ISyE/Math/CS/Stat 525 – Linear Optimization
Spring 2021

Assignment 0

This assignment will not be graded and there is no due date. Do not hand in the solutions.

This assignment is designed so that you can self-evaluate your Math skills and your knowledge of Linear Algebra. You can refresh your Math skills by studying the document "Introduction to mathematical arguments" by Michael Hutchings, which is available in Canvas. Regarding Linear Algebra, in particular students must have working knowledge of set theory, vectors and matrices, matrix inversion, subspaces and bases, and affine subspaces. You can strengthen your knowledge of Linear Algebra by revising any textbook or lecture notes on the topic. A nice overview is presented in Section 1.5 "Linear algebra background and notation" of the recommended textbook for the course, which is "Introduction to Linear Optimization" by D. Bertsimas and J.N. Tsitsiklis. After completing the assignment, check your answers using the solutions available in Canvas. Then revise the concepts that were not clear to you. Note that understanding basic proof techniques and having a working knowledge of linear algebra are essential in this course!

Test your Math skills

Exercise 1 .... ***0 points***

Rewrite each of the following sentences by using mathematical notation:

(a) For each integer x, the double of x is even.

> **Solution:** $\forall x \in \mathbb{Z}$, $2x$ is even.

(b) An integer x is odd if there exists an integer y such that $x = 2y + 1$.

> **Solution:** $\forall x \in \mathbb{Z}, \text{if } \exists y \in \mathbb{Z} \mid x = 2y + 1, \text{then } x \text{ is odd.}$

(c) For each subset of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that has cardinality at least 5, there exists one element of the subset that is even.

> **Solution:** $\forall A \subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}$ with $|A| \geq 5$, then $\exists x \in A \mid x$ is even.

Exercise 2 .... ***0 points***

Negate the following statements:

(a) Each student of 525 is at least 22 years old.

> **Solution:** There exists at least one student of 525 that is less than 22 years old.

(b) In all math classes there exists a student that is a genius.

> **Solution:** There exists a math class where no student is a genius.

(c) In each lecture, the teacher arrives late or leaves early.

Solution: There exists at least one lecture where the teacher arrives on time and leaves on time.

Exercise 3 . . . . **_0 points_**

Disprove the following conjectures by hexibiting a counterexample:

(a) Conjecture 1: Every positive integer is equal to the sum of two integer squares.

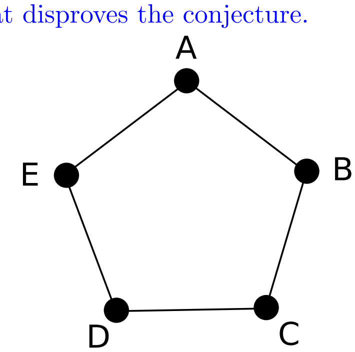
Solution: 3 is a positive integer that cannot be expressed as the sum of two integer squares. In fact, 0, 1 and -1 are the only integers whose squares are at most 3, and (3-0) and (3-1) are not the square of any integer.

(b) Conjecture 2: Each subset of $\{1, 2, \ldots, 8\}$ that has cardinality at least 3 contains a number that is a multiple of 2 or of 3.

Solution: $\{1, 5, 7\}$ is a subset of cardinality 3 such that each element of the subset is neither a multiple of 2, nor of 3.

(c) Conjecture 3: At any party with at least 5 people, there are always three people that mutually know each other, or three people that do not know each other. (Note: A, B and C mutually know each other if any two persons among A, B and C know each other. Similarly, A, B and C do not know each other if any two persons among A, B and C are strangers.)

Solution: Consider a party with 5 people $\{A, B, C, D, E\}$. We represent each person as a dot, and we trace a line connecting two dots if and only if the corresponding persons know each other. Below is an instance that disproves the conjecture.



Exercise 4 . . . . **_0 points_**

Consider the following true statement:

If an integer is a multiple of 4, then it is a multiple of 2.

(a) Being a multiple of 4 is a necessary or sufficient condition for being a multiple of 2?

Solution: It is a sufficient condition.

(b) Being a multiple of 2 is a necessary or sufficient condition for being a multiple of 4?

Solution: It is a necessary condition.

Exercise 5 . . . . **_0 points_**

(a) A $\Rightarrow$ B means that A is necessary or sufficient for B?

Solution: Sufficient.

(b) A $\Leftarrow$ B means that A is necessary or sufficient for B?

Solution: Necessary.

Exercise 6 . . . . *0 points*

(a) Prove the following: $\forall x, y \in \mathbb{Z}$, if one among x and y is even, then xy is even.

