SOLUTION FOR HW1

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1. Problem 1

Memoryless implies

$$\forall m, n \in \mathbb{Z}^+ : \Pr(X > m + n \mid X > m) = \Pr(X > n)$$

or

$$\forall m, n \in \mathbb{Z}^+ : \frac{\Pr(X > m + n \cap X > m)}{\Pr(X > m)} = \frac{\Pr(X > m + n)}{\Pr(X > m)} = \Pr(X > n)$$

or

$$\forall m, n \in \mathbb{Z}^+ : \Pr(X > m + n) = \Pr(X > m) \Pr(X > n)$$

define survival function as $\forall t \in \mathbb{Z}^+ : G(t) = \Pr(X > t)$

$$G(m + n) = G(m)G(n)$$

it can be seen that

$$\forall x \in \mathbb{Z}^+ : G(x) = G(1)^x$$

from the fact that $\Pr(X = \cdot)$ is a probability distribution on $\mathbb{Z}^+$:

$$\Pr(X = 1) + \Pr(X > 1) = \Pr(X = 1) + G(1) = 1$$

thus

$$\forall x \in \mathbb{Z}^+ : G(x) = (1 - \Pr(X = 1))^x$$

thus

$$\Pr(X = n) = \Pr(X > n - 1) - \Pr(X > n) = G(n - 1) - G(n)$$
$$= (1 - \Pr(X = 1))^{n-1} - (1 - \Pr(X = 1))^n = \Pr(X = 1)(1 - \Pr(X = 1))^{n-1}$$

it is a geometric distribution by definition.

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2. Problem 2

part1. This can be viewed as *Coupon collector's problem*
let X_i $(i \in [1,2])$ be the kids needed to be give birth after kids of $i-1$ genders have existed, is geometric random variable

$$\forall i \in [1,2] : p_i = 1 - \frac{i-1}{n} = 1 - \frac{i-1}{2}$$
$$\forall i \in [1,2] : \mathbf{E}(X_i) = \frac{1}{p_i} = \frac{2}{2-i+1}$$
$$\begin{aligned} \mathbf{E}(X) = & \ \mathbf{E}(X_1) + \mathbf{E}(X_2) \\ = & \ \tfrac{2}{2} + \tfrac{2}{1} \\ = & \ 3 \end{aligned}$$

because the possibilities of male and female are the same

$$\mathbf{E}(male) = \mathbf{E}(female) = \frac{3}{2}$$

part2.

$$\forall i \in [1,2] : p_i = 1 - \frac{i-1}{n} = 1 - \frac{i-1}{2}$$
$$\mathbf{E}(X_1) = \sum_{i=1}^{1} i p_1 = 1 * 1 = 1$$
$$\mathbf{E}(X_2) = \sum_{i=1}^{4} i(1-p_2)^{i-1} p_2 = \sum_{i=1}^{4} i \left(\frac{1}{2}\right)^i = \frac{1}{2} + 2 * \frac{1}{4} + 3 * \frac{1}{8} + 4 * \frac{1}{16} = \frac{13}{8}$$

the other possible outcome X_3 is the 5 children are the same gender

$$\mathbf{E}(X_3) = 4 * \left(\frac{1}{2}\right)^4 = \frac{1}{4}$$
$$\mathbf{E}(X) = \mathbf{E}(X_1) + \mathbf{E}(X_2) + \mathbf{E}(X_3) = \frac{23}{8}$$

because the possibilities of male and female are the same

$$\mathbf{E}(male) = \mathbf{E}(female) = \frac{23}{16}$$