Lecture 6: Introduction

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Outline

1. Positive Definite Matrices
   - Definition and example
   - Characterization of SPD

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Positive Definite Matrices, defined

Definition

$A_{m\times m}$ is symmetric: $A = A'$. Then:

- A is positive definite if $(Av, v) \geq 0$, for every vector $v \in R^m$.
- A is strictly positive definite if $(Av, v) > 0$, for every vector $v \neq 0$.

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Exercise (not so simple!)

Let A be a symmetric positive definite matrix. Then TFCAE

- A is strictly positive definite
- A is invertible

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Example

Let A be any matrix (possibly not even square). Then $A'A$ is (symmetric) positive definite:

$$(A'Av, v) = ... \geq 0$$

2×2 positive definite matrices

Question: Given a symmetric matrix 2×2

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Note that, for an scalar t,

$$(A(tv), tv) = t^2(Av, v).$$

So, it is enough to check positive definiteness for

- $v = e_1$
- $v = [t, 1]'$, $t \in R$.

$$(Ae_1, e_1) = a.$$

For $v = [t\ 1]'$,

$$(Av, v) = at^2 + 2bt + c.$$

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... so $a > 0$, and $b^2 - ac < 0$, i.e., $\det(A) > 0$. So we get that A is SPD if and only if

- $a > 0$
- $\det(A) > 0$.

SPD, eigenvalues, and minors

A is $m \times m$, $A = A'$. Then TFCAE:

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2. $\sigma(A) > 0$.
3. All the main principal minors of A are positive.

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Proof: $(2) \implies (1)$. We assume $\sigma(A) > 0$. By Schur, $A = QDQ'$.
Since $\sigma(A)$ are the diagonal of D, D has positive diagonal entries.
Let v be an non-zero vector. $w := Q'v$ is also non-zero.

$$(Av, v) = (QDQ'v, v) = (DQ'v, Q'v) = (Dw, w) = \sum_{i=1}^{m} w(i)^2 D(i,i) > 0.$$

In the last step we used the fact that all the $D(i,i)$ are positive, and one of the $w(i)$'s is not zero.

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3. All the main principal minors of A are positive.

Proof: $(1) \implies (2)$. Let (λ, v) be an eigenpair of A. By (1),

$$(Av, v) > 0.$$

So

$$0 < (Av, v) = (\lambda v, v) = \lambda(v, v),$$

which implies that λ is positive, too.