ISyE/Math/CS/Stat 525 – Linear Optimization
Spring 2021

Assignment 0

This assignment will not be graded and there is no due date. Do not hand in the solutions.

This assignment is designed so that you can self-evaluate your Math skills and your knowledge of Linear Algebra. You can refresh your Math skills by studying the document "Introduction to mathematical arguments" by Michael Hutchings, which is available in Canvas. Regarding Linear Algebra, in particular students must have working knowledge of set theory, vectors and matrices, matrix inversion, subspaces and bases, and affine subspaces. You can strengthen your knowledge of Linear Algebra by revising any textbook or lecture notes on the topic. A nice overview is presented in Section 1.5 "Linear algebra background and notation" of the recommended textbook for the course, which is "Introduction to Linear Optimization" by D. Bertsimas and J.N. Tsitsiklis. After completing the assignment, check your answers using the solutions available in Canvas. Then revise the concepts that were not clear to you. Note that understanding basic proof techniques and having a working knowledge of linear algebra are essential in this course!

Test your Math skills

Exercise 1 . . . . *0 points*

Rewrite each of the following sentences by using mathematical notation:

(a) For each integer x, the double of x is even.

(b) An integer x is odd if there exists an integer y such that $x = 2y + 1$.

(c) For each subset of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that has cardinality at least 5, there exists one element of the subset that is even.

Exercise 2 . . . . *0 points*

Negate the following statements:

(a) Each student of 525 is at least 22 years old.

(b) In all math classes there exists a student that is a genius.

(c) In each lecture, the teacher arrives late or leaves early.

Exercise 3 . . . . *0 points*

Disprove the following conjectures by hexibiting a counterexample:

(a) Conjecture 1: Every positive integer is equal to the sum of two integer squares.

(b) Conjecture 2: Each subset of $\{1, 2, \ldots, 8\}$ that has cardinality at least 3 contains a number that is a multiple of 2 or of 3.

(c) Conjecture 3: At any party with at least 5 people, there are always three people that mutually know each other, or three people that do not know each other. (Note: A, B and C mutually know each other if any two persons among A, B and C know each other. Similarly, A, B and C do not know each other if any two persons among A, B and C are strangers.)

Exercise 4 . . . . *0 points*

Consider the following true statement:

If an integer is a multiple of 4, then it is a multiple of 2.

(a) Being a multiple of 4 is a necessary or sufficient condition for being a multiple of 2?

(b) Being a multiple of 2 is a necessary or sufficient condition for being a multiple of 4?

Exercise 5 .... ***0 points***

(a) A $\Rightarrow$ B means that A is necessary or sufficient for B?

(b) A $\Leftarrow$ B means that A is necessary or sufficient for B?

Exercise 6 .... ***0 points***

(a) Prove the following: $\forall x, y \in \mathbb{Z}$, if one among x and y is even, then xy is even.

(b) As a consequence, prove that $\forall x \in \mathbb{Z}$, $x(x+1)$ is even.

Exercise 7 .... ***0 points***

Prove the following: For every integer x, if x is odd, then there exists an integer y such that $x^2 = 8y+1$.

Exercise 8 .... ***0 points***

Prove by contradiction that $\sqrt{2} + \sqrt{6} < \sqrt{15}$

Exercise 9 .... ***0 points***

Imagine four cards placed on the table, each with a letter on one face and a number on the other one:

A D 9 5

You are given the rule which states: "if A is on a card, then 5 is on its other side". Which card(s) need to be turned over to check whether the rule holds?

Exercise 10 .... ***0 points***

Prove the following: Let $x \in \mathbb{Z}$. If x^2 is odd, then x is odd.
Hint: use the contrapositive.

Exercise 11 .... ***0 points***

Prove by induction the following statement: For each positive integer x, $2^{3x+1} + 5$ is a multiple of 7.

Exercise 12 .... ***0 points***

Prove by induction that $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$.

Test your Linear Algebra

Exercise 1 .... ***0 points***

Compute, if it exists, the matrix product AB for the following matrices:

(a) $A = \begin{bmatrix} 3 & 4 & 0 \\ 2 & 7 & 1 \\ 6 & 5 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 & 8 \\ 5 & 7 & 0 \\ 6 & 4 & 3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 3 & 5 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

Exercise 2 .... ***0 points***

Let $x = (3, 4, 5)$, $y = (1, 3, -3)$, and $z = (2, 1, -1)$.

(a) Compute the inner products $x'y$ and $x'z$.

(b) Are x and y orthogonal? What about x and z?

(c) Compute the norm of x, y and z.

Exercise 3 .... ***0 points***

True or false (justify your answers).

(a) $\|x\| \cdot \|y\| \geq |x'y|$ for any two vectors x and y in $\mathbb{R}^n$.

(b) $x'x \geq 0$ for each $x \in \mathbb{R}^n$.

(c) For any two matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$, $AB = BA$.

(d) For any two matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times k}$, $(AB)' = A'B'$.

(e) For any two matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times k}$, $(AB)' = B'A'$.

(f) For any two symmetric matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$, $(AB)' = A'B'$.

(g) Let $A_1, \ldots, A_n$ be the columns of a matrix A, i.e.: $A = \begin{bmatrix} | & | & & | \\ A_1 & A_2 & \ldots & A_n \\ | & | & & | \end{bmatrix}$, and let $x = (x_1, x_2, \ldots, x_n)$. We have $Ax = A_1x_1 + A_2x_2 + \cdots + A_nx_n$.

(h) A square matrix A is invertible if and only if its determinant is non-zero.

(i) A square matrix A has zero determinant if and only if its rows are linearly dependent.

Exercise 4 .... ***0 points***

Is the following matrix invertible?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ -1 & 0 & 1 \end{bmatrix}$$

If A^{-1} exists, compute it. Find all solutions of the system $Ax = b$ for $b = (2, 1, 1)$.

Exercise 5 .... ***0 points***

Recall that vectors $x^1, \ldots, x^k$ are said to be linearly independent if $\sum_{j=0}^{k} a_j x^j = 0$ implies $a_j = 0$ for every $j = 1, \ldots, k$. Show that the vectors in a given finite collection are linearly independent if and only if none of the vectors can be expressed as a linear combination of the others.

Exercise 6 .... ***0 points***

Let $x = (-2, -1, 3, 4)$ and $y = (-8, 2, -2, 1)$.

(a) Are x and y linearly independent?

(b) Define a linear combination z of x and y.

(c) Are x, y and z linearly independent?

Exercise 7 .... ***0 points***

Prove that the following two definitions of *linear function* are equivalent:

Definition 1: A function $f : \mathbb{R}^n \to R$ is *linear* if:

1. $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^n$
2. $f(\lambda x) = \lambda f(x)$ for all $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$

Definition 2: A function $f : \mathbb{R}^n \to R$ is *linear* if it can be written as:

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n,$$

where $c_1, c_2, \ldots, c_n$ are real numbers.