ISyE/CS/Math 728: Integer Optimization
Polyhedral Theory

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Overview

Some fundamental definitions/theory useful for IP

- Representations of polyhedra (Minkowski-Weil)
- Fundamental theorem of integer programming
- Complexity of separation vs. optimization over a mixed-integer set

Minkowski-Weil for Cones

Polyhedral cones

- A set $C \subseteq \mathbb{R}^n$ is a polyhedral cone if C is the intersection of a finite number of half-spaces containing the origin on their boundaries:

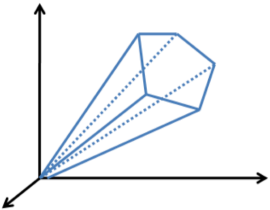
$$C := \{x \in \mathbb{R}^n : Ax \leq 0\}$$

for some $m \times n$ matrix A.

- Given a vector $r \in C \setminus \{0\}$, the half line

$$\text{cone}(r) = \{\lambda r : \lambda \geq 0\}$$

is contained in C, and it is called a ray of C.



Basic Cone Properties

If C is a polyhedral cone, then:

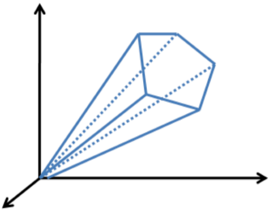
- If $r \in C$, then $\lambda r \in C$ for all $\lambda \geq 0$
- If $r_1, r_2 \in C$, then $r_1 + r_2 \in C$

Finitely generated cones

- A set $C \subseteq \mathbb{R}^n$ is a finitely generated cone if there exist vectors $r^1, \dots, r^k \in \mathbb{R}^n$ such that C is the set of all conic combinations of vectors $r^1, \dots, r^k$:

$$C = \{x \in \mathbb{R}^n : x = \sum_{j=1}^{k} \lambda_j r^j,\ \lambda_j \geq 0, \forall j = 1, \dots, k\}.$$

- We write $C = \text{cone}(r^1, \dots, r^k)$, and we say that $r^1, \dots, r^k$ are the generators of C, and that C is the cone generated by $r^1, \dots, r^k$.



Minkowski-Weyl Theorem for cones

Theorem 3.11 (Minkowski-Weyl Theorem for Cones).
A subset of $\mathbb{R}^n$ is a finitely generated cone if and only if it is a polyhedral cone.

We'll skip the proof (see CCZ book for details)

Minkowski-Weyl Theorem for cones

- Slightly stronger result

Proposition 3.12.
Given a rational matrix $A \in \mathbb{R}^{m \times n}$, there exist rational vectors $r^1, \ldots, r^k \in \mathbb{R}^n$ such that

$$\{x : Ax \leq 0\} = \text{cone}(r^1, \ldots, r^k).$$

Conversely, given rational vectors $r^1, \ldots, r^k \in \mathbb{R}^n$, there exists a rational matrix $A \in \mathbb{R}^{m \times n}$ such that

$$\text{cone}(r^1, \ldots, r^k) = \{x : Ax \leq 0\}.$$

- By scaling property of rays in a cone, we can further assume $r^1, \ldots, r^k$ are integral

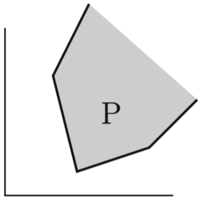
Minkowski-Weil for Polyhedra

Polyhedra

- A set $P \subseteq \mathbb{R}^n$ is a <u>polyhedron</u> if P is the intersection of a finite number of half-spaces. That is,

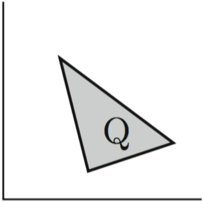
$$P := \{x \in \mathbb{R}^n : Ax \leq b\}$$

for some $m \times n$ matrix A and m vector b.



Polytopes

- A subset Q of $\mathbb{R}^n$ is a polytope if Q is the convex hull of a finite set of vectors in $\mathbb{R}^n$.



