Lemma Π_1 and Π_2 are equivalent if and only if:

(i) For every feasible solution to Π_1, there exists a feasible solution to Π_2, with cost equal or lower, and

(ii) For every feasible solution to Π_2, there exists a feasible solution to Π_1, with cost equal or lower.

Proof.

"$\Rightarrow$" If Π_1 and Π_2 are both infeasible there is nothing to prove. Thus we may assume that Π_1 and Π_2 have the same optimal cost.

(i) holds because the optimal solution to Π_2 has cost equal or lower than every feasible solution to Π_1.

(ii) holds symmetrically

"$\Leftarrow$" If Π_1 and Π_2 are both infeasible we are done, so assume that one is feasible, wlog Π_1.

(i) implies that also Π_2 is feasible.

Assume by contradiction that one of the two problems has strictly lower optimal cost, wlog Π_1. Then, $\exists\, x^*$ feasible to Π_1 with cost strictly lower than any feasible solution to Π_2.

This contradicts (i).

Hence the optimal cost of the two problems is the same. $\square$

Observation The two problems in Example 1.4 are equivalent.

$$\Pi_1: \quad \begin{array}{ll} \text{minimize} & 2x_1 + 4x_2 \\ \text{subject to} & x_1 + x_2 \geq 3 \\ & 3x_1 + 2x_2 = 14 \\ & x_1 \geq 0, \end{array} \qquad \Pi_2: \quad \begin{array}{ll} \text{minimize} & 2x_1 + 4x_2^+ - 4x_2^- \\ \text{subject to} & x_1 + x_2^+ - x_2^- - x_3 = 3 \\ & 3x_1 + 2x_2^+ - 2x_2^- = 14 \\ & x_1, x_2^+, x_2^-, x_3 \geq 0. \end{array}$$

Proof.

We use the Lemma.

(i) Let (x_1, x_2) be any feasible solution to Π_1.

Define

$$x_2^+ := \begin{cases} x_2 & \text{if } x_2 \geq 0 \\ 0 & \text{if } x_2 < 0 \end{cases} \qquad x_2^- := \begin{cases} 0 & \text{if } x_2 \geq 0 \\ -x_2 & \text{if } x_2 < 0. \end{cases}$$

and $x_3 := x_1 + x_2 - 3$.

Note that $x_2^+ - x_2^- = x_2$.

the vector (x_1, x_2^+, x_2^-, x_3) is a feasible solution to Π_2 because

$$x_1 + (x_2^+ - x_2^-) - x_3 = \cancel{x_1} + \cancel{x_2} - (\cancel{x_1} + \cancel{x_2} - 3) = 3$$

$$3x_1 + 2x_2^+ - 2x_2^- = 3x_1 + 2x_2 = 14$$

$$x_1, x_2^+, x_2^-, x_3 \geq 0.$$

the cost of (x_1, x_2^+, x_2^-, x_3) is

$$2x_1 + 4x_2^+ - 4x_2^- = 2x_1 + 4x_2$$

hence equal to the cost of (x_1, x_2).

(ii) Let (x_1, x_2^+, x_2^-, x_3) be any feasible solution to Π_2.

Define $x_2 := x_2^+ - x_2^-$.

the vector (x_1, x_2) is a feasible solution to Π_1

because

$$x_1 + x_2 = x_1 + x_2^+ - x_2^- \geq 3$$

$$3x_1 + 2x_2 = 3x_1 + 2x_2^+ - 2x_2^- = 14$$

$$x_1 \geq 0.$$

the cost of (x_1, x_2) is

$$2x_1 + 4x_2 = 2x_1 + 4x_2^+ - 4x_2^-$$

hence equal to the cost of (x_1, x_2^+, x_2^-, x_3) $\square$