Homework 1: Problems (Due March 21)

Remark: for [programing] problems, you are supposed to write a brief report about the solution with code attached. You are also asked to submit a source code.

(1) Consider a floating-point system

$$x = \pm\left(d_0 + \frac{d_1}{\beta} + \frac{d_2}{\beta^2} + \cdots + \frac{d_{p-1}}{\beta^{p-1}}\right)\beta^E$$

where $0 \le d_i \le \beta - 1$, $i = 0, \ldots, p-1$, and $L \le E \le U$

Prove the following arguments:

- Total number of normalized floating-point numbers is

  $$2(\beta - 1)\beta^{p-1}(U - L + 1) + 1$$

- Smallest positive normalized number: $\text{UFL} = \beta^L$
- Largest floating-point number: $\text{OFL} = \beta^{U+1}(1 - \beta^{-p})$

(2) Exercise 1.4

Consider the problem of evaluating the function sin(x), in particular, the propagated data error, i.e., the error in the function value due to a perturbation h in the argument x.

(a) Estimate the absolute error in evaluating sin(x).
(b) Estimate the relative error in evaluating sin(x).
(c) Estimate the condition number for this problem.
(d) For what values of the argument x is this problem highly sensitive?

(3) [Programming], adapted from Computer problem 1.7

(a). Write a program to compute an approximate value for the derivative of a function using the finite-difference formula

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

Test your program using the function $\cos x$ for x = 1. Determine the error by comparing with the square of the built-in function $-\sin x$. Plot the magnitude of the error as a function of h, for $h = 10^{-k}$, $k = 0, \cdots, 16$. You should use log scale for h and for the magnitude of the error. Is there a minimum value for the magnitude of the error? How does the corresponding value for h compare with the rule of thumb $h = \sqrt{\epsilon_{\text{mach}}}$ derived in Example 1.3?

(b) Repeat the exercise using the centered difference approximation

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

(4) [Programming] Computer problem 1.9

(a). Write a program to compute the exponential function e^x using the infinite series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

(b). Summing in the natural order, what stopping criterion should you use?

(c). Test your program for

$$x = \pm 1, \pm 5, \pm 10, \pm 15, \pm 20$$

and compare your results with the built-in function $\exp(x)$.

(5) Prove the following arguments, which indicates the equivalence of vector norms for $\boldsymbol{x} \in \mathbf{R}^n$

$$\|\boldsymbol{x}\|_2 \leq \|\boldsymbol{x}\|_1 \leq \sqrt{n}\|\boldsymbol{x}\|_2$$
$$\|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_2 \leq \sqrt{n}\|\boldsymbol{x}\|_\infty$$

(6) Exercise 2.4

2.4. (*a*) Show that the following matrix is singular.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

(*b*) If $\boldsymbol{b} = [2 \quad 4 \quad 6]^T$, how many solutions are there to the system $\boldsymbol{Ax} = \boldsymbol{b}$?

(c) Compute the matrix norm of the matrix A (in 1 and ∞ norm)

(d) What is the condition number of the matrix?

(7) [Programming]

(a) Implement Algorithm 2.4 (LU Factorization by Gaussian Elimination with partial pivoting). You can use Example 2.16 to verify your program.

(b) Use the program above to compute the inverse of the matrix $\boldsymbol{A}$ in the example via solving $\boldsymbol{Ax_i} = \boldsymbol{e_i}$ for $1 \leq i \leq 3$, where e_i is the i-th unit vector whose i-th entry is equal to one and all the rest are zeros.

(c) Compute the condition number of the matrix $\boldsymbol{A}$ (in 1 and ∞ norm) based on the inverse you obtained in (b).

(8) [Programming] Computer problem 2.3. Note: you can use Gaussian-elimination program with partial pivoting (the previous problem) or Gaussian-elimination program available on Blackboard to solve the problem. You are not asked to use sparse solvers. More details can be found in the textbook.