Lecture 8: QR-factorization, continued

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February 12, 2021

Outline

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1. QR factorization, computational details
   - The first step
   - The second step
   - The general step

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Creating the H_1 matrix

- Input: The original matrix $A_0 := A$.
- Output: The matrix H_1, the matrix $A_1 := H_1 A_0$.
- We need the first column of A_1 to be “good":

$$A_1 e_1 = a e_1, \quad \text{for some real number } a.$$

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With $x := A_0 e_1$ the first column of A_0, we need $y := A_1 e_1$ to be 'good': of the form $a e_1$.
Since $A_1 = H_1 A_0$, we have that $y = H_1 x$:

$$y = A_1 e_1 = (H_1 A_0) e_1 = H_1 (A_0 e_1) = H_1 :$$

That's just rudimentary linear algebra: if $C = AB$ then A maps to the columns of B to the colomns of C, and B maps the rows of A to the rows of C

Creating the H_1 matrix

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Since $A_1 = H_1 A_0$, we have that $y = H_1 x$:
We can find such a matrix H_1 provided

$$||x||_2 = ||y||_2 = |a|,$$

so we choose

$$a = \pm ||x||_2.$$

Creating the H_1 matrix

Summary of the first step

- $x := A_0 e_1$. Set $y := ||x||_2 e_1$.
- $w := \frac{x-y}{||x-y||_2}$.
- Define $H_1 := I - 2ww'$. Then $A_1 = H_1 A_0$.

Creating the matrix H_2

- Input: The matrix A_1, whose first column is “good".
- Output: The matrix H_2, the matrix $A_2 := H_2 A_1$.
- We need the first two columns of A_2 to be “good":

1. First, we will make sure that
$$A_1 e_1 = A_2 e_1.$$
2. Second, $x := A_1 e_2$, then $y := A_2 e_2$ is $H_2 x$. So, we need first to choose y, and then to construct H_2 so it maps x to y.
3. We choose $y(1) = x(1)$ - this ensures that $A_1 e_1 = A_2 e_1$.
4. We choose $y(2) := ||x(2:m)||_2$. This ensure that $||x||_2 = ||y||_2$.
5. $y(i) = 0$ for $i \geq 0$, since y should be ‘good’: A suitable second column in an upper triangular matrix.

Creating the matrix H_2

Summary of the second step

- $x := A_1 e_2$. Choose
$$y = x(1)e_1 + ||x(2:m)||_2 e_2.$$
- $w := \frac{x-y}{||x-y||_2}$.
- $H_2 := I - 2ww'$, $A_2 = H_2 A_1 = H_2 H_1 A_0$.

Creating the matrix H_j

- Input: The matrix A_{j-1}, whose first $j-1$ columns are "good".
- Output: The matrix H_j, the matrix $A_j := H_j A_{j-1}$.

1. $x := A_{j-1} e_j$.
2. $$y = \sum_{i=1}^{j-1} x(i) e_i + ||x(j:m)||_2 e_j.$$
3. $w := \frac{x-y}{||x-y||_2}$.
4. Define $H_j := I - 2ww'$. Then $A_j = H_j A_{j-1}$.

Justifying the process

Question: How do we know that

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This is straight linear algebra argument, based on the following:

- For $i < j$, $w(i) = 0$, since $x(i) = y(i)$.
- Therefore, the first $j-1$ columns of $2ww'$ are zero. So, the first $j-1$ columns of H_j are identity.
- If v is one of the first $j-1$ columns of A_{j-1}, then $v(k) = 0$, $k \geq j$.
- Conclusion: $H_j v = v$, for such v.

Mini demo!!