ISyE/Math/CS/Stat 525 – Linear Optimization
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Homework 2

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Exercise 1

(a) $(1/2, 1, 1/2, 0, 0)$ is not a basic solution.
$(1, 2, 0, 0, 0)$ is a basic solution. It is degenerate basic feasible solution. Bases:$\{A_1, A_2, A_3\}, \{A_1, A_2, A_4\}, \{A_1, A_2, A_5\}$.
$(1, 0, 0, 1, 0)$ is not a basic solution.

(b) By Theorem 2.3, x^* is a vertex if and only if x^* is a basic feasible solution.
$(1/2, 1, 1/2, 0, 0)$ is not a vertex.
$(1, 2, 0, 0, 0)$ is a vertex. $f(x) = x_3 + x_4 + x_5$.
$(1, 0, 0, 1, 0)$ is not a vertex.

Exercise 2

First of all, we prove that for any $0 \leqslant \lambda \leqslant 1$, $\lambda u + (1-\lambda)v \in P$, by definition, u, v satisfies

$$a_i' u \geqslant b_i, i = 1, \cdots, m$$

$$a_i' v \geqslant b_i, i = 1, \cdots, m$$

thus,

$$a_i'(\lambda u + (1-\lambda)v) = \lambda a_i' u + (1-\lambda) a_i' v \geqslant b_i, i = 1, \cdots, m$$

It means any elements in L is in P.
Next, we prove that, for any $0 \leqslant \lambda \leqslant 1$, $a_i' z = b_i, i = 1, \cdots, n-1$. u, v are distinct basic feasible solutions, so they satisfies

$$a_i' u = b_i, i = 1, \cdots, n-1$$

$$a_i' v = b_i, i = 1, \cdots, n-1$$

then,

$$a_i'(\lambda u + (1-\lambda)v) = \lambda a_i' u + (1-\lambda) a_i' v = b_i, i = 1, \cdots, n-1$$

It means, $\{\lambda u + (1-\lambda)v : 0 \leqslant \lambda \leqslant 1\} \subseteq \{z \in P : a_i' z = b_i, i = 1, \cdots, n-1\}$. Consider $A = [a_1' a_2' \cdots a_{n-1}']$, $A(z-v) = Az - Av = 0.A$ is invertible, so $z = v$ for first $n-1$ elements. The same as z and u. This means z is the linear combination of u and v. So, we proved $L = \{z \in P : a_i' z = b_i, i = 1, \cdots, n-1\}$.

Exercise 4

(a) Consider the polyhedron, by Theorem 2.5. We can find a new polyhedron $Q1$ such that

$$Q = Q1 = \{(\lambda_1', \cdots, \lambda_m') : \sum_{i=1}^{m} \lambda_i A_i = y, \lambda_1, \cdots, \lambda_n \geqslant 0\}$$

The rest λ are zeros. Then, we have constructed the coefficients $\lambda_1, \cdots, \lambda_n \geqslant 0$.

(b) By Corollary 2.6, The convex hull of a finite number of vectors is a polyhedron. Similar to the proof of (a), we need an extra λ to make sure $\sum_{i=1}^{n} \lambda_i = 1$.