Lecture 5: Introduction

Amos Ron

University of Wisconsin - Madison

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Outline

1. The Singular Value Decomposition
   - Defining the SVD, 2×2 case
   - Derivation

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SVD

A is any matrix 2×2.

Definition

The SVD of A is a decomposition

$$A = U\Sigma V',$$

where

- U and V are orthogonal 2×2.
- Σ is diagonal with non-negative diagonal entries.
- The columns of U are left singular vectors of A.
- The columns of V are right singular vectors of A.
- The diagonal entries of Σ are singular values of A.

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Then, if $A = U\Sigma V'$, then also $A = (UR)\Sigma(RV')$.

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Then, if $A = U\Sigma V'$, then also $A = (UR)\Sigma(RV')$.
Since UR and RV' are still orthogonal, this is another SVD of A.
Up to this triviality (i.e., the mulitiplication of the singular vectors by -1), the SVD is unique whenever $\Sigma(1,1) > \Sigma(2,2)$.

'Reverse engineering' on the SVD

Assume that $A = U\Sigma V'$ as before.

First fundamental computation

$$AA' = (U\Sigma V')(U\Sigma V')' = (U\Sigma V')(V\Sigma U') = U\Sigma(V'V)\Sigma U' = U\Sigma^2 U',$$

and

$$A'A = V\Sigma^2 V'.$$

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Conclusion

- The left singular vectors are the eigenvectors of AA'.
- The right singular vectors are the eigenvectors of $A'A$.
- The squares of the singular values are the eigenvalues of $A'A$ and of AA'.

‘Reverse engineering’ on the SVD

Assume that $A = U\Sigma V'$ as before.

Second fundamental computation

$$A' = (U\Sigma V')' = V\Sigma U'.$$

So,

- The left singular vectors of A are the right singular vectors of A'.
- The left singular vectors of A' are the right singular vectors of A.
- The singular values of A and A' are the same.

'Reverse engineering' on the SVD

Assume that $A = U\Sigma V'$ as before.

Third fundamental computation

$$AV = U\Sigma(V'V) = U\Sigma,$$

i.e., $Av_1 = \Sigma(1,1)u_1$, $Av_2 = \Sigma(2,2)u_2$.
Similarly,

$$A'U = V\Sigma.$$

What are we really looking for?

The SVD reads as

$$AV = U\Sigma.$$

It therefore find two vector v_1, v_2 (The columns of V) such that

- v_1, v_2 is an orthonormal basis for $\mathbb{R}^2$.
- $(Av_1, Av_2) = 0$.

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Indeed, if you find such v_1, v_2, you may define

$$u_i := \frac{Av_i}{||Av_i||_2}.$$

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We get then

$$AV = U\Sigma,$$

with V, U as above (orthonormal then), and Σ diagonal, with diagonal entries $||Av_i||_2$.

Derviation

We take v_1, v_2 the orthonormal eigenbasis of $A'A$.
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So, (λ_i, v_i), is an eigenpair of $A'A$, and we want Av_i to be an eigenpair of AA':

$$(AA')(Av_i) = A(A'A)v_i = A(\lambda_i v_i) = \lambda_i A v_i.$$

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The above works if A is non-sigular, and $\lambda_1 \neq \lambda_2$

Demo #3