Projection of Polyhedra

Definition: Projection

If $Q \subseteq \mathbb{R}^n \times \mathbb{R}^p$, the projection of Q onto $\mathbb{R}^n$ is the set:

$$\text{proj}_x(Q) = \{x \in \mathbb{R}^n : \exists w \in \mathbb{R}^p \text{ s.t. } (x, w) \in Q\}.$$

Fact

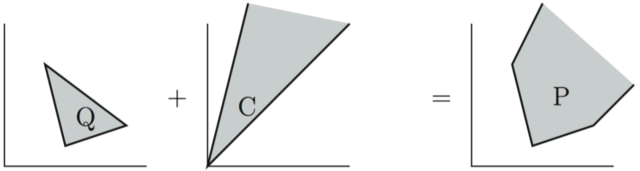
A projection of a polyhedron is a polyhedron.

- Proof: Fourier Motzkin elimination

Minkowski-Weyl Theorem for polyhedra

Theorem 3.13 (Minkowski-Weyl Theorem).
A subset P of $\mathbb{R}^n$ is a polyhedron if and only if $P = Q + C$ for some polytope $Q \subset \mathbb{R}^n$ and finitely generated cone $C \subseteq \mathbb{R}^n$.

- Provides an alternative representation of a polyhedron.
- We prove only the forward direction. Reverse follows from Farkas lemma (LP duality)
- Both directions can be derived from the Minkowski-Weyl Theorem for cones (see CCZ book)



Corollary

Corollary 3.14 (Minkowski-Weyl Theorem for Polytopes).
A set $Q \subseteq \mathbb{R}^n$ is a polytope if and only if Q is a bounded polyhedron.

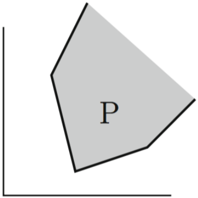
Lineality Space and Recession Cone

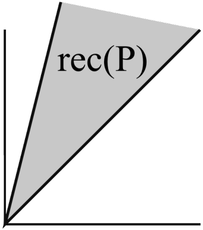
Recession Cone

- Given a nonempty polyhedron P, the recession cone of P is the set

$$\text{rec}(P) := \{r \in \mathbb{R}^n : x + \lambda r \in P,\ \forall x \in P,\ \forall \lambda \in \mathbb{R}_+\}.$$

- We will refer to the rays of $\text{rec}(P)$ as the rays of the polyhedron P.



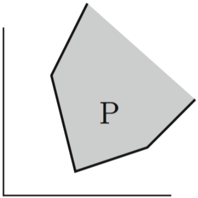


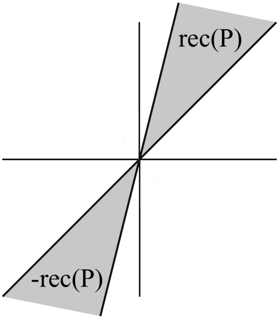
Lineality Space

- The lineality space of P is the set

$$\operatorname{lin}(P) := \{r \in \mathbb{R}^n : x + \lambda r \in P,\ \forall x \in P,\ \forall \lambda \in \mathbb{R}\}.$$

- Note that $\operatorname{lin}(P) = \operatorname{rec}(P) \cap -\operatorname{rec}(P)$.





- When $\operatorname{lin}(P) = \{0\}$, we say that P is pointed.
- Geometrically, a nonempty polyhedron is pointed when it does not contain any line.
- If $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, then $\operatorname{lin}(P) = \{0\} \Leftrightarrow \operatorname{rank}(A) = n$

Lineality Space

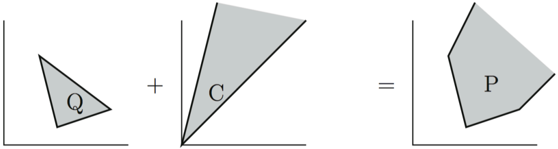
We can now understand better Minkowski-Weyl:

Proposition 3.15.
Let

$$P := \{x \in \mathbb{R}^n : Ax \leq b\} = \mathrm{conv}(v^1, \dots, v^p) + \mathrm{cone}(r^1, \dots, r^q)$$

be a nonempty polyhedron. Then

$$\mathrm{rec}(P) = \{r \in \mathbb{R}^n : Ar \leq 0\} = \mathrm{cone}(r^1, \dots, r^q),$$
$$\mathrm{lin}(P) = \{r \in \mathbb{R}^n : Ar = 0\}.$$



Vertices

Definition: Vertex (aka Extreme point)
$x \in P$ is an vertex of P if there does not exist $x^1, x^2 \in P$ and $\lambda \in (0,1)$ such that $x^1 \neq x^2$ and $x = \lambda x^1 + (1-\lambda)x^2$

Theorem
Let $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ be a pointed polyhedron, and let $\bar{x} \in P$. The following statements are equivalent.

