728 HW3. ZIJIE ZHANG, KECHUN WANG

1. We can prove this by Theorem 4.6 (Ghouila-Houri)

Assume that any k-column submatrix of A admits an equitable bicoloring.

Case $k=2$, it is trivial.

Case $k+1$, we can always find a k-col submatrix of A which is the same as the left k cols in $(k+1)$-col submatrix. In this condition, the rows end with 0 must be append with 0. So we only care about the rows end with 1. We have the following conditions

$$\begin{array}{c} \\ \begin{bmatrix} * & * & * & 1 \\ * & * & * & 1 \end{bmatrix} \end{array} \begin{array}{c} \text{Sum} \\ 1 \\ -1 \end{array} \Rightarrow \begin{array}{c} \\ \begin{bmatrix} * & * & 1 & 1 & 1 & \cdots & \cdots & 1 \\ * & * & 0 & 0 & 1 & \cdots & & 1 \end{bmatrix} \end{array} \begin{array}{c} \text{sum} \\ 1 \\ -1 \end{array}$$

If the $(k+1)$th column is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

$$\begin{array}{c} \\ \left[\begin{array}{cccccccc|c} * & * & 1 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ * & * & 0 & 0 & 1 & \cdots & & 1 & 1 \end{array}\right] \end{array} \begin{array}{c} \text{sum} \\ 0 \\ 1 \end{array}$$

We found that this operation has no effect on the rest of the rows

So Case $k+1$ is true

In conclusion, any matrix A with this property is TU.

2. Consider the constraint matrix of P.

Set

$$A_{(n+1)\times(n+1)} = \begin{bmatrix} I & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & -1 \\ 0 & & \ddots & & 1 & -1 \\ 0 & 0 & \cdots & & 0 & 1 \end{bmatrix}$$

$$b = [0 \; 0 \cdots\cdots 0 \; 1]'$$

$P = \{ x \in \mathbb{R}^{n+1} ; \; Ax \le b, \; x \ge 0 \}$, b is integral vector.

We need to prove that A is TU.

Case 1: $|I| = 1$ Case 2: $\begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{vmatrix} = 0$.

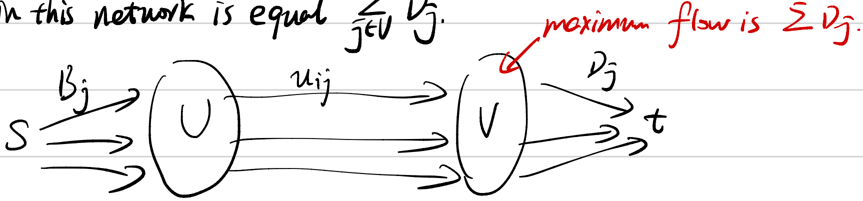
Case 3: $\begin{vmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = -1$ Case 4: $\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 1$

We can check every possible square submatrix of A. and their determinants are +1, -1 or 0. Then, A is TU.

We proved that P is an integral

3.1. The maximum flow in the network is $\sum_{j\in V} D_j$. means the flow from $S \to U \to V \to t$ is $\sum_{j\in V} D_j$. There exists a flow such that satisfies all of the constraints. The feasible flow to the transportation problem is the same as the maximum flow in this network.

On the other have, if we know there exists a feasible flow to the transportation problem. then $\sum_{i\in U} x_{ij} = D_j$, this flow uses all the capacity of all the arcs from U to V. This implies the maximum flow is smaller or equal to $\sum_{i\in U} x_{ij} = D_j$. So we have proved that a feasible flow to the transportation problem exists if and only if the maximum flow in this network is equal $\sum_{j\in V} D_j$.



