ISyE/Math/CS/Stat 525 – Linear Optimization
Spring 2021

Assignment 2 – Chapter 2

Due date: February 27 at 11:59pm.

Instructions and policy: **Undergraduate students** should handle in the five exercises that are marked with [U]. **Graduate students** should handle in the five exercises that are marked with [G]. All other exercises are optional for keen students and should not be handled in. The assignment should be submitted electronically in Canvas. Late submission policy: 20% of total points will be deducted per hour. Each student is encouraged to solve all the exercises in the assignment to practice for the exams.

Students are strongly encouraged to work in groups of two on homework assignments. To find a partner you can post on the "Discussions" section in Canvas. Only one file should be submitted for both group members. In order to submit the assignment for your group please follow these steps in Canvas: Step 1. Click on the "People" tab, then on "Groups", and join one of the available groups named "Assignments Group 1", "Assignments Group 2", ...; Step 2. When also your partner has joined the same group, one of the two can submit the assignment by clicking on the "Assignments" tab, then on the assignment to be submitted, and finally on "Submit assignment". The submission will count for everyone in your group.

Groups must work independently of each other, may not share answers with each other, and solutions must not be copied from the internet or other sources. If improper collaboration is detected, *all groups* involved will automatically receive a 0. Students must properly give credit to any outside resources they use (such as books, papers, etc.). In doing these exercises, you must justify all of your answers and cite every result that you use. You are not allowed to share any content of this assignment.

Exercise 1 [U][G] .... ***10 points***

Consider the polyhedron defined by the following system of linear inequalities:

$$\begin{array}{rcrcrcrcrcl} -x_1 & + & x_2 & + & x_3 & + & 2x_4 & - & 2x_5 & = & 1 \\ 3x_1 & + & 2x_2 & + & 7x_3 & - & x_4 & + & 4x_5 & = & 7 \\ -2x_1 & + & 4x_2 & + & 6x_3 & + & 2x_4 & - & x_5 & = & 6 \\ & & & & & & & & x & \geq & 0. \end{array}$$

Consider the three vectors $(1/2, 1, 1/2, 0, 0)$, $(1, 2, 0, 0, 0)$ and $(1, 0, 0, 1, 0)$. For each vector:

(a) (5 points) Say if the vector is a basic solution and, if so, specify if the basic solution is degenerate and determine all the bases associated to it.

(b) (5 points) Say if the vector is a vertex and, if so, determine an objective function that is uniquely minimized at the vertex.

Exercise 2 [G] .... ***10 points***

Consider the polyhedron $P = \{x \in \mathbb{R}^n : a_i'x \geq b_i, i = 1, \ldots, m\}$. Suppose that u and v are *distinct* basic feasible solutions that satisfy $a_i'u = a_i'v = b_i, i = 1, \ldots, n-1$, and assume that the vectors $a_1, \ldots, a_{n-1}$ are linearly independent (this implies that u and v are adjacent basic feasible solutions). Let $L = \{\lambda u + (1-\lambda)v : 0 \leq \lambda \leq 1\}$ be the segment that joins u and v. Prove that $L = \{z \in P : a_i'z = b_i, i = 1, \ldots, n-1\}$. (*Hint: Consider the one-dimensional set* $G = \{z \in \mathbb{R}^n : a_i'z = b_i, i = 1, \ldots, n-1\}$.)

Exercise 3 .... ***0 points***

A mapping f is called *affine* if it is of the form $f(x) = Ax + b$, where A is a matrix and b is a vector. Let P and Q be polyhedra in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. We say that P and Q are *isomorphic* if there exist affine mappings $f : P \mapsto Q$ and $g : Q \mapsto P$ such that $g(f(x)) = x$ for all $x \in P$, and $f(g(y)) = y$ for all $y \in Q$. (Intuitively, isomorphic polyhedra have the same shape.)

(a) If P and Q are isomorphic, show that there exists a one-to-one correspondence between their extreme points. In particular, if f and g are as above, show that x is an extreme point of P if and only if $f(x)$ is an extreme point of Q.

(b) **(Introducing slack variables leads to an isomorphic polyhedron)** Let $P = \{x \in \mathbb{R}^n \mid Ax \geq b,\ x \geq 0\}$, where A is a matrix of dimensions $k \times n$. Let $Q = \{(x, z) \in \mathbb{R}^{n+k} \mid Ax - z = b,\ x \geq 0,\ z \geq 0\}$. Show that P and Q are isomorphic.

