# Equilibrium in Cumulative n-person Repeated Portfolio Game

복잡계 2016년 가을 컨퍼런스


## 조남운

고등과학원

# Non Cooperative Games

| | Simple | Complex |
|---|---|---|
| **# of Players** | 2-Person Game | n-Person Game |
| **Repeated?** | One-shot Game | Repeated Game |
| **Strategy** | Pure Strategy | Mixed Strategy |
| **# of Actions** | Discrete | Continuous |

# Equilibrium Concepts for Games

- Static
  - Pair of dominant strategies
  - Subgame Perfect Equilibrium
  - Nash Equilibrium
- Dynamic
  - Evolutionary Stable Equilibrium

# Focus

- Static: Finding Equilibria
- Dynamic: Stability of Equilibria

# Nash Equilibria

- Nash proved the existence of the pair of best responses when strategy is stochastic: Mixed Strategy
- Mixed strategy can be introduced WITHOUT modifying any game rules

# Evolutionary Game Theory

- Evolutionary Game Theory consider the dynamics of strategies in multi player repeated games
- Replicator Dynamics is significant
  - Successful strategy is increasing
  - Unsuccessful strategy is decreasing

# Replicator Dynamics

- What is the proper increase functional form in repeated game?
  - There is no natural agreement
- In this paper, a rule is proposed in certain form of game

# Main Setting

- n-person infinitely repeated game
- Continuous action: Portfolio
- Payoff is cumulative and multiplicative
  - Payoff of previous game become endowment of next game
  - Payoff is calculated based on initial endowment
- Global state determines individual payoff
  - EG is dependent on local state

