Lecture 4: Introduction

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Outline

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1. Matrix Norms
   - Characterizing the ∞-norm
   - Characterizing the 2-norm

2. Positive definite matrices
   - Definition and example

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∞-norm

Theorem: Computing the ∞-norm

1. For an A $m \times n$,
$$||A||_1 = ||A'||_\infty.$$
2. Let $b'_1, \ldots, b'_m$ be the rows of A. Then
$$||A||_\infty = \max_{1 \le i \le m} ||b_i||_1.$$

∞-norm

Theorem: Computing the ∞-norm

1. For an A $m \times n$,
$$||A||_1 = ||A'||_\infty.$$
2. Let $b_1', \ldots, b_m'$ be the rows of A. Then
$$||A||_\infty = \max_{1 \le i \le m} ||b_i||_1.$$

Comment: The equivalence of the two conditions above follows directly from the characterization of the 1-norm.
Comment: Assertion (2) above can be proved directly, using a similar approach (but with different details) to the prooof of the 1-norm case.

∞-norm

We show how to prove $||A||_1 = ||A'||_\infty$ directly from basic Linear Algebra principles.
Step I: Show that, for any $v \in R^m$,

$$||v||_1 = \max\{(v, w) : ||w||_\infty = 1\}, \quad \text{and}$$

$$||v||_\infty = \max\{(v, w) : ||w||_1 = 1\}.$$

Step II: Since $||A||_1 = \max\{||Av||_1 : ||v||_1 = 1\}$, it follows that

$$||A||_1 = \max\{(Av, w) : ||v||_1 = 1,\ ||w||_\infty = 1\}.$$

Step III: Since $||A'||_\infty = \max\{||A'w||_\infty : ||w||_\infty = 1\}$, it follows that

$$||A'||_\infty = \max\{(A'w, v) : ||v||_1 = 1,\ ||w||_\infty = 1\}.$$

The 2-norm of a matrix

Some basics:

$$||v||_2^2 = (v, v).$$

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Whatever A is, $A'A$ is symmetric, and its eigenvalues are non-negative.

$$(BC)' = C'B' \implies (A'A)' = A'A'' = A'A.$$

The 2-norm of a matrix

Some basics:

$$||v||_2^2 = (v, v).$$

Whatever A is, $A'A$ is symmetric, and its eigenvalues are non-negative.

$$(A'A)v = \lambda v \implies \lambda ||v||_2^2 = (\lambda v, v) = (A'Av, v) = (Av, Av) = ||Av||_2^2,$$

$$\implies \lambda = \frac{||Av||_2^2}{||v||_2^2} \geq 0.$$

Also:

$$\lambda \leq ||A||_2^2.$$

The 2-norm of a matrix

Whatever A is, $A'A$ is symmetric, and its eigenvalues are non-negative.

Definition

- A left singular vector of A is an eigenvector of $A'A$.
- An $s \geq 0$ is a singular value of A is $s^2 \in \sigma(A'A)$.

Notation (spectral radius): A square:

$$\rho(A) := \max\{|\lambda| : \lambda \in \sigma(A)\}.$$

So:

$$||A||_2 \geq \sqrt{\rho(A'A)}.$$

Characterizing the 2-norm

Theorem: Chracterizing the 2-norm

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Proof: We already saw that $||A||_2 \geq \sqrt{(}\rho(A'A))$.
Now, Let $v \in R^m$, such that $||v||_2 = 1$, and $||A||_2 = ||Av||_2$. Let

$$A'A = QDQ'$$

be the Schur decomposition of $A'A$. Then

$$||A||_2^2 = ||Av||_2^2 = (Av, Av) = (A'Av, v) =$$
$$(QDQ'v, v) = (DQ'v, Q'v).$$

Denote $w := Q'v$. Since Q' is orthogonal, $||w||_2 = ||v||_2 = 1$.

Characterizing the 2-norm

$$||A||_2^2 = ||Av||_2^2 = (Av, Av) = (A'Av, v) =$$
$$(QDQ'v, v) = (DQ'v, Q'v).$$

Denote $w := Q'v$. Since Q' is orthogonal, $||w||_2 = ||v||_2 = 1$.
So,

$$||A||_2^2 = (Dw, w) = \sum_{i=1}^{m} D(i,i)w(i)^2 \leq \sum_{i=1}^{m} \rho(A'A)w(i)^2 = \rho(A'A) \sum_{i=1}^{m} w(i)^2 =$$

So, $||A||_2 \leq \sqrt{(}\rho(A'A))$.

Demo #2

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Definition of Positive Definiteness