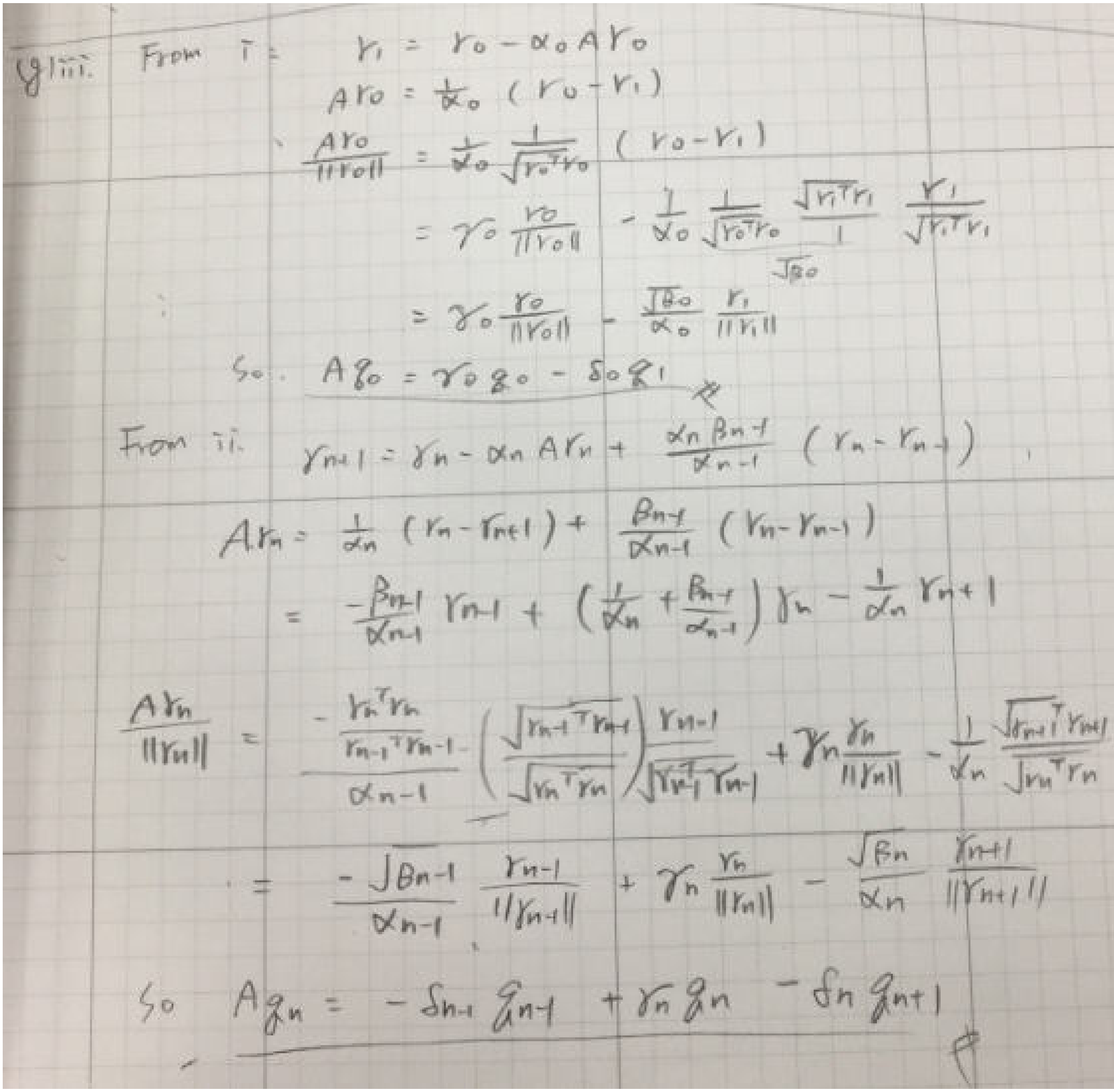


Figure 10: Problem A (g) ii.



