Lecture 2: Introduction

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Outline

1. e-vectors, e-values
   - Definition, example
   - Diagonalizability

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Eigenpairs

Definition

A $m \times m$. (λ, v), $\lambda \in C$, $v \in C^m \backslash 0$, is eigenpair of A, if

$$Av = \lambda v.$$

The set of all eigenvalues is the spectrum

$$\sigma(A)$$

of A.

Note: A real valued matrix might have complex eigenvalues!
Reminder: The characteristic polynomial of A is

$$p_A(t) := \det(A - tI).$$

$$\lambda \in \sigma(A) \quad \iff \quad p_A(\lambda) = 0.$$

Eigenpairs

The eigenvectors of the matrix A is any linearly independent maximal set of eigenvectors. The cardinality of such set depends only on A.

Eigenpairs

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}.$$

Then:

- $p_A(t) = (t-2)(t-3) - 2 = t^2 - 5t + 4$
- This implies that $\sigma(A) = \{1, 4\}$.

Eigenpairs

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}.$$

Then:

- The matrix $A - 4I$ must be singular. Indeed,

$$A - 4I = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}$$

- Every non-zero vector in $\ker(A - 4I)$ is an eigenvector. Since $\dim\ker(A - 4I) = 1$, we select only one eigenvector from this null space. For example, we can choose $v_1 =$ .... Then, $(4, v_1)$ is an eigenpair.

Eigenpairs

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}.$$

Then:

- The matrix $A - I$ must be singular. Indeed,

$$A - I = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

- Every non-zero vector in $\ker(A - I)$ is an eigenctor. Since $\dim \ker(A - I) = 1$, we select only one eigenvector from this null space. For example, $v_2 =$ .... Then, $(1, v_2)$ is an eigenpair.

Eigenpairs

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}.$$

Then:

- Since eigenvectors associated with different e-values are always linearly independent, (v_1, v_2) are independent, hence form a basis for R^2.

Diagonalizability

Definition: Diagonalizability

A $m \times m$ is diagonalizable is there exists a basis for C^m made of e-vectors of A

Diagonalizability

Theorem:

A is square. TFCAE:

1. A is diagonalizable
2. There exist a matrix P and a diagonal matrix D such that
$$A = PDP^{-1}.$$
3. Another equivalent condition deferred.

Proof of Theorem

$(2) \implies (1)$: We have $AP = PD$. We prove that each column of P is an eigenvector of A. This proves (1), since the columns of any $m \times m$ invertible matrix form a basis for C^m.
The jth column of P is Pe_j. Now:

$$A(Pe_j) = (AP)e_j = (PD)(e_j) = P(De_j) = P(D(j,j)e_j) = D(j,j)(Pe_j).$$

So, $(D(j,j), Pe_j)$ is an eigenpair of A.

Proof of Theorem

$(1) \implies (2)$ We are given m eigenpairs (λ_j, v_j), with $(v_1, \ldots, v_m)$ a basis for C^m. Let P be the matrix whose columns are $v_1, \ldots, v_j$, and let D be the diagonal matrix whose diagonal is $\lambda_1, \ldots, \lambda_m$. We show that $A = PDP^{-1}$ by showing that $AP = PD$, i.e., by showing that, for every j,

$$(AP)e_j = (PD)e_j.$$

Now,

$$(AP)e_j = A(Pe_j) = Av_j = \lambda_j v_j = P(\lambda_j e_j) = P(De_j) = (PD)e_j.$$