ISyE/Math/CS/Stat 525
Linear Optimization

1. Introduction

Prof. Alberto Del Pia
University of Wisconsin-Madison

Based on the book *Introduction to Linear Optimization* by D. Bertsimas and J.N. Tsitsiklis

Outline

Sec. 1.1 We introduce linear progrmming (LP) problems.

Sec. 1.2 We present some examples.

Sec. 1.3 We consider some classes of optimization problems involving nonlinear functions that can be reduced to LP problems.

Sec. 1.4 We solve a few simple examples of LP problems and obtain some basic geometric intuition on the nature of the problem.

Notation

- A $m \times n$ matrix A is an array of real numbers a_{ij}:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

- The transpose of A is the $n \times m$ matrix

$$A' = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}.$$

Notation

- A n-dimensional (column) vector x is a $n \times 1$ matrix

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = (x_1, x_2, \ldots, x_n).$$

- A n-dimensional row vector x is a $1 \times n$ matrix.
- The inner product of two n-dimensional vectors x and y is

$$x' y = \sum_{i=1}^{n} x_i y_i.$$

1.1 Variants of the linear programming problem

A linear programming problem (Example 1.1)

$$\begin{array}{rll} \text{minimize} & 2x_1 - x_2 + 4x_3 & \text{objective function} \\ \text{subject to} & \left.\begin{array}{l} x_1 + x_2 + x_4 \le 2 \\ 3x_2 - x_3 = 5 \\ x_3 + x_4 \ge 3 \\ x_1 \ge 0 \\ x_3 \le 0 \end{array}\right\} & \text{constraints} \end{array}$$

- x_1, x_2, x_3, x_4 are the decision variables.
- The objective function is linear and it can be written as $c'x$, where $c = (2, -1, 4, 0)$.
- The constraints are linear equalities and inequalities, and can be written in the form $a'x = b$, $a'x \le b$, or $a'x \ge b$.
- Example: The first constraint is of the form $a'x \le b$, with $a = (1, 1, 0, 1)$, and $b = 2$.

General linear programming (LP) problem

$$\begin{array}{rll} \text{minimize} & c'x & \text{objective function} \\ \text{subject to} & \left.\begin{array}{ll} a_i'x \geq b_i & i \in M_1 \\ a_i'x \leq b_i & i \in M_2 \\ a_i'x = b_i & i \in M_3 \\ x_j \geq 0 & j \in N_1 \\ x_j \leq 0 & j \in N_2. \end{array}\right\} & \text{linear constraints} \end{array}$$

- $x_1, x_2, \ldots, x_n$ are the decision variables.
- $c'x$: objective function, where $c \in \mathbb{R}^n$ is a given cost vector.
- M_1, M_2, M_3: given finite index sets.
  $\forall i \in M_1 \cup M_2 \cup M_3$, we are given a vector $a_i \in \mathbb{R}^n$ and a scalar $b_i \in \mathbb{R}$, that define a linear constraint.
- N_1, N_2: given subsets of $\{1, \ldots, n\}$.
  If $j \notin N_1 \cup N_2$, we say that x_j is a free variable.

General linear programming (LP) problem

$$\begin{array}{rll}
\text{minimize} & c'x & \text{objective function} \\
\text{subject to} & \left.\begin{array}{ll}
a_i'x \geq b_i & i \in M_1 \\
a_i'x \leq b_i & i \in M_2 \\
a_i'x = b_i & i \in M_3 \\
x_j \geq 0 & j \in N_1 \\
x_j \leq 0 & j \in N_2.
\end{array}\right\} & \text{linear constraints}
\end{array}$$

- A feasible solution is a vector x satisfying all of the constraints.
- The feasible set is the set of all feasible solutions.
- The problem is infeasible if the feasible set is empty.
- The cost of a feasible solution x is $c'x$.
- An optimal solution is a feasible solution x^* that minimizes the objective function. Formally, $c'x^* \leq c'x$, $\forall x$ feasible.
- The optimal cost is the value $c'x^*$.

General linear programming (LP) problem

$$\left.\begin{array}{lll} \text{minimize} & c'x & \\ \text{subject to} & a_i'x \geq b_i & i \in M_1 \\ & a_i'x \leq b_i & i \in M_2 \\ & a_i'x = b_i & i \in M_3 \\ & x_j \geq 0 & j \in N_1 \\ & x_j \leq 0 & j \in N_2. \end{array}\right\}$$

objective function

linear constraints

- If $\forall K \in \mathbb{R}$, $\exists x$ feasible with $c'x \leq K$, we say that the optimal cost is $-\infty$, and that the problem is unbounded.
- Note that there is no need to study maximization problems separately:

$$\begin{array}{lll} \max & c'x & \\ \text{s.t.} & x \in S \end{array} \quad = \quad - \begin{array}{ll} \min & (-c)'x \\ \text{s.t.} & x \in S. \end{array}$$

A simpler form

The feasible set in a general LP problem can be expressed exclusively in terms of inequality constraints of the form

$$a_i' x \geq b_i.$$

In fact:

- $a_i' x = b_i \quad \Leftrightarrow \quad a_i' x \leq b_i,\ a_i' x \geq b_i.$
- $a_i' x \leq b_i \quad \Leftrightarrow \quad -a_i' x \geq -b_i.$
- $x_j \geq 0$ is a special case of $a_i' x \geq b_i$.
- $x_j \leq 0$ is a special case of $a_i' x \leq b_i$.

A simpler form

- Suppose that there is a total of m constraints of the form

$$a_i' x \geq b_i, \qquad i = 1, \ldots, m.$$

- Let $b = (b_1, \ldots, b_m)$, and let A be the $m \times n$ matrix

$$A = \begin{bmatrix} — & a_1' & — \\ & \vdots & \\ — & a_m' & — \end{bmatrix}.$$

- Then, the m constraints can be expressed compactly in the form

$$Ax \geq b.$$

- The LP problem can then be written

$$\begin{aligned} \text{minimize} \quad & c' x \\ \text{subject to} \quad & Ax \geq b. \end{aligned}$$

Example 1.2

Let's write the LP problem in Example 1.1 in this simpler form.

$$\begin{aligned} \text{minimize} \quad & 2x_1 - x_2 + 4x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_4 \leq 2 \\ & 3x_2 - x_3 = 5 \\ & x_3 + x_4 \geq 3 \\ & x_1 \geq 0 \\ & x_3 \leq 0 \end{aligned}$$

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\begin{array}{rl}
\text{minimize} & 2x_1 - x_2 + 4x_3 \\
\text{subject to} & x_1 + x_2 + x_4 \leq 2 \\
& 3x_2 - x_3 = 5 \\
& x_3 + x_4 \geq 3 \\
& x_1 \geq 0 \\
& x_3 \leq 0
\end{array}
\qquad
\begin{array}{rl}
\text{minimize} & 2x_1 - x_2 + 4x_3 \\
\text{subject to} & -x_1 - x_2 - x_4 \geq -2 \\
& 3x_2 - x_3 \geq 5 \\
& -3x_2 + x_3 \geq -5 \\
& x_3 + x_4 \geq 3 \\
& x_1 \geq 0 \\
& -x_3 \geq 0
\end{array}
$$

Example 1.2

Let's write the LP problem in Example 1.1 in this simpler form.

$$\begin{aligned} \text{minimize} \quad & c'x \\ \text{subject to} \quad & Ax \geq b, \end{aligned}$$

with $c = (2, -1, 4, 0)$,

$$A = \begin{bmatrix} -1 & -1 & 0 & -1 \\ 0 & 3 & -1 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

and $b = (-2, 5, -5, 3, 0, 0)$.

$$\begin{aligned} \text{minimize} \quad & 2x_1 - x_2 + 4x_3 \\ \text{subject to} \quad & -x_1 - x_2 - x_4 \geq -2 \\ & 3x_2 - x_3 \geq 5 \\ & -3x_2 + x_3 \geq -5 \\ & x_3 + x_4 \geq 3 \\ & x_1 \geq 0 \\ & -x_3 \geq 0 \end{aligned}$$

Standard form problems

Standard form problems

A LP problem of the form

$$\begin{aligned} \text{minimize} \quad & c'x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

is said to be in standard form.