g iii. From i:

$$r_1 = r_0 - \alpha_0 A r_0$$

$$A r_0 = \frac{1}{\alpha_0}(r_0 - r_1)$$

$$\frac{A r_0}{\|r_0\|} = \frac{1}{\alpha_0}\frac{1}{\sqrt{r_0^T r_0}}(r_0 - r_1)$$

$$= \gamma_0 \frac{r_0}{\|r_0\|} - \frac{1}{\alpha_0}\underbrace{\frac{1}{\sqrt{r_0^T r_0}}\frac{\sqrt{r_1^T r_1}}{1}}_{\sqrt{\beta_0}}\frac{r_1}{\sqrt{r_1^T r_1}}$$

$$= \gamma_0 \frac{r_0}{\|r_0\|} - \frac{\sqrt{\beta_0}}{\alpha_0}\frac{r_1}{\|r_1\|}$$

So $\underline{A q_0 = \gamma_0 q_0 - \delta_0 q_1}$ #

From ii.

$$r_{n+1} = r_n - \alpha_n A r_n + \frac{\alpha_n \beta_{n-1}}{\alpha_{n-1}}(r_n - r_{n-1})$$

$$A r_n = \frac{1}{\alpha_n}(r_n - r_{n+1}) + \frac{\beta_{n-1}}{\alpha_{n-1}}(r_n - r_{n-1})$$

$$= \frac{-\beta_{n-1}}{\alpha_{n-1}} r_{n-1} + \left(\frac{1}{\alpha_n} + \frac{\beta_{n-1}}{\alpha_{n-1}}\right) r_n - \frac{1}{\alpha_n} r_{n+1}$$

$$\frac{A r_n}{\|r_n\|} = \frac{-\frac{r_n^T r_n}{r_{n-1}^T r_{n-1}}}{\alpha_{n-1}}\left(\frac{\sqrt{r_{n-1}^T r_{n-1}}}{\sqrt{r_n^T r_n}}\right)\frac{r_{n-1}}{\sqrt{r_{n-1}^T r_{n-1}}} + \gamma_n \frac{r_n}{\|r_n\|} - \frac{1}{\alpha_n}\frac{\sqrt{r_{n+1}^T r_{n+1}}}{\sqrt{r_n^T r_n}}$$

$$= \frac{-\sqrt{\beta_{n-1}}}{\alpha_{n-1}}\frac{r_{n-1}}{\|r_{n-1}\|} + \gamma_n \frac{r_n}{\|r_n\|} - \frac{\sqrt{\beta_n}}{\alpha_n}\frac{r_{n+1}}{\|r_{n+1}\|}$$

So $\underline{A q_n = -\delta_{n-1} q_{n-1} + \gamma_n q_n - \delta_n q_{n+1}}$ #

Figure 11: Problem A (g) iii.

