Borel - Cantelli lemma: $A_1, A_2, \ldots$ are events with $\sum_{n=1}^{\infty} P(A_n) < \infty$ then $P(A_n \text{ i.o.}) = 0$.

Which one of these 0-1 sequences were generated by fair coin flips?

(1) 1,0,1,0,1,0,0,1,0,1,0,1,0,1,0,1,0,0,0,1,0,1,0,1,0,1,0,1,1,0,1,0,1,1,0,0,0,1,0,1,0,0,0,1,0,1,0
,1,0,1,0,1,1,0,1,0,1,0,1,0,0,1,0,1,0,1,0,1,0,1,1,1,0,1,0,0,1,0,1,0,0,0,1,0,1,0,0,1,0,1,0,0,0,
1,1,0,1,0,0,1

(2) 0,1,0,0,1,0,0,0,1,0,1,0,0,0,0,0,0,1,1,1,1,0,0,0,0,0,0,0,0,1,0,1,1,1,0,0,0,0,1,1,1,0,0,0,0,1,0
,0,0,0,0,0,1,0,1,1,1,0,0,0,0,1,0,1,1,1,0,0,0,0,1,1,0,1,1,0,0,1,0,0,0,0,0,0,0,1,0,1,1,0,1,0,1,1,
0,1,0,0,0,0,1

Answer: the 2^{nd} one

Ex: $X_1, X_2, \ldots$ i.i.d Bernoulli $\left(\frac{1}{2}\right)$

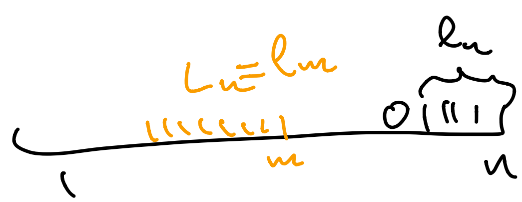
H: 1 T: 0

L_n: largest head-run among the first n random variables.

$0 \le L_n \le n$

Then $$P\left(\frac{L_n}{\log_2 n} \to 1 \right) = 1$$

Proof: ℓ_n : largest head-run still alive



$\ell_n \le L_n$

$$P(\ell_n \geq k) = \frac{1}{2^k}$$

$1 \le k \le n$

$\varepsilon > 0$ is fixed

$$P\left(\ell_n \geq (1+\varepsilon)\log_2 n\right) \le \underbrace{n^{-1-\varepsilon}}_{\text{summable in } n}$$

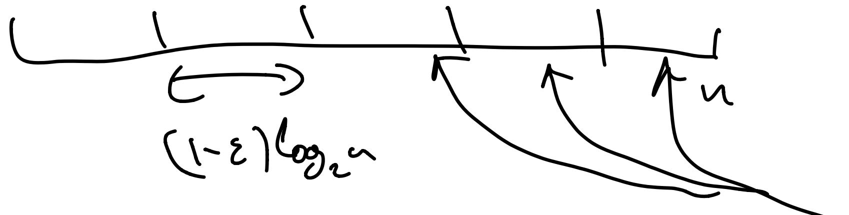
$$P\left(\ell_n < (1+\varepsilon)\log_2 n \text{ eventually}\right) = 1$$

$$\parallel$$

$$P\left(\limsup_{n\to\infty} \frac{L_n}{\log_2 n} \le 1+\varepsilon\right)$$

For the lower bound:

$$P(L_n \le (1-\varepsilon)\log_2 n)$$



$$P(L_n \le (1-\varepsilon)\log_2 n) \le P(\text{no special block of all 1s}) \le$$

$$\le \Big(1 - \underbrace{\frac{1}{2^{(1-\varepsilon)\log_2 n}}}_{n^{-1+\varepsilon}}\Big)^{\frac{n}{\log_2 n(1-\varepsilon)}} \le \underbrace{e^{-\frac{n^{\varepsilon}}{\log_2 n}\cdot c}}_{\text{summable}}$$

$$1 - x \le e^{-x}$$

By B-C lemma: $\liminf \frac{L_n}{\log_2 n} \ge 1-\varepsilon$ with probability 1.

This implies $\frac{L_n}{\log_2 n} \xrightarrow{a.s.} 1$

2nd Borel-Cantelli lemma

If $A_1, A_2, \dots$ are independent and $\sum_{n=1}^{\infty} P(A_n) = \infty$ then $P(A_n \text{ i.o.}) = 1$.

Proof: $\{A_n \text{ i.o.}\}^c = \{A_n^c$ holds for n large enough$\}$

$$= \bigcup_m \bigcap_{n=m}^{\infty} A_n^c$$

$$P\left(\bigcap_{n=m}^{M} A_n^c\right) = \prod_{n=m}^{M} \left(1-P(A_n)\right) \le e^{-\sum_{n=m}^{M} P(A_n)} \xrightarrow{M\to\infty} 0$$

$\bigcap_{n=m}^{M} A_n^c$: decreasing sequence in M

Then

$$P\left(\bigcap_{n=m}^{\infty} A_n^c\right) = 0$$

$$P\left(\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} A_n^c\right) \le \sum_{m=1}^{\infty} P\left(\bigcap_{n=m}^{\infty} A_n^c\right) = 0$$