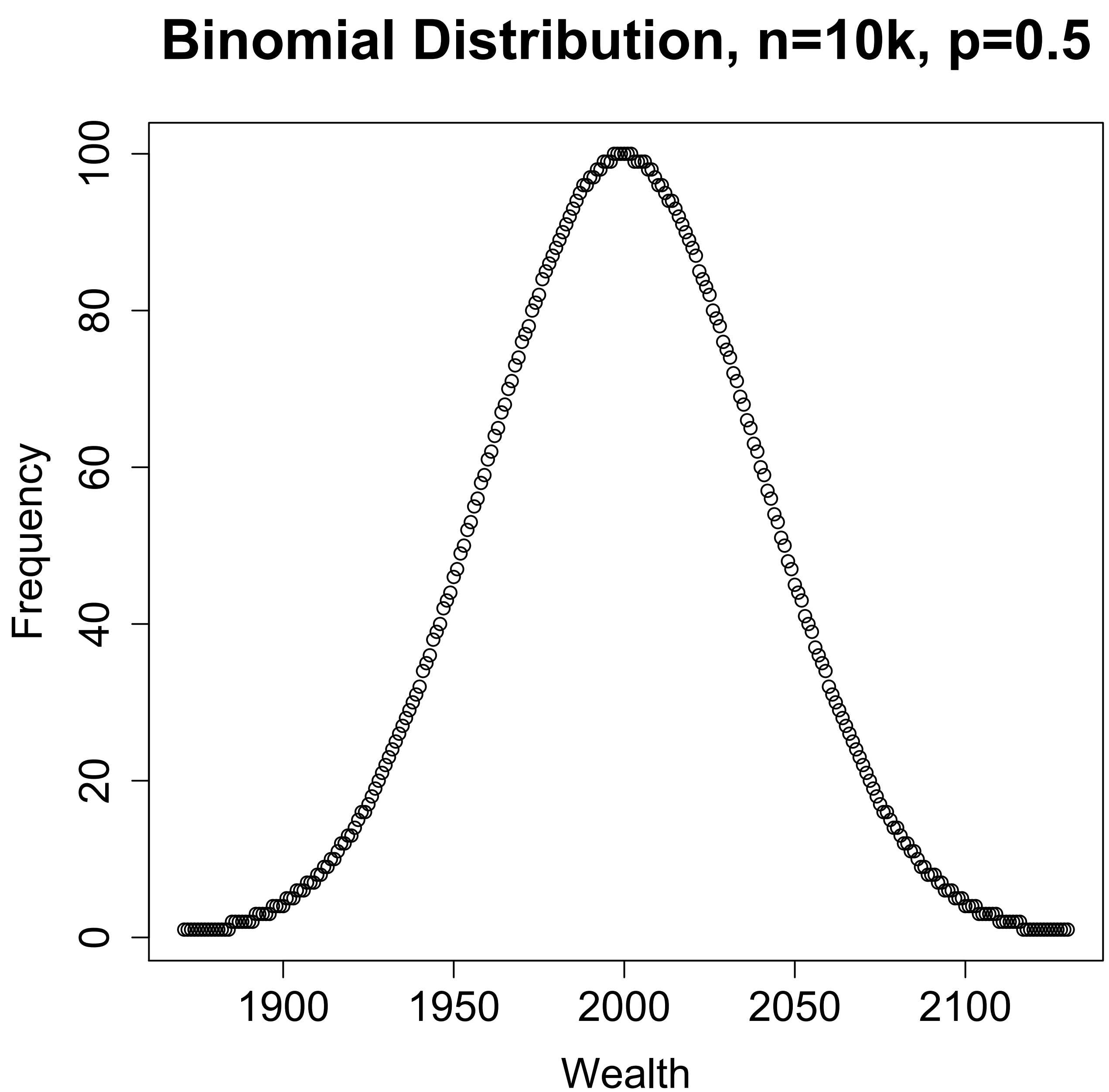
# Dynamics

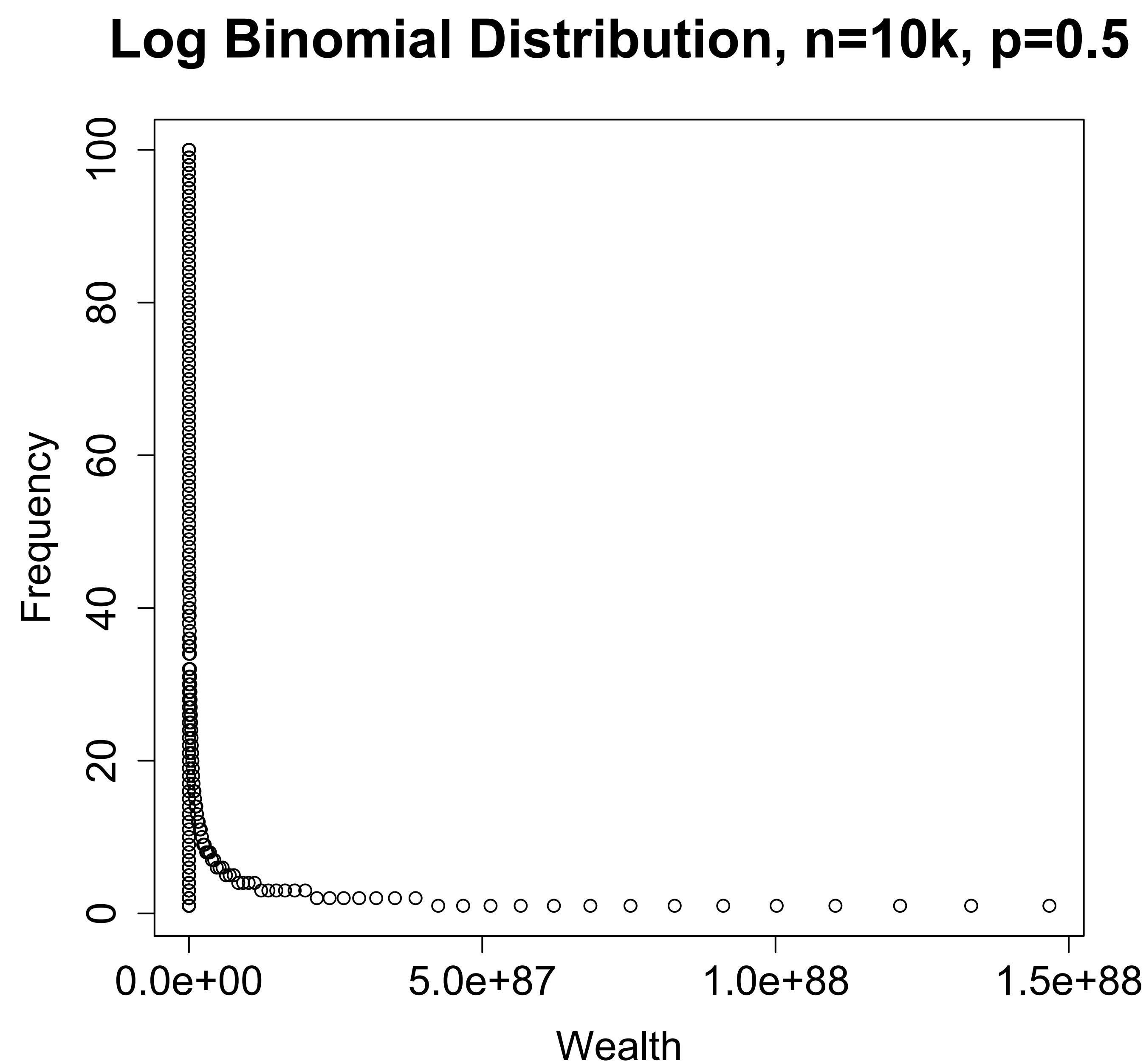
- Players do NOT change their strategy
  - Similar to Evolutionary Game Theory
  - Tractable
- Replication dynamics depend on relative payoff

# Example: Massive Coin Flip Game

- Same initial endowment, Same Strategy
- Coin Flip
  - Head: $W(t+1) = 1.1*W(t)$
  - Tail: $W(t+1) = W(t)$
- Result: Log binomial wealth distribution
  - Fair condition, unequal outcome

# Log Binomial Distribution



# Famous multi-person portfolio Games

- Public Goods Game - Extension of Prisoner's Dilemma Game
- Minority Game
- In this presentation, the extension of Hawk-Dove Game is proposed

# Strategy Types

- All-in Strategy
  - Pure Strategy NE
  - Mixed Strategy NE
- Portfolio Strategy
  - 2-person Game
  - n-person Game

# Modified Hawk Dove Game (n=2, All in action)

| ΔW(t) | Safe Action | Risky Action |
| --- | --- | --- |
| **Safe Action** | $r_S W_t, \quad r_S W_t$ | $r_S W_t, \quad r_R W_t$ |
| **Risky Action** | $r_R W_t, \quad r_S W_t$ | $-W_t, \quad -W_t$ |

# Pure Strategy NE

- Two person: $j = 1, 2$
- Pure strategy: all-in to safe asset or risky asset
- NE: $(Safe_2, Risky_1)$ & $(Safe_1, Risky_2)$

Table: Payoff matrix of two-person binary choice investment game

| | $Safe_2$ | $Risky_2$ |
|---|---|---|
| $Safe_1$ | $(r^S W_1, r^S W_2)$ | $(\mathbf{r^S W_1, r^R W_2})$ |
| $Risky_1$ | $(\mathbf{r^R W_1, r^S W_2})$ | $(-W_1, -W_2)$ |

*Note: Agent $j$'s strategy $Safe_j$ & $Risky_j$ means $(\alpha_j, \beta_j) = (1, 0)$ & $(0, 1)$ strategy, respectively.*

# Mixed Strategy NE

- $\tilde{\beta}_j$: probability of $j$'s risky asset strategy (all-in to risky asset)
- Three Nash equilibria $(\tilde{\beta}_1^*, \tilde{\beta}_2^*)$:

$\rightarrow (1,0), (0,1)$ & $(1 - \frac{r^S+1}{r^R+1}, 1 - \frac{r^S+1}{r^R+1})$

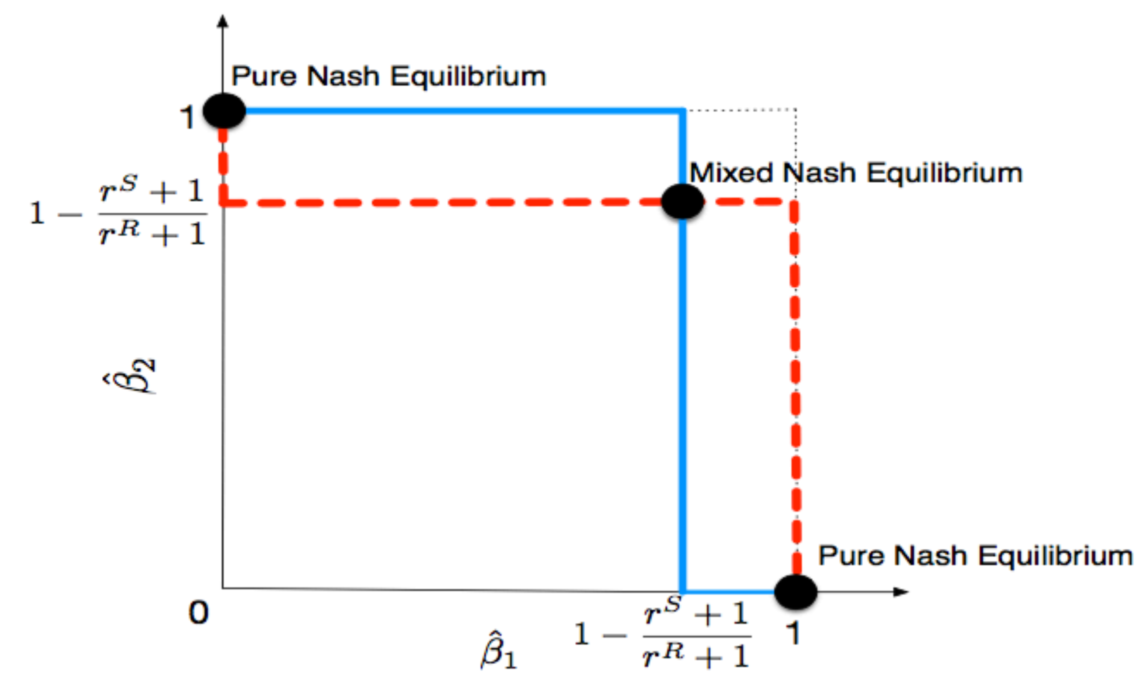



Figure: Mixed Strategy NE in 2-person game

# Portfolio Strategy (2-person Game)

- $\beta_j$: $j$'s portfolio of risky asset
- Internal solution of $(\beta_1, \beta_2)$: aggregated risky asset ratio $= \bar{R}$
- Corner solution of $(\beta_1, \beta_2)$:
  - $\overline{AB}$: (0, 0.8-1) for $\bar{R} = 0.8$, (0-0.6, 1) for $\bar{R} = 0.4$
  - $\overline{CD}$: (0.8-1, 0) for $\bar{R} = 0.8$, (1, 0-0.6) for $\bar{R} = 0.4$
  - other's strategy guarantees s crisis state: best response is 0

