Lecture 3: Introduction

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Outline

1. Diagonalizability
   - General square case
   - The Symmetric case, the Schur decomposition

2. Norms
   - Vector Norms
   - Matrix Norms

Outline

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Diagonalizability

Definition: Diagonalizability

A $m \times m$ is diagonalizable is there exists a basis for C^m made of e-vectors of A

Diagonalizability

Theorem:

A is square. TFCAE:

1. A is diagonalizable
2. There exist a matrix P and a diagonal matrix D such that

$$A = PDP^{-1}.$$

3. Another equivalent condition deferred.

Proof of Theorem

$(2) \implies (1)$: We have $AP = PD$. We prove that each column of P is an eigenvector of A. This proves (1), since the columns of any $m \times m$ invertible matrix form a basis for C^m.
The jth column of P is Pe_j. Now:

$$A(Pe_j) = (AP)e_j = (PD)(e_j) = P(De_j) = P(D(j,j)e_j) = D(j,j)(Pe_j).$$

So, $(D(j,j), Pe_j)$ is an eigenpair of A.

Proof of Theorem

$(1) \implies (2)$ We are given m eigenpairs (λ_j, v_j), with $(v_1, \ldots, v_m)$ a basis for C^m. Let P be the matrix whose columns are $v_1, \ldots, v_j$, and let D be the diagonal matrix whose diagonal is $\lambda_1, \ldots, \lambda_m$. We show that $A = PDP^{-1}$ by showing that $AP = PD$, i.e., by showing that, for every j,

$$(AP)e_j = (PD)e_j.$$

Now,

$$(AP)e_j = A(Pe_j) = Av_j = \lambda_j v_j = P(\lambda_j e_j) = P(De_j) = (PD)e_j.$$

The symmetric case

Reminder: A is symmetric whenever $A = A'$.

Theorem: Spectral rudiments of a symmetric matrix

Assume $A = A'$. Then:

- $\sigma(A) \subset R$.
- A is diagonalizable.
- There is an A-eigenbasis which is also an orthonormal basis.
- The Schur Decomposition: A is orthogonally diagonalizable:

$$A = QDQ' = QDQ^{-1},$$

with Q orthogonal and D diagonal.

Demo #1

Outline

Definition of Norm

Definition: Norm

Let

$$||\cdot||$$

be an assignment from R^m to

$$R_+ := \{c \in R \mid c \geq 0\} :$$

$$R^m \ni v \mapsto ||v|| \in R_+.$$

This assignment is a norm if the following conditions are valid:

- $||v|| = 0$ if and only if $v = 0$.
- For $c \in R$, $v \in R^m$, we have $||cv|| = |c|||v||$.
- For $v, w \in R^m$, $||v + w|| \leq ||v|| + ||w||$.

Definition of Norm

Example: The 1-norm, mean-norm, ℓ_1-norm...

$$||v||_1 := \sum_{i=1}^{m} |v(i)|.$$

Definition of Norm

Example: The 2-norm, Euclidean-norm, ℓ_2-norm, the least square norm...

$$||v||_2 := \sqrt{\sum_{i=1}^{m} |v(i)|^2}.$$

Definition of Norm

Example: The ∞-norm, max-norm, ℓ_∞-norm, uniform norm...

$$||v||_\infty := \max_{1\le i\le m} |v(i)|.$$

Definition of Norm

Example: The p-norm, ℓ_p-norm, $1 \leq p < \infty$

$$||v||_p := \left(\sum_{i=1}^{m} |v(i)|^p \right)^{1/p}.$$

Definition of matrix norms

A is $m \times n$, maps thus R^n to R^m.
We choose a norm, $||\cdot||$, for the domain,
and a norm $||\cdot||'$ for the range.

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Definition: Matrix norm

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If the norms $||\cdot||$ and $||\cdot||'$ are both p-norms for the same p, we denote the matrix norm as $||A||_p$.

The 1-norm of a matrix

Theorem: computing $||A||_1$

Let $A_{m\times n}$ with columns $a_1, \ldots, a_n$. Then

$$||A||_1 = \max_{1\le i\le n} ||a_i||_1$$

The 1-norm of a matrix

Theorem: computing $||A||_1$

Let $A_{m\times n}$ with columns $a_1,\ldots,a_n$. Then

$$||A||_1 = \max_{1\le i\le n} ||a_i||_1 =: X$$

Proof: We need to show that $||A||_1 \le X$, and $||A||_1 \ge X$.
First, for any $1 \le j \le m$, $||e_j||_1 = 1$, therefore

$$||a_j||_1 = ||Ae_j||_1 \le ||A||_1.$$

Therefore,

$$X \le ||A||_1.$$

The 1-norm of a matrix

Theorem: computing $||A||_1$

Let $A_{m\times n}$ with columns $a_1,\ldots,a_n$. Then

$$||A||_1 = \max_{1\le i\le n} ||a_i||_1 =: X$$

Now, let $v \in R^n$, $||v||_1$. Then

$$||Av||_1 = ||\sum_{i=1}^{n} v(i)a_i||_1 \le \sum_{i=1}^{n} ||v(i)a_i||_1 = \sum_{i=1}^{n} |v(i)|||a_i||_1$$

The 1-norm of a matrix

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$$\le \sum_{i=1}^{n} |v(i)|X$$

The 1-norm of a matrix

Now, let $v \in R^n$, $||v||_1$. Then

$$||Av||_1 = ||\sum_{i=1}^n v(i)a_i||_1 \le \sum_{i=1}^n ||v(i)a_i||_1 = \sum_{i=1}^n |v(i)|||a_i||_1$$

$$\le \sum_{i=1}^n |v(i)|X$$

$$= X\sum_{i=1}^n |v(i)| = X||v||_1 = X.$$

Therefore, $||A||_1 \le X$.