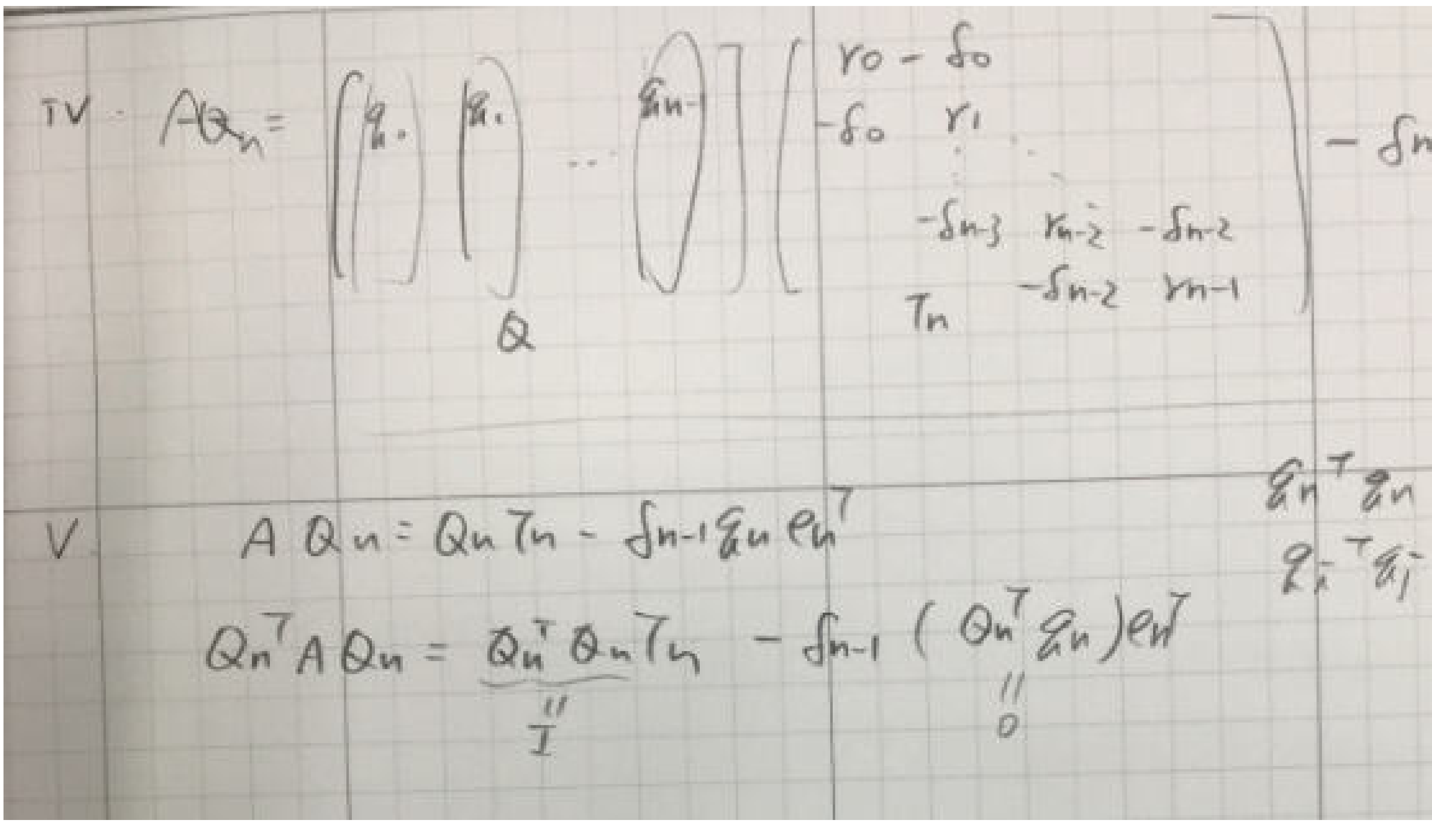


Figure 12: Problem A (g) iv. and v.

2 Problem B

The smallest value of N is 100 to make sure the uniform norm within 10^{-2}.

Double click HW2.exe in the folder of HW2/x64/Release. Two function, ErrorOfLinearInterpolation and ErrorVsGridSpacing2DWave, will be executed. Then result will be displayed on screen, shown as Figure 13 .
(Github repository is https://github.com/ychang253/HW2.git)

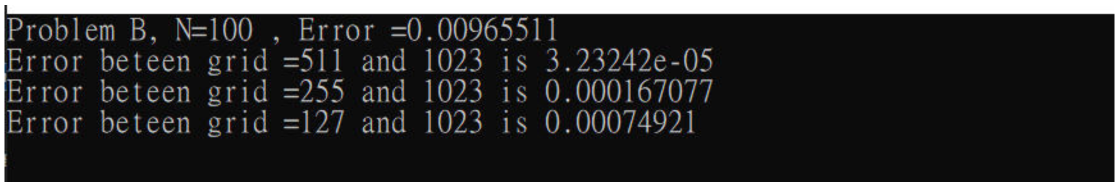


Figure 13: Screen output

3 Problem C

3.1 (a)

For $n+1 \geq 2$.

$$U_{i,j}^{n+1} = 2U_{i,j}^{n} - U_{i,j}^{n-1} + \frac{\Delta t^2}{h^2}(U_{i-1,j}^{n} + U_{i+1,j}^{n} + U_{i,j-1}^{n} + U_{i,j+1}^{n} - 4U_{i,j}^{n})$$

For $n+1 = 1$. To know $U_{i,j}^{-1}$ by the relation: $\frac{U_{i,j}^{1} - U_{i,j}^{-1}}{2\Delta t} = f(x)f(y)$. Then the $U_{i,j}^{1}$ can be determined.