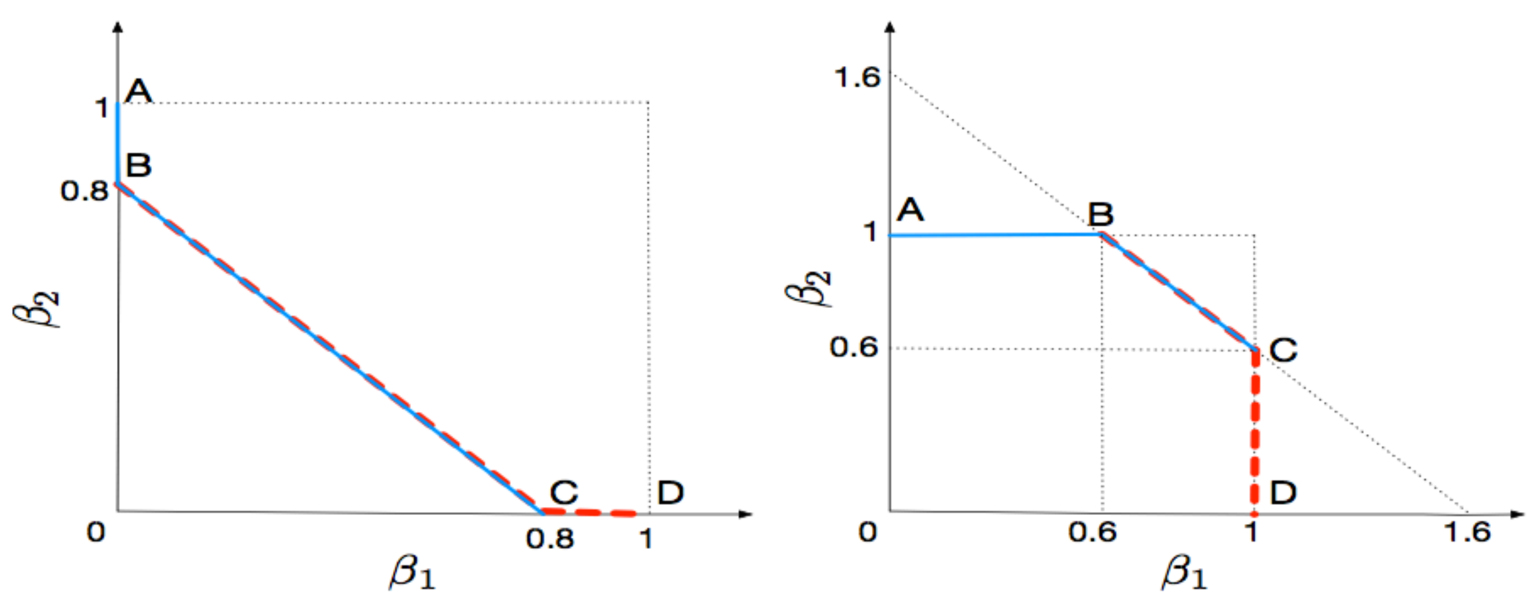



Figure: Best response of 2-person game when $\bar{R} = 0.8$ (Left) $\bar{R} = 0.4$ (Right)

# Nash Bargaining Game

- Portfolio HD Game is similar to Nash Bargaining Game (Nash 1950)
- 2 person game
  - p1: propose u1
  - p2: accept / reject
- Feasible agreement lies on the u1 + u2 = 1
- Difference: Portfolio HD Game's feasible agreement is weighted sum of cumulative payoff

| | u1 |
|---|---|
| Accept | $u_1, \ 1 - u_1$ |
| Reject | $0, \quad 0$ |

# Notations

- $W_{jt}$ : Cumulative Payoff of infinitely many Player (j) at Round (t)
- $W_R$: Quantity of Risk Strategy
- $W_S$: Quantity of Safe (risk free) Strategy
- $r_R$ : Return rate of Risky Action
- $r_S$ : Return rate of Safe (risk free) Action
- $\beta$ : Rate of Risk Strategy

$$\beta_{jt} := W_{Rjt}/W_{jt}$$

# Portfolio Strategy in n-person Game

- Assumptions (for tractability)

  - same initial endowment

$$W_0(j) = w_0, \quad \forall j \in [0,1]$$

  - portfolio $j$ as continuous variable: $j \in [0,1]$

  - portfolio is uniformly distributed: $\beta(j) = j, \quad \forall j$

  - after first crisis, all states are reset to initial condition

# Multiplicative & Cumulative Payoff Scheme

- Payoff is Multiplicative: Payoff is defined as the rate of endowment

$$\Pi_{it} = \Delta W_{it} = r_{it} W_{it}$$

- Payoff is Cumulative: Endowment is the payoff of previous game

$$W_{t+1} = (1 + r) W_t$$

# Global State

- Two global state: Normal / Crisis
  - Normal: $r_R > r_S$
  - Crisis: $r_R < r_S$

# Return Rate

- r_S is constant
- r_R is dependent on the global state
  - Normal State $r_R > r_S$
  - Crisis State $r_R = -1 < r_S$

$$
\begin{aligned}
\Delta W^R_{j,t+1} &= r^R \cdot W^R_{j,t} \cdot \Psi(\bar{\beta}_t \leq \bar{R}) \\
&= \begin{cases} 0, & \text{if } \beta_t > \bar{R} \in [0,1] \\ r^R \cdot W^R_{j,t}, & \text{otherwise} \end{cases}
\end{aligned}
$$

# Determinant of Global State

- β(t) > R ⟶ crisis state
- otherwise, normal state

$$\beta_t := \frac{\sum_{j=1}^{N} W_{j,t}^{R}}{\sum_{j=1}^{N} (W_{j,t}^{S} + W_{j,t}^{R})}$$

- For given $\bar{R} \in [0, 1]$,
- current state is '*crisis state*': $\beta_t > \bar{R}$
- otherwise, current state is '*normal state*'

- Assumption: prob(crisis) is monotonic increasing with regard to risky asset ratio (Allen (2000); Minsky (1992))

