ISyE/Math/CS/Stat 525 – Linear Optimization
Spring 2021

Assignment 3 – Chapter 3 part 1

Due date: March 20 at 11:59pm.

Instructions and policy: **Undergraduate students** should handle in the five exercises that are marked with [U]. **Graduate students** should handle in the five exercises that are marked with [G]. All other exercises are optional for keen students and should not be handled in. The assignment should be submitted electronically in Canvas. Late submission policy: 20% of total points will be deducted per hour. Each student is encouraged to solve all the exercises in the assignment to practice for the exams.

Students are strongly encouraged to work in groups of two on homework assignments. To find a partner you can post on the "Discussions" section in Canvas. Only one file should be submitted for both group members. In order to submit the assignment for your group please follow these steps in Canvas: Step 1. Click on the "People" tab, then on "Groups", and join one of the available groups named "Assignments Group 1", "Assignments Group 2", ...; Step 2. When also your partner has joined the same group, one of the two can submit the assignment by clicking on the "Assignments" tab, then on the assignment to be submitted, and finally on "Submit assignment". The submission will count for everyone in your group.

Groups must work independently of each other, may not share answers with each other, and solutions must not be copied from the internet or other sources. If improper collaboration is detected, *all groups* involved will automatically receive a 0. Students must properly give credit to any outside resources they use (such as books, papers, etc.). In doing these exercises, you must justify all of your answers and cite every result that you use. You are not allowed to share any content of this assignment.

Exercise 1 .... ***0 points***

Recall that a set $S \subset \mathbb{R}^n$ is said to be *convex* if for any $x, y \in S$, and any $\lambda \in [0,1]$, we have $\lambda x + (1-\lambda)y \in S$.

Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a convex function and let $S \subset \mathbb{R}^n$ be a convex set. Let x^* be an element of S. Suppose that x^* is a local optimum for the problem of minimizing $f(x)$ over S; that is, there exists some $\varepsilon > 0$ such that $f(x^*) \le f(x)$ for all $x \in S$ for which $\|x - x^*\| \le \varepsilon$. Prove that x^* is a global minimum; that is, $f(x^*) \le f(x)$ for all $x \in S$.

Exercise 2 [U][G] .... ***10 points***

Let $P = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1,\ x \ge 0\}$ and consider the vector $x = (0,0,1)$. Find the set of feasible directions at x.

Exercise 3 [U][G] .... ***10 points***

Consider the problem of minimizing $c'x$ over a polyhedron P. Prove the following:

(a) (5 points) A feasible solution x is optimal if and only if $c'd \ge 0$ for every feasible direction d at x.

(b) (5 points) A feasible solution x is the unique optimal solution if and only if $c'd > 0$ for every nonzero feasible direction d at x.

Exercise 4 [U][G] .... ***10 points***

Let x be an element of the standard form polyhedron $P = \{x \in \mathbb{R}^n | Ax = b, x \ge 0\}$. Prove that a vector $d \in \mathbb{R}^n$ is a feasible direction at x if and only if $Ad = 0$ and $d_i \ge 0$ for every i such that $x_i = 0$.

Exercise 5 .... ***0 points***

Consider the problem of minimizing $c'x$ over the set $P = \{x \in \mathbb{R}^n : Ax = b,\ Dx \le f,\ Ex \le g\}$. Let $x^* \in P$ be such that $Dx^* = f$, $(Ex^*)_i < g_i$ for all i. Show that the set of feasible directions at x^* is $\{d \in \mathbb{R}^n : Ad = 0,\ Dd \le 0\}$.

Exercise 6 [G] .... ***10 points***

Let x be a basic feasible solution associated with some basis matrix B. Prove the following:

(a) If the reduced cost of every nonbasic variable is positive, then x is the unique optimal solution.

(b) If x is the unique optimal solution and is nondegenerate, then the reduced cost of every nonbasic variable is positive.

Exercise 7 [U] .... ***10 points***

Consider a feasible solution x to the standard form problem

$$\begin{aligned} \text{minimize} \quad & c'x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0, \end{aligned}$$

and let $Z = \{i : x_i = 0\}$. Show that x is an optimal solution if and only if the linear programming problem

$$\begin{aligned} \text{minimize} \quad & c'd \\ \text{subject to} \quad & Ad = 0 \\ & d_i \geq 0, \qquad i \in Z, \end{aligned} \tag{1}$$

has an optimal cost of zero.

Exercise 8 .... ***0 points***

Consider a linear programming problem in standard form and recall that for a basic feasible solution x and a feasible direction d at x we have

$$c'd = \sum_{i \in N} \bar{c}_i d_i, \tag{2}$$

where N is set of nonbasic indices at x and $\bar{c}$ is the vector of reduced costs at x.

Suppose that x^* is an optimal basic feasible solution, and consider an optimal basis associated with x^*. Let B and N be the set of basic and nonbasic indices, respectively. Let I be the set of nonbasic indices i for which the corresponding reduced costs are zero.

(a) Show that if I is empty, then x^* is the only optimal solution.

(b) Show that x^* is the unique optimal solution if and only if the following linear programming problem has an optimal value of zero:

$$\begin{aligned} \text{maximize} \quad & \sum_{i \in I} x_i \\ \text{subject to} \quad & Ax = b \\ & x_i = 0, \qquad i \in N \setminus I, \\ & x_i \geq 0, \qquad i \in B \cup I. \end{aligned}$$

Exercise 9 [U][G] .... ***10 points***

Consider the problem

$$\begin{aligned} \text{minimize} \quad & -2x_1 - x_2 \\ \text{subject to} \quad & x_1 - x_2 \leq 2 \\ & x_1 + x_2 \leq 6 \\ & x_1, x_2 \geq 0. \end{aligned}$$

(a) (2 points) Convert the problem into standard form and construct a basic feasible solution at which $(x_1, x_2) = (0, 0)$.

(b) (5 points) Carry out the full tableau implementation of the simplex method, starting with the basic feasible solution of part (a).

(c) (3 points) Draw a graphical representation of the problem in terms of the original variables x_1, x_2, and indicate the path taken by the simplex algorithm.