# Assignment 4

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# 1

## a

Since $-5 < 0$, the solutions to this ODE is asymptotically stable.

## b

Since $|1 + h\lambda| = 1.5 > 1$, Euler's method is not stable for this ODE.

## c

$$
\begin{aligned}
y_{k+1} \simeq y_k + hy'_k = y_k - 5hy_k = (1 - 5h)y_k \quad (1)\\
y(0.5) = (1 - 5 \cdot 0.5)y(0) = -1.5 \quad (2)
\end{aligned}
$$

## d

Since $\left|\frac{1}{1-h\lambda}\right| = \frac{2}{7} < 1$, the backward Euler's method is not stable for this ODE.

## e

$$
\begin{aligned}
y_{k+1} &\simeq y_k + hy'_{k+1} = y_k - 5hy_{k+1} &(3)\\
\Leftrightarrow y_{k+1} &= \frac{1}{1+5h} y_k &(4)\\
y(0.5) &= \frac{1}{1+5 \cdot 0.5} y(0) = \frac{2}{7} &(5)
\end{aligned}
$$

# 2

## a

$$
\begin{aligned}
y_{k+1} \simeq y_k + hy'_{k+1} = y_k - hy^2_{k+1} \quad (6)\\
\Leftrightarrow g(y_{k+1}) := hy^2_{k+1} + y_{k+1} - y_k = 0 \quad (7)
\end{aligned}
$$

## b

$$
\begin{aligned}
\frac{\partial g(y_{k+1})}{\partial y_{k+1}} &= 2hy_{k+1} + 1 &(8)\\
g(y_{k+1})\left(\frac{\partial g(y_{k+1})}{\partial y_{k+1}}\right)^{-1} &= \frac{hy^2_{k+1} + y_{k+1} - y_k}{2hy_{k+1} + 1} &(9)
\end{aligned}
$$

To solve (7),

$$
y^{\text{new}}_{k+1} \leftarrow y_k - \frac{hy^2_{k+1} + y_{k+1} - y_k}{2hy_{k+1} + 1} \quad (10)
$$

### c

$$y_1^{\text{init}} \leftarrow y_0 + hy_0' = y_0 - hy_0^2 = 0.9 \tag{11}$$

### d

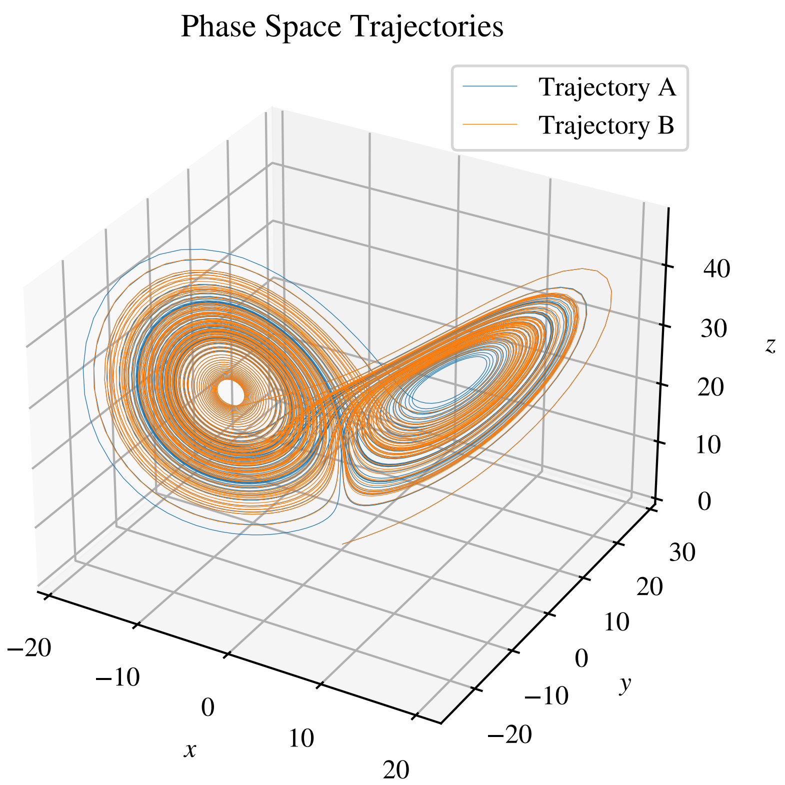
$$y_1^{[1]} \leftarrow y_0 - \frac{hy_1^2 + y_1 - y_0}{2hy_1 + 1} = 0.9161 \tag{12}$$

