SOLUTION MANUAL FOR *PROBABILISTIC METHOD AND RANDOM GRAPHS*

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ABSTRACT. In this note, we present solutions for selected homework problems for course *Probabilistic method and random graphs*: http://z14120902.github.io/pm.html.

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1. HW1

1. Memoryless implies

$$\forall m,n \in \mathbb{Z}^+ : \mathbf{P}(X > m+n \mid X > m) = \mathbf{P}(X > n)$$

or

$$\forall m,n \in \mathbb{Z}^+ : \frac{\mathbf{P}(X > m+n \cap X > m)}{\mathbf{P}(X > m)} = \frac{\mathbf{P}(X > m+n)}{\mathbf{P}(X > m)} = \mathbf{P}(X > n)$$

or

$$\forall m,n \in \mathbb{Z}^+ : \mathbf{P}(X > m+n) = \mathbf{P}(X > m)\mathbf{P}(X > n)$$

the rest will be easy.

2.

1. Recall *Coupon collector's problem*, there are 2 coupons in this case

$$\mathbf{E}X = 1+2 = 3$$

2.

- after first kid, success within 5 kids: $\sum_{i=1}^{4} \frac{i}{2^i} = \frac{13}{8}$
- after first kid, fail within 5 kids: $4\frac{1}{2^4} = \frac{1}{4}$

$\mathbf{E}X = 1 + \frac{13}{8} + \frac{1}{4} = \frac{23}{8}$

2. HW2

1.

$$\mathbf{P}(X \ge (1+\delta)\mu_H) = \mathbf{P}(e^{\lambda X} \ge e^{\lambda(1+\delta)\mu_H}) \le \frac{\mathbf{E}[e^{\lambda X}]}{e^{\lambda(1+\delta)\mu_H}} = \frac{\mathbf{E}[e^{\lambda \sum_{i=1}^{n} X_i}]}{e^{\lambda(1+\delta)\mu_H}} = \frac{e^{\sum_{i=1}^{n}(p_i e^{\lambda}+(1-p_i))}}{e^{\lambda(1+\delta)\mu_H}} \le \frac{e^{\sum_{i=1}^{n} p_i(e^{\lambda}-1)}}{e^{\lambda(1+\delta)\mu_H}} = \frac{e^{\mu(e^{\lambda}-1)}}{e^{\lambda(1+\delta)\mu_H}} \le \frac{e^{\mu_H(e^{\lambda}-1)}}{e^{\lambda(1+\delta)\mu_H}} = \left(\frac{e^{(e^{\lambda}-1)}}{e^{\lambda(1+\delta)}}\right)^{\mu_H}$$

the rest will be easy.

2. we need the following

$$\mathbf{E}[e^{\lambda a_i X_i}] = p_i e^{\lambda a_i} + 1 - p_i = 1 + p_i(e^{\lambda a_i} - 1) \le \underline{e^{p_i(e^{\lambda a_i}-1)} \le e^{p_i a_i(e^{\lambda}-1)}}$$

or

$$e^{\lambda a_i} - 1 \le a_i(e^{\lambda} - 1)$$

or

$$\frac{e^{\lambda a_i} - 1}{a_i} \le \frac{e^{\lambda * 1} - 1}{1}$$

this is slope of line through $(x, e^{\lambda x})$ and $(0,1)$, which is obvious via plot of function $e^{\lambda x}$, the rest will be easy.

3.

1.

$$\mathbf{E}f(Z) = \sum_i p_i f(z_i) = \sum p_i f(z_i * 1 + (1-z_i) * 0) \le \sum_i p_i z_i f(1) + p_i(1-z_i) f(0) = pf(1) + (1-p)f(0) = \mathbf{E}f(X)$$

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2. Omit

3. HW4

1. the exact probability is

$$\mathbf{P}(n\ bins\ m\ balls\ max\ load = 1) = (1-\frac{1}{n})(1-\frac{2}{n})\dots(1-\frac{m-1}{n})$$

1. we need $\mathbf{P}(n\ bins\ m\ balls\ max\ load = 1) \leq \frac{1}{e}$ or

$$\mathbf{P}(n\ bins\ m\ balls\ max\ load = 1) \leq e^{-\frac{1}{n}} e^{-\frac{2}{n}} \dots e^{-\frac{m-1}{n}} = e^{-\frac{m(m-1)}{2n}} \leq \frac{1}{e}$$

we can calculate m

2. we need $\mathbf{P}(n\ bins\ m\ balls\ max\ load = 1) \geq \frac{1}{2}$ or

$$\mathbf{P}(n\ bins\ m\ balls\ max\ load = 1) \geq e^{-\frac{1}{n}-\frac{1}{n^2}} e^{-\frac{2}{n}-\frac{2^2}{n^2}} \dots e^{-\frac{m-1}{n}-\frac{(m-1)^2}{n^2}} = e^{-\frac{m(m-1)}{2n}-\frac{(m-1)m(2m-1)}{6n^2}} \geq \frac{1}{2}$$

we can calculate m

2.

