Assignment 3

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1

Data points $(-1, 1), (0, 0), (1, 1)$.

a

Since there are 3 data points, $\phi_1(t) = 1, \phi_2(t) = x, \phi_3(t) = t^2, \theta = [a, b, c]^\top$.

$$\Phi_{i,j} = \phi_j(i) \tag{1}$$

$$\Phi = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \tag{2}$$

$$\Phi\theta = y \Leftrightarrow \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{3}$$

Thus, $p(t) = \phi(t)^\top \theta = t^2$.

b

As for Lagrange basis,

$$\phi_i(t) = \prod_{0 \le j \le n \land i \ne j} \frac{t - t_j}{t_i - t_j} \tag{4}$$

$$\phi_1(t) = \frac{t-0}{-1-0} \frac{t-1}{-1-1} = \frac{t(t-1)}{2} \tag{5}$$

$$\phi_2(t) = \frac{t+1}{0+1} \frac{t-1}{0-1} = -t^2 + 1 \tag{6}$$

$$\phi_3(t) = \frac{t+1}{1+1} \frac{t-0}{1-0} = \frac{t(t+1)}{2} \tag{7}$$

$$p(t) = \phi(t)^\top y = t^2 \tag{8}$$

c

As for Newton basis, $\theta = [a, b, c]^\top$,

$$\phi_i(t) = \prod_{0 \le j \le n \land i \ne j} (t - t_j) \tag{9}$$

$$\phi_1(t) = 1 \tag{10}$$

$$\phi_2(t) = t + 1 \tag{11}$$

$$\phi_3(t) = t(t+1) \tag{12}$$

$$\theta_1 = y_1 = 1 \tag{13}$$

$$\theta_{i+1} = \frac{y_{i+1} - p_i(t_{i+1})}{\phi_{i+1}(t_{i+1})} \tag{14}$$

Thus, $\theta = [1, -1, 1]^\top, p(t) = \phi(t)^\top \theta = t^2$.

d

$p(t) = t^2$ is always obtained.

e

As for linear interpolating,

$$p(t) = \begin{cases} -t, -1 \leq t < 0 \\ t, 0 \leq t \leq 1 \end{cases} \tag{15}$$

f

As for cubic natural spline interpolation,

$$p(t) = \begin{cases} p_1(t) = \alpha_1 + \alpha_2 t + \alpha_3 t^2 + \alpha_4 t^3, -1 \leq t < 0 \\ p_2(t) = \beta_1 + \beta_2 t + \beta_3 t^2 + \beta_4 t^3, 0 \leq t \leq 1 \end{cases} \tag{16}$$

Interpolate the data at the endpoints,

$$p_1(-1) = \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 = 1 \tag{17}$$

$$p_1(0) = \alpha_1 = 0 \tag{18}$$

$$p_2(0) = \beta_1 = 0 \tag{19}$$

$$p_2(1) = \beta_1 + \beta_2 + \beta_3 + \beta_4 = 1 \tag{20}$$

The first derivative of the interpolating function is continuous at $t = 0$,

$$\frac{\partial p_1(t)}{\partial t}_{|t=0} = \alpha_2 = \beta_2 = \frac{\partial p_2(t)}{\partial t}_{|t=0} \tag{21}$$

The second derivative of the interpolating function is continuous at $t = 0$,

$$\frac{\partial^2 p_1(t)}{\partial t^2}_{|t=0} = 2\alpha_3 = 2\beta_3 = \frac{\partial^2 p_2(t)}{\partial t^2}_{|t=0} \tag{22}$$

A natural spline has second derivative equal to zero at the endpoints,

$$\frac{\partial^2 p_1(t)}{\partial t^2}_{|t=-1} = 2\alpha_3 - 6\alpha_4 = 0 \tag{23}$$

$$\frac{\partial^2 p_2(t)}{\partial t^2}_{|t=1} = 2\beta_3 + 6\beta_4 = 0 \tag{24}$$