Thm: Suppose that $X_1, X_2, \ldots$ are iid with $E\left[|X_1|\right] = \infty$. $S_n = X_1 + \ldots + X_n$

$$P\left(\lim_{n\to\infty} \frac{S_n}{n} \text{ exists and finite}\right) = 0.$$

Proof: ① $P(|X_n| \ge n \text{ i.o}) = 1$

$$\underset{\infty}{\overset{\parallel}{E|X_1|}} = \int_0^{\infty} P(|X_1| \ge x)\, dx \le \underset{\infty}{\overset{\parallel}{\sum_{n=0}^{\infty} P(|X_1| \ge n)}}$$

$P(|X_1| \ge x)$: non-increasing in x

$$\sum_{n=0}^{\infty} P(|X_1| \geq n) = 1 + \sum_{n=1}^{\infty} P(\underbrace{|X_n| \geq n}_{A_n}) = \infty$$

so by the 2^{nd} B-C lemma $P(|X_n| \geq n \text{ i.o}) = 1$

(2) If $\lim \frac{S_n}{n}$ is finite then

$$\underbrace{\frac{S_n}{n} - \frac{S_{n+1}}{n+1}}_{} \longrightarrow 0$$

$S_{n+1} = S_n + X_{n+1}$

$$\underbrace{\frac{S_n}{n(n+1)}}_{\downarrow \atop 0} - \underbrace{\frac{X_{n+1}}{n+1}}_{\text{does not converge to } 0}$$

This implies that $P(\lim \frac{S_n}{n} \in \mathbb{R}) = 0$.

Strong LLN

Thm: $X_1, X_2, \ldots$ iid with $E|X_1| < \infty$

Then $\frac{S_n}{n} \xrightarrow{a.s.} E[X_1]$.

Proof (Etemadi)

We truncate the random variables

$$Y_n = X_n \mathbb{1}(|X_n| \le n)$$

$$T_n = \sum_{k=1}^{n} Y_k$$

$$\sum_{k=1}^{\infty} P(X_k \neq Y_k) = \sum_{k=1}^{\infty} P(|X_k| > k) < \infty$$

$$\sim E[|X_1|]$$

$$E|X_1| = \int_0^\infty P(|X_1| \ge x)\,dx \ge \sum_{k=1}^{\infty} P(|X_1| \ge k)$$

B-C 1. implies that

$$P(X_k = Y_k \text{ if } k \text{ is large enough}) = 1$$

hence $\frac{S_n - T_n}{n} \xrightarrow{a.s.} 0$.

It is enough to show that $\frac{T_n}{n} \xrightarrow{a.s.} E(X_1)$

We may assume that $X_k \ge 0$.

$(X_k = X_k^+ - X_k^-)$

$$\boxed{\text{Lemma: } \sum_{k=1}^{\infty} \frac{1}{k^2} \operatorname{Var} Y_k \le 4 E|X_1| < \infty}$$

$$\le \sum_{k=1}^{\infty} \frac{1}{k^2} E Y_k^2$$

Let $\alpha > 1$, $k(n) = \lfloor \alpha^n \rfloor$.

We prove $\frac{T_{k(n)}}{k(n)} \xrightarrow{a.s.} E X_1$

$$\sum_{n=1}^{\infty} P\left(\left|T_{k(n)} - E T_{k(n)}\right| \geq \varepsilon k(n)\right) \leq$$

$$\leq \sum_{n=1}^{\infty} \frac{\operatorname{Var} T_{k(n)}}{\varepsilon^2 k(n)^2} = \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{k(n)^2} \sum_{m=1}^{k(n)} \operatorname{Var} Y_m$$

$$= \varepsilon^{-2} \sum_{m=1}^{\infty} \operatorname{Var} Y_m \underbrace{\sum_{k(n) \geq m} k(n)^{-2}}_{\text{geometric series}} \leq$$

$$\leq C_\alpha \varepsilon^{-2} \sum_{m=1}^{\infty} \operatorname{Var} Y_m \cdot \frac{1}{m^2} < \infty$$

$$P\left(\frac{\left|T_{k(n)} - E T_{k(n)}\right|}{k(n)} < \varepsilon \text{ for } n \text{ large enough} \right) = 1$$

$$\frac{T_{k(n)} - E T_{k(n)}}{k(n)} \xrightarrow{a.s.} 0$$

$$E Y_k \to E X_1$$ dominated convergence thm

$$\frac{E T_{k(n)}}{k(n)} \to E X_1$$

$$\frac{T_{k(n)}}{k(n)} \xrightarrow{a.s.} E[X_1]$$

If $k(n) \le m \le k(n+1)$

$$\frac{T_{k(n)}}{k(n+1)} \le \frac{T_m}{m} \le \frac{T_{k(n+1)}}{k(n)}$$

$$\frac{k(n)}{k(n+1)} \longrightarrow \alpha$$

$$\underbrace{\frac{1}{\alpha} E[X_1] \le \liminf \frac{T_m}{m} \le \limsup \frac{T_m}{m} \le \alpha \cdot E[X_1]}$$

$\alpha > 1$ can be arbitrary so with $\alpha = 1 + \frac{1}{\ell}$

$$E[X_1] \le \liminf \frac{T_m}{m} \le \limsup \frac{T_m}{m} \le E[X_1]$$

$$P\left(\lim \frac{T_n}{n} = E[X_1] \right) = 1$$