Solution: Suppose without loss of generality that x is even. Then there exists an integer k such that $x = 2k$. This implies that $xy = 2ky = 2(ky)$ Since ky is the product of two integers, it is an integer. Thus xy can be expressed as $2\bar{k}$, where $\bar{k}$ is an integer equal to ky. this proves that xy is even.

(b) As a consequence, prove that $\forall x \in \mathbb{Z}$, $x(x+1)$ is even.

Solution: Now let x be an arbitrary integer. If x is even, then $x(x+1)$ is even. If x is odd, then $x+1$ is even, and again $x(x+1)$ is even.

Exercise 7 . . . . *0 points*

Prove the following: For every integer x, if x is odd, then there exists an integer y such that $x^2 = 8y + 1$.

Solution: Let x be an odd integer. Then $x = 2k + 1$ for some integer k. As a consequence $x^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4k(k+1) + 1$. Since $k(k+1)$ is even (see previous question), we have that $\frac{k(k+1)}{2}$ is integer. Thus $x^2 = 4k(k+1)+1 = 8\frac{k(k+1)}{2}+1$ and the claim is proven by setting $y = \frac{k(k+1)}{2}$.

Exercise 8 . . . . *0 points*

Prove by contradiction that $\sqrt{2} + \sqrt{6} < \sqrt{15}$

Solution: Suppose by contradiction that $\sqrt{2} + \sqrt{6} \geq \sqrt{15}$. Then $8 + 2\sqrt{12} = (\sqrt{2} + \sqrt{6})^2 \geq 15$. This implies $2\sqrt{12} \geq 7$, thus $48 \geq 49$, a contradiction.

Exercise 9 . . . . *0 points*

Imagine four cards placed on the table, each with a letter on one face and a number on the other one:

A D 9 5

You are given the rule which states: "if A is on a card, then 5 is on its other side". Which card(s) need to be turned over to check whether the rule holds?

Solution: A and 9.

Exercise 10 .... *0 points*

Prove the following: Let $x \in \mathbb{Z}$. If x^2 is odd, then x is odd.
Hint: use the contrapositive.

> **Solution:** We prove the equivalent statement: Let $x \in \mathbb{Z}$. If x is even, then x^2 is even. Suppose that x is even, thus $x = 2k$ for some integer k. We get $x^2 = 4k^2 = 2(2k^2)$. Since $2k^2$ is an integer, we have proved that x^2 is even.

Exercise 11 .... *0 points*

Prove by induction the following statement: For each positive integer x, $2^{3x+1} + 5$ is a multiple of 7.

> **Solution:** The base case is $x = 1$. In this case we get $2^4 + 5 = 21 = 3 \cdot 7$. Now suppose that the statement holds for $x \geq 1$, i.e. , $2^{3x+1} = 7k$ for some integer k. We show that the statement also holds for $x + 1$. Let $\alpha = 2^{3(x+1)+1} + 5$. Note that α is a multiple of 7 if and only if $\alpha + 7 \cdot 5$ is a multiple of 7. Thus we now prove that $\alpha + 7 \cdot 5 = 2^{3(x+1)+1} + 5 + 7 \cdot 5 = 2^{3(x+1)+1} + 40$ is a multiple of 7. We have
>
> $$\begin{aligned} \alpha + 7 \cdot 5 &= 2^{3(x+1)+1} + 40 \\ &= 2^{3x+1} \cdot 2^3 + 5 \cdot 2^3 \\ &= (2^{3x+1} + 5) \cdot 2^3 \\ &= 7k \cdot 2^3. \end{aligned}$$

Exercise 12 .... *0 points*

Prove by induction that $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$.

> **Solution:** We prove the result by induction on n. The base case is $n = 1$, as we obtain the identity $2 = 2$. Now we assume by induction that the claim is true for $n \geq 1$, and we prove it for $n + 1$. We obtain the following:
>
> $$\begin{aligned} \sum_{k=1}^{n+1} k(k+1) \sum_{k=1}^{n} k(k+1) &+ (n+1)(n+2) \\ &= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) \\ &= (n+1)(n+2)(\frac{n}{3} + 1) \\ &= \frac{(n+1)(n+2)(n+3)}{3}. \end{aligned}$$

Test your Linear Algebra

Exercise 1 .... ***0 points***

Compute, if it exists, the matrix product AB for the following matrices:

(a) $A = \begin{bmatrix} 3 & 4 & 0 \\ 2 & 7 & 1 \\ 6 & 5 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 & 8 \\ 5 & 7 & 0 \\ 6 & 4 & 3 \end{bmatrix}$

Solution: $AB = \begin{bmatrix} 26 & 34 & 24 \\ 45 & 57 & 19 \\ 79 & 75 & 69 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}$

Solution: $AB = \begin{bmatrix} 2 & 6 & 10 \\ 3 & 9 & 15 \\ 4 & 12 & 20 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 3 & 5 \end{bmatrix}$

Solution: $AB = \begin{bmatrix} 8 & 4 \\ 6 & 10 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

Solution: The matrix product does not exist since the number of columns of matrix A is different from the number of rows of matrix B.

