Lecture 14: Underdetermined least squares

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Outline

1. A comment on overdetermined least squares

2. Underdetermined systems
   - The characterization theorem
   - The normal equation algorithm
   - Solving via QR-factorization

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Systems that have the same least square solution

Assume that A is $m \times n$ and B is $m \times n$, and

$$\text{range(A)} = \text{range(B)}.$$

Then, the least square solution to $Ax = b$, and $By = b$ is the same in the sense that

$$Ax^* = By^*.$$

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Then, the least square solution to $Ax = b$, and $By = b$ is the same in the sense that

$$Ax^* = By^*.$$

Moreover, x^* is the solution of the system

$$B'Ax = B'b,$$

and y^* is the solution of the system

$$A'By = A'b.$$

Systems that have the same least square solution

The QR-factorization algorithm creates Q_1 such that

$$\mathrm{range(A)} = \mathrm{range(Q_1)}.$$

So, indeed, we can solve

$$Q_1' Ax = Q_1' b.$$

If you are not interested in what x^* is but only in Ax^*, you may solve

$$Q_1 z = b.$$

The solution is $z^* = Q_1' b$, and then

$$Ax^* = Q_1 Q_1' b.$$

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The problem defined

Underdetermined least squares

Input: System $n \times m$

$$A'x = b,$$

for which there is more than one exact solution.
Output: $x^* \in R^n$ such that

- $A'x^* = b$.
- If $A'x = b$, then $||x^*||_2 \leq ||x||_2$.

The characterization theorem

Characterizing x^*

If $y \in R^n$ such that:

- $A'y = b$
- $y \perp \ker(A')$

Then $y = x^*$.

The characterization theorem

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Proof: Let x be such that $Ax = b$. Need to show

$$||y||_2 \leq ||x||_2.$$

Note: $A'(x-y) = A'x - A'y = b - b = 0$, therefore $(x-y, y) = 0$.
Then:

$$||x||_2^2 = ||(x-y)+y||_2^2 = ||x-y||_2^2 + ||y||_2^2 \geq ||y||_2^2.$$

Linear algebra theorem

B $n \times m$ is any matrix $v \in R^m$.

Then: $v \perp \ker B$ if and only if $v \in \mathrm{range}(\mathrm{B}')$.

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Proof: We only prove the 'if'. So, assume $v \in \mathrm{range(B')}$:

$$v = B'w.$$

Let $x \in \ker(B)$. Need to show that $(v, x) = 0$:

$$(v, x) = (B'w, x) = \ldots$$

Linear algebra theorem

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$$(v, x) = (B'w, x) = (w, Bx) = (w, 0) = 0.$$

Linear algebra theorem

Take $B = A'$, then $B' = A$, therefore

Another characterization theorem

Given $A'x = b$, A $n \times m$, with solutions. Assume $y \in R^m$ such that

- $A'y = b$.
- $y \in \mathrm{range(A)}$.

Then $y = x^*$.

The algorithm

$A'x = b$, there are solutions

- Assume $x \in \mathrm{range}(\mathrm{A})$, i.e., $x = At$.
- Then: $A'At = b$.
- The normal equation has solutions: Always, $\mathrm{range}(\mathrm{A'A}) = \mathrm{range}(\mathrm{A'})$, and by assumption $b \in \mathrm{range}(\mathrm{A'})$. Let t^* be such solution.
- $x^* = At^*$.

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Underdetermined via QR-factorization

We factor $A = Q_1R_1$ as before.
Since $\mathrm{range}(Q_1) = \mathrm{range}(A)$, then

$$x \in \mathrm{range}(A)$$

is the same as $x = Q_1t$:

$$A'Q_1t = b.$$

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$$A' Q_1 t = b.$$

$A = Q_1 R_1$, so

$$A' Q_1 = R_1' Q_1' Q_1 = R_1'.$$

So, we solve

$$R_1' t = b.$$

Underdetermined via QR-factorization

Solving UD least squares via QR-factorization

Given $A'x = b$:

- Q-factor $A = Q_1 R_1$.
- Solve $R_1' t = b$.
- $x^* = Q_1 t$.