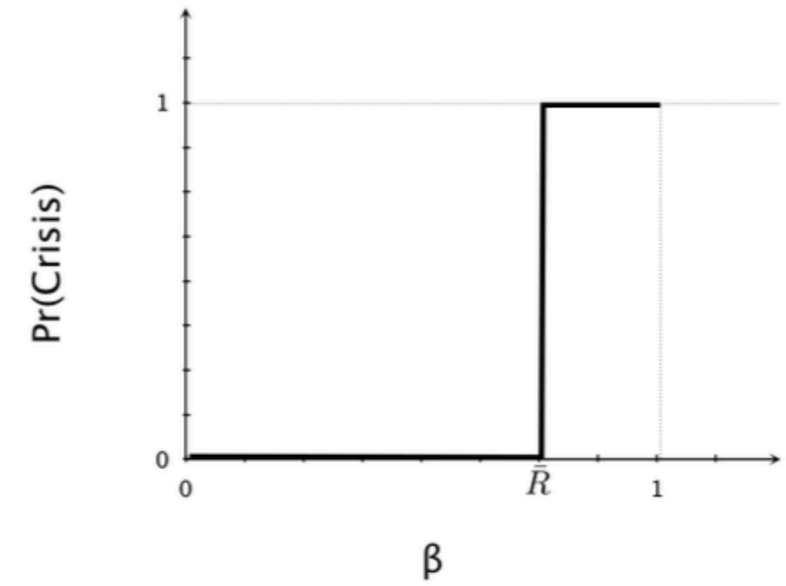



Figure: Prob. of crisis state w.r.t. $\beta$, binary condition

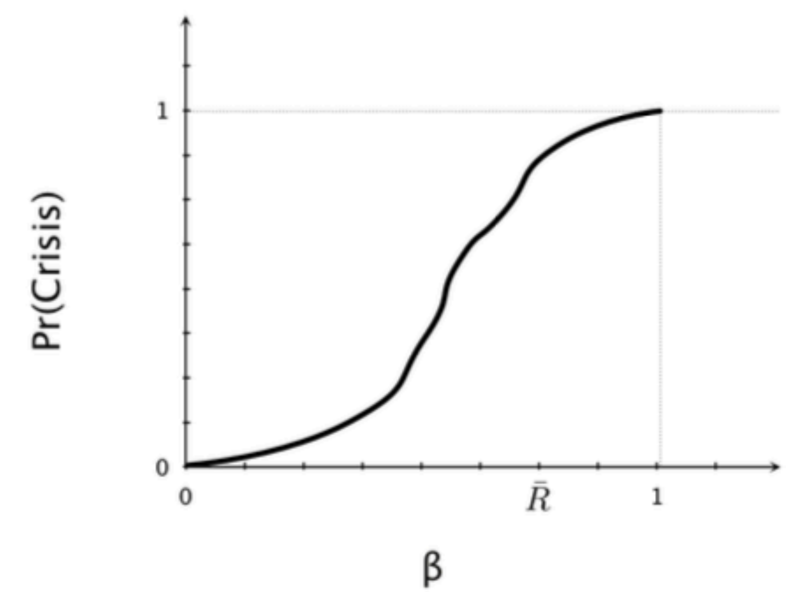



Figure: Prob. of crisis state w.r.t. $\beta$, more realistic condition

# Formal Statement

- We express $W_t^R(j)$ and $W_t^S(j)$ in terms of $j$:

$$W_t^R(j) = \beta(j)W_t(j) = jW_t(j), \quad W_t^S(j) = (1-j)W_t(j)$$

- Assumption of a fixed strategy in all portfolios

$$(1-\beta(j))(1+r^S) + \beta(j)(1+r^R) = 1 + r^S + \beta(j)(r^R - r^S)$$

$$W_{t+1}(j) = \begin{cases} (1 + r^S + j(r^R - r^S))W_t(j), & \text{if } \ \beta_t \leq \bar{R} \text{ (normal state)} \\ (1-j)(1+r^S)W_t(j), & \text{otherwise (crisis state)} \end{cases}$$

# Formal Analysis

- Aggregated risky asset ratio at period t ($\beta_t$, $\omega_t$):

$$\beta_t := \frac{\bar{W}_t^R}{\bar{W}_t} = \frac{\int_j W_t^R(j)dj}{\int_j W_t(j)dj}$$

$$\omega_t := \frac{\bar{W}_t^R}{\bar{W}_t^S} = \frac{\beta_t}{1-\beta_t}$$

- Nash Equilibrium: $\beta_t = \bar{R}$ $(= 0.8)$, $\omega_t = 4$

# Difference with Evolutionary Game

- Most Evolutionary Games depend on sum of individual payoff
  - This model is product of individual payoff rate
- Most Evolutionary Games consider local interaction
  - This model considers global state

