# Lecture 1: Introduction

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# Outline

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# Column maps - synthesis

- We are given a matrix $A$ $m \times n$. All matrices in our course are real valued.
- A matrix is a linear map: $A : R^n \to R^m, v \mapsto Av$.
- There are two ways to view the linear map $A$. Those are are mathematically equivalent, but conceptually very different.
- The synthesis/column map view of $A$ partitions it into its columns: $\{a_1, \ldots, a_n\} \subset R^m$. Then:

$$Av = \sum_{i=1}^{n} v(i) a_i.$$

# Column maps - analysis

Reminder: transposition: $A' : R^m \to R^n$, with

$$A'(i,j) = A(j,i).$$

Reminder: inner product. For $v, w \in R^n$,

$$(v,w) := v'w = \sum_{i=1}^{n} v(i)w(i).$$

Reminder: The most important property of $A'$: if $v \in R^n,\ w \in R^m$, then

$$(Av, w) = (v, A'w).$$

# Column maps - analysis

- The alternative is to slice the matrix into its rows $\{b'_1, \ldots, b'_m\} \subset R^n$.
- Then $(Av)(i) = (b_i, v)$.

# Rank

## Definition: Rank

Let $A$ is $m \times n$. Known: $\mathrm{ran}A$ is a subspace of $R^m$.

$$\mathrm{rank}A := \dim(\mathrm{ran}A).$$

## Theorem

$$\mathrm{rank}A = \mathrm{rank}A'$$

# Invertibility

Now, $A$ is square, $m \times m$.

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**Definition: invertibility**

$A$ is invertible if there exists $A^{-1}$ $m \times m$ such that

$$A^{-1}A = I,$$

with $I$ the $m \times m$ identity matrix.

# Invertibility

**Theorem: Equivalent conditions to invertibility**

- $A$ $m \times m$ is invertible
- $\det A \neq 0$.
- $\text{rank} A = m$.
- $\ker A = \{0\}$.
- The columns of $A$ are l.i.
- $0$ is not an eigenvalue of $A$.
- .... add your own here.

# Representing a vector with a basis

Standard Problem: $(w_1, \ldots, w_m)$ is a basis for $R^m$.
$v \in R^m$. How to find $c \in R^m$ such that:

$$v = \sum_{i=1}^{m} c(i) w_i?$$

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Solution: $A$ is $m \times m$ with columns $w_1, \ldots, w_m$.
Then:

$$v = Iv = (AA^{-1})v = A(A^{-1}v).$$

Since our basis comprises the columns of $A$,

$$c = A^{-1}v,$$

so that

$$c(i) = (v, b_i),$$

with $b_i'$ the $i$'th row of $A^{-1}$.

# Orthgonality

## Definition

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Note: The relation $Q'Q = I$, implies that, with $q_1, \ldots, q_m$ the columns of $Q$,

$$(q_i, q_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Note: $q_1, \ldots, q_m$ are the rows of $Q^{-1}$, hence

$$v = \sum_{i=1}^{m} (q_i, v) q_i, \quad \forall v \in R^m.$$

# Orthgonality

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## Theorem

$Q$ is orthogonal, $v, w \in R^m$. Then:

$$(Qv, Qw) = (v, w).$$

# Outline

# Eigenpairs

## Definition

$A$ $m \times m$. $(\lambda, v)$, $\lambda \in C$, $v \in R^m \backslash 0$, is eigenpair of $A$, if

$$Av = \lambda v.$$