On Nonsingular M-Matrices

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ABSTRACT

We extend to nonsingular M-matrices the following result by G. Sierksma: If S is a nonsingular irreducible M-matrix and if x and $y \neq 0$ satisfy $Sx = y$, with $x_i > 0$ whenever $y_i < 0$, then all the coordinates in x are positive. This theorem has several corollaries dealing with bounds on solutions and their relative errors, which we also generalize.

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1. INTRODUCTION

It is a standard fact in the Perron-Frobenius theory that if T is a nonnegative irreducible matrix, then there exists a unique (up to constants) eigenvector corresponding to the spectral radius of T, such that all its coordinates are positive. This theorem is a basic tool in the input-output Leontief model; a complete analysis of this theorem for a general nonnegative T can be found in [1].

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Let now $s > r(T) :=$ spectral radius of T, and consider the nonsingular M-matrix $S := sI - T$; another useful result for the Leontief model is that the solution of $Sx = y$ satisfies $x \gg 0$ (i.e. $x_i > 0$ *for all* i), whenever $y > 0$ (i.e. $y \neq 0$ and $y_i \geqslant 0$ *for all* i). This result has been extended by G. Sierksma in the following remarkable way (see Theorem 6 in [2]):

If $Sx = y \neq 0$ *and* $x_i > 0$ *whenever* $y_i < 0$, *then* $x \gg 0$.

Our aim is to extend Sierksma's result as well as some of its corollaries to the case where S is a general nonsingular M-matrix.

2. ON POSITIVE SOLUTIONS

We assume from now on that S is defined as above with a nonnegative T, but not necessarily irreducible. N will denote the set of positive integers not greater than n, the order of S. $G(S)$ is the directed graph of S.

We say that a nonempty subset K of $G(S)$ is a *nucleus* if it is a strongly connected component of $G(S)$. For a nucleus K, N_K denotes the set of indices involved in K; for a vector y, y_K is the subvector whose indices are in N_K; analogously, S_K denotes the corresponding principal submatrix of S. Note that S_K is maximal irreducible.

It will also be assumed throughout that the vectors x and y satisfy

$$Sx = y, \tag{2.1}$$

and we define

$$N_+(x) := \{i \in N : x_i > 0\}.$$

THEOREM 2.1. *If* $y_K \neq 0$ *for each nucleus* K, *and if* $x_i > 0$ *whenever* $y_i < 0$, *then* $x \gg 0$.

Proof. Observe that by performing a permutation similarity on S we can bring S to a block lower triangular form, where the diagonal blocks are the principal submatrices of S indexed by (the vertices of) the strongly connected components of S. Let the diagonal block S_{ii} correspond to the nucleus K_i. We now show that for every i we have

$$N_{K_i} \subseteq N_+(x). \tag{2.2}$$

Assume to the contrary that (2.2) does not hold, and let i be the minimal positive integer for which

$$N_{K_i} \not\subseteq N_+(x). \tag{2.3}$$

If $N_{K_i} \cap N_+(x) = \varnothing$, then it follows that $y_{K_i} > 0$. Since S is in block lower triangular form, and since i is the minimal positive integer for which (2.3) holds, it now follows that

$$S_{K_i} x_{K_i} \geqslant y_{K_i}. \tag{2.4}$$

If K_i is a single node and $T_{K_i} = 0$, then we trivially have $x_{K_i} \gg 0$. Else, S_{K_i} is an irreducible matrix, and by applying the result mentioned in the introduction to (2.4) we obtain $x_{K_i} \gg 0$, contradicting our assumption (2.3). Therefore, we have

$$N_{K_i} \cap N_+(x) \neq 0. \tag{2.5}$$

By (2.3) and (2.5) let $r, t \in K_i$ be such that $r \notin N_+(x)$ and $t \in N_+(x)$. Since T_{K_i} is an irreducible nonnegative matrix, there exists a positive integer p such that $(T^p)_{rt} > 0$. Let p be the minimal such integer. We use the asterisk * to denote subvectors and principal submatrices indexed by the complement of $N_+(x)$. Clearly, $y^* \geqslant 0^*$ and $x^* \leqslant 0^*$. If $p = 1$, then, in view of $T_{rt} > 0$, we have