# Evolutionary Game Theoretic Approach

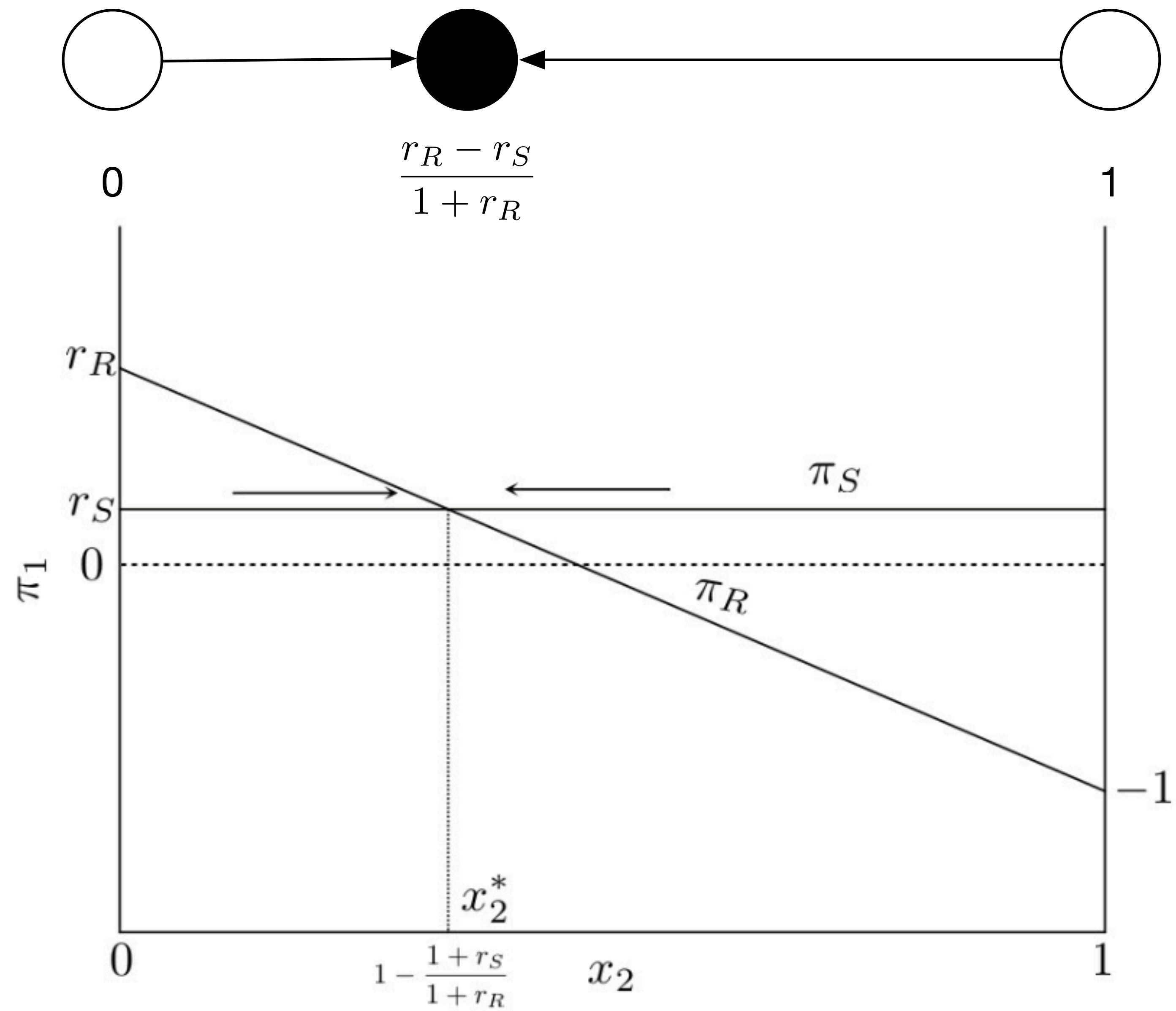
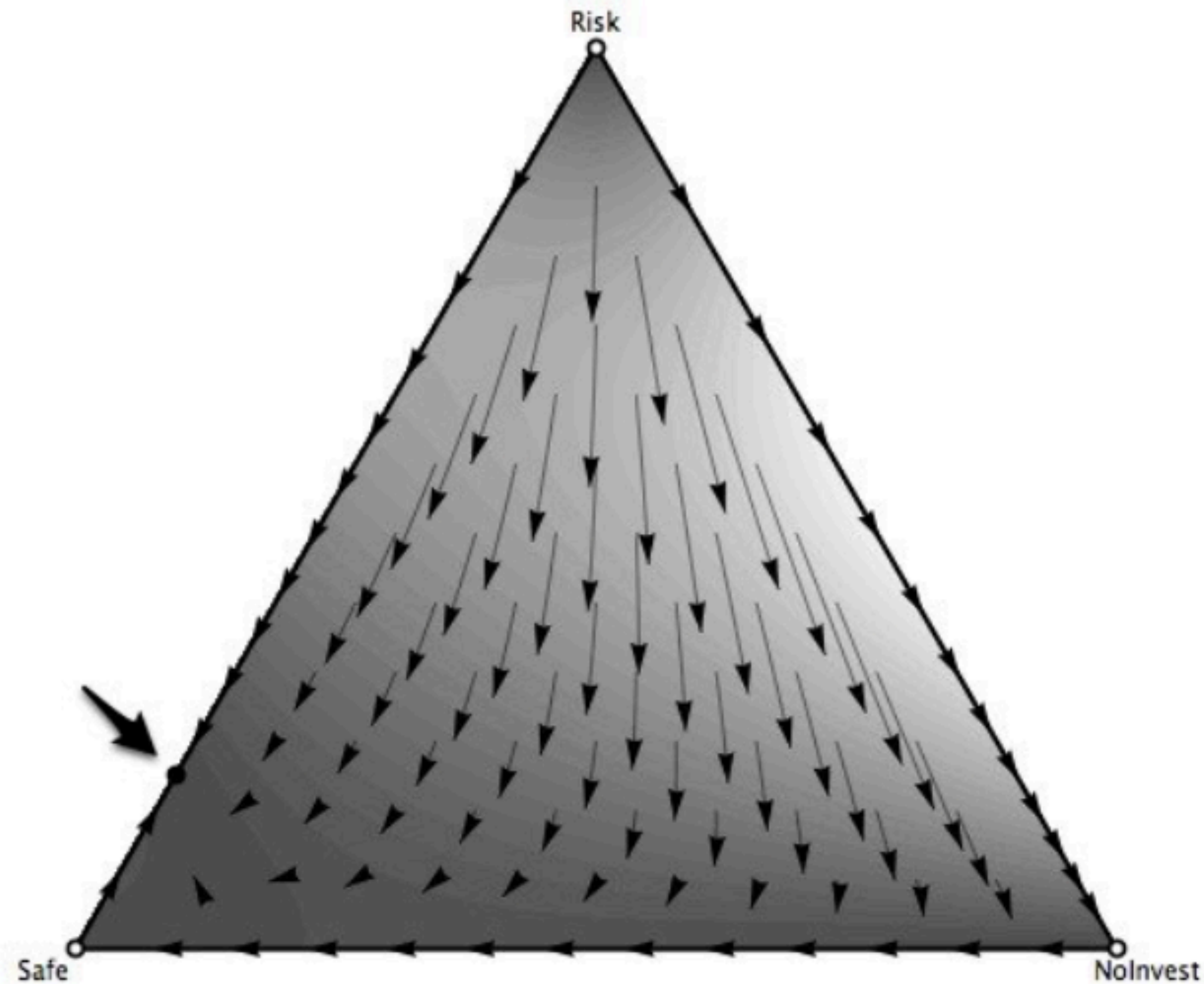



Figure 1.5: The rate of return $\pi_1$ of each investment strategy with regard to $x_2$ at the local crisis case

