ISyE/Math/CS 728
Integer Optimization

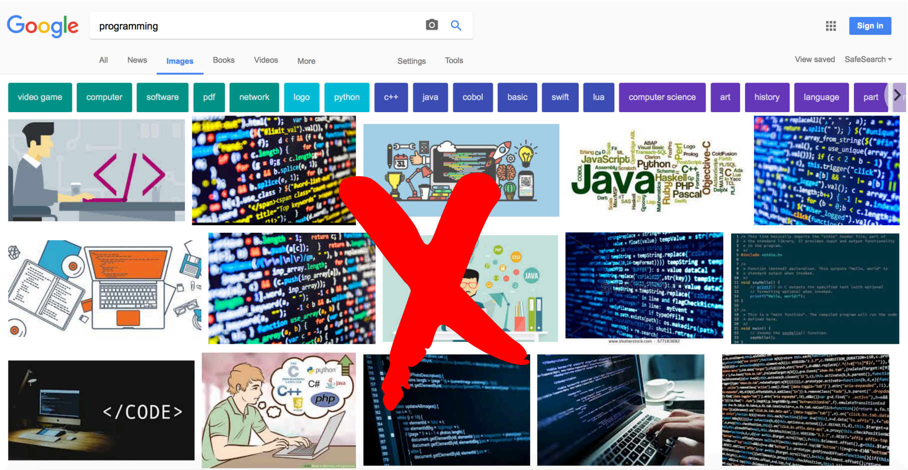
Course Introduction

Prof. Jim Luedtke
University of Wisconsin-Madison

Significant use from the book *Integer Programming* by M. Conforti, G. Cornuéjols, and G. Zambelli

Integer Programming ~~Programming~~ Optimization

Programming



It is NOT about programming languages...

Optimization

It is about making OPTIMAL decisions.

Lately "Programming" is getting replaced by "Optimization".

What is Integer Optimization about?

- Integer Optimization is about decision making with linear functions and with integers.
- More precisely:
  - All the functions that we consider (objective and constraints) are linear functions.
  - Some of the decisions can take only certain integer values.

An Integer Optimization problem (a.k.a. integer program)

$$
\begin{array}{rl}
\text{minimize} & 2x_1 - x_2 + 4x_3 \\
\text{subject to} & x_1 + x_2 + x_4 \leq 2 \\
 & 3x_2 - x_3 = 5 \\
 & x_3 + x_4 \geq 3 \\
 & x_1 \geq 0 \\
 & x_3 \leq 0 \\
 & x_1, x_2 \quad \text{integral (i.e., } x_1, x_2 \in \mathbb{Z}\text{).}
\end{array}
$$

Why is optimization with integers important?

First answer

Many decisions involve deciding a quantity that is indivisible:

- Number of airplanes to produce.
- Number of floors in a building.
- What about number of cents to invest in a stock?

Sometimes a continuous approximation is good enough.

Why is optimization with integers important?

Slam dunk answer

We can use $0, 1$ (binary) variables for a variety of purposes:

- Modeling yes/no decisions.
- Enforcing logical conditions.
- Modeling fixed costs.
- Modeling piecewise linear functions.

Usually a continuous approximation is not good enough.

The usefulness of binary variables

Suppose for a set of elements to choose from, $i \in S$, we have the binary variables:

$$x_i = \begin{cases} 1 & \text{if element } i \text{ chosen} \\ 0 & \text{otherwise.} \end{cases}$$

How can we model these restrictions with linear inequalities?

1. If we do choose i, we must choose j.
2. We must either choose both i and j, or neither.
3. We can choose at most t items from a set S of items.
4. If we choose any one item in a set S, we must choose k ($k \notin S$).
5. If we choose all items in a set S, we must choose k ($k \notin S$).

Example: The assignment problem

- There are n jobs to be performed by n workers.
- We know the cost c_{ij} of assigning job i to worker j for each pair i, j.
- What is the cheapest way of assigning one job to each of the n workers?
- This problem can be easily written as an Integer Optimization problem.