1. $\mathbf{P}(X=n) = e^{-\mu}\frac{\mu^n}{n!}$

$$\mathbf{P}(Y=k) = \sum_{n=k}^{\infty} \mathbf{P}(X=n)\binom{n}{k} p^k (1-p)^{n-k} = \sum_{n=k}^{\infty} e^{-\mu}\frac{\mu^n}{n!}\binom{n}{k} p^k(1-p)^{n-k}$$

$$= \frac{e^{-\mu}p^k}{k!}\sum_{n=k}^{\infty}\frac{\mu^n(1-p)^{n-k}}{(n-k)!} = \frac{e^{-\mu}(\mu p)^k}{k!}\sum_{n=k}^{\infty}\frac{\mu^{n-k}(1-p)^{n-k}}{(n-k)!} = \frac{e^{-\mu}(\mu p)^k}{k!}e^{\mu(1-p)} = e^{-\mu p}\frac{(\mu p)^k}{k!}$$

Z can be proved likewise

2.

$$\mathbf{P}(Y=k_1, Z=k_2) = \mathbf{P}(X=k_1+k_2)\binom{k_1+k_2}{k_1} p^{k_1}(1-p)^{k_2} = e^{-\mu}\frac{\mu^{k_1+k_2}}{(k_1+k_2)!}\frac{(k_1+k_2)!}{k_1!k_2!}p^{k_1}(1-p)^{k_2} = e^{-\mu p}\frac{(\mu p)^{k_1}}{k_1!}e^{-\mu(1-p)}\frac{(\mu(1-p))^{k_2}}{k_2!} = \mathbf{P}(Y=k_1)\mathbf{P}(Z=k_2)$$

3.

1. <u>**for** $n=1$</u>, there are no other students

$$\mathbf{P}(2\ students\ same\ birthday) = 0$$

<u>**for** $n \in \{2,\dots,365\}$</u>
the probability of max load is 1 is

$$\mathbf{P}(max\ load = 1) = (1-\frac{1}{365})(1-\frac{2}{365})\dots(1-\frac{n-1}{365})$$

thus

$$\mathbf{P}(2\ students\ same\ birthday) = 1-(1-\frac{1}{365})(1-\frac{2}{365})\dots(1-\frac{n-1}{365})$$

<u>**for** $n \in \{365, 366, \dots\}$</u>

$$\mathbf{P}(2\ students\ same\ birthday) = 1$$

2. <u>**for** $n=1$</u>, there are no other students

$$\mathbf{P}(existing\ another\ students\ same\ birthday) = 0$$

<u>**for** $n \in \{2,\dots,365\}$</u>

$$\mathbf{P}(no\ other\ students\ same\ birthday) = \left(\frac{364}{365}\right)^{n-1}$$

$$\mathbf{P}(existing\ another\ students\ same\ birthday) = 1-\left(\frac{364}{365}\right)^{n-1}$$

4. $\mathbf{P}(X \geq x) \leq \frac{\mathbf{E}e^{\lambda X}}{e^{\lambda x}} = \frac{e^{\mu(e^x-1)}}{e^{\lambda x}}$. let $\lambda = \ln(\frac{x}{\mu})$

4. HW5

1. Covered in the class, omit.

2.

1. probability of a bin with load 1

$$\mathbf{P}(X_i = 1) = \binom{b}{1}\frac{1}{n}\left(1 - \frac{1}{n}\right)^{b-1}$$

the expected balls will be served

$$\mathbf{E}X = n\mathbf{P}(X_i = 1) = b\left(1 - \frac{1}{n}\right)^{b-1}$$

thus, expected number of balls at the start of the next round $b - b\left(1 - \frac{1}{n}\right)^{b-1}$

2.

$$x_{j+1} = x_j - x_j\left(1 - \frac{1}{n}\right)^{x_j - 1} = x_j\left[1 - \left(1 - \frac{1}{n}\right)^{x_j - 1}\right]$$

consider $f(x) = (1 - \frac{1}{n})^{x+1} - (1 - \frac{x}{n})$, we can get $x_{j+1} \le \frac{x_j^2}{n}$ or $\ln x_{j+1} \le 2\ln x_j - \ln n$ the rest will be easy.

3. Recall *Poisson Approximation Theorem*. $\mathbf{P}(X_1 \ne 0 \cap \cdots \cap X_n \ne 0 \mid \sum X_i = k)$ is the same probability of <u>all bins are not empty</u> in k balls into n bins model, the probability increases with k for the obvious reason.

4. Recall *Poisson Approximation Theorem*. $\lim_{n\to\infty}\mathbf{P}(\mathscr{E}|X = m + \sqrt{2m\ln m}) - \mathbf{P}(\mathscr{E}|X = m - \sqrt{2m\ln m}) \le$ the probability of at least one empty bin after throwing $m - \sqrt{2m\ln m}$ balls but at least one among the next $2\sqrt{2m\ln m}$ balls goes into that bin

$$\le \lim_{n\to\infty}\frac{2\sqrt{2m\ln m}}{n} = \lim_{n\to\infty}\frac{2\sqrt{2n\ln n\ln(n\ln n)}}{n} = \lim_{n\to\infty}\frac{2\sqrt{2n(\ln n)^2 + 2n\ln n\ln\ln n}}{n} \sim \lim_{n\to\infty}\frac{\ln n}{\sqrt{n}} \to 0$$

5. HW6

1.

1. $m = n, \lambda = 1$: $e\sqrt{n}\left(e^{-1}\frac{1^1}{1!}\right)^n \le \sqrt{n}e^{1-n}$

2. $\frac{n!}{n^n}$

2. [Mitzenmacher and Upfal, 2005] Theorem 5.10

6. HW7

1. for any graph G over n vertices, we need to prove for model $\mathscr{G}_n$ and $\mathscr{G}_{n,\frac{1}{2}}$

$$\mathbf{P}_{\mathscr{G}_n}(G) = \mathbf{P}_{\mathscr{G}_{n,\frac{1}{2}}}(G) = \frac{1}{2^{\binom{n}{2}}}$$

2. [ERDdS and R&WI, 1959] Theorem 1

References

[ERDdS and R&WI, 1959] ERDdS, P. and R&WI, A. (1959). On random graphs i. *Publ. Math. Debrecen*, 6:290–297.

[Mitzenmacher and Upfal, 2005] Mitzenmacher, M. and Upfal, E. (2005). *Probability and computing: Randomized algorithms and probabilistic analysis*. Cambridge university press.