# EGT Approach

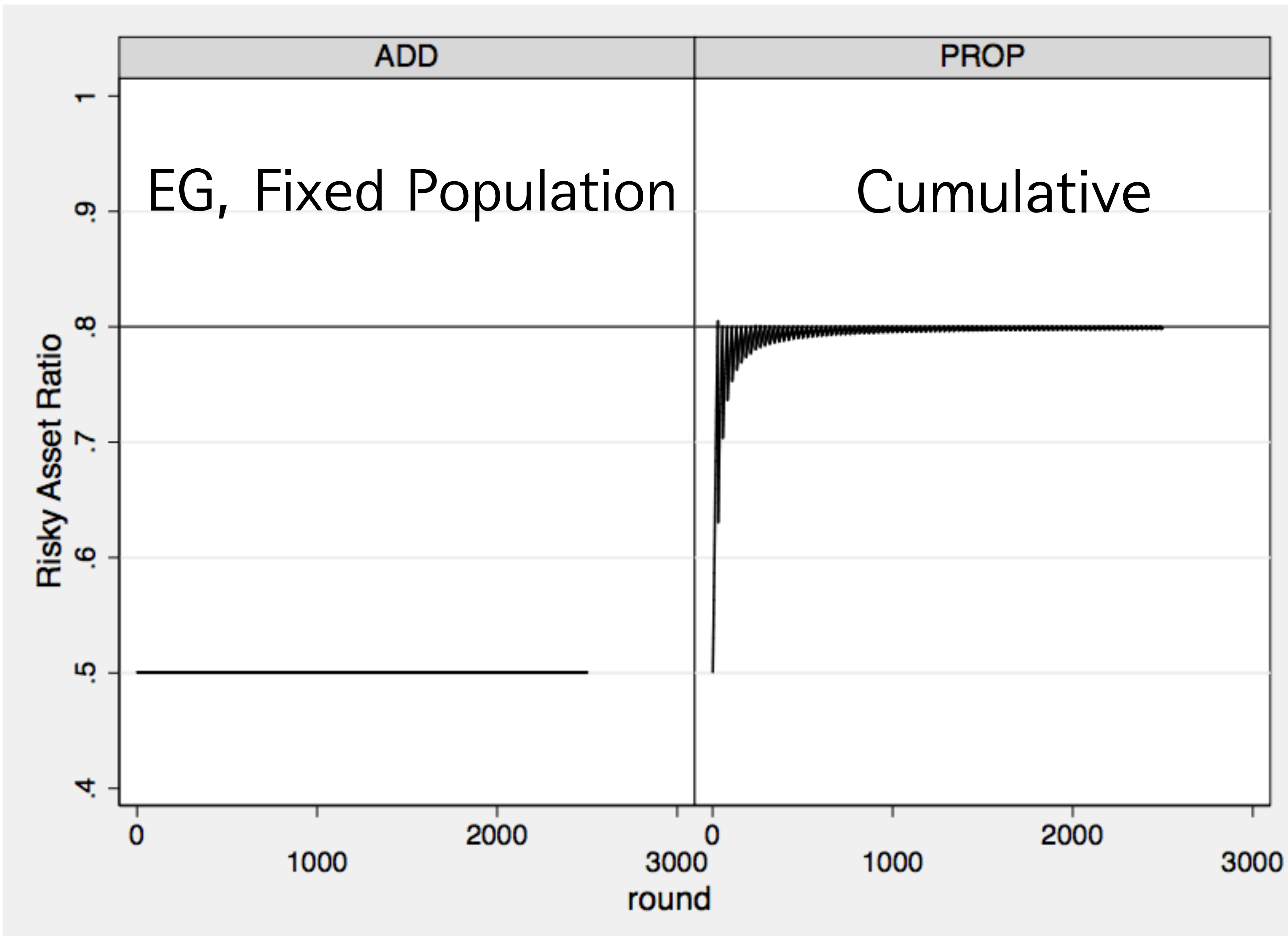


# Cumulative & Multiplicative Game

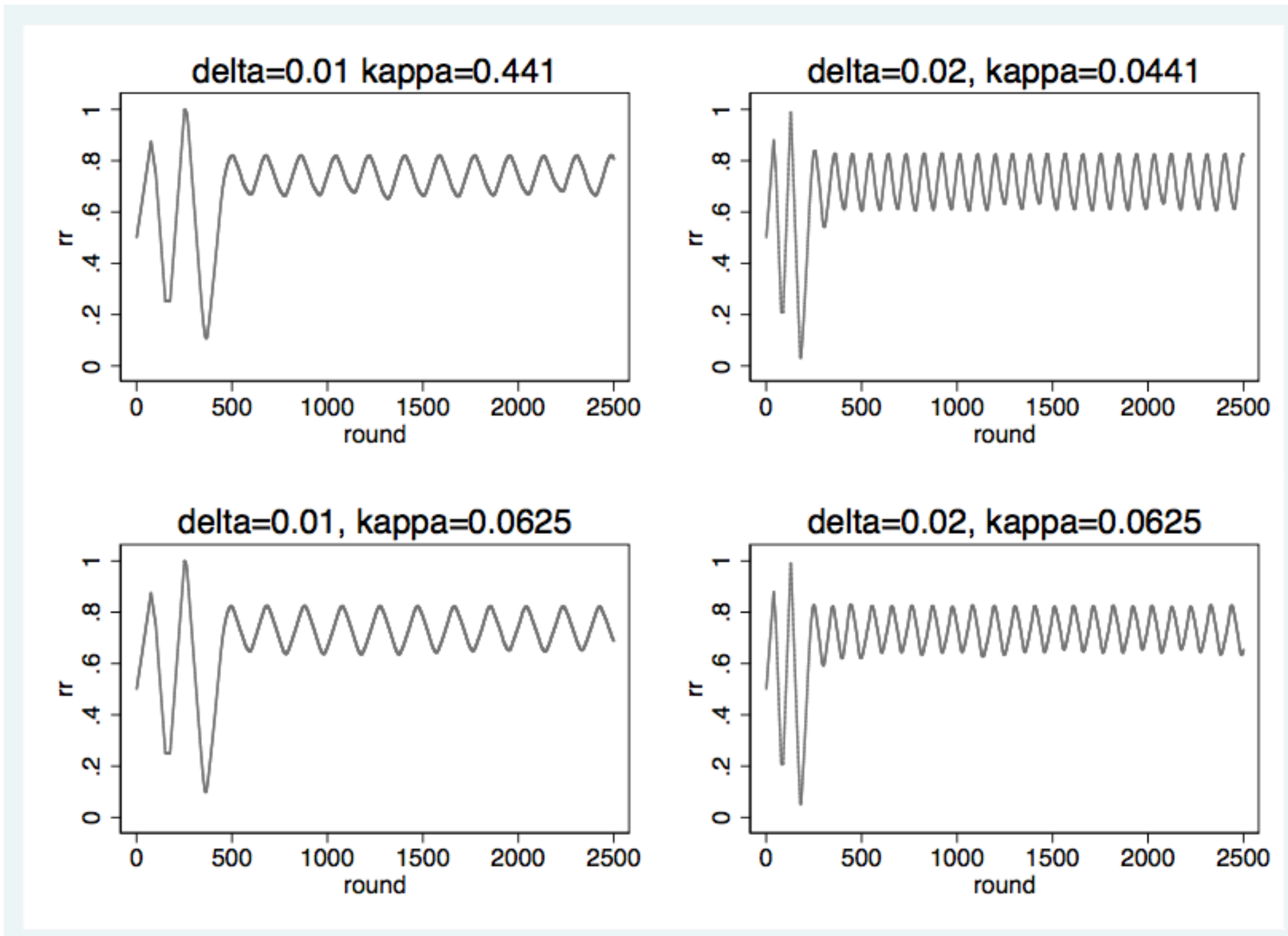
- We can calculate the deterministic dynamics when the strategy is fixed
- Average strategy converges to R: threshold of macro state

$$\Delta\tilde{R}_t = \begin{cases} \frac{(r_R - r_S)(\Sigma_{t-1} - \tilde{R}^2_{t-1})}{1 + r_S + (r_R - r_S)\tilde{R}_{t-1}} > 0, & \text{if } \tilde{R}_{t-1} \leq \bar{R} \\ \frac{\tilde{R}^2_{t-1} - \Sigma_{t-1}}{1 - \tilde{R}_{t-1}} < 0, & \text{if } \tilde{R}_{t-1} > \bar{R} \end{cases}$$