Assignment problem: Formulation

Decision variables:

- $x_{ij} = 1$ if job i is assigned to worker j, $= 0$ otherwise

Objective: Minimize total cost of assignments

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

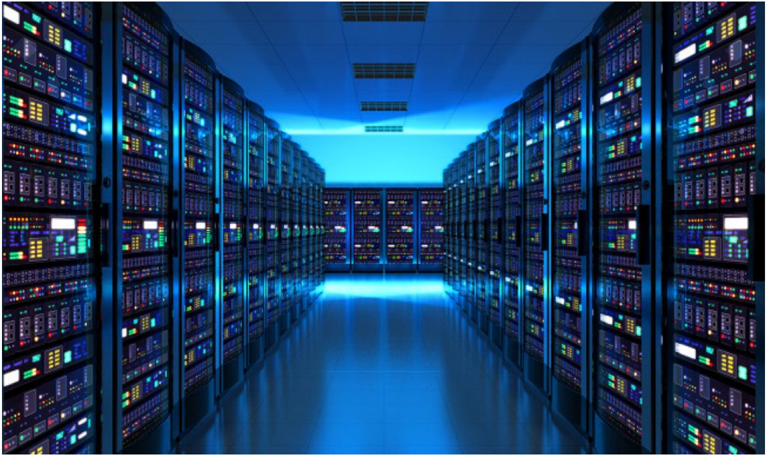
Constraints: Each job assigned to one worker

$$\sum_{j=1}^{n} x_{ij} = 1, \quad \forall i = 1, \ldots, n$$

Constraints: Each workder is assigned one job

$$\sum_{i=1}^{n} x_{ij} = 1, \quad \forall j = 1, \ldots, n$$

Can growing computing power help?



Can growing computing power help?

- Consider the assignment problem.
- Simple: check each possible assignment and select the best.
- But how long will it take? Let's use an IBM BlueGene/L:

#*people*	Solutions to check	Time
3	6	0
10	3.6×10^6	0
15	1.3×10^{12}	1 sec
30	2.7×10^{32}	14 billion years[a]
70	1.2×10^{100} [b]	

[a] That's four times the age of the universe!

[b] That's more than the number of particles in the observable universe!

So how hard is Integer Optimization?

- We will see that this particular integer program can be solved very efficiently (by solving a linear program).
- Could we be more clever and solve all integer programs efficiently? (After all, we can solve linear programs...)
- Solving general integer programs can be much more difficult than solving linear programs.
  - There is a whole theory (complexity) that supports this claim.
  - You will learn a tiny bit of about this.

Integer optimization is everywhere!

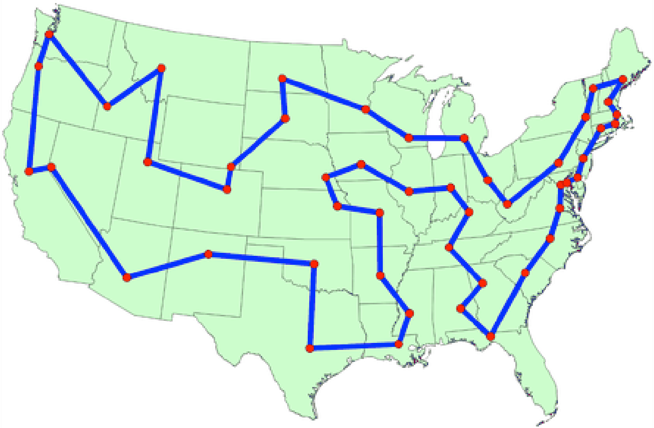
The knapsack problem

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} c_i x_i \\ \text{s. t.} \quad & \sum_{i=1}^{n} a_i x_i \leq b \\ & x \in \{0,1\}^n. \end{aligned}$$

- You are choosing what to bring in your backpack.
- You can carry at most b pounds.
- You have n possible items.
- Item i would give benefit of value c_i and weight a_i.
- What items to pack?

The traveling salesman problem (TSP)

- Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city and returns to the origin city?



More examples

- Design a supply chain network: decide where to open facilities and which customers they should serve.
- Portfolio optimization: limit number of stocks held.
- Stochastic optimization with risk: ensure percent of outcomes which are "bad" is small.
- LTL vehicle routing: which loads should be combined on trucks and in what sequence should the loads be delivered?
- Sports scheduling: assign games to days.
- Chemical processing network design: which process types should be used and how should they be connected?
- Machine learning: sparse linear regression, Bayesian network structure learning, sparse PCA

Couse details

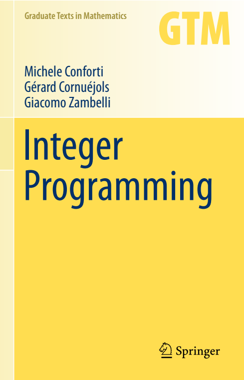
Purpose of this course

- Present the mathematical foundations of Integer Optimization.
- Understand cases that can be solved efficiently.
- Emphasis on the techniques that are most successful in current software implementations: convexification and enumeration.

What this course is NOT about:

- We won't spend much time on how to model problems as Integer Optimization problems. (See ISyE 524)
  - We study what makes a model "good".
  - We will see some advanced modeling techniques.
- We won't cover combinatorial algorithms (see ISyE 425) or specialized algorithms for "easy" integer optimization problems (see CS 577).