$$U_{i,j}^{1} = U_{i,j}^{0} + f(x)f(y)\Delta t + \frac{\Delta t^2}{2h^2}(U_{i-1,j}^{0} + U_{i+1,j}^{0} + U_{i,j-1}^{0} + U_{i,j+1}^{0} - 4U_{i,j}^{0})$$

Comparing to the result of reference grid 1023 X 1023 at time = 1 sec, the log-log plot of the error vs. grid spacing is shown as Figure 14 . The slope is close to 2 meaning current method is second-order accurate. Simulation result of time = 1 s is shown as Figure 15.

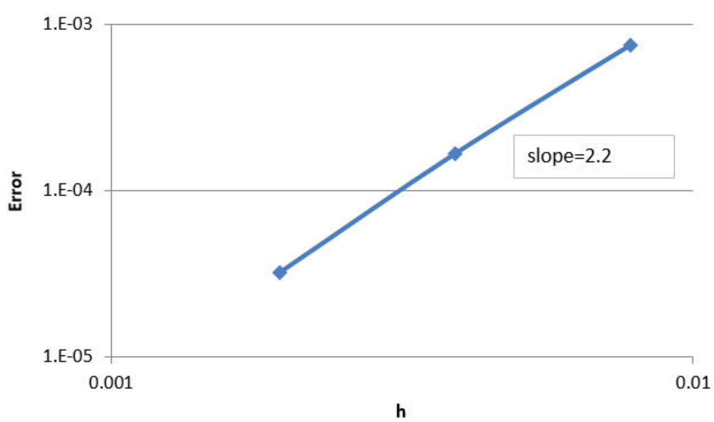


Figure 14: Error vs. Grid spacing

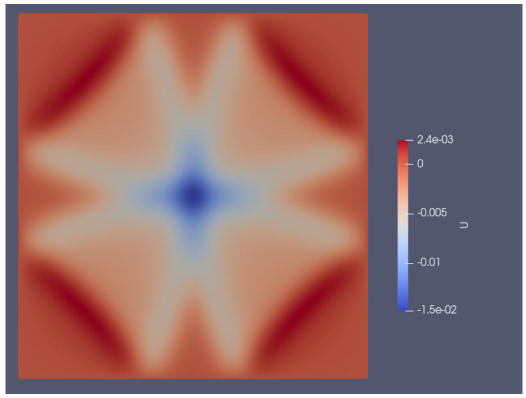


Figure 15: Simulation result at time=1s

3.2 (b)

$$y''(t) = \lambda y$$

$$y^{n+1} - (2 + \lambda \Delta t^2) y^n + y^{n-1} = 0$$

$$\rho^2 - (2 + \lambda \Delta t^2)\rho + 1 = 0$$

$$\rho = (1 + \frac{\lambda \Delta t^2}{2}) \pm \sqrt{(1 + \frac{\lambda \Delta t^2}{2})^2 - 1}$$

If we want $|\rho| \leq 1$, then $\text{Im}((1 + \frac{\lambda \Delta t^2}{2})) = 0$ and $-1 \leq \text{Re}((1 + \frac{\lambda \Delta t^2}{2})) \leq 1$, and finally we can get $-4 \leq \lambda \Delta t^2 \leq 0$. Plot in complex plane is shown as Figure 16.

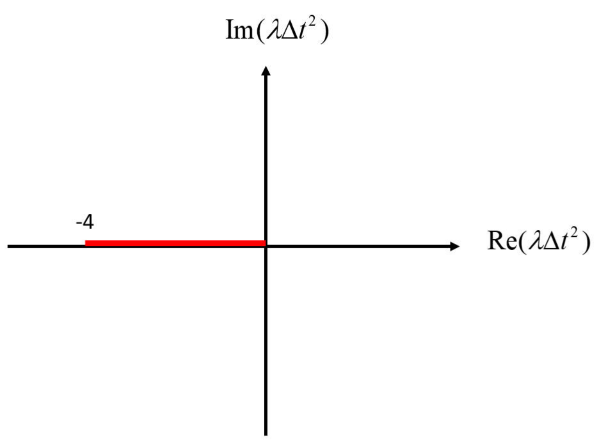


Figure 16: Stability region

3.3 (c)

Method of line

$$U'(t) = AU(t)$$

Since A is 2D Laplacian, Max$(\lambda_A) = \frac{-8}{\Delta x^2}$, conbinate with result from (a), then CFL condition $\frac{\Delta t}{\Delta x} \leq \sqrt{\frac{1}{2}}$ is obtained.

