Homework 3: Problems (Due May 16)

Remark: for [programing] problems, you are supposed to write a brief report about the solution with code attached. You are also asked to submit a source code. Exercises and Computer Problems are taken from the textbook.

(1) Given three data points $(-1,1)$, $(0,0)$ and $(1,1)$ [30 points]

(a) Determine the interpolating polynomial of degree two using the monomial basis

(b) Determine the interpolating polynomial of degree two using the Lagrange basis

(c) Determine the interpolating polynomial of degree two using the Newton basis

(d) Show that the three representations in (a)-(c) give the same polynomial

(e) Determine the piecewise linear interpolating polynomial

(f) Determine the cubic natural spline interpolation (Hint: Example 7.6)

(2) Show the two Legendre polynomials $P_1(x) = x$ and $P_2(x) = 3x^2 - 1$ that defined on interval $[-1,1]$ are orthogonal to each other. [5 points]

(3) Recall the kth Chebyshev polynomial of first kind, defined on interval $[-1,1]$ is given by $T_k(t) = \cos(k \cdot \arccos(t))$ [10 points]

(a) Show the orthogonality condition below

$$\int_{-1}^{1} \frac{T_i(t)T_j(t)}{\sqrt{1-t^2}}\, dt = \delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

(b) Show that $T_3(t) = 4t^3 - 3t$. [Hint: you are supposed to figure out the relation between $\cos 3\theta$ and $\cos\theta$ firstly]

(4) [Programming] Reproduce Figures 7.7 and 7.8 in the textbook (also see below) [20 points]

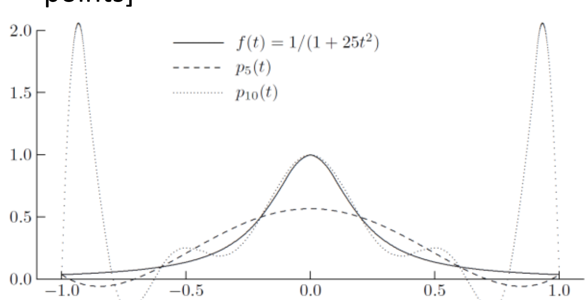


Figure 7.7: Interpolants of Runge's function at equally spaced points.

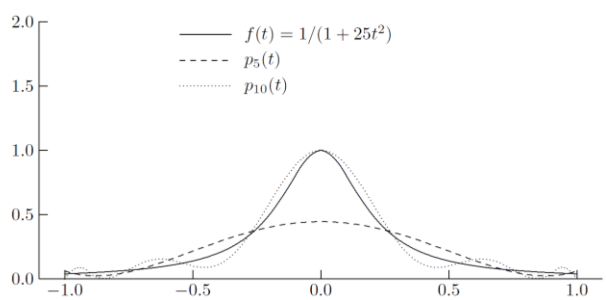


Figure 7.8: Interpolants of Runge's function at Chebyshev points.

Here p_5 and p_{10} represents polynomials of degree 5 and 10. In Figure 7.7, equally spaced interpolation points are used. In Figure 7.8, Chebyshev points are used as the interpolation points.

(5) Consider the integral $\int_0^1 x^3 dx$ [30 points]

(a) Compute the approximate value of the integral first by midpoint rule and then trapezoid rule;

(b) Use the difference between these two results to estimate the error in each of them;

(c) Combine the two results to obtain the Simpson's rule approximation to the integral;

(d) Would you expect the result from Simpson's rule to be exact for this problem? Why?

(e) Compute the approximate value of the integral by Gauss quadrature. Is the result exact? Why?

(f) Using the composite midpoint quadrature rule, compute the approximate value for the integral using a mesh size $h = 0.5$.

(6) Suppose that Lagrange interpolation at a given set of nodes $x_1, x_2, \cdots, x_n$ is used to derive a quadrature rule. Prove that the corresponding weights are given by the integrals of the Lagrange basis functions, $w_i = \int_a^b l_i(x)dx\,, i = 1, \cdots, n$. [5 points]

(7) [Programming]: Calculate the integral $\int_0^3 e^x dx$: [30 points]

(a) Apply composite <u>midpoint</u> integration rule with mesh sizes $h = 2^i$ for $i = 1, \cdots, 10$ to compute the approximate value of the integral. Calculate the error for each choice of h and then plot the error against the h in log-log scales.

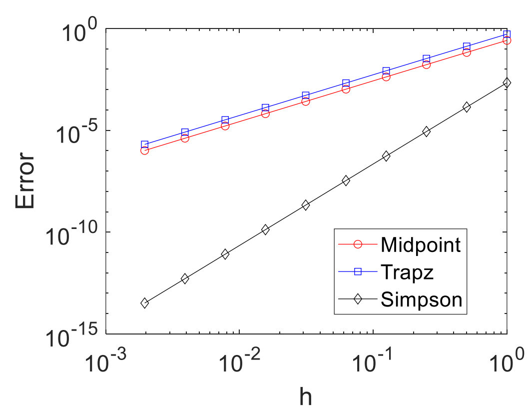
(b) Apply composite <u>trapezoid</u> integration rule with mesh sizes $h = 2^i$ for $i = 1, \cdots, 10$ to compute the approximate value of the integral. Calculate the error for each choice of h and then plot the error against the h in log-log scales.

(c) Apply composite <u>Simpson's</u> integration rule with mesh sizes $h = 2^i$ for $i = 1, \cdots, 10$ to compute the approximate value of the integral. Calculate the error for each choice of h and then plot the error against the h in log-log scales.

(d) Apply composite <u>3-points Gauss</u> integration rule with mesh sizes $h = 2^i$ for $i = 1, \cdots, 10$ to compute the approximate value of the integral. Calculate the error for each choice of h and then plot the error against the h in log-log scales.

(e) For each case above, you can fit the error as a function of h in the form $Error = Ch^{order}$. Use the linear least square method to estimate the fitted parameters C and $order$.

Hint: the plots should be similar to the one below



(8) Exercises 8.12 and 8.13 [20 points]

8.12. The forward difference formula

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

and the backward difference formula

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}$$

are both first-order accurate approximations to the first derivative of a function $f\colon \mathbb{R} \to \mathbb{R}$. What order accuracy results if we average these two approximations? Support your answer with an error analysis.

8.13. Given a sufficiently smooth function $f\colon \mathbb{R} \to \mathbb{R}$, use Taylor series to derive a second-order accurate, one-sided difference approximation to $f'(x)$ in terms of the values of $f(x)$, $f(x+h)$, and $f(x+2h)$.