$$S^* x^* > y^*. \tag{2.6}$$

Since S^* is a nonsingular M-matrix, we get from (2.6) the contradiction $x^* > 0^*$. Suppose now that $p > 1$. Since

$$(T^m)_{rt} = 0 \qquad \forall r \notin N_+(x), \quad \forall t \in N_+(x) \text{ and } m < p,$$

we get

$$(T^m y)_r \geqslant 0 \qquad \text{for} \quad r \notin N_+(x). \tag{2.7}$$

Since

$$(s^pI - T^p)x = \left(\sum_{j=1}^{p} s^{p-j}T^{j-1}\right)(SI - T)x = \left(\sum_{j=1}^{p} s^{p-j}T^{j-1}\right)y,$$

(2.7) implies that

$$s^p x_r - \sum_j (T^p)_{rj} x_j \geqslant 0 \qquad \text{for} \quad r \notin N_+(x). \tag{2.8}$$

The choice of p and (2.8) finally imply that

$$[s^pI^* - (T^p)^*]x^* > 0. \tag{2.9}$$

Now S is a nonsingular M-matrix, whence the same holds for $s^pI^* - (T^p)^*$, which applied in (2.9) yields the contradiction $x^* > 0$.

Let us finally prove that $N = N_+(x)$. If we had $N \neq N_+(x)$, consider $r \notin N_+(x)$. Equation (2.2) implies that there exists a path in $G(S)$ leading from r to a nucleus K, whence there exists a minimal positive p and an index t such that $(T^p)_{rt} > 0$. If we use the asterisk as above, once again we have $y^* \geqslant 0^*$ and $x^* \leqslant 0^*$, and by reasoning in the same way we have already done, we get the contradiction $x^* > 0^*$.

Thus, $N_+(x) = N$ and the proof is complete. ■

REMARK 2.2. Our proof of Theorem 2.1 is very similar to the proof of Theorem 6 in [2] handling the irreducible case. In [2], the author applies his Proposition 2 to the matrix T_*^α. Proposition 2 holds in general only for irreducible matrices, while, although T is irreducible, T_*^α can be reducible. Thus technically, the proof of Theorem 6 in [2] is incorrect. Our result, generalizing Theorem 6 in [2] to the general (reducible) case, settles this point.

3. BOUNDS FOR THE SOLUTION

We assume in this section that S is diagonally dominant and set

$$N_-(y) := N_+(-y).$$

THEOREM 3.1. *The following propositions hold*:

(i) *If* $N_K \cap N_+(y) \neq \varnothing$ *for each nucleus* K, *then*

$$x_i \leqslant \max\{0, \max\{x_j : j \in N_+(y)\}\} \qquad \forall i \in N.$$

(ii) *If* $N_K \cap N_-(y) \neq \varnothing$ *for each nucleus* K, *then*

$$\min\{0, \min\{x_j : j \in N_-(y)\}\} \leqslant x_i \qquad \forall i \in N.$$

(iii) *If the hypotheses in* (i) *and* (ii) *hold and there exist* j_1 *and* j_2 *such that* $x_{j_1} < 0$ *and* $x_{j_2} > 0$, *then we have*

$$\min\{x_j : j \in N_-(y)\} \leqslant x_i \leqslant \max\{x_j : j \in N_+(y)\} \qquad \forall i \in N.$$

REMARK 3.2. The diagonal dominance of S in the theorem above is not redundant, as the following example with irreducible T puts in evidence. Consider

$$T := \begin{pmatrix} 0 & 0.5 \\ 1.5 & 0 \end{pmatrix}, \qquad y_1 := \tfrac{3}{8}, \qquad y_2 := -\tfrac{1}{4}, \quad \text{and} \quad s := 1.$$

The solution of (2.1) is given by $x_1 = 1$, $x_2 = \frac{5}{4}$; and (i) in Theorem 3.1 does not hold.