Interpretation of a standard form problem

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- Let $x \in \mathbb{R}^n$ and let $A = \begin{bmatrix} | & | & & | \\ A_1 & A_2 & \dots & A_n \\ | & | & & | \end{bmatrix}$.
- The vector Ax can be written as

$$Ax = A_1x_1 + A_2x_2 + \cdots + A_nx_n = \sum_{j=1}^{n} A_jx_j.$$

- Thus the constraints $Ax = b$ can be written as

$$\sum_{j=1}^{n} A_jx_j = b.$$

Interpretation of a standard form problem

$$\begin{aligned} \text{minimize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} A_j x_j = b \\ & x_j \geq 0 \qquad\qquad j \in \{1, \ldots, n\} \end{aligned}$$

- $A_1, \ldots, A_n$ can be interpreted as resource vectors.
- b is a target vector to “synthesize”.
- To do so, we mix a non-negative amount x_j of each resource A_j.
- c_j is the unit cost of the jth resource.
- The goal is to synthesize b minimizing the cost $\sum_{j=1}^{n} c_j x_j$.

Example 1.3 (The diet problem)

- There are n different ingredients (our resources) and a target ideal food that we want to synthesize.
- There are m different nutrients.
- We are given the following table with the nutritional content of a unit of each ingredient.

	ingr 1	$\cdots$	ingr n
nutr 1	a_{11}	$\cdots$	a_{1n}
$\vdots$	$\vdots$		$\vdots$
nutr m	a_{m1}	$\cdots$	a_{mn}

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- There are n different ingredients (our resources) and a target ideal food that we want to synthesize.
- There are m different nutrients.
- We are given the following table with the nutritional content of a unit of each ingredient.
- We are given the nutritional contents of the ideal food.

	ingr 1	$\cdots$	ingr n	ideal food
nutr 1	a_{11}	$\cdots$	a_{1n}	b_1
$\vdots$	$\vdots$		$\vdots$	$\vdots$
nutr m	a_{m1}	$\cdots$	a_{mn}	b_m

Example 1.3 (The diet problem)

- There are n different ingredients (our resources) and a target ideal food that we want to synthesize.
- There are m different nutrients.
- We are given the following table with the nutritional content of a unit of each ingredient.
- We are given the nutritional contents of the ideal food.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

- Let A be the $m \times n$ matrix with entries a_{ij}.
- Let b be the m-dimensional vector with entries b_i.
- Note that the jth column A_j of this matrix represents the nutritional content of the jth ingredient.

Example 1.3 (The diet problem)

$$\begin{aligned} \text{minimize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} A_j x_j = b \\ & x_j \geq 0 \qquad\qquad j \in \{1, \ldots, n\} \end{aligned}$$

- We then interpret the standard form problem as the problem of mixing nonnegative quantities x_j of the available ingredients, to synthesize the ideal food at minimal cost.

Example 1.3 (The diet problem)

- In a variant of this problem, the vector b specifies the minimal requirements of an adequate diet.
- In this case, the constraints $Ax = b$ are replaced by $Ax \geq b$, and the problem is not in standard form.

$$\begin{aligned} \text{minimize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} A_j x_j \geq b \\ & x_j \geq 0 \qquad\qquad j \in \{1, \dots, n\} \end{aligned}$$

Reduction to standard form

Reduction to standard form

The standard form problem

$$\begin{aligned} \text{minimize} \quad & c'x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0, \end{aligned}$$

is a special case of the general form

$$\begin{aligned} \text{minimize} \quad & c'x \\ \text{subject to} \quad & a_i'x \geq b_i \quad i \in M_1 \\ & a_i'x \leq b_i \quad i \in M_2 \\ & a_i'x = b_i \quad i \in M_3 \\ & x_j \geq 0 \qquad j \in N_1 \\ & x_j \leq 0 \qquad j \in N_2. \end{aligned}$$

- The converse is also true: a general LP problem can be transformed into an equivalent problem in standard form. Let's see how!

Reduction to standard form

- We show how to transform a general LP problem into an equivalent problem in standard form.

$$\begin{array}{rlll} \text{minimize} & c'x & & \\ \text{subject to} & a_i'x \geq b_i & i \in M_1 & \\ & a_i'x \leq b_i & i \in M_2 & \\ & a_i'x = b_i & i \in M_3 & \\ & x_j \geq 0 & j \in N_1 & \\ & x_j \leq 0 & j \in N_2 & \end{array} \quad \longrightarrow \quad \begin{array}{rl} \text{minimize} & c'x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

- Thanks to this result, we will only need to develop methods to solve standard form problems.

Reduction to standard form

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 & x_j \leq 0 & j \in N_2 &
\end{array}
\quad \longrightarrow \quad
\begin{array}{rl}
\text{minimize} & c'x \\
\text{subject to} & Ax = b \\
 & x \geq 0
\end{array}
$$

- Thanks to this result, we will only need to develop methods to solve standard form problems.
- The problem transformation involves two steps:
  - (a) Elimination of nonpositive and free variables.
  - (b) Elimination of inequality constraints.

Reduction to standard form

(a.1) Elimination of a nonpositive variable x_j, $j \in N_2$.

- We replace x_j by

$$-x_j'.$$

- x_j' is a new variable on which we impose the sign constraint

$$x_j' \geq 0.$$

Reduction to standard form

(a.2) Elimination of a free variable.

- Idea: Any real number x_j can be written as the difference of two nonnegative numbers. Example: $-5 = 0 - (5)$.

Reduction to standard form

(a.2) Elimination of a free variable.

- Idea: Any real number x_j can be written as the difference of two nonnegative numbers. Example: $-5 = 0 - (5)$.
- Given a free variable x_j, we replace it by

$$x_j^+ - x_j^- .$$

- x_j^+ and x_j^- are new variables on which we impose the sign constraints

$$x_j^+ \geq 0 \quad \text{and} \quad x_j^- \geq 0.$$

Reduction to standard form

(b.1) Elimination of inequality constraints $\leq$

- Consider an inequality constraint of the form

$$\sum_{j=1}^{n} a_{ij}x_j \ \leq \ b_i.$$

- We introduce a new variable s_i and the standard form constraints

$$\begin{aligned} \sum_{j=1}^{n} a_{ij}x_j \ + \ s_i &= b_i \\ s_i &\geq 0. \end{aligned}$$

- Such a variable s_i is called a slack variable.

Reduction to standard form

(b.2) Elimination of inequality constraints $\geq$

- Consider an inequality constraint of the form

$$\sum_{j=1}^{n} a_{ij}x_j \;\geq\; b_i.$$

- We introduce a new variable s_i and the standard form constraints

$$\begin{aligned} \sum_{j=1}^{n} a_{ij}x_j \;-\; s_i &= b_i \\ s_i &\geq 0. \end{aligned}$$

- Such a variable s_i is called a surplus variable.

Example 1.4

The problem

$$\begin{aligned} \text{minimize} \quad & 2x_1 + 4x_2 \\ \text{subject to} \quad & x_1 + x_2 \geq 3 \\ & 3x_1 + 2x_2 = 14 \\ & x_1 \geq 0, \end{aligned}$$

is equivalent to the standard form problem

$$\begin{aligned} \text{minimize} \quad & 2x_1 + 4x_2^+ - 4x_2^- \\ \text{subject to} \quad & x_1 + x_2^+ - x_2^- - x_3 = 3 \\ & 3x_1 + 2x_2^+ - 2x_2^- = 14 \\ & x_1, x_2^+, x_2^-, x_3 \geq 0. \end{aligned}$$

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- What does it mean that two problems are equivalent?

Equivalence of optimization problems

- Consider two minimization problems: Π_1 and Π_2.
- Each of them could be, for example, a LP problem.
- Π_1 and Π_2 are equivalent if they are either both infeasible, or they have the same optimal cost.

Equivalence of optimization problems

- Consider two minimization problems: Π_1 and Π_2.
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- Π_1 and Π_2 are <u>equivalent</u> if they are either both infeasible, or they have the same optimal cost.

Lemma

Π_1 and Π_2 are equivalent if and only if:

(i) For every feasible solution to Π_1, there exists a feasible solution to Π_2, with cost equal or lower, and

(ii) For every feasible solution to Π_2, there exists a feasible solution to Π_1, with cost equal or lower.

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- For maximization problems we should replace “or lower” with “or higher”.
- Let’s prove the lemma!

