ISyE/CS 728 – Integer Optimization
Spring 2021
Assignment #3

Due Date: March 11, 1pm.

The assignment should be submitted electronically in pdf format (except for code files) in Canvas. Late submission policy: 20% of total points will be deducted per hour.

Students are strongly encouraged to work in groups of two on homework assignments. Only one set of solutions should be submitted for both group members. In order to submit the assignment for your group please follow these steps in Canvas. Step 1: Click on the "People" tab, then on "Assignments Groups", and join one of the available groups; Step 2: When also your partner has joined the same group, one of the two can submit the assignment by clicking on the "Assignments" tab, then on the assignment to be submitted, and finally on "Submit assignment". The submission will count for everyone in your group. If you prefer to be in no group, then just skip Step 1.

Groups must work independently of each other, may not share answers with each other, and solutions must not be copied from the internet or other sources. If improper collaboration is detected, *all groups* involved will automatically receive a 0. Students must properly give credit to any outside resources they use (such as books, papers, etc.). In doing these exercises, you must justify all of your answers and cite every result that you use. You are not allowed to share any content of this assignment.

1 Totally Unimodular

1.1 Problem (2 points)

Consider a 0-1 matrix A with the following property: For all rows $i = 1, \ldots, m$, if $a_{ij} = a_{ik} = 1$ for $k > j + 1$, then $a_{it} = 1$ for all $j < t < k$. Prove that any matrix A with this property is TU.

2 An Integral Polyhedron

Consider the polyhedron

$$P = \{x \in \mathbf{R}^{n+1}_+ : \ x_i \leq x_{n+1}, \ \forall i = 1, ..., n \text{ and } x_{n+1} \leq 1\}.$$

2.1 Problem (2 points)

Show that P is an integral polyhedron. (Hint: Use totally unimodularity of the constraint matrix.)

3 Transportation Problem Feasibility

Consider a transportation problem with a set of suppliers, U, with capacity $B_i \geq 0, i \in U$, a set of customers, V, with demand $D_j \geq 0, j \in V$, capacities u_{ij}, for $i \in U, j \in V$, representing the maximum amount of product that can be sent from supplier $i \in U$ to customer $j \in V$. We are interested in determining whether or not a feasible flow from suppliers to customers exists, i.e., if there exists x_{ij} for $i \in U, j \in V$, such that:

$$\sum_{j \in V} x_{ij} \le B_i, \quad i \in U \tag{1}$$

$$\sum_{i \in U} x_{ij} = D_j, \quad j \in V \tag{2}$$

$$0 \le x_{ij} \le u_{ij}, \quad i \in U, j \in V \tag{3}$$

3.1 Problem (1 points)

Consider a network with nodes s, t, U, V, with arcs (s, i) for $i \in U$, having capacity B_i, arcs (j, t) for $j \in V$ having capacity D_j, and (i, j) for $i \in U, j \in V$, having capacity u_{ij}. Show that a feasible flow to the transportation problem exists if and only if the maximum flow in this network is equal to $\sum_{j \in V} D_j$.

3.2 Problem (2 points)

Using the result from part (a) and the max-flow min-cut theorem, show that a feasible flow to the transportation problem exists if and only if the following inequalities are satisfied:

$$\sum_{i \in U \setminus S} B_i + \sum_{i \in S} \sum_{j \in T} u_{ij} \ge \sum_{j \in T} D_j, \quad \forall S \subseteq U, T \subseteq V. \tag{4}$$

4 Bin packing as matching

4.1 Problem (2 points)

Consider the bin packing problem having n items with weights $a_1, \dots, a_n \in (0, 1)$ and bin capacity of 1. The bin packing problem is to find the minimum number of bins required to pack these items. Show how this problem can be formulated and solved as a maximum cardinality matching problem if $a_i > 1/3$ for all $i = 1, \dots, n$. Your solution should specify the data for the matching problem you would need solved (i.e., the graph) and how the resulting solution would be mapped to a solution of the bin packing problem. You should also argue why your formulation is correct.

5 Convex hull of a Minkowski Sum

Recall that the Minkowski sum, $S = A + B$ of two sets A and B is defined as:

$$S = \{z = x + y : x \in A, y \in B\}.$$

Recall that a set A is convex if for all $x, y \in A$ and $\lambda \in (0, 1)$ the point $\lambda x + (1 - \lambda) y \in A$.

5.1 Problem (1 points)

Show that if A and B are convex sets, then their Minkowski sum $A + B$ is convex.

5.2 Problem (2 points)

For arbitrary (not necessarily convex) sets A and B show that

$$\mathrm{conv}(A + B) = \mathrm{conv}(A) + \mathrm{conv}(B).$$

Hint: the inclusion $\subseteq$ is a consequence of part 5.1 and the fact (from definition of a convex hull) that if C is any convex set containing a set S, then $\mathrm{conv}(S) \subseteq C$.