(i) $\bar{x}$ is a vertex.

(ii) $\bar{x}$ satisfies at equality n linearly independent inequalities of $Ax \leq b$.

Extreme Rays

Recall: $r \in \mathbb{R}^n$ is a ray of P if $x + \lambda r \in P$ for all $x \in P$ and $\lambda \geq 0$

Definition: Extreme Ray

$r \in \mathbb{R}^n$ is an extreme ray of P if r is a ray of P, and there does not exist rays r^1 and r^2 of P and $\lambda \in (0,1)$ such that $r^1 \neq \alpha r^2$ for any $\alpha \geq 0$ and $r = \lambda r^1 + (1-\lambda) r^2$.

Extreme Rays

Theorem

Let $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ be a pointed polyhedron, and let $\bar{r}$ be a ray of P. The following are equivalent.

(i) $\bar{r}$ is an extreme ray of P.

(ii) $\bar{r}$ satisfies at equality $n-1$ linearly independent inequalities of $Ax \leq 0$.

Minkowski-Weil (again!)

Minkowski-Weil Theorem for Pointed Polyhedra

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a non-empty *pointed* polyhedron. Then,

$$P = \left\{ \begin{array}{ll} x \in \mathbb{R}^n : & x = \sum_{i=1}^{t} \lambda_i x^i + \sum_{j=1}^{k} \mu_j r^j \\ & \text{for some } \lambda \in \mathbb{R}^t_+, \mu \in \mathbb{R}^k_+ \text{ with} \\ & \sum_{i=1}^{t} \lambda_i = 1 \end{array} \right\}.$$

where $x^1, \ldots, x^t$ are the (finitely many) extreme points of P and $r^1, \ldots, r^k$ are the (finitely many) extreme rays of P.

See handwritten example

The Fundamental Theorem of IP

- Consider a mixed integer linear set

$$S := \{(x, y) \in \mathbb{Z}^n \times \mathbb{R}^p : Ax + Gy \leq b\},$$

  where matrices A, G and vector b have rational entries.
- We will see that S admits a perfect formulation that is a rational polyhedron.

The Fundamental Theorem of IP

Meyer's theorem (Theorem 4.30).
Given rational matrices A, G and a rational vector b, let

$$P := \{(x, y) : Ax + Gy \leq b\},$$
$$S := \{(x, y) \in P : x \text{ integral}\}.$$

1. There exist rational matrices A', G' and a rational vector b' such that

$$\mathrm{conv}(S) = \{(x, y) : A'x + G'y \leq b'\}.$$

2. If S is nonempty, the recession cones of $\mathrm{conv}(S)$ and P coincide.

The Fundamental Theorem of IP

First, we prove Meyer's theorem when P is bounded:

Lemma.
Given rational matrices A, G and a rational vector b, let

$$P := \{(x, y) : Ax + Gy \leq b\},$$
$$S := \{(x, y) \in P : x \text{ integral}\}.$$

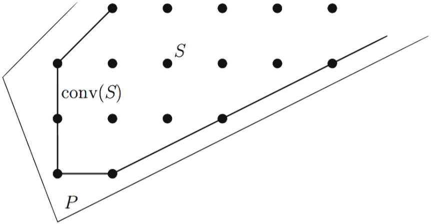
If P is bounded, then there exist rational matrices A', G' and a rational vector b' such that

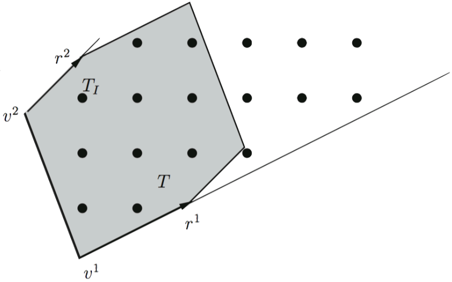
$$\text{conv}(S) = \{(x, y) : A'x + G'y \leq b'\}.$$

Let's prove it!

The Fundamental Theorem of IP

- The unbounded case is more complex.
- We need one more idea:





Let's prove Meyer's theorem!

The Fundamental Theorem of IP

Remark

In Meyer's theorem, if matrices A, G are not rational, then $\text{conv}(S)$ may not be a polyhedron.