Exercise 4 [U][G] .... ***10 points***

Let $A_1, \ldots, A_n$ be a collection of vectors in $\mathbb{R}^m$.

(a) (5 points) Let

$$C = \left\{ \sum_{i=1}^{n} \lambda_i A_i : \lambda_1, \ldots, \lambda_n \geq 0 \right\}.$$

Show that if $y \in C$, then there exist coefficients $\lambda_1, \ldots, \lambda_n \geq 0$ such that (i) *at most* m of the coefficients are nonzero, and (ii) $y = \sum_{i=1}^{n} \lambda_i A_i$.

Hint: Consider the polyhedron

$$Q = \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \sum_{i=1}^{n} \lambda_i A_i = y, \lambda_1, \ldots, \lambda_n \geq 0 \right\}.$$

(b) (5 points) Let P be the convex hull of the vectors A_i, i.e.

$$P = \left\{ \sum_{i=1}^{n} \lambda_i A_i : \sum_{i=1}^{n} \lambda_i = 1, \lambda_1, \ldots, \lambda_n \geq 0 \right\}.$$

Show that if $y \in P$, then there exist coefficients $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$, such that (i) *at most* $m+1$ of the coefficients are nonzero, and (ii) $y = \sum_{i=1}^{n} \lambda_i A_i$.

Exercise 5 .... ***0 points***

Consider the standard form polyhedron $P = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}$, and assume that the rows of the matrix A are linearly independent.

(a) Suppose that two different bases lead to the same basic solution. Show that the basic solution is degenerate.

(b) Consider a degenerate basic solution. Is it true that it corresponds to two or more distinct bases? Prove it or give a a counterexample.

(c) Suppose that a basic solution is degenerate. Is it true that there exists a distinct adjacent basic solution which is degenerate? Prove it or give a a counterexample.

Exercise 6 .... ***0 points***

Consider the standard form polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}$. Suppose that the matrix A, of dimension $m \times n$, has linearly independent rows, and that all basic feasible solutions are nondegenerate. Let x be a vector in P that has exactly m positive components.

(a) Show that x is a basic feasible solution.
Hint: Look at the proof of Theorem 2.6 $(b) \to (a)$ in the book. The idea of the proof has been explained in class.

(b) Show that the result of part (a) is false if the nondegeneracy assumption is removed.

Exercise 7 .... ***0 points***

Let P be a bounded polyhedron in $\mathbb{R}^n$, let a be a vector in $\mathbb{R}^n$, and let b be some scalar. We define

$$Q = \{x \in P \mid a'x = b\}.$$

Show that every extreme point of Q is either an extreme point of P or a convex combination of two adjacent extreme points of P. *Hint*: Look at the proof of Theorem 2.6 $(b) \to (a)$ in the book. The idea of this proof has been explained in class.

Exercise 8 [U] .... ***10 points***

Consider the set $\{x \in \mathbb{R}^n \mid x_1 = x_2 = \cdots = x_{n-1} = 0,\ 0 \leq x_n \leq 1\}$. Could this be the feasible set of a problem in standard form in $\mathbb{R}^n$?

Exercise 9 [U][G] .... ***10 points***

We know that every linear program can be transformed into an equivalent linear program in standard form. We also know that a nonempty polyhedron in standard form has at least one extreme point. We are then tempted to conclude that every nonempty polyhedron has at leat one extreme point. What is wrong with this argument?

Exercise 10 [U][G] .... ***10 points***

Consider the polyhedron P defined by the following system of 5 linear inequalities in 3 variables x_1, x_2, x_3.

$$
\begin{array}{llll}
2x_1 & -5x_2 & +4x_3 & \leq 10 \\
3x_1 & -6x_2 & +3x_3 & \leq 9 \\
5x_1 & +10x_2 & -x_3 & \leq 15 \\
-x_1 & +5x_2 & -2x_3 & \leq -7 \\
-3x_1 & +2x_2 & +6x_3 & \leq 12
\end{array}
$$

Apply the Fourier-Motzkin elimination algorithm to P to compute $\Pi_1(P)$ by eliminating first variable x_3 and then variable x_2. Is P empty?