COROLLARY 3.3. *The following propositions hold*:

(i) *If* $N_K \cap N_+(y) \neq \varnothing$ *for each nucleus* K, *and if for some* m

$$x_m = \max\{x_j : j \in N_+(y)\} \quad \textit{and} \quad x_m \geqslant 0,$$

then

$$y_m \geqslant 0.$$

(ii) *If* $N_K \cap N_-(y) \neq \varnothing$ *for each nucleus* K, *and if for some* m

$$x_m = \min\{x_j : j \in N_-(y)\} \quad \textit{and} \quad x_m \leqslant 0,$$

then

$$y_m \leqslant 0.$$

REMARK 3.4. Theorem 3.1 and Corollary 3.3 generalize Theorem 7 and Corollaries 8 and 9 in [2].

4. BOUNDS FOR THE RELATIVE ERRORS

We assume now that y in (2.1) satisfies

$$y \geqslant 0 \quad \text{and} \quad y_K \neq 0 \quad \text{for each nucleus } K. \tag{4.1}$$

Note that (4.1) implies that $x \gg 0$. Let Δy be a vector such that if $y_i = 0$ then $\Delta y_i \geqslant 0$, and consider Δx such that

$$S\,\Delta x = \Delta y.$$

THEOREM 4.1. *The following propositions hold*:

(i) *If* $N_K \cap N_+(\Delta y) \neq \varnothing$ *for each nucleus* K, *then*

$$(\Delta x_i/x_i) \leqslant \max\{0, \max\{\Delta x_j/x_j : j \in N_+(\Delta y)\}\} \quad \forall i \in N.$$

(ii) *If* $N_K \cap N_-(\Delta y) \neq \varnothing$ *for each nucleus* K, *then*

$$\min\{0, \min\{\Delta x_j/x_j : j \in N_-(\Delta y)\}\} \leqslant (\Delta x_i/x_i) \quad \forall i \in N.$$

REMARK 4.2. Sierksma's proof of Theorem 4.1 for irreducible S implicitly assumes that $y + \Delta y \gg 0$ (See Theorem 21 in [2]); with that assumption, the same proof works for a general nonsingular M-matrix S. The general case treated here can then be obtained in the following way: Consider, for positive real t, $y(t) := y + tu$, with $u_i := 1$ for all i, and let $x(t)$ be such that

$$Sx(t) = y(t).$$

$y(t) + \Delta y$ has all its coordinates positive, and the inequalities in Theorem 4.1 hold for $x(t)$ and Δx. By letting t tend to 0, we get the final conclusion.

REMARK 4.3. Theorem 25 in [2] states that if $\Delta y \neq 0$ and $\Delta y_i = 0$, then for such i, strict inequalities are valid in Theorem 4.1 (with irreducible S). In order for this to be true, a stronger hypothesis on T is necessary (such as $t_{jk} > 0$ for all $j \neq k$), as the following example shows: Consider a positive real $t < 1$, and define

$$T := \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & t \\ t & 0 & 0 \end{pmatrix}.$$

T is irreducible and its spectral radius is t. If we take

$$x_i := \frac{1}{1-t} \quad \text{for } 1 \leqslant i \leqslant 3, \quad \text{and} \quad s := 1,$$

then $Sx = u$, with u as in Remark 4.2. If we now define

$$\Delta u_1 := -\tfrac{1}{2}, \qquad \Delta u_2 := 0, \quad \text{and} \quad \Delta u_3 := t/2,$$

then the solution for $S\Delta x = \Delta u$ is given by

$$\Delta x_1 = -\tfrac{1}{2}, \qquad \Delta x_2 = 0, \quad \text{and} \quad \Delta x_3 = 0.$$

But we have $N_+(\Delta u) = \{3\}$, and (i) in Theorem 4.1 does not hold with strict inequality for $i := 2$.

COROLLARY 4.4. *Let there be j_1 and j_2 such that $x_{j_1} < 0$ and $x_{j_2} > 0$. If for each nucleus K*

$$N_K \cap N_+(\Delta y) \neq \varnothing \quad \textit{and} \quad N_K \cap N_-(\Delta y) \neq \varnothing,$$

then we have

$$\min\left\{\frac{\Delta x_j}{x_j} : j \in N_-(\Delta y)\right\} \leqslant \frac{\Delta x_i}{x_i} \leqslant \max\left\{\frac{\Delta x_j}{x_j} : j \in N_+(\Delta y)\right\} \qquad \forall i \in N.$$

REMARK 4.5. Corollary 4.4 generalizes Corollary 23 in [2].

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