3.2. By the max-flow min-cut theorem,

$$\max\{v(x) : x \text{ is feasible } s,t\text{-flow}\} = \min\{h(C) : C \text{ is an } s\text{-}t\text{-cut}\}$$

From (a), we can find a maximum flow in the Network. Suppose there is a s-t cut. $\sum_{i\in U\setminus S} B_i + \sum_{i\in S}\sum_{j\in T} u_{ij}$ is a min-cut.

$$\sum_{i\in U\setminus S} B_i + \sum_{i\in S}\sum_{j\in T} u_{ij} \geq \sum_{j\in T} D_j \quad \forall S\subseteq U, T\subseteq V.$$

5.1 pf: for $z_1, z_2 \in A+B$, and $\lambda \in (0,1)$

Suppose $z_i = x_i + y_i$, $x_i \in A$ $y_i \in B$, $i=1,2$, Then

$\lambda z_1 + (1-\lambda) z_2 = \lambda(x_1+y_1) + (1-\lambda)(x_2+y_2)$

$= [\lambda x_1 + (1-\lambda)x_2] + [\lambda y_1 + (1-\lambda) y_2]$ $(*)$

Since A and B are convex sets

$x_1, x_2 \in A \Rightarrow \lambda x_1 + (1-\lambda) x_2 \in A$

$y_1, y_2 \in B \Rightarrow \lambda y_1 + (1-\lambda) y_2 \in B$

so $(*) \in A+B$ which means $\lambda z_1 + (1-\lambda) z_2 \in A+B$

$\Rightarrow$ $A+B$ is convex

5.2 pf: 1° By the proof of 5.1, we know that

$conv(A) + conv(B)$ is also convex set

Since $A \subseteq conv(A)$, $B \subseteq conv(B)$

$\Rightarrow A+B \subseteq conv(A) + conv(B)$

So $conv(A+B) \subseteq conv(A) + conv(B)$ by definition of a convex hull

2° for any $z \in conv(A) + conv(B)$

suppose $z = x+y$, $x \in conv(A)$, $y \in conv(B)$

suppose $x = \sum_{i=1}^{n} \alpha_i x_i$, $y = \sum_{j=1}^{m} \beta_j y_j$ $x_i \in A$, $y_j \in B$

And $\sum_i \alpha_i = \sum_j \beta_j = 1$

Then $x+y = \sum_{i=1}^{n} \alpha_i x_i + \sum_{j=1}^{m} \beta_j y_j$

$= \sum_{j=1}^{m} \beta_j (x + y_j)$ $(*)$

And we know $x + y_j = \sum_{i=1}^{n} \alpha_i x_i + y_j$

$= \sum_{i=1}^{n} \alpha_i (x_i + y_j)$

Since $x_i + y_j \in A+B$ for $i=1,2,\cdots,n$

$\Rightarrow$ $x+y_j \in \text{conv}(A+B)$

Then by $(*)$, we know $x+y = \sum_{j=1}^{m} \beta_j (x+y_j)$

$x+y \in \text{conv}(\text{conv}(A+B)) = \text{conv}(A+B)$

So $\text{conv}(A) + \text{conv}(B) \subseteq \text{conv}(A+B)$

$\Rightarrow$ $\text{conv}(A) + \text{conv}(B) = \text{conv}(A+B)$

4.1 Define graph $G=(V,E)$ $V=\{v_1, \cdots, v_n\}$

There is an edge between v_i, v_j if and only if $a_i + a_j \leq 1$

Then we need to find a matching of maximum cardinality $M \subseteq E$, suppose $\text{card}(M) = m$

Finally we put the items i, j in one bin if the edge connected v_i, v_j is in M and we put each of the rest of the items in a different bin.

$\Rightarrow$ At this time we need $m + (n-2m) = n-m$ bins

pf of the algorithm:

Since $a_i > \frac{1}{3}$ $\Rightarrow$ $\forall i,j,k$ $a_i + a_j + a_k > 1$ which means we can just put one or two items in one bin.

And in the algorithm, there is an edge between v_i, v_j means we can put item i and j in one bin

So by solving the maximum cardinality matching problem, we find a solution using the most number of bins containing two items.

· So the bins containing one items are the Least.

⇒ The solution help us to find the minimum number of bins and a way to pack these items.