From (17) to (24),

$$[\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4]^\top = \left[0, 0, \frac{3}{2}, \frac{1}{2}, 0, 0, \frac{3}{2}, -\frac{1}{2}\right]^\top \tag{25}$$

Thus,

$$p(t) = \begin{cases} p_1(t) = \frac{1}{2}t^3 + \frac{3}{2}t^2, -1 \leq t < 0 \\ p_2(t) = -\frac{1}{2}t^3 + \frac{3}{2}t^2, 0 \leq t \leq 1 \end{cases} \tag{26}$$

2

Notice that $[-1,1]$ is a symmetric interval.

$$P_1(x)P_2(x)w(x) = (3x^3 - x)w(x) \tag{27}$$

is odd function when $w(x)$ is even.

That is, when $w(x)$ is even,

$$\int_{-1}^{1} P_1(x)P_2(x)w(x)\mathrm{d}x = 0 \tag{28}$$

3

$$T_k(x) = \cos(k \arccos(t)), t \in [-1,1] \tag{29}$$

a

$$\begin{aligned}
\int_{-1}^{1} \frac{T_i(t)T_j(t)}{\sqrt{1-t^2}}\mathrm{d}t &= \int_{-\pi}^{0} \frac{T_i(\cos x)T_j(\cos x)}{\sqrt{1-\cos^2 x}}\mathrm{d}\cos x & (30)\\
&= \int_{-\pi}^{0} \frac{\cos ix \cos jx}{-\sin x}(-\sin x \mathrm{d}x) & (31)\\
&= \int_{-\pi}^{0} \cos ix \cos jx \mathrm{d}x & (32)
\end{aligned}$$

When $i \neq j$,

$$\int_{-\pi}^{0} \cos ix \cos jx \mathrm{d}x = \int_{-\pi}^{0} \frac{1}{2}[\cos{(i+j)t} + \cos{(i-j)t}]\mathrm{d}x = 0 \tag{33}$$

When $i = j$,

$$\begin{aligned}
\int_{-\pi}^{0} \cos ix \cos jx \mathrm{d}x &= \int_{-\pi}^{0} \cos^2 ix \mathrm{d}x & (34)\\
&= \int_{-\pi}^{0} \frac{1+\cos 2ix}{2}\mathrm{d}x & (35)\\
&= \frac{\pi}{2} & (36)
\end{aligned}$$

Thus,

$$\int_{-1}^{1} \frac{T_i(t)T_j(t)}{\sqrt{1-t^2}}\mathrm{d}t = \frac{\pi}{2}\delta_{i,j} \tag{37}$$