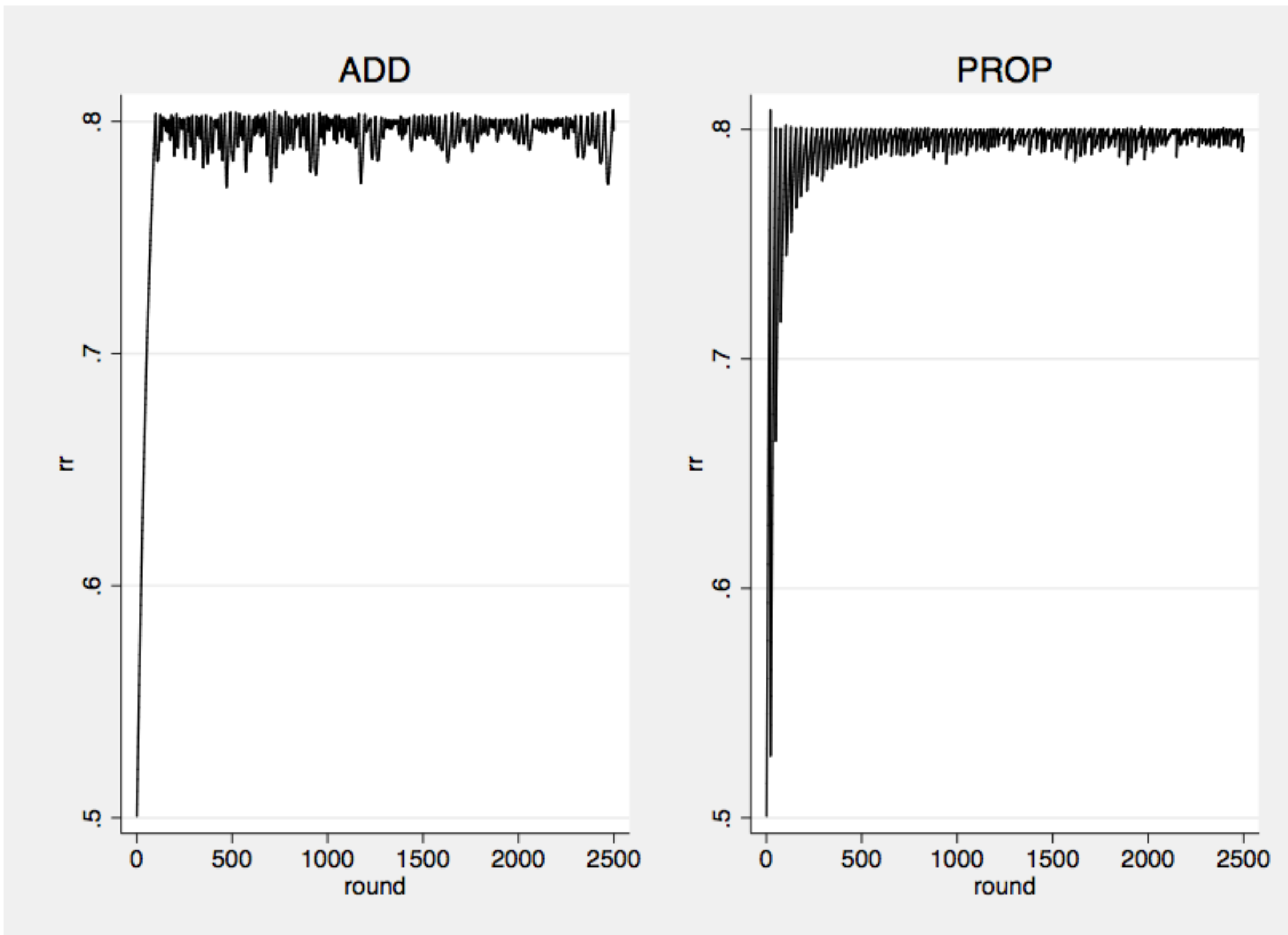
# Average Strategy (Fixed Population)