Exercise 2 .... ***0 points***

Let $x = (3, 4, 5)$, $y = (1, 3, -3)$, and $z = (2, 1, -1)$.

(a) Compute the inner products $x'y$ and $x'z$.

Solution: $x'y = 3 + 12 - 15 = 0$ and $x'z = 6 + 4 - 5 = 5$.

(b) Are x and y orthogonal? What about x and z?

Solution: x and y are orthogonal, since $x'y = 0$. Instead, x and z are not.

(c) Compute the norm of x, y and z.

Solution: $\|x\| = \sqrt{9 + 16 + 25} = 5\sqrt{2}$, $\|y\| = \sqrt{1 + 9 + 9} = \sqrt{19}$, $\|z\| = \sqrt{4 + 1 + 1} = 6$.

Exercise 3 .... ***0 points***

True or false (justify your answers).

(a) $\|x\| \cdot \|y\| \geq |x'y|$ for any two vectors x and y in $\mathbb{R}^n$.

Solution: True. This is the *Schwarz inequality*. Equality holds if and only if one of the two vectors is a scalar multiple of the other.

(b) $x'x \geq 0$ for each $x \in \mathbb{R}^n$.

Solution: True. $x'x = \sum_{i=1}^n (x_i)^2$, that is nonnegative since $(x_i)^2 \geq 0$ for all $i = 1, \ldots, n$.

(c) For any two matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$, $AB = BA$.

Solution: False. For example the statement is not true for $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(d) For any two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times k}$, $(AB)' = A'B'$.

Solution: False. The matrix product $A'B'$ might even not exist if $m \neq k$. If $m = k$, but $m \neq n$, the product exists, and is a $n \times n$ matrix, while $(AB)'$ is a $k \times m$ matrix. Finally, also when $m = n = k$, $(AB)' = A'B'$ might not hold, as in the following case: $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 3 & 5 \end{bmatrix}$, $(AB)' = \begin{bmatrix} 8 & 6 \\ 4 & 10 \end{bmatrix}$, $A'B' = \begin{bmatrix} 8 & 3 \\ 8 & 10 \end{bmatrix}$.

(e) For any two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times k}$, $(AB)' = B'A'$.

Solution: True.

$$[(AB)']_{ij} = [AB]_{ji} = \sum_{\ell=1}^n [A]_{j\ell}[B]_{\ell i}.$$

$$[B'A']_{ij} = \sum_{\ell=1}^n [B']_{i\ell}[A']_{\ell j} = \sum_{\ell=1}^n [A]_{j\ell}[B]_{\ell i}.$$

(f) For any two symmetric matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$, $(AB)' = A'B'$.

Solution: False.

$$\begin{aligned} (AB)' &= (A'B')' = BA, \\ A'B' &= AB, \end{aligned}$$

thus the statement is not true in general, but it is true if and only if $AB = BA$.

(g) Let $A_1, \ldots, A_n$ be the columns of a matrix A, i.e.: $A = \begin{bmatrix} | & | & & | \\ A_1 & A_2 & \ldots & A_n \\ | & | & & | \end{bmatrix}$, and let $x = (x_1, x_2, \ldots, x_n)$. We have $Ax = A_1x_1 + A_2x_2 + \cdots + A_nx_n$.

Solution: True. Let e_i be the i-th unit vector. Then $x = x_1e_1 + \cdots + x_ne_n$.

$$Ax = A(x_1e_1 + \cdots + x_ne_n) = x_1(Ae_1) + \cdots + x_n(Ae_n) = \sum_{\ell=1}^n x_iA_i.$$

(h) A square matrix A is invertible if and only if its determinant is non-zero.

Solution: True. See Theorem 1.2 in the textbook.

(i) A square matrix A has zero determinant if and only if its rows are linearly dependent.

Solution: True. See Theorem 1.2 in the textbook.

Exercise 4 .... ***0 points***

Is the following matrix invertible?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ -1 & 0 & 1 \end{bmatrix}$$

If A^{-1} exists, compute it. Find all solutions of the system $Ax = b$ for $b = (2, 1, 1)$.

Solution: Yes, the matrix is invertible since its determinant is -6.

$$A^{-1} = \begin{bmatrix} 1/6 & 1/6 & -1/2 \\ 2/3 & -1/3 & 1 \\ 1/6 & 1/6 & 1/2 \end{bmatrix}$$

The unique solution of $Ax = b$ is $(0, 2, 1)$.