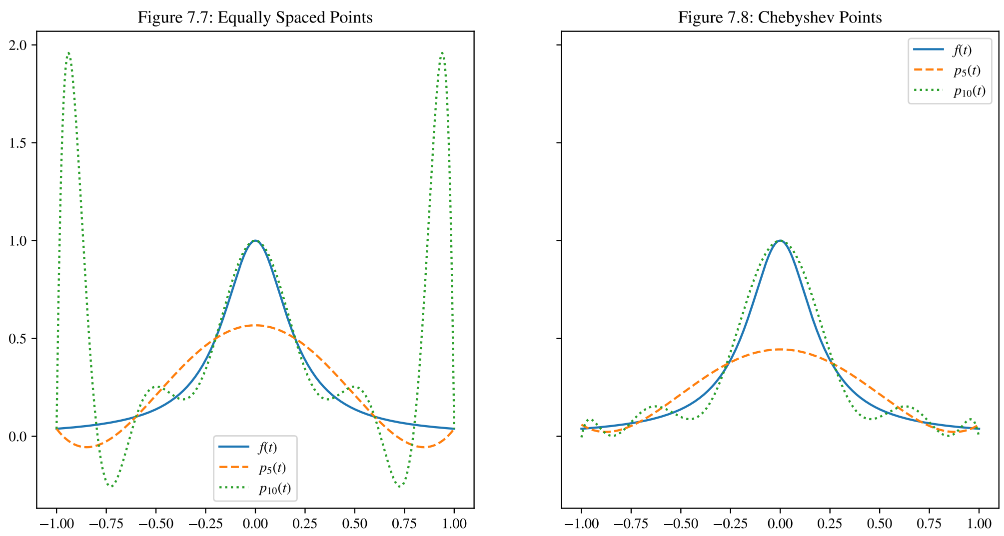
4

Core Code

```
def chebyshev_nodes(n, a, b):
    return 0.5 * (a + b) + 0.5 * (b - a) * np.cos(np.pi * (2 *
np.arange(1, n + 1) - 1) / (2 * n))

# Lagrange polynomial interpolation
def polynomial_interpolation(points, x_values):
    x_points, y_points = points
    polynomial = np.zeros_like(x_values)
    n = len(x_points)
    for i in range(n):
        term = y_points[i]
        for j in range(n):
            if i != j:
                term *= (x_values - x_points[j]) / (x_points[i] -
x_points[j])
        polynomial += term
    return polynomial
```

Result



5

a

$$M(f) = (b-a)f\left(\frac{a+b}{2}\right) = 1 \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{8} \tag{38}$$

$$T(f) = \frac{b-a}{2}(f(a)+f(b)) = \frac{1}{2}(0+1) = \frac{1}{2} \tag{39}$$

b

$$E(f) = \frac{T(f) - M(f)}{3} = \frac{1}{8} \tag{40}$$

$$-2E(f) = -\frac{1}{4} \tag{41}$$

c

$$S(f) = \frac{2M(f) + T(f)}{3} = \frac{1}{4} \tag{42}$$

d

Yes, since when $I(f) = S(f)$, $\frac{\partial^3 R(f)}{\partial x^3} = 0$, which means the Simpson rule is of degree 3.

e

$$\int_0^1 x^3 \mathrm{d}x = A_0 f(x_0) + A_1 f(x_1) \tag{43}$$

There are 4 parameters.

$$\int_0^1 1 = 1 = A_0 + A_1 \tag{44}$$

$$\int_0^1 x = \frac{1}{2} = A_0 x_0 + A_1 x_1 \tag{45}$$

$$\int_0^1 x^2 = \frac{1}{3} = A_0 x_0^2 + A_1 x_1^2 \tag{46}$$

$$\int_0^1 x^3 = \frac{1}{4} = A_0 x_0^3 + A_1 x_1^3 \tag{47}$$

The solution is $[A_0, A_1, x_0, x_1]^\top = [\frac{1}{2}, \frac{1}{2}, \frac{3-\sqrt{3}}{6}, \frac{3+\sqrt{3}}{6}]^\top$

$$G(f) = A_0 f(x_0) + A_1 f(x_1) = \frac{1}{4} \tag{48}$$

The result is exact since $G(f)$ is of degree $4 - 1 = 3$.

f

$$M_2(f) = (0.5 - 0)f(0.25) + (1 - 0.5)f(0.75) = 0.21875 \tag{49}$$

6

$$f(x) = L_n(x) = \sum_{0 \le k \le n} f(x_k) l_k(x) \tag{50}$$

$$I(f) = \int_a^b L_n(x) \mathrm{d}x = \int_a^b \sum_{0 \le k \le n} f(x_k) l_k(x) \mathrm{d}x \tag{51}$$