Example 1.4

The problem Π_1

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is equivalent to the standard form problem Π_2

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- Given the feasible solution

  $$(x_1, x_2) = (6, -2)$$

  to Π_1, we obtain the feasible solution

  $$(x_1, x_2^+, x_2^-, x_3) = (6, 0, 2, 1)$$

  to Π_2, which has the same cost.

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- Conversely, given the feasible solution

  $$(x_1, x_2^+, x_2^-, x_3) = (8, 1, 6, 0)$$

  to Π_2, we obtain the feasible solution

  $$(x_1, x_2) = (8, -5)$$

  to Π_1 with the same cost.

Example 1.4

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- Let's formally show that these two problems are equivalent!

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- Let's formally show that these two problems are equivalent!
- Exercise: Show that our transformation always yields an equivalent problem.

General form or standard form?

- We will often use the general form

$$Ax \geq b$$

  to develop the theory of LP.

- We will use the standard form

$$Ax = b, \ x \geq 0$$

  when it comes to algorithms, since it is computationally more convenient.

1.2 Examples of LP problems

A production problem

A production problem

- We can produce n different goods using m different materials.
- Let b_i, $i = 1, \ldots, m$, be the available amount of the ith material.
- The jth good, $j = 1, \ldots, n$, requires a_{ij} units of the ith material and results in a revenue of c_j per unit produced.
- We need to decide how much of each good to produce in order to maximize its total revenue.

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- Decision variables:
  Let x_j, $j = 1, \ldots, n$, be the amount of the jth good.

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- We need to decide how much of each good to produce in order to maximize its total revenue.
- Decision variables:
  Let x_j, $j = 1, \ldots, n$, be the amount of the jth good.
- Formulation:

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i & i = 1, \ldots, m \\ & x_j \geq 0 & j = 1, \ldots, n. \end{array}$$

Multiperiod planning of electric power capacity

Multiperiod planning of electric power capacity

- A state wants to plan its electricity capacity for the next T years.
- The demand for electricity during year $t = 1, \ldots, T$ is of d_t megawatts.

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- There is a capital cost per megawatt of the capacity that becomes operational at the beginning of year t. For coal plants it is c_t, and for nuclear plants is n_t.
- Coal plants last for 20 years, nuclear plants last for 15 years.

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- There is a capital cost per megawatt of the capacity that becomes operational at the beginning of year t. For coal plants it is c_t, and for nuclear plants is n_t.
- Coal plants last for 20 years, nuclear plants last for 15 years.
- No more than 20% of the total capacity should ever be nuclear.
- We want to find a least cost capacity expansion plan.

Multiperiod planning of electric power capacity

- Decision variables:
  Let x_t and y_t be the amount of coal (respectively, nuclear) capacity brought on line at the beginning of year t.
- Objective function:

$$\text{minimize} \quad \sum_{t=1}^{T}(c_t x_t + n_t y_t).$$

Multiperiod planning of electric power capacity

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- Objective function:

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- It will be useful (but not necessary) to introduce some more decision variables:
  Let w_t and z_t be the total coal (respectively, nuclear) capacity available in year t.

Multiperiod planning of electric power capacity

Constraints:

- Coal plants last for 20 years:

$$w_t = \sum_{s=\max\{1,t-19\}}^{t} x_s \qquad t = 1, \ldots, T.$$

- Nuclear plants last for 15 years:

$$z_t = \sum_{s=\max\{1,t-14\}}^{t} y_s \qquad t = 1, \ldots, T.$$

Multiperiod planning of electric power capacity

- The available capacity must meet the forecasted demand:

$$w_t + z_t + e_t \geq d_t \qquad t = 1, \ldots, T.$$

- No more than 20% of the total capacity should ever be nuclear:

$$\frac{z_t}{w_t + z_t + e_t} \leq 0.2 \qquad t = 1, \ldots, T$$

$$\Updownarrow$$

$$0.8z_t - 0.2w_t \leq 0.2e_t \qquad t = 1, \ldots, T.$$

Multiperiod planning of electric power capacity

Formulation:

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{t=1}^{T}(c_t x_t + n_t y_t) & \\
\text{subject to} & \displaystyle w_t - \sum_{s=\max\{1,t-19\}}^{t} x_s = 0 & t = 1, \ldots, T \\
& \displaystyle z_t - \sum_{s=\max\{1,t-14\}}^{t} y_s = 0 & t = 1, \ldots, T \\
& w_t + z_t \geq d_t - e_t & t = 1, \ldots, T \\
& 0.8 z_t - 0.2 w_t \leq 0.2 e_t & t = 1, \ldots, T \\
& x_t, y_t, w_t, z_t \geq 0 & t = 1, \ldots, T.
\end{array}
$$

Multiperiod planning of electric power capacity

Formulation:

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\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{t=1}^{T}(c_t x_t + n_t y_t) & \\
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 & x_t, y_t, w_t, z_t \geq 0 & t = 1, \ldots, T.
\end{array}
$$

Question: How would the formulation look like if we did not introduce variables w_t and z_t?

A scheduling problem

A scheduling problem

- A hospital wants to make a weekly night shift schedule for its nurses.
- The demand for nurses for the night shift on day j is an integer d_j, $j = 1, \ldots, 7$.
- Every nurse works 5 days in a row on the night shift.
- The problem is to find the minimal number of nurses the hospital needs to hire.

A scheduling problem

Decision variables:

- We could try using a decision variable y_j equal to the number of nurses that work on day j.
- But we would not be able to capture the constraint that every nurse works 5 days in a row.

- We define x_j as the number of nurses starting their week on day j.

A scheduling problem

Formulation:

$$\begin{array}{rll}
\text{minimize} & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 & \\
\text{subject to} & x_1 + x_4 + x_5 + x_6 + x_7 \geq d_1 & \\
& x_1 + x_2 + x_5 + x_6 + x_7 \geq d_2 & \\
& x_1 + x_2 + x_3 + x_6 + x_7 \geq d_3 & \\
& x_1 + x_2 + x_3 + x_4 + x_7 \geq d_4 & \\
& x_1 + x_2 + x_3 + x_4 + x_5 \geq d_5 & \\
& x_2 + x_3 + x_4 + x_5 + x_6 \geq d_6 & \\
& x_3 + x_4 + x_5 + x_6 + x_7 \geq d_7 & \\
& x_j \geq 0 & j = 1, \ldots, 7 \\
& x_j \text{ integer} & j = 1, \ldots, 7.
\end{array}$$

A scheduling problem

- This would be a LP problem, except for the constraints

$$x_j \text{ integer} \qquad j = 1, \ldots, 7.$$

- We actually have an integer linear programming problem; see course ISyE/Math/CS 728 - Integer Optimization.
- What can we say about this problem without taking 728?

A scheduling problem

- This would be a LP problem, except for the constraints

$$x_j \text{ integer} \qquad j = 1, \ldots, 7.$$

- We actually have an integer linear programming problem; see course ISyE/Math/CS 728 - Integer Optimization.
- What can we say about this problem without taking 728?
- Let's ignore ("relax") the integrality constraints. We obtain the so-called LP relaxation of the original problem.

A scheduling problem

$$\begin{array}{rll}
\text{minimize} & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 & \\
\text{subject to} & x_1 + x_4 + x_5 + x_6 + x_7 \geq d_1 & \\
& x_1 + x_2 + x_5 + x_6 + x_7 \geq d_2 & \\
& x_1 + x_2 + x_3 + x_6 + x_7 \geq d_3 & \\
& x_1 + x_2 + x_3 + x_4 + x_7 \geq d_4 & \\
& x_1 + x_2 + x_3 + x_4 + x_5 \geq d_5 & \\
& x_2 + x_3 + x_4 + x_5 + x_6 \geq d_6 & \\
& x_3 + x_4 + x_5 + x_6 + x_7 \geq d_7 & \\
& x_j \geq 0 & j = 1, \ldots, 7 \\
& \sout{x_j \text{ integer}} & \sout{j = 1, \ldots, 7}.
\end{array}$$

- The optimal cost will be less than or equal to the optimal cost of the original problem. Why?

A scheduling problem

$$
\begin{array}{rll}
\text{minimize} & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 & \\
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 & x_3 + x_4 + x_5 + x_6 + x_7 \geq d_7 & \\
 & x_j \geq 0 & j = 1, \ldots, 7
\end{array}
$$

~~x_j integer~~ ~~$j = 1, \ldots, 7$~~.