3.4 (d)

$$U^n = e^{i(k_1 j_1 \Delta x + k_2 j_2 \Delta x - \omega n \Delta t)}$$

$$\frac{4}{\Delta t^2}\sin^2(\frac{\omega \Delta t}{2}) = \frac{4}{\Delta x^2}(\sin^2(\frac{k_1 \Delta x}{2}) + \sin^2(\frac{k_2 \Delta x}{2}))$$

$$\sin^2(\frac{\omega \Delta t}{2}) = \frac{\Delta t^2}{\Delta x^2}(\sin^2(\frac{k_1 \Delta x}{2}) + \sin^2(\frac{k_2 \Delta x}{2}))$$

For real ω, right hand side must less than 1, so $\frac{2\Delta t^2}{\Delta x^2} \leq 1$. Finally, CFL condition $\frac{\Delta t}{\Delta x} \leq \sqrt{\frac{1}{2}}$ is the same as result from (b).

3.5 (e)

$$u_{tt} + \frac{1}{12}\Delta t^2 u_{tttt} + O(\Delta t^4) = u_{xx} + \frac{1}{12}\Delta x^2 u_{xxxx} + u_{yy} + \frac{1}{12}\Delta x^2 u_{yyyy} + O(\Delta x^4)$$

$$u_{tttt} = u_{xxxx} + u_{yyyy} + 2u_{xxyy}$$

Modified equation will be :

$$u_{tt} = u_{xx} + u_{yy} + \frac{1}{12}(\Delta x^2 - \Delta t^2)(u_{xxxx} + u_{yyyy}) - \frac{1}{6}\Delta t^2 u_{xxyy}$$

Fourier series : If $u_{tt} = u_{xx}$, then $\hat{u}_{tt}(\xi, t) = -\xi^2 \hat{u}(\xi, t)$ and $\hat{u}(\xi, t) = e^{-i\xi t}\hat{u}_0(\xi, t)$.
If $u_{tt} = u_{xxxx}$, then $\hat{u}_{tt}(\xi, t) = -\xi^4 \hat{u}(\xi, t)$ and $\hat{u}(\xi, t) = e^{-i\xi^2 t}\hat{u}_0(\xi, t)$.

Since modified equation u_{tt} is a combination of u_{xx} and u_{xxxx}, then the solution will be in this form

$$u(x, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{u}_0(\xi, t) e^{-i\xi(x - (a_1 + a_2\xi)t)}$$

$\omega = (a_1 + a_2\xi)$ is real, which mean the equation is dispersive.

4 Problem D

$$U_{i,j}^{n+1} = 2U_{i,j}^n - U_{i,j}^{n-1} + \Delta t^2 A U^n$$

$$U_{i,j}^{n+1} = (2I + \Delta t^2 A)U^n - U_{i,j}^{n-1} = B(\Delta t)U^n - U_{i,j}^{n-1}$$

Consider the LTE $U_{i,j}^{n+1} = B(\Delta t)U^n - U_{i,j}^{n-1} + \Delta t^2 \tau^n$ and $E^n = U^n - u^n$, then $E^n = B^n E^0 - \Delta t^2 \sum_{m=1}^{N} B^{n-m}\tau^{m-1}$.

$$\|E\|^n \leq \|B\|^n \|E_0\| + \Delta t^2 \sum_{m=1}^{N} \|B\|^{n-m} \|\tau^{m-1}\|$$

Weak-stability

$$\|E\|^n \leq C_T \|E_0\| + \Delta t^2 N C_T \text{max} \|\tau^{m-1}\|$$

$E_0 \to 0$ if initial condition is well handled, and $\tau \to 0$ if $\Delta t \to 0$