$$= \sum_{0 \le k \le n} f(x_k) \int_a^b l_k(x) \mathrm{d}x \tag{52}$$

$$= \sum_{0 \le k \le n} w_k f(x_k) \tag{53}$$

7

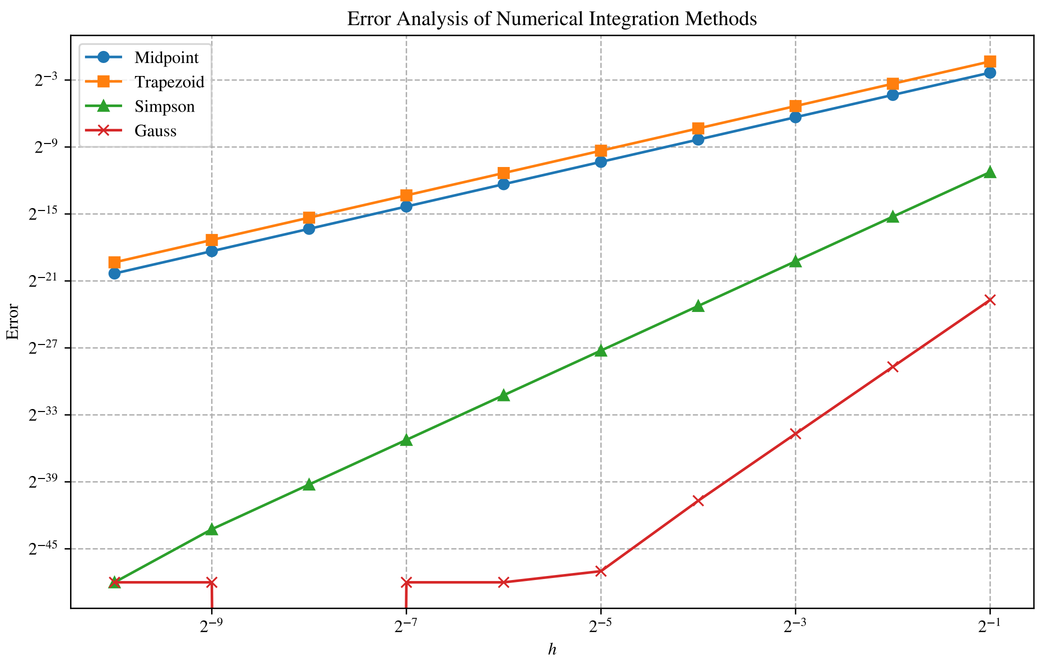
$$e = C \cdot h^n \Leftrightarrow \log e = \log C + n \log h \tag{54}$$

Output

```
# (C, order)
0.7923224542232744 1.9992933732410973
1.587137649897742 1.9995965300241643
0.007343895022866783 4.041227723532598
9.372066198851523e-06 5.99563201077984
```

That is, the dominant term in error of composite midpoint or Trapezoid rules is $O(h^2)$, the dominant term in error of the composite Simpson rule is $O(h^4)$ and the dominant term in error of the composite 3-points Gauss rules is $O(h^6)$.

Result



8

8.12

Average them,

$$\frac{\frac{f(x+h)-f(x)}{h}+\frac{f(x)-f(x-h)}{h}}{2}=\frac{f(x+h)-f(x-h)}{2h} \tag{55}$$

By Taylor series,

$$f(x+h)=f(x)+f'(x)h+\frac{1}{2}f''(x)h^2+O(h^3) \tag{56}$$

$$f(x-h)=f(x)-f'(x)h+\frac{1}{2}f''(x)h^2+O(h^3) \tag{57}$$

Thus,

$$\frac{f(x+h)-f(x-h)}{2h}=f'(x)+O(h^2) \tag{58}$$

, which is second-order accurate.

8.13

$$f(x+h)=f(x)+f'(x)h+\frac{1}{2}f''(x)h^2+O(h^3) \tag{59}$$

$$f(x+2h)=f(x)+2f'(x)h+2f''(x)h^2+O(h^3) \tag{60}$$

Consider the linear composition $af(x)+bf(x+h)+cf(x+2h)$,

$$\frac{b}{2}+2c=0 \tag{61}$$

Suppose $b=4, c=-1$, then $a=-b-c=-3$ and $b+2c=2$.

That is,

$$f'(x)=\frac{-3f(x)+4f(x+h)-f(x+2h)}{2h}+O(h^2) \tag{62}$$

, which is second-order accurate.