# 3

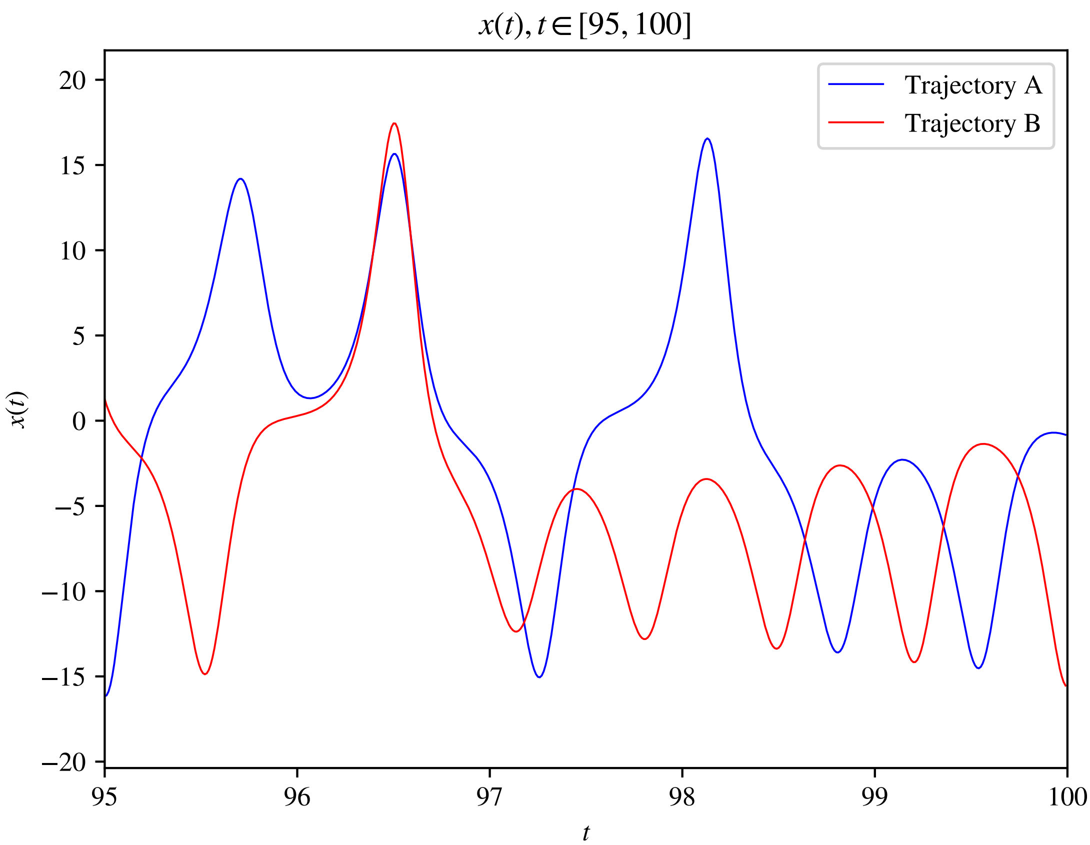
## Core Code

```
# Classical fourth-order Runge-Kutta method
def runge_kutta_4(f, y0, t):
    n = len(t)
    y = np.zeros((n, len(y0)))
    y[0] = y0
    dt = t[1] - t[0]
    for i in range(1, n):
        k1 = dt * f(y[i - 1], t[i - 1])
        k2 = dt * f(y[i - 1] + 0.5 * k1, t[i - 1] + 0.5 * dt)
        k3 = dt * f(y[i - 1] + 0.5 * k2, t[i - 1] + 0.5 * dt)
        k4 = dt * f(y[i - 1] + k3, t[i - 1] + dt)
        y[i] = y[i - 1] + (k1 + 2 * k2 + 2 * k3 + k4) / 6
    return y
```

### a & b


Phase Space Trajectories
Trajectory A
Trajectory B
40
30
20
10
0
z
30
20
10
0
−10
−20
y
−20
−10
0
10
20
x

**c**


$x(t), t \in [95, 100]$
Trajectory A
Trajectory B
20
15
10
5
0
−5
−10
−15
−20
$x(t)$
95
96
97
98
99
100
$t$

# 4

## a

$$
\begin{aligned}
u_{k+1} &= u_k + hu'_k && (13)\\
u(b) &= u(a) + hu'(a) && (14)\\
\Leftrightarrow \beta &= \alpha + (b-a)u'(a) && (15)
\end{aligned}
$$

## b

$$
u'(a) = \frac{\beta - \alpha}{b - a} \tag{16}
$$

# 5

## a

$$
\begin{aligned}
u_{k+1} &= u_k + h_k \cdot \frac{u'_k + u'_{k+1}}{2} && (17)\\
\Leftrightarrow u(1) &= u(0) + 1 \cdot \frac{u'(0) + u'(1)}{2} && (18)\\
\Leftrightarrow 1 &= 1 + 1 \cdot \frac{s_0 + s_1}{2} && (19)\\
\Leftrightarrow s_0 + s_1 &= 0 && (20)\\
u'_{k+1} &= u'_k + h_k \cdot \frac{u''_k + u''_{k+1}}{2} && (21)\\
\Leftrightarrow u'(1) &= u'(0) + 1 \cdot \frac{u''(0) + u''(1)}{2} && (22)\\
\Leftrightarrow s_1 &= s_0 + 1 \cdot \frac{-4-4}{2} && (23)\\
\Leftrightarrow -s_0 + s_1 &= -4 && (24)
\end{aligned}
$$