- If the optimal solution to the LP relaxation happens to be integer, then it is also an optimal solution to the original problem. Why?

A scheduling problem

$$
\begin{array}{rll}
\text{minimize} & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 & \\
\text{subject to} & x_1 + x_4 + x_5 + x_6 + x_7 \geq d_1 & \\
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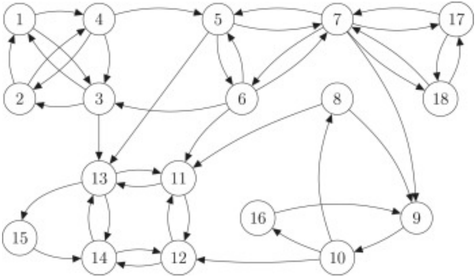
& x_3 + x_4 + x_5 + x_6 + x_7 \geq d_7 & \\
& x_j \geq 0 & j = 1, \ldots, 7 \\
& \cancel{x_j \text{ integer}} & \cancel{j = 1, \ldots, 7}.
\end{array}
$$

- If it is not integer, we can obtain a feasible solution to the original problem by rounding each x_j upwards.
- But this solution is not necessarily optimal!

Choosing paths in a communication network

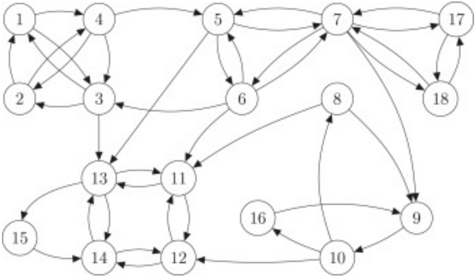
Choosing paths in a communication network

- Consider a communication network $G = (N, A)$.
- N is the set of nodes, $|N| = n$.
- A is the set of communication links that connect the nodes.
- A link allowing one-way transmission from node i to node j is described by an ordered pair (i, j).



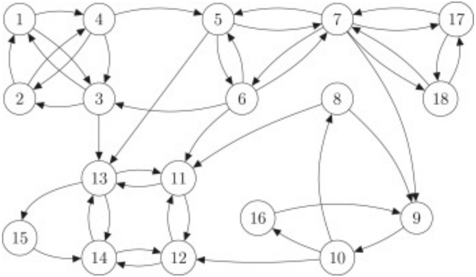
Choosing paths in a communication network

- Each link $(i, j) \in A$ can carry up to u_{ij} bits per second.
- There is a positive charge c_{ij} per bit transmitted along (i, j).
- Each node k generates data at the rate of $b^{k\ell}$ bits per second, that have to be transmitted to node ℓ.



Choosing paths in a communication network

- Data can be transmitted either through a direct link (k, ℓ) or by tracing a sequence of links.
- Data with the same origin and destination can be split and transmitted along different paths.
- The problem is to choose paths along which all data reach their intended destinations, while minimizing the total cost.



Choosing paths in a communication network

- Decision variables:
  We introduce variables $x_{ij}^{k\ell}$ indicating the amount of data with origin k and destination ℓ that traverse link (i, j).
- Objective function:

$$\text{minimize} \quad \sum_{(i,j)\in A} \sum_{k=1}^{n} \sum_{\ell=1}^{n} c_{ij} x_{ij}^{k\ell}.$$

Choosing paths in a communication network

Constraints:

- The amount of data is always nonnegative:

$$x_{ij}^{k\ell} \geq 0 \qquad\qquad (i,j) \in A, \quad k,\ell = 1,\ldots,n.$$

- The total traffic through a link (i,j) cannot exceed the link's capacity:

$$\sum_{k=1}^{n}\sum_{\ell=1}^{n} x_{ij}^{k\ell} \leq u_{ij} \qquad\qquad (i,j) \in A.$$

Choosing paths in a communication network

- The last constraint is a flow conservation constraint at node i for data with origin k and destination ℓ.
- Let $b_i^{k\ell}$ be the net flow at node i (flow that exits i minus flow that enters i), of data with origin k and destination ℓ.

$$b_i^{k\ell} = \begin{cases} b^{k\ell} & \text{if } i = k \\ -b^{k\ell} & \text{if } i = \ell \\ 0 & \text{otherwise.} \end{cases}$$

- We can now write the flow conservation constraint

$$\underbrace{\sum_{j|(i,j)\in A} x_{ij}^{k\ell}}_{\text{flow that exits } i} - \underbrace{\sum_{j|(j,i)\in A} x_{ji}^{k\ell}}_{\text{flow that enters } i} = b_i^{k\ell} \qquad i, k, \ell = 1, \ldots, n.$$

Choosing paths in a communication network

Formulation:

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{(i,j)\in A}\sum_{k=1}^{n}\sum_{\ell=1}^{n} c_{ij}x_{ij}^{k\ell} & \\
\text{subject to} & \displaystyle\sum_{j|(i,j)\in A} x_{ij}^{k\ell} - \sum_{j|(j,i)\in A} x_{ji}^{k\ell} = b_i^{k\ell} & i,k,\ell = 1,\dots,n \\
 & \displaystyle\sum_{k=1}^{n}\sum_{\ell=1}^{n} x_{ij}^{k\ell} \le u_{ij} & (i,j)\in A \\
 & x_{ij}^{k\ell} \ge 0 & (i,j)\in A,\;\; k,\ell = 1,\dots,n.
\end{array}
$$

Choosing paths in a communication network

- A similar problem arises when we consider a transportation company that wishes to transport several commodities from their origins to their destinations through a network.
- This problem is known as the multicommodity flow problem, with the traffic corresponding to each origin-destination pair viewed as a different commodity.

Choosing paths in a communication network

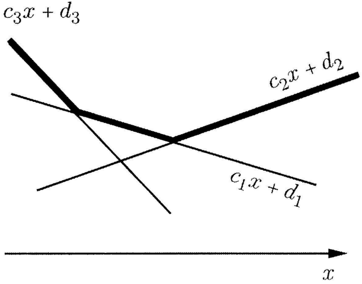
- There is a version of this problem, known as the minimum cost network flow problem, in which we do not distinguish between different commodities.
- Instead, we are given the amount b_i of external supply or demand at each node i, and the objective is to transport material from the supply nodes to the demand nodes, at minimum cost.
- The network flow problem contains as special cases some important problems such as:
  - The shortest path problem.
  - The maximum flow problem.
  - The assignment problem.
- See course ISyE/Math/CS 425 - Introduction to Combinatorial Optimization.

1.3 Piecewise linear convex functions

Piecewise linear convex functions

- We consider an important class of nonlinear optimization problems that can be cast as LP problems.
- Let $c_1, \ldots, c_m$ be vectors in $\mathbb{R}^n$, let $d_1, \ldots, d_m$ be scalars, and consider the function $f\colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) = \max_{i=1,\ldots,m} (c_i' x + d_i).$$



Piecewise linear convex functions

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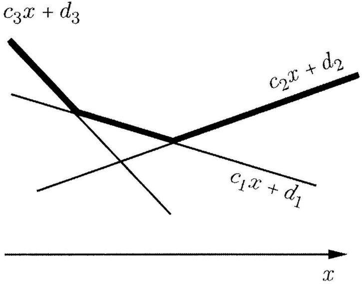
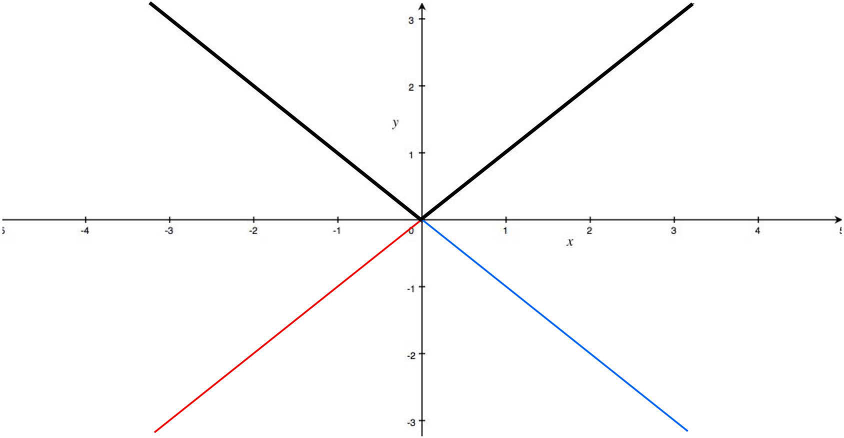


- A function of this form is called a piecewise linear convex function.