Exercise 5 .... ***0 points***

Recall that vectors $x^1, \ldots, x^k$ are said to be linearly independent if $\sum_{j=0}^{k} a_j x^j = 0$ implies $a_j = 0$ for every $j = 1, \ldots, k$. Show that the vectors in a given finite collection are linearly independent if and only if none of the vectors can be expressed as a linear combination of the others.

Solution: Consider a collection of K vectors $x^1, \ldots, x^K$.

- Let us prove first that if $x^1, \ldots, x^K$ are linearly independent, then none of the vectors can be expressed as a linear combination of the others. By contradiction, assume that it is possible to write one of these vectors, let it be x^K, as a linear combination of the others:

$$x^K = \sum_{k=1}^{K-1} a_k x^k,$$

where $a_k \in \mathbb{R}$ for $k = 1, \ldots, K$. Therefore we have that

$$\sum_{k=1}^{K-1} a_k x^k - x^K = 0.$$

Thus, we found a linear combination of $x^1, \ldots, x^K$ equal to 0, in which some coefficients are nonzero (the coefficient of x^K is -1). This is a contradiction, since we are assuming that $x^1, \ldots, x^K$ are linearly independent.

- Now let us assume that none of the vectors $x^1, \dots, x^K$ can be expressed as a linear combination of the others. We want to show that $x^1, \dots, x^K$ are linearly independent.

  Suppose, by contradiction, that $x^1, \dots, x^K$ are linearly dependent, i.e. there exist real numbers $a_1, \dots, a_K$ not all of them zero, such that $\sum_{k=1}^K a_k x^k = 0$. Let $h \in \{1, \dots, K\}$ be such that $a_h \neq 0$, and for each $k \neq h$ in $\{1, \dots, K\}$ define $\bar{a}^k = -a^k / a^h$. Then we have

  $$x^h = \sum_{k \in \{1, \dots, K\}, k \neq h} \bar{a}_k x^k.$$

  Thus vector x^h can be expressed as a linear combination of the other vectors of our collection, a contradiction.

Exercise 6 .... *0 points*

Let $x = (-2, -1, 3, 4)$ and $y = (-8, 2, -2, 1)$.

(a) Are x and y linearly independent?

Solution: Yes. In fact, y cannot be expressed as a scalar multiple of x.

(b) Define a linear combination z of x and y.

Solution: $z = \lambda^1 x + \lambda^2 y$, where $\lambda^1, \lambda^2 \in \mathbb{R}$. For example $z = 2x - y = (4, -4, 8, 7)$.

(c) Are x, y and z linearly independent?

Solution: No, by definition, since z can be expressed as a linear combination of x and y.

Exercise 7 .... *0 points*

Prove that the following two definitions of *linear function* are equivalent:

Definition 1: A function $f : \mathbb{R}^n \to R$ is *linear* if:

1. $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^n$
2. $f(\lambda x) = \lambda f(x)$ for all $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$

Definition 2: A function $f : \mathbb{R}^n \to R$ is *linear* if it can be written as:

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n,$$

where $c_1, c_2, \dots, c_n$ are real numbers.

Solution:

- We first prove that if f satisfies Definition 1, then it also satisfies Definition 2.

  Let e_i be the i-th unit vector. Then $x = x_1 e_1 + \cdots + x_n e_n$. Thus:

  $$f(x) = f(x_1 e_1 + \cdots + x_n e_n) \underset{\text{(a)}}{=} f(x_1 e_1) + \cdots + f(x_n e_n) \underset{\text{(b)}}{=} f(e_1) x_1 + \cdots + f(e_n) x_n,$$

  and we can set $c_i = f(e_i)$ for $i = 1, \dots, n$.

- We now prove that if f satisfies Definition 2, then it also satisfies Definition 1. In fact:

    1.

    $$\begin{aligned} f(x+y) &= c_1(x_1+y_1) + c_2(x_2+y_2) + \cdots + c_n(x_n+y_n) \\ &= (c_1x_1 + c_2x_2 + \cdots + c_nx_n) + (c_1y_1 + c_2y_2 + \cdots + c_ny_n) = f(x) + f(y) \end{aligned}$$

    2.

    $$\begin{aligned} f(\lambda x) &= c_1(\lambda x_1) + c_2(\lambda x_2) + \cdots + c_n(\lambda x_n) \\ &= \lambda c_1x_1 + \lambda c_2x_2 + \cdots + \lambda c_nx_n \\ &= \lambda(c_1x_1 + c_2x_2 + \cdots + c_nx_n) = \lambda f(x) \end{aligned}$$