CS513, Spring 21
Prof. Ron

HW #4, Factor = 1

Due Feb. 20, 2021

(Q1, Factor=.25) Let B be any invertible matrix.
(a) Prove that (λ, v) is an eigenpair of B if and only if $(1/\lambda, v)$ is an eigenpair of B^{-1} (check examples first if you feel confused).
(b) Now, let A be an invertible matrix. Use the claim in (a) in order to find a formula for $\|A^{-1}\|_2$ in terms of singular values of A. Explain. Hint: We assigned different notations to the matrix that appears in (a) and the matrix that appears in (b) for a reason!

(Q2, Factor=.75) Let A be a *symmetric* matrix with the following special property:

$$\lambda \in \sigma(A) \quad \Longrightarrow -\lambda \notin \sigma(A).$$

(Since $\sigma(A) \subset \mathbb{R}$, the above simply says that A does not have two different eigenvalues whose square is the same).
(a) For a 6×6 A of your choice, use `Matlab` in order to find the spectrum and eigenvectors of A and the spectrum and eigenvectors of $A'A = A^2$ (your matrix A must be non-singular and cannot have any zero entry, and make sure that it satisfies the special condition above; any generic symmetric matrix will satisfy that condition). Based on that example, conjecture a general connection between the spectrum and the eigenvectors of a symmetric A that satisfies the above special condition and the spectrum and eigenvectors of $A'A$. The conjecture should be of the form '(λ, v) is an eigenpair of $A'A$ if and only if .... is an eigenpair of A'). If you do not find any reasonable conjecture to make, run more examples. However, turn in the `Matlab` output of *one* of your tests only.)
(b) Prove your conjecture from (a). Note that there are two parts in the proof (the 'if' part and the 'only if' part).
(c) In view of the above, state a theorem that connects the eigenvalues of a symmetric A to its singular values. Check your theorem against the matrix

$$\begin{pmatrix} -4 & 72 \\ 72 & -46 \end{pmatrix}. \tag{אא}$$

(d) Provide an example that shows that your theorem in (a) does *not* hold, in general, for symmetric matrices that do not satisfy the special condition above (Hint: you may find such example even in 2D, and the matrix may be chosen to be sensationally simple!)