Lecture 7: QR-factorization

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February 10, 2021

Outline

1. QR factorization
   - The big picture
   - Householder matrices

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The goal and a flow chart

Goal

$A_{m\times m}$ is invertible. We want to find matrices $Q_{m\times m}$ and $R_{m\times m}$ s.t.:

- Q is orthogonal.
- R is upper triangular (all entries below the diagonal are 0).
- $A = QR$.

The goal and a flow chart

We will create a sequence of matrices

$$A_0 \to A_1 \to \ldots \to A_{m-1},$$

with $A_0 := A$.
Each A_j is partially upper triangular:

$$(A_j e_i)(k) = 0, \quad 1 \le i \le j,\ k > i :$$

The first j columns of A_j are “good".

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$$(A_j e_i)(k) = 0, \quad 1 \le i \le j,\ k > i :$$

The first j columns of A_j are “good".
The last matrix, A_{m-1} is upper triangular. This is R.

Flow chart continues

We have $m-1$ steps, each one is as follows:

- Input: A_{j-1}, $1 \leq j \leq m-1$.
- Output: $A_j = H_j A_{j-1}$.
- The matrix H_j is orthogonal and symmetric (so, $H_j^2 = I$).

Flow chart continues

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Therefore

$$A_{m-1} = H_{m-1} H_{m-2} \ldots H_1 A_0,$$

hence

$$A_0 = (H_1 H_2 \ldots H_{m_1}) A_{m-1}.$$

Defining

$$Q := H_1 \ldots H_{m_1},$$

we get

$$A = QR.$$

Mini demo!!

Householder matrices introduced

Definition: Householder matrix

Every matrix H_w of the form

$$H_w := I - 2ww',$$

$w \in R^m$, $||w||_2 = 1$, is called Householder.

Householder matrices introduced

Theorem: basic properties of Householder

Given a Householder matrix $H := H_w$,

1. H is symmetric: $H = H'$.
2. H is orthogonal: $H^{-1} = H'$.
3. H is self-invertible: $H^{-1} = H$.
4. $Hw = -w$.
5. If (v, w) then $Hv = v$.
6. If $x, y \in R^m$ such that

$$||x||_2 = ||y||_2, \quad \text{and} \quad w = \frac{x - y}{||x - y||_2},$$

then $Hx = y$.

Householder matrices introduced

$H = H'$:

$$H' = (I - 2ww')' = I - 2(ww')' = I - 2w''w' = I - 2ww' = H.$$

Householder matrices introduced

$H = H^{-1}$:

$$HH = (I - 2ww')(I - 2ww') = I - 4ww' + 4ww'ww' = ...$$

Householder matrices introduced

$H = H^{-1}$:

$$HH = I - 4ww' + 4ww'ww' = I - 4ww' + 4w(w'w)w' = I - 4ww' + 4ww' = I.$$

Householder matrices introduced

$Hw = -w$:

$$Hw = (I - 2ww')w = w - 2w(w'w) = w - 2w = -w.$$

Householder matrices introduced

$Hv = v$, in case $(v, w) = 0$:

$$Hv = (I - 2ww')v = v - 2w(w'v) = v - 0 = v.$$

Householder matrices introduced

$$||x||_2 = ||y||_2, \quad \text{and} \quad w = \frac{x-y}{||x-y||_2},$$

then $Hx = y$:

- $x - y = cw$, $c \in R$, therefore

$$H(x-y) = H(cw) = cHw = -cw = y - x.$$

-

$$(x+y, x-y) = (x,x) - (x,y) + (y,x) - (y,y) = \ldots$$

Householder matrices introduced

$$||x||_2 = ||y||_2, \quad \text{and} \quad w = \frac{x-y}{||x-y||_2},$$

then $Hx = y$:

- $x - y = cw$, $c \in R$, therefore

$$H(x-y) = H(cw) = cHw = -cw = y - x.$$

- $$(x+y, x-y) = (x,x) - (x,y) + (y,x) - (y,y) = 1 - 1 = 0.$$

  Therefore, $H(x+y) = x+y$.
- $H(2x) = H((x-y)+(x+y)) = H(x-y) + H(x+y) =$
  $y - x + x + y = 2y$.