Piecewise linear convex functions

- A simple example is the absolute value function defined by

$$f(x) = |x| = \max\{x, -x\}.$$



Piecewise linear convex constraints

- Suppose that we are given a constraint of the form

$$\underbrace{\max_{i=1,\ldots,m} (c_i' x + d_i)}_{\text{piecewise linear convex}} \leq h.$$

- Such a constraint can be rewritten using only linear inequalities as

$$c_i' x + d_i \leq h \qquad\qquad i = 1, \ldots, m.$$

Example

$$
\begin{aligned}
\text{minimize} \quad & x_1 + x_2 \\
\text{subject to} \quad & \max\{x_1 + 2x_2,\ 2x_1 + x_2\} \leq 2 \\
& x_1 \geq 0 \\
& x_2 \geq 0
\end{aligned}
$$

is equivalent to the LP problem

$$
\begin{aligned}
\text{minimize} \quad & x_1 + x_2 \\
\text{subject to} \quad & x_1 + 2x_2 \leq 2 \\
& 2x_1 + x_2 \leq 2 \\
& x_1 \geq 0 \\
& x_2 \geq 0
\end{aligned}
$$

Piecewise linear convex constraints

- Question: What if instead we have a constraint of the form

$$\underbrace{\max_{i=1,\ldots,m}(c_i'x + d_i)}_{\text{piecewise linear convex}} \geq h \quad ?$$

Piecewise linear convex objective functions

- We now consider a generalization of LP, where the objective function is piecewise linear convex:

$$\begin{aligned} \text{minimize} \quad & \max_{i=1,\dots,m} (c_i' x + d_i) \\ \text{subject to} \quad & Ax \geq b. \end{aligned}$$

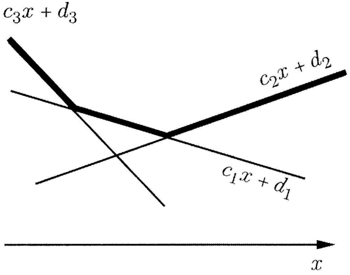
Piecewise linear convex objective functions

- Idea: for a given vector x, the value

$$\max_{i=1,\dots,m}(c_i'x+d_i)$$

  is equal to the smallest number z such that

$$z \geq \max_{i=1,\dots,m}(c_i'x+d_i) \quad \Leftrightarrow \quad z \geq c_i'x+d_i \quad \forall i=1,\dots,m.$$



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- For this reason, the optimization problem is equivalent to the LP problem

  $$\begin{aligned} \text{minimize} \quad & z \\ \text{subject to} \quad & z \geq c_i'x + d_i \quad i = 1, \ldots, m \\ & Ax \geq b. \end{aligned}$$

  where the decision variables are z and x.

Piecewise linear convex objective functions

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$$\begin{array}{rl} \text{minimize} & z \\ \text{subject to} & z \geq c_i'x + d_i \quad i = 1,\ldots,m \\ & Ax \geq b. \end{array}$$

where the decision variables are z and x.

Exercise: Show equivalency, and note that the same argument does not go through for maximization problems.

Example

$$\begin{array}{rl} \text{minimize} & \max\{2x_1 + 4x_2,\ 2x_1 + x_2\} \\ \text{subject to} & x_1 + x_2 \geq 3 \\ & 3x_1 + 2x_2 = 14 \\ & x_1 \geq 0 \end{array}$$

is equivalent to the LP problem

$$\begin{array}{rl} \text{minimize} & z \\ \text{subject to} & z \geq 2x_1 + 4x_2 \\ & z \geq 2x_1 + x_2 \\ & x_1 + x_2 \geq 3 \\ & 3x_1 + 2x_2 = 14 \\ & x_1 \geq 0. \end{array}$$

Problems involving absolute values

Problems involving absolute values

Consider a problem

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n} c_i \left| x_i \right| \\ \text{subject to} \quad & Ax \geq b, \end{aligned}$$

where $c_i \geq 0$ for every $i = 1, \ldots, n$.

- The objective function can be shown to be piecewise linear convex (exercise).
- However, it is a bit involved to express it in the form

$$\max_{j=1,\ldots,m} (c_j' x + d_j).$$

- Thus we give a more direct formulation.

Problems involving absolute values

- We observe that $|x_i|$ is the smallest number z_i that satisfies

$$x_i \leq z_i \qquad \text{and} \qquad -x_i \leq z_i.$$

- We obtain the equivalent LP problem

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n} c_i z_i \\ \text{subject to} \quad & Ax \geq b \\ & x_i \leq z_i \qquad i = 1, \ldots, n \\ & -x_i \leq z_i \quad i = 1, \ldots, n. \end{aligned}$$

Problems involving absolute values

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$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n} c_i z_i \\ \text{subject to} \quad & Ax \geq b \\ & x_i \leq z_i \qquad i = 1, \ldots, n \\ & -x_i \leq z_i \quad i = 1, \ldots, n. \end{aligned}$$

Exercise: Show equivalency, and note that we need both assumptions that we are minimizing, and that $c_i \geq 0$ for every $i = 1, \ldots, n$.

Example 1.1

$$\begin{array}{rl} \text{minimize} & 2\,|x_1| + x_2 \\ \text{subject to} & x_1 + x_2 \geq 4 \end{array}$$

is equivalent to the LP problem

$$\begin{array}{rl} \text{minimize} & 2z_1 + x_2 \\ \text{subject to} & x_1 + x_2 \geq 4 \\ & x_1 \leq z_1 \\ & -x_1 \leq z_1. \end{array}$$

Data fitting

Data fitting

- We are given m data points of the form

$$(a_i, b_i), \quad i = 1, \dots, m,$$

  where $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$.
- We wish to predict the value of the variable b from knowledge of the vector a.

Data fitting

- We are given m data points of the form

  $$(a_i, b_i), \quad i = 1, \ldots, m,$$

  where $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$.
- We wish to predict the value of the variable b from knowledge of the vector a.
- In such a situation, one often uses a linear model of the form

  $$b = a'x,$$

  where x is a parameter vector to be determined.

Data fitting

- Given a particular parameter vector x, the residual, or prediction error, at the ith data point is defined as

$$\left| b_i - a_i' x \right| .$$

- Given a choice between alternative models, one should choose a model that "explains" the available data as best as possible, i.e., a model that results in small residuals.

Data fitting

- One possibility is to minimize the largest residual:

$$\begin{aligned} \text{minimize} \quad & \max_{i=1,\ldots,m} \left| b_i - a_i' x \right| \\ \text{subject to} \quad & x \in \mathbb{R}^n. \end{aligned}$$

- We have a piecewise linear convex objective function.
- An equivalent LP formulation is:

$$\begin{aligned} \text{minimize} \quad & z \\ \text{subject to} \quad & b_i - a_i' x \leq z \qquad i = 1, \ldots, m \\ & -b_i + a_i' x \leq z \quad i = 1, \ldots, m. \end{aligned}$$

Data fitting

- A different approach is to minimize the sum of all the residuals:

$$\begin{array}{rl} \text{minimize} & \displaystyle\sum_{i=1}^{m} \left| b_i - a_i' x \right| \\ \text{subject to} & x \in \mathbb{R}^n. \end{array}$$

- $|b_i - a_i' x|$ is the smallest number z_i that satisfies

$$b_i - a_i' x \leq z_i \qquad \text{and} \qquad -b_i + a_i' x \leq z_i.$$

- We obtain the equivalent formulation

$$\begin{array}{rll} \text{minimize} & z_1 + \cdots + z_m & \\ \text{subject to} & b_i - a_i' x \leq z_i & i = 1, \ldots, m \\ & -b_i + a_i' x \leq z_i & i = 1, \ldots, m. \end{array}$$

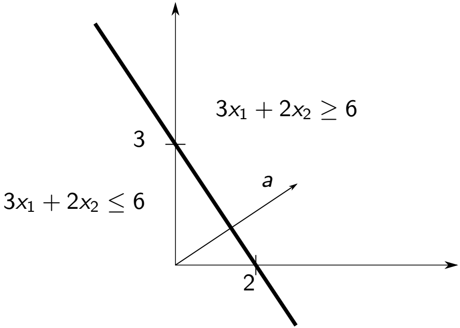
1.4 Graphical representation and solution

Graphical representation and solution: two variables

- In the Cartesian plane the equation

$$a_1x_1 + a_2x_2 = b$$

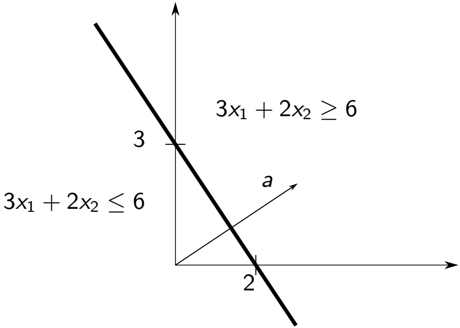
is a line that partitions the plane into two halfspaces.



