

EQUIVALENCE OF PROBABILITY TAIL BOUND AND LAPLACE TRANSFORM DECAY FOR SUBGAUSSIAN VARIABLES

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Definition 1 (Subgaussian random variable). *A real-valued random variable X is σ -subgaussian if*

$$\exists \sigma > 0, \forall s \in \mathbb{R} : \log \mathbf{E} e^{sX} \leq \frac{s^2 \sigma^2}{2}$$

Claim 2. *subgaussian tail bound implies Laplace transform decay.*

Proof. Given subgaussian random variable X

$$\log \mathbf{E} e^{sX} \leq \frac{s^2 \sigma^2}{2}$$

By Markov inequity

$$\Pr(f - \mathbf{E}f \geq t) \leq \inf_{s>0} \frac{\mathbf{E} e^{s(f - \mathbf{E}f)}}{e^{st}} = \inf_{s>0} e^{\log \mathbf{E} e^{s(f - \mathbf{E}f)} - st} \leq \inf_{s>0} e^{\frac{s^2 \sigma^2}{2} - st} = e^{-\frac{t^2}{2\sigma^2}}$$

□

Claim 3. *Laplace transform decay implies subgaussian tail bound.*

Proof. Suppose we have a random variable with Laplace transform decay

$$\Pr(|X - \mathbf{E}X| \geq t) \leq 2e^{-ct^2}$$

$\forall a \in (0, c)$

$$\begin{aligned} & \mathbf{E} e^{a(X - \mathbf{E}X)^2} \\ &= 1 + \mathbf{E} \int_0^{|X - \mathbf{E}X|} d(e^{at^2}) \\ &= 1 + \mathbf{E} \int_0^{|X - \mathbf{E}X|} 2ate^{at^2} dt \\ &= 1 + \mathbf{E} \int_0^\infty 2ate^{at^2} \mathbf{1}(|X - \mathbf{E}X| > t) dt \\ &= 1 + \int_0^\infty 2ate^{at^2} \Pr(|X - \mathbf{E}X| > t) dt \\ &\leq 1 + \int_0^\infty 2ate^{at^2} 2e^{-ct^2} dt \\ &= 1 + 4a \int_0^\infty te^{-(c-a)t^2} dt \\ &= 1 + \frac{2a}{-(c-a)} \int_0^\infty e^{-(c-a)t^2} d(-(c-a)t^2) \\ &= 1 + \frac{2a}{-(c-a)} [e^{-(c-a)t^2} \Big|_0^\infty] \\ &= 1 + \frac{2a}{c-a} \\ &= \frac{c+a}{c-a} \end{aligned}$$

$$\begin{aligned}
& \mathbf{E}e^{s(X-\mathbf{E}X)} \\
&= 1 + \mathbf{E} \sum_{i \geq 0} \frac{(s(X-\mathbf{E}X))^{i+2}}{(i+2)!} \\
&= 1 + \mathbf{E} \sum_{i \geq 0} s^2 (X - \mathbf{E}X)^2 \frac{(s(X-\mathbf{E}X))^i}{(i+2)!} \\
&= 1 + \mathbf{E} \sum_{i \geq 0} s^2 (X - \mathbf{E}X)^2 \frac{(s(X-\mathbf{E}X))^i}{i!} \left(\frac{1}{i+1} - \frac{1}{i+2} \right) \\
&= 1 + \mathbf{E} \sum_{i \geq 0} s^2 (X - \mathbf{E}X)^2 \frac{(s(X-\mathbf{E}X))^i}{i!} \int_0^1 (y^i - y^{i+1}) dy \\
&= 1 + \int_0^1 (1-y) \mathbf{E}[s^2 (X - \mathbf{E}X)^2 \sum_{i=0}^\infty \frac{(s(X-\mathbf{E}X))^i}{i!} y^i] dy \\
&= 1 + \int_0^1 (1-y) \mathbf{E}[s^2 (X - \mathbf{E}X)^2 e^{ys(X-\mathbf{E}X)}] dy \\
&\leq 1 + \int_0^1 (1-y) \mathbf{E}[s^2 (X - \mathbf{E}X)^2 e^{|s(X-\mathbf{E}X)|}] dy \\
&\leq 1 + \frac{s^2}{2} \mathbf{E}[(X - \mathbf{E}X)^2 e^{\frac{s^2}{2a} + \frac{a(X-\mathbf{E}X)^2}{2}}] \\
&= 1 + \frac{s^2}{2} \mathbf{E}[(X - \mathbf{E}X)^2 e^{\frac{s^2}{2a}} e^{\frac{a(X-\mathbf{E}X)^2}{2}}] \\
&= 1 + \frac{s^2}{2} e^{\frac{s^2}{2a}} \mathbf{E}[(X - \mathbf{E}X)^2 e^{\frac{a(X-\mathbf{E}X)^2}{2}}] \\
&= 1 + \frac{s^2}{a} e^{\frac{s^2}{2a}} \mathbf{E}[a(X-\mathbf{E}X)^2 e^{\frac{a(X-\mathbf{E}X)^2}{2}}] \\
&\leq 1 + \frac{s^2}{a} e^{\frac{s^2}{2a}} \mathbf{E}[e^{a(X-\mathbf{E}X)^2}] \\
&\leq 1 + \frac{(c+a)s^2}{(c-a)a} e^{\frac{s^2}{2a}} \\
&\leq \left(1 + \frac{(c+a)s^2}{(c-a)a} \right) e^{\frac{s^2}{2a}} \\
&\leq e^{\frac{(c+a)s^2}{(c-a)a}} e^{\frac{s^2}{2a}} \\
&\leq e^{\frac{a+3c}{2a(c-a)} s^2}
\end{aligned}$$

let $t = \frac{a}{c} \in (0, 1)$

$$\mathbf{E}e^{s(X-\mathbf{E}X)} \leq \inf_t e^{\frac{t+3}{t(1-t)} \frac{s^2}{2c}}$$

$\min f(t) = \frac{t+3}{t(1-t)}$ s.t. $t \in (0, 1)$

$$\begin{aligned}
f' &= \frac{t(1-t) - (1-2t)(t+3)}{t^2(1-t)^2} = \frac{t^2 + 6t - 3}{t^2(1-t)^2} = \frac{(t - (-3 + 2\sqrt{3}))(t - (-3 - 2\sqrt{3}))}{t^2(1-t)^2} \\
\min \frac{t+3}{t(1-t)} &= \frac{t+3}{t(1-t)} \Big|_{-3+2\sqrt{3}} = \frac{\sqrt{3}(2\sqrt{3}+3)(2+\sqrt{3})}{3}
\end{aligned}$$

thus

$$\mathbf{E}e^{s(X-\mathbf{E}X)} \leq e^{\frac{\sqrt{3}(2\sqrt{3}+3)(2+\sqrt{3})}{6} \frac{s^2}{c}} \leq e^{\frac{7s^2}{c}}$$

this is a subgaussian tail bound. □