# Introducing Mimic Process

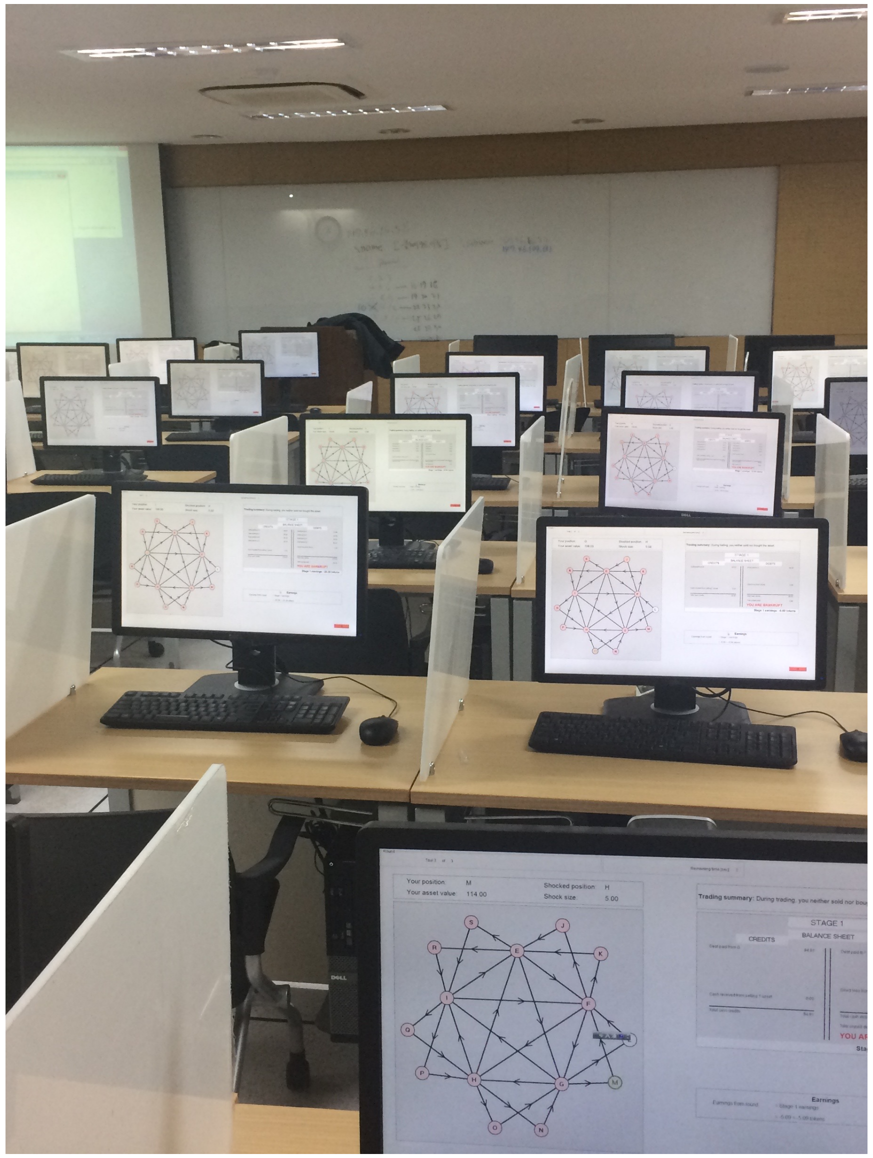


# Introducing Mimic & Mutation

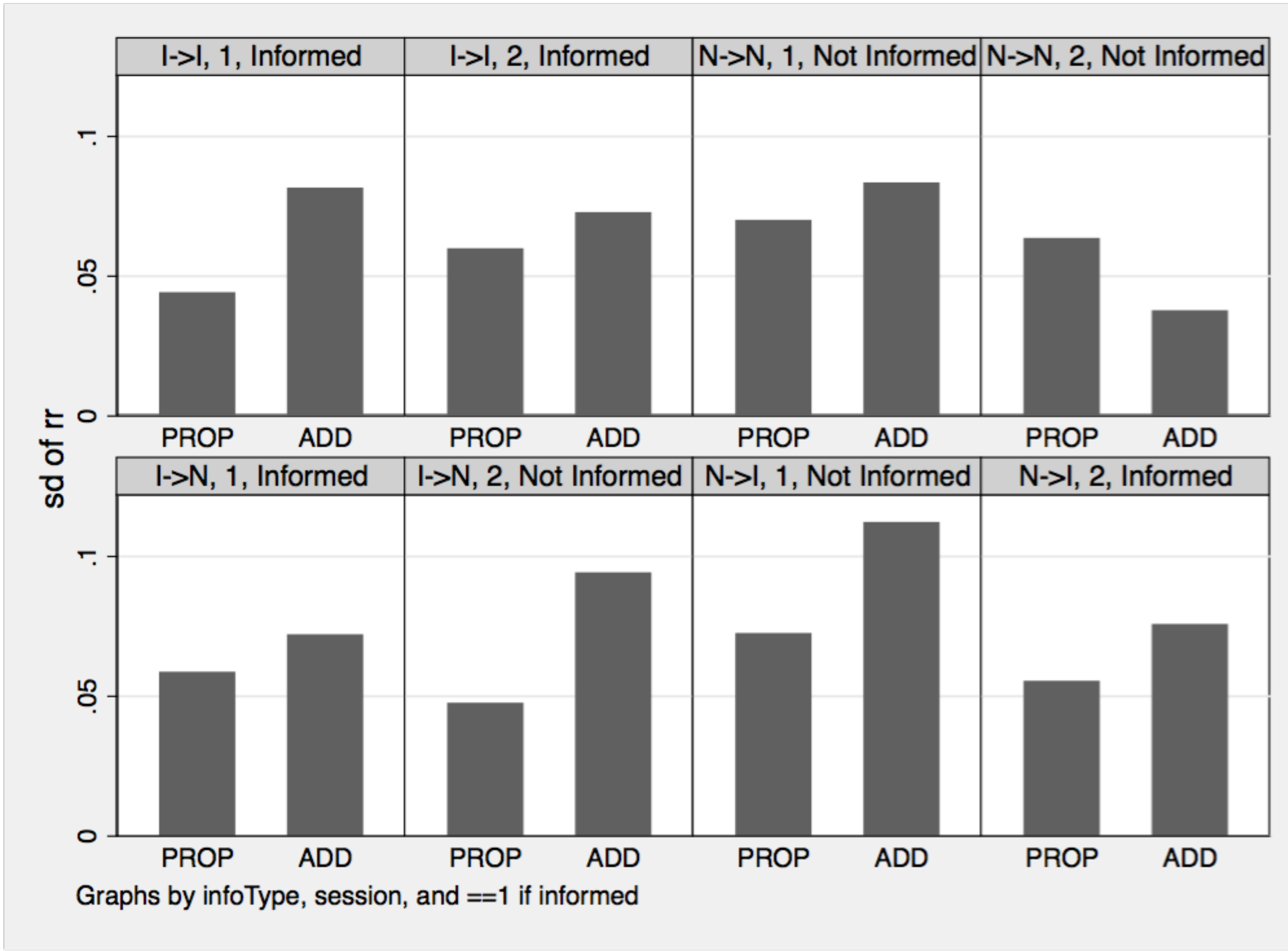


# Lab Experiment

| 항목 | 내용 |
| --- | --- |
| 참가자 | 서울대 학부생 |
| 기간 | 2014 - 2015 |
| 참가자 수 | 140 명 |
| 그룹 수 | 8 |
| 평균보상 | 10,000 원 |



# Experimental Result



# Public Goods Game

- β(j): Contribution Rate of infinitely many player j

$$j \in [0, 1]$$

- r: MPCR (Marginal Per Capita Return)
- Each Player set their endowment to:
  - Keep (1-β(j))W(j,t) → remains
  - Contribute β(j)W(j,t) → aggregated and grow at the rate of MPCR → return to players EVENLY

# PGG and Prisoner's Dilemma

- PGG is n-person, portfolio version of Prisoner's Dilemma
- If n=2, PGG is equivalent to PD Game
- With cumulative payoff scheme, PGG can be analyzed

# Cumulative Portfolio Strategy PGG

- Fixed Strategy
  - β(j) is distributed uniformly
  - β(j) = j in continuous form
- Payoff of previous game becomes the endowment of next game

$$\Delta W_{it} = \underbrace{-iW_{it}}_{\text{contribution}} + \underbrace{(1+r)\int_j jW_{jt}\mathrm{d}j\mathrm{d}i}_{\text{return from aggregated contribution}}$$

# PGG Dynamics

$$W_{it} \approx (1-i)^t + \sum_{\tau}^{t} \frac{1}{(\tau+1)(\tau+2)}(1+r)\mathrm{d}i(1-i)^{(t-\tau)}$$

- W(i,t) is decreasing with i, increasing with t

$$\lim_{t\to\infty} W_{it} = 0 \quad \forall i \neq 0$$

$$\lim_{t\to\infty} W_{it} > 0 \quad \text{if} \quad i = 0$$

Nam-Un, Cho/Center for Distributive Justice/ mailto:namun@snu.ac.kr

# Portfolio Minority Game

- Two actions: U and D
  - Portfolio Strategy: Player i's Rate of U := β(i) = i (if strategy is uniformly distributed)
  - WU(i)=iW(i)
  - WD(i)=(1-i)W(i)
  - rW>rL

# Formal Representation

$$\int_i iW_{it}\mathrm{d}t < \frac{1}{2}\int_i W_{it}\mathrm{d}t \quad \Rightarrow \quad W_{it+1} = (ir_W + (1-i)r_L)W_{it}$$

$$\int_i iW_{it}\mathrm{d}t > \frac{1}{2}\int_i W_{it}\mathrm{d}t \quad \Rightarrow \quad W_{it+1} = (ir_L + (1-i)r_W)W_{it}$$

$$\int_i iW_{it}\mathrm{d}t = \frac{1}{2}\int_i W_{it}\mathrm{d}t \quad \Rightarrow \quad W_{it+1} = \frac{r_W + r_L}{2}W_{it}$$

# Market Dynamics as Replicator Dynamics

- Replicator Dynamics in population are the result of Evolutionary Processes
- In economic systems, wealth weighted population is important
- 가중평균의 가중치가 되는 초기 자본량은 이전까지의 게임의 누적적 결과 → 더 성공적인 전략을 취해왔던 플레이어의 전략이 더 높은 영향을 미침
- payoff 자체가 fitness 로 작동

# Hayek Hypothesis

- 시장을 일종의 Voting System 으로 해석해본다면..
- 자본금이 많은 공급/수요자의 가치평가가 더 크게 반영되는 Valuation System
- 기존 자본금의 취득이 더 성공적인 Valuation에 의한 결과였다면 가중평균된 Market Value (=Price)는 의미있는 값일 수 있음

# Conclusion

- There seems to convergence toward one(some) of NE in portfolio strategy with cumulative and multiplicative payoff scheme
  - Extended HD Game
  - PGG
  - (MG)
- This scheme seems to be natural for replicator dynamics

# Further Study

- General proof of this type of games

# 감사합니다!