SP21 COMPSCI 513 - Homework 1

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Proof.

(a) By definition,

$$\|Q\|_2 = max \left\{ \frac{\|Qv\|_2}{\|v\|_2} : v \neq 0 \right\} = 1$$

For $\|v\|_2 = 1$, $\|v\|_2^2 = 1$,

$$\|Q\|_2^2 = 1 = \|v\|_2^2 = \|Qv\|_2^2 = (Qv, Qv) = (Q'Qv, v)$$

$Q'Q$ is symmetric, by the schur decomposition, $Q'Q = SDS'$. Where S is orthogonal and D is diagonal.

$$1 = (SDS'v, v) = (DS'v, S'v)$$

Denote $w = S'v$, we have $(Dw, w) = 1$.

$$1 = \sum D_{ii} w_i^2 = \sum w_i^2$$

$$\sum (D_{ii} - 1) w_i^2 = 0$$

So, $D = I$. 1 is the only eigenvalue of $Q'Q$.

(b) From the proof of (a), $D = I$ means

$$Q'Q = SS'$$

Here S is orthogonal, Q must also be orthogonal.

$\square$