Complexity of separation

Review of separation

Separation problem

Let $P \subseteq \mathbb{R}^n$ be a polyhedron. The Separation Problem for P is: Given $x^* \in \mathbb{R}^n$, is $x^* \in P$? If not, find an inequality $\pi x \leq \pi_0$ satisfied by all points in P, but violated by x^*.

- For MIP, $S = \{(x, y) \in \mathbb{Z}^n_+ \times \mathbb{R}^p_+ : Ax + Gy \leq b\}$
- $P = \text{conv}(S)$ is a polyhedron
- An inequality is valid for S if and only if it is valid for $\text{conv}(S)$
  - Exercise: Prove this!
- So, the separation problem over $\text{conv}(S)$ is exactly the problem of finding valid inequalities for the set S

Complexity of separation

Efficient separation implies efficient optimization
If the Separation Problem for a polyhedron P can be solved in polynomial time, then the linear program $\max\{cx : x \in P\}$ can be solved in polynomial time using the ellipsoid method.

- Finding valid inequalities for a MIP is as hard as solving the MIP!
- What about the reverse? Could solving a MIP be easier than separating?

Complexity of separation (2)

- Suppose we have an efficient algorithm for solving MIP: $\max\{cx : x \in S\} = \max\{cx : x \in \text{conv}(S)\}$
- Now consider the separation problem: Given x^*, determine if $x^* \in \text{conv}(S)$ and if not, provide a valid inequality that cuts it off
- Let $\Pi \subseteq \mathbb{R}^{n+1}$ be the set of *all* valid inequalities for $\text{conv}(S)$
  - $(\pi, \pi_0) \in \Pi$ if and only if $\pi x \leq \pi_0$ is valid for $\text{conv}(S)$
  - The set Π is called the polar of the polyhedron $\text{conv}(S)$

Proposition

The polar of any polyhedron P is a polyhedral cone.

- Proof provides an explicit representation of the polar

Separation as optimization

Separation problem: $\max\{\pi x^* - \pi_0 : (\pi, \pi_0) \in \Pi\}$

- If no valid inequality violated by x^* exists, optimal value $= 0$
- Otherwise, problem is unbounded (because Π is a cone)

Inequalities are invariant to positive scaling: for any $\lambda > 0$

$$\pi x \leq \pi_0 \Leftrightarrow (\lambda\pi)x \leq (\lambda\pi_0)$$

Thus by (e.g.,) taking $\lambda = 1/\|(\pi, \pi_0)\|_1$, we can *normalize* Π as

$$\bar{\Pi} = \{(\pi, \pi_0) \in \Pi : \|(\pi, \pi_0)\|_1 = 1\}$$

- For any $(\pi, \pi_0) \in \Pi$ there exists $(\bar{\pi}, \bar{\pi}_0)$ in $\bar{\Pi}$ that defines an equivalent inequality
- Any norm can be used for normalization: $\| \cdot \|_1$ is convenient as $\bar{\Pi}$ is a polyhedron

Complexity of separation

Separation problem: $\max\{\pi x^* - \pi_0 : (\pi, \pi_0) \in \bar{\Pi}\}$

- If no valid inequality violated by x^* exists, optimal value $= 0$
- Otherwise, violated inequality is found

Since $\bar{\Pi}$ is a polyhedron, we can solve this by the ellipsoid algorithm if we can solve the separation problem over Π.

- Given $(\hat{\pi}, \hat{\pi}_0)$ is it in Π? If not, provide an inequality that cuts it off from Π
- Solve: $\hat{z} = \max\{\hat{\pi}x : x \in S\}$.
- If $\hat{z} \leq \hat{\pi}_0$, then $\hat{\pi}x \leq \hat{\pi}_0$ for all $x \in \text{conv}(S)$, hence in $(\hat{\pi}, \hat{\pi}_0) \in \Pi$
- Else if $\hat{z} < \infty$, find a solution $\hat{x} \in S$ with $\hat{\pi}\hat{x} > \hat{\pi}_0$, inequality $\pi\hat{x} \leq \pi_0$ is valid for Π and separates $(\hat{\pi}, \hat{\pi}_0)$ from Π
- Else if $\hat{z} = \infty$, find a ray $\hat{r}$ of $\text{conv}(S)$ with $\hat{\pi}\hat{r} > 0$, inequality $\pi\hat{r} \leq 0$ separates $(\hat{\pi}, \hat{\pi}_0)$ from Π

So, if we can optimize efficiently, we can always separate efficiently!