Graphical representation and solution: two variables

- Each halfspace contains the vectors that satisfy the inequality

$$a_1x_1 + a_2x_2 \ \geq \ b \qquad \text{or} \qquad a_1x_1 + a_2x_2 \ \leq \ b.$$

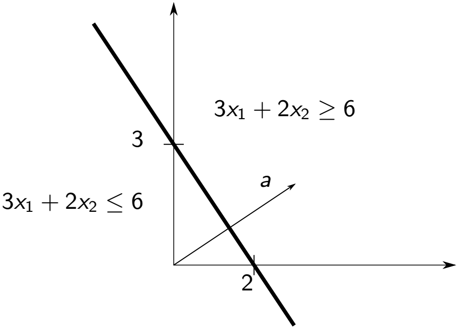


Graphical representation and solution: two variables

- Consider the family of parallel lines

$$a_1 x_1 + a_2 x_2 = b,$$

  where $a_1, a_2 \in \mathbb{R}$ are fixed and $b \in \mathbb{R}$ is a parameter.
- The vector (a_1, a_2) is orthogonal to the lines of the family, and points in the direction where b increases.



Example 1.6:

$$\begin{aligned} \text{minimize} \quad & -x_1 - x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leq 3 \\ & 2x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0. \end{aligned}$$

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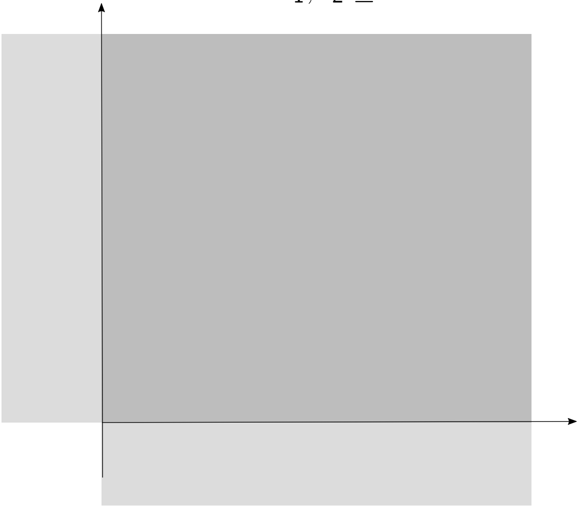

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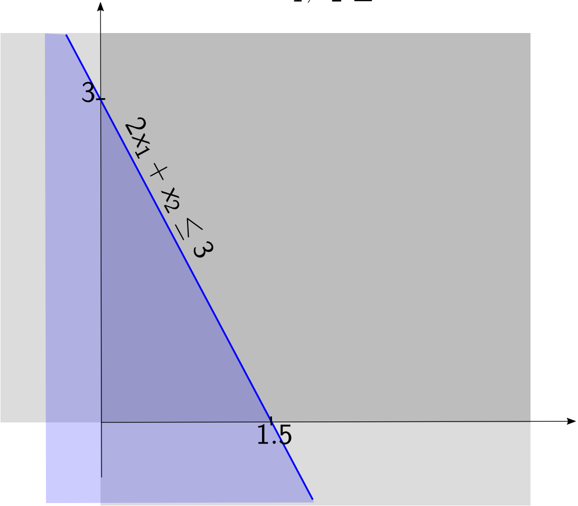

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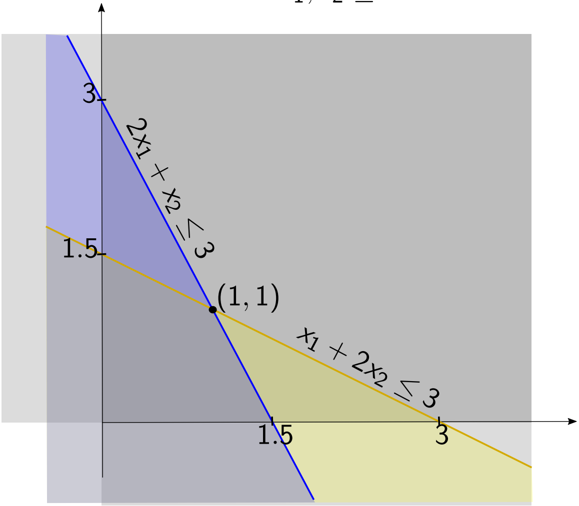

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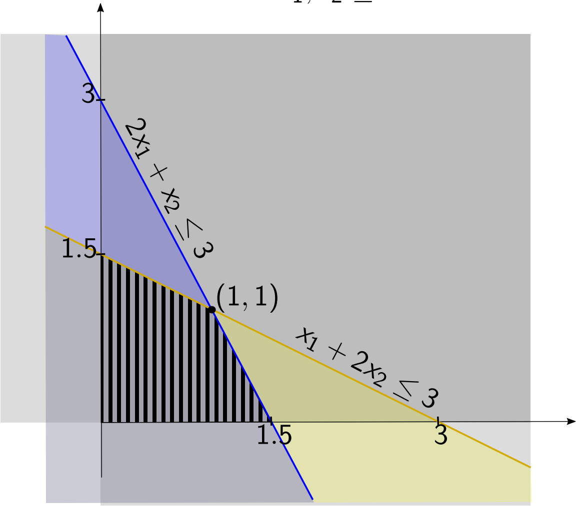
$$\begin{array}{rl} \text{minimize} & -x_1 - x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 3 \\ & 2x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0. \end{array}$$



Example 1.6:

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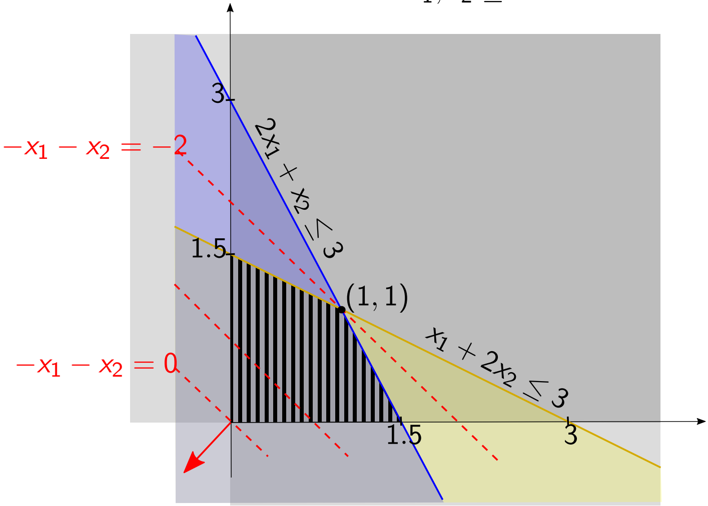


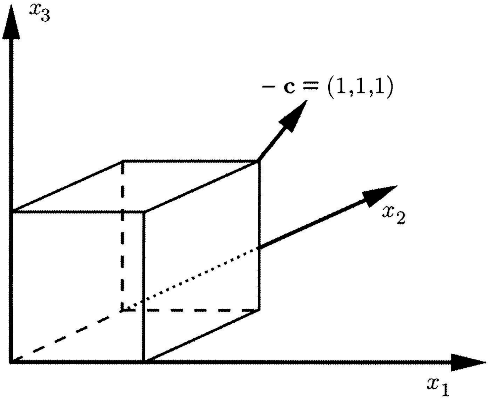
Graphical representation and solution

For a problem in three dimensions:

- The same approach can be used except that the set of points with the same value of $c'x$ is a plane, instead of a line.
- This plane is again perpendicular to the vector c.
- The objective is to slide this plane as much as possible in the direction of $-c$, as long as we do not leave the feasible set.