(20)(24) could be easily solved,

$$
s_0 = 2, s_1 = -2 \tag{25}
$$

Then,

$$
\begin{aligned}
u'(0.5) &= u'(0) + 0.5 \cdot \frac{u''(0) + u''(0.5)}{2} = 2 - 2 = 0 && (26)\\
u(0.5) &= u(0) + 0.5 \cdot \frac{u'(0) + u'(0.5)}{2} = 1 + 0.5 = 1.5 && (27)
\end{aligned}
$$

## b

$$
\begin{aligned}
u'_k &= \frac{u_{k+1} - u_{k-1}}{2h} && (28)\\
u''_k &= \frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} && (29)\\
u''(0.5) &= \frac{u(1) - 2u(0.5) + u(0)}{0.5^2} && (30)\\
&= \frac{2 - 2u(0.5)}{0.25} = -4 && (31)
\end{aligned}
$$

That is,

$$u(0.5) = 1.5 \tag{32}$$

### c

Suppose

$$u(t) = x_1 t^2 + x_2 t + x_3 \tag{33}$$

, then

$$u'(t) = 2x_1 t + x_2 \tag{34}$$

$$u''(t) = 2x_1 \tag{35}$$

The following equations would be obtained,

$$u''(t) = 2x_1 = -4 \tag{36}$$

$$u(0) = x_3 = 1 \tag{37}$$

$$u(1) = x_1 + x_2 + x_3 = 1 \tag{38}$$

Easily solved,

$$[x_1, x_2, x_3]^\top = [-2, 2, 1]^\top \tag{39}$$

Thus,

$$u(t) = -2t^2 + 2t + 1 \tag{40}$$

$$u(0.5) = 1.5 \tag{41}$$

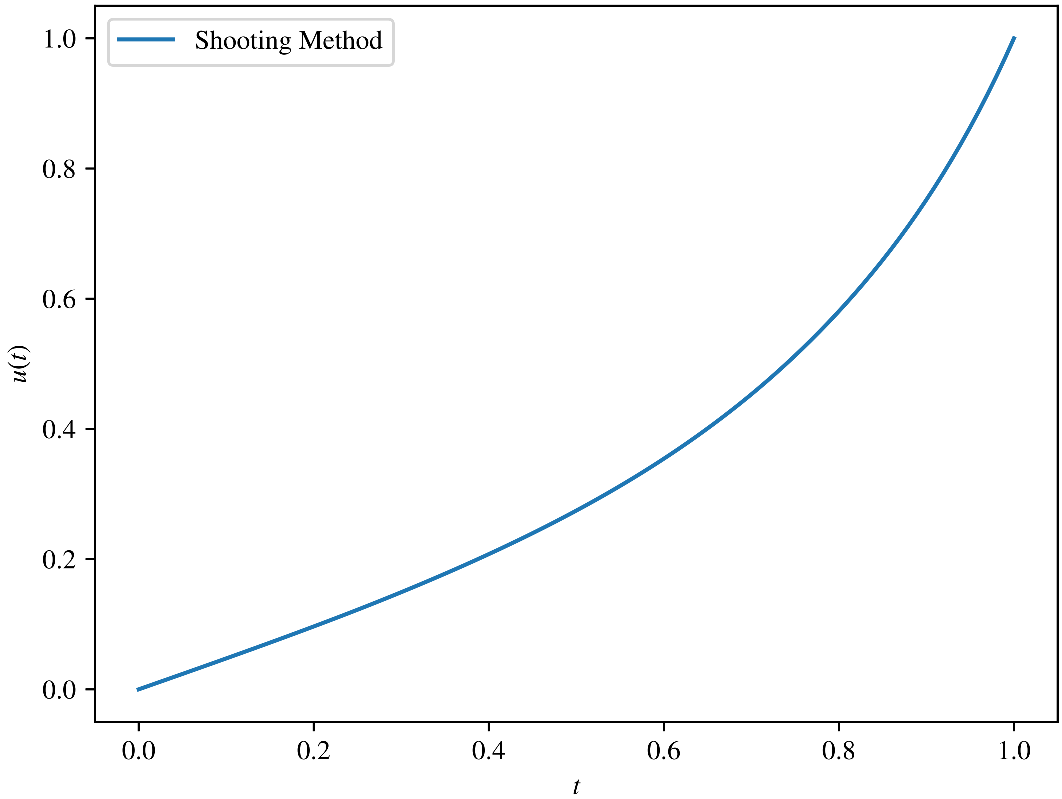
# 6

### a

## Core Code

```
def ode_system(t, y):
    u, up = y
    return [up, 10 * u**3 + 3 * u + t**2]

# 定义目标函数，用于寻找合适的初始斜率
def shooting_function(up0):
    if isinstance(up0, np.ndarray):
        up0 = up0[0]
    sol = solve_ivp(ode_system, [0, 1], [0, up0], t_eval=[1])
    return sol.y[0, -1] - 1

# 寻找合适的初始斜率
up0_guess = 1.0  # 初始猜测
sol = root(shooting_function, up0_guess)
up0_optimal = sol.x[0]
```

## Result