Recommended textbook



Integer Programming,
by Michele Conforti, Gérard Cornuéjols, and Giacomo Zambelli (2014).

- Winner of the 2015 INFORMS Lanchester Prize for best contribution to Operations Research in the past 3 years.
- I will try to use consistent notation, and provide references to related sections of this book
- Some parts of course will not closely follow the book

What we will cover (subject to change)

- Algorithmic frameworks: Branch & bound and cutting planes
- Modeling techniques, and comparing models
- Integer programing software
- Perfect formulations
- Decomposition methods
- Polyhedral theory and strong valid inequalities
- General purpose valid inequalities

Warning on course difficulty

This is a Ph.D.-level course $\Rightarrow$ Much of the material in this course is deep.

- Requires serious math.
- Proofs will be done by me and by you.

Warning on course difficulty

- “Hard but rewarding”

Prerequisites

Essential background that will be assumed:

- Working knowledge of linear algebra.
  See Section 1.5 of the 525 textbook: "Introduction to Linear Optimization" by D. Bertsimas, and J.N. Tsitsiklis. (6 pages)
  - E.g., set theoretic notation, vectors and matrices, matrix inversion, subspaces and bases, affine subspaces, linear independence, and the rank of a matrix.
- Knowledge of Linear Optimization.
  See 525 textbook.
  - E.g., geometry of linear programming, the simplex method, duality theory, network flow problems, complexity of linear programming, and the ellipsoid method.
- Basic proof techniques.
  See Introduction to mathematical arguments in Canvas.
- Mathematical notation.
  See Math symbols cheat sheet in Canvas.

Other Optimization courses

- ISyE/CS 719: Stochastic Programming.
- ISyE/CS 723: Dynamic Programming and Associated Topics.
- ISyE/Math/CS/Stat 726: Nonlinear Optimization I.
- ISyE/CS 727: Convex Analysis.
- ISyE/CS 730: Nonlinear Optimization II.
- ISyE/Math/CS 728: Integer Optimization.
- CS 733: Computational Methods for Large Sparse Systems.
- ISyE 823: Special Topics in Operations Research.

If interested in research in Optimization, Madison is a great place to be :)

Technical details

Class Overview

- Office Hours:
  - Tuesday and Thursday: 10:45–11:45am, right after lecture in same zoom room
- Course home page in Canvas:
  - https://canvas.wisc.edu/courses/230933
  - Syllabus, lecture slides, introductory material, homework assignments and solutions, grades.
  - Use Q&A forum in Piazza module to post questions about the course.
- See syllabus in Canvas for more info

Expectations

I am expected to...

- Teach lectures.
- Be at my office hours.
- Guide your learning process (homework).
- Give you feedback on how you are doing in a timely fashion.

You are expected to...

- Learn.
- Attend lectures and participate. (Ask questions!)
- Do the homework.
- Know and follow academic conduct guidelines.
- Follow online lecture etiquette (e.g., stay muted unless called on, feel free to use chat)

Homework

- There will be approximately one assignment every 2 weeks.
- Strongly encouraged to work in groups of 2 people.
- Submit pdf in Canvas (one submission per group).
- Assignments may not be completely graded.
- Complete solutions will be published in Canvas.
- TA Rui Chen (rchen234@wisc.edu) will deal with assignments and related questions.
- TA office hours:
  - Monday 1-2pm, Wednesday 2-3pm, same zoom link.

$*$ Notify TA this week if you have a conflict with both his office hours.

Grading

Components of grade:

- Homework (25%)
- Midterm (35%) : **March 12, 1pm – March 16, 1pm.**
- Option: Final exam or course project (35%) : **May 2, 1pm – May 4 1pm.**
- Class participation (5%)

Exams will be take-home and may contain a computational implementation component

Optional Project

Students have the option to do a final project in place of final exam

- **Not** allowed to do both!

Projects can/should be done with a partner

- But different partner than homework!

If you wish to do a project, must submit proposed project topic by April 9 at the latest.

- Suggested to submit earlier for approval so you can start earlier

About me...

- B.S., ISyE, UW-Madison, 2001.
- M.S., OR, GA Tech, 2003.
- Ph.D., ISyE, GA Tech, 2007.
- Fall 2007-Summer 2008: IBM Research
- Research Areas: Integer programming (linear and nonlinear), stochastic programming, applications
- Married. Three children, Rowan (13), Cameron (11), Remy (7). One dog, Bowie.
- Interesting: Did cross-country bicycle tour after college
- Hobbies: Mountain biking, board games, kids

About you...

Please complete the Student Survey quiz in Canvas by Jan 31!

Questions about the course?