Example 1.7

$$\begin{array}{rl} \text{minimize} & -x_1 - x_2 - x_3 \\ \text{subject to} & 0 \le x_1 \le 1 \\ & 0 \le x_2 \le 1 \\ & 0 \le x_3 \le 1. \end{array}$$



Graphical representation and solution

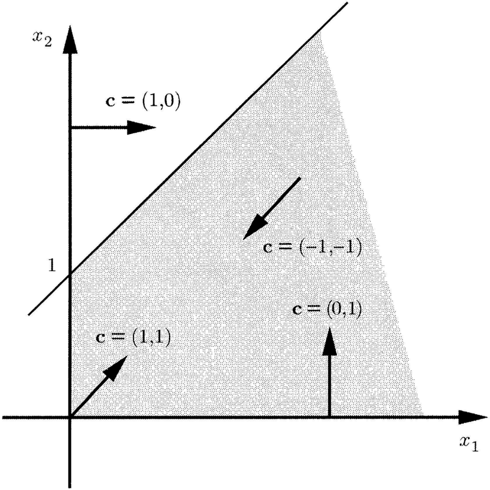
- In both of the preceding examples, the feasible set is bounded, (does not extend to infinity), and the problem has a unique optimal solution.
- This is not always the case and we have some additional possibilities.
- Let's see them!

Example 1.8

- Consider the feasible set in $\mathbb{R}^2$ defined by the constraints

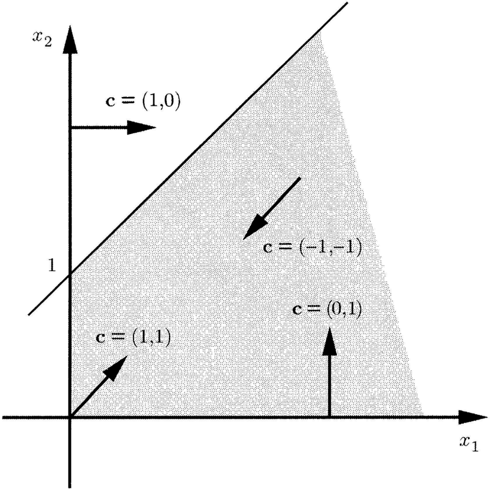
$$\begin{aligned} -x_1 + x_2 &\leq 1 \\ x_1 &\geq 0 \\ x_2 &\geq 0. \end{aligned}$$

- We consider different cost vectors c.



Example 1.8

$$\begin{aligned} \text{minimize} \quad & x_1 + x_2 \\ \text{subject to} \quad & -x_1 + x_2 \leq 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0. \end{aligned}$$



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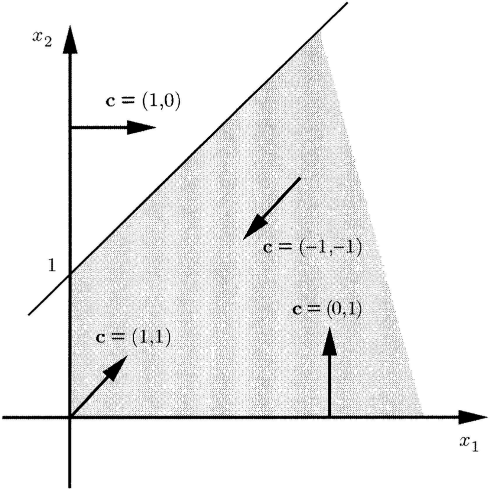
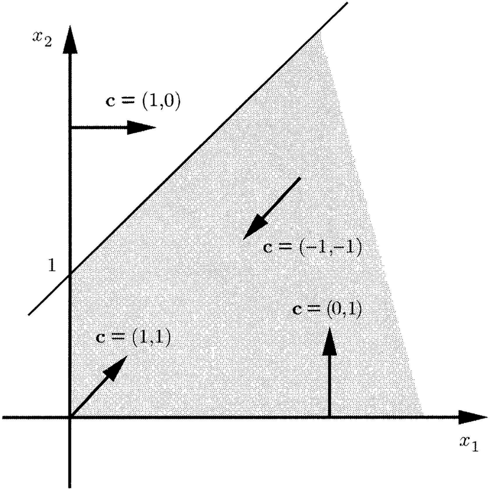
- $x = (0, 0)$ is the unique optimal solution.



Example 1.8

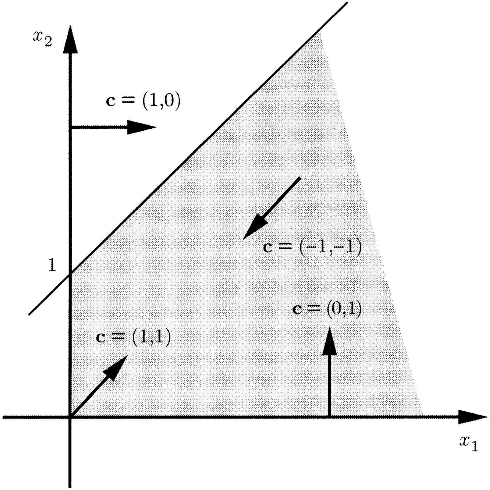
$$\begin{aligned} \text{minimize} \quad & x_1 \\ \text{subject to} \quad & -x_1 + x_2 \leq 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0. \end{aligned}$$



Example 1.8

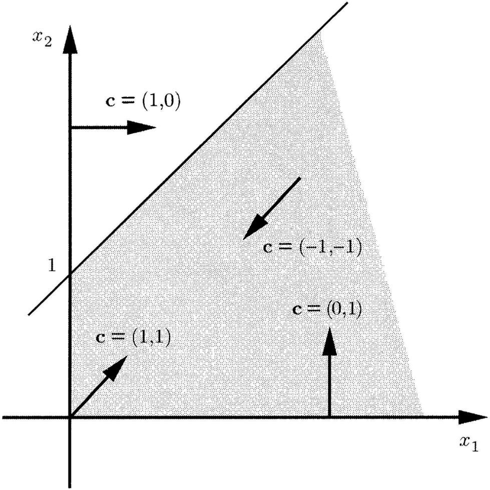
$$\begin{aligned} \text{minimize} \quad & x_1 \\ \text{subject to} \quad & -x_1 + x_2 \leq 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0. \end{aligned}$$

- There are multiple optimal solutions.
- The set of optimal solutions is bounded.



Example 1.8

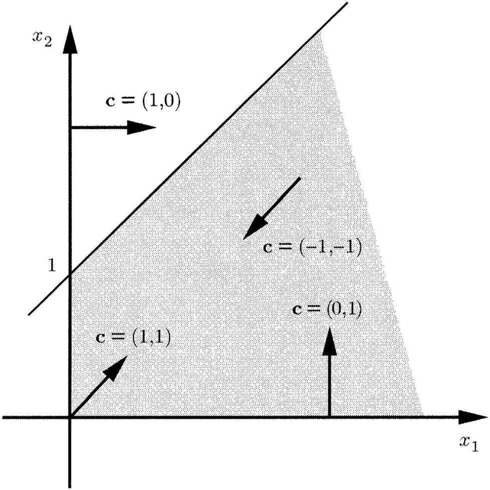
$$\begin{aligned} \text{minimize} \quad & x_2 \\ \text{subject to} \quad & -x_1 + x_2 \leq 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0. \end{aligned}$$



Example 1.8

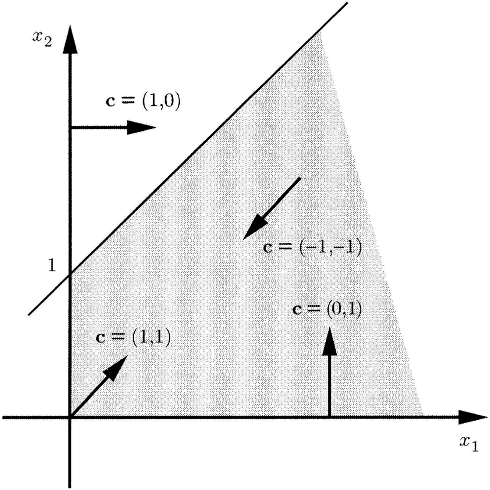
$$\begin{aligned} \text{minimize} \quad & x_2 \\ \text{subject to} \quad & -x_1 + x_2 \leq 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0. \end{aligned}$$

- There are multiple optimal solutions.
- The set of optimal solutions is unbounded.



Example 1.8

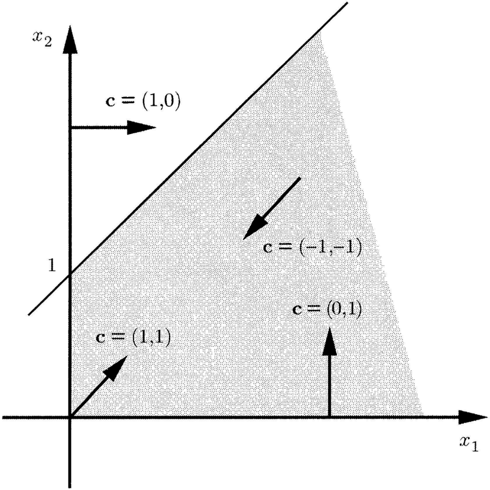
$$\begin{aligned} \text{minimize} \quad & -x_1 - x_2 \\ \text{subject to} \quad & -x_1 + x_2 \leq 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0. \end{aligned}$$



Example 1.8

$$\begin{aligned} \text{minimize} \quad & -x_1 - x_2 \\ \text{subject to} \quad & -x_1 + x_2 \leq 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0. \end{aligned}$$

- We can obtain a sequence of feasible solutions whose cost converges to $-\infty$.
- We say that the optimal cost is $-\infty$ and that the problem is unbounded.



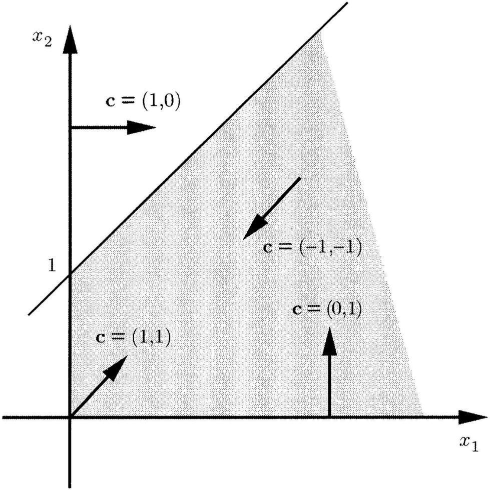
Example 1.8

If we impose the additional constraint

$$x_1 + x_2 \leq -2$$

we obtain the feasible set

$$\begin{aligned} -x_1 + x_2 &\leq 1 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \\ x_1 + x_2 &\leq -2. \end{aligned}$$



Example 1.8

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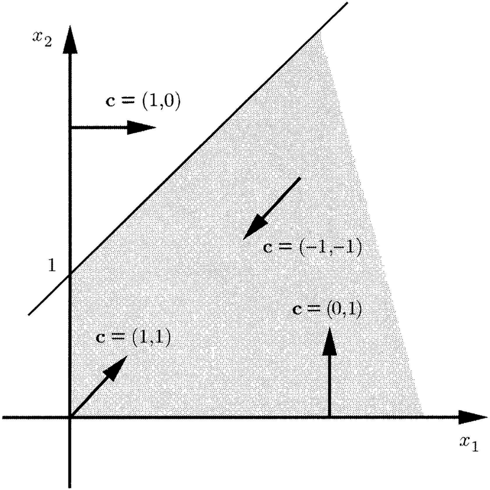
we obtain the feasible set

$$\begin{aligned} -x_1 + x_2 &\leq 1 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \\ x_1 + x_2 &\leq -2. \end{aligned}$$

- No feasible solution exists.



Graphical representation and solution

In Example 1.8 we have the following possibilities:

(a) There exists a unique optimal solution.

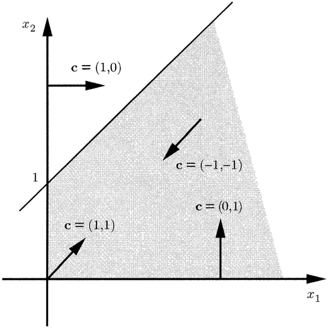
(b) There exist multiple optimal solutions; in this case, the set of optimal solutions can be either bounded or unbounded.

(c) The optimal cost is $-\infty$, and no feasible solution is optimal.

(d) The feasible set is empty.

Graphical representation and solution

- In the examples that we have considered, if the problem has at least one optimal solution, then an optimal solution can be found among the corners of the feasible set.



- In Chapter 2, we will show that this is a general feature of LP problems, as long as the feasible set has at least one corner.

Visualizing standard form problems

Visualizing standard form problems

How do we visualize standard form problems?

$$\begin{aligned} \text{minimize} \quad & c'x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Visualizing standard form problems

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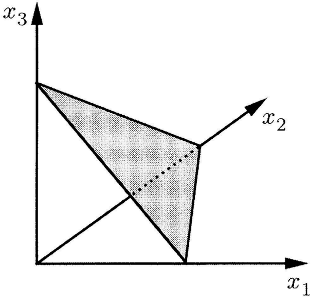
$$\begin{aligned} \text{minimize} \quad & c'x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- If the dimension n of the vector x is at most three we know how.



Example:

$$x_1 + x_2 + x_3 = 1$$
$$x_1, x_2, x_3 \geq 0$$

Visualizing standard form problems

How do we visualize standard form problems?

$$\begin{aligned} \text{minimize} \quad & c'x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- However, when $n \leq 3$, the feasible set does not have much variety and does not provide enough insight into the general case. Why?
- Thus we wish to visualize standard form problems even if the dimension n of the vector x is greater than three.

Visualizing standard form problems

Suppose that we have a standard form problem

$$\begin{aligned} \text{minimize} \quad & c'x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0, \end{aligned}$$

and that the matrix A has dimensions $m \times n$.

- In particular, the decision vector x is of dimension n and we have m equality constraints.
- We assume that $m \leq n$ and that the constraints $Ax = b$ force x to lie on an $(n - m)$-dimensional set (A has full rank).

Visualizing standard form problems

Suppose that we have a standard form problem

$$\begin{aligned} \text{minimize} \quad & c'x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0, \end{aligned}$$

and that the matrix A has dimensions $m \times n$.

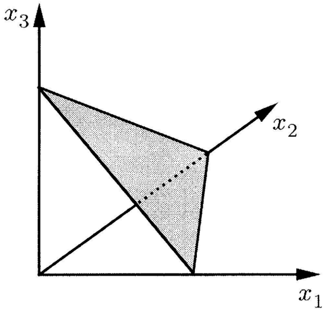
- If we "stand" on that $(n - m)$-dimensional set and ignore the m dimensions orthogonal to it, the feasible set is only constrained by the linear inequality constraints $x_i \geq 0$, $i = 1, \ldots, n$.
- In particular, if $n - m = 2$, the feasible set can be drawn as a two-dimensional set defined by n linear inequality constraints.

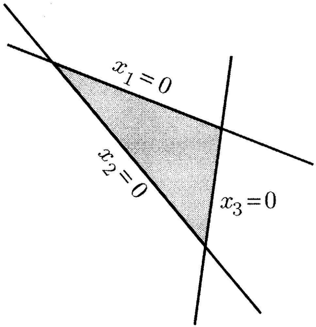
Visualizing standard form problems

Example: Consider the feasible set in $\mathbb{R}^3$

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 1 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

and note that $n = 3$ and $m = 1$.





Visualizing standard form problems

Example: Consider the feasible set in $\mathbb{R}^3$

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

and note that $n = 3$ and $m = 1$.

- Algebraically, we use the equations to reduce the number of variables.
- For example, if we substitute $x_3 = 1 - x_1 - x_2$ we obtain

$$\begin{aligned} 1 - x_1 - x_2 &\geq 0 \\ x_1, x_2 &\geq 0 \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} x_1 + x_2 &\leq 1 \\ x_1, x_2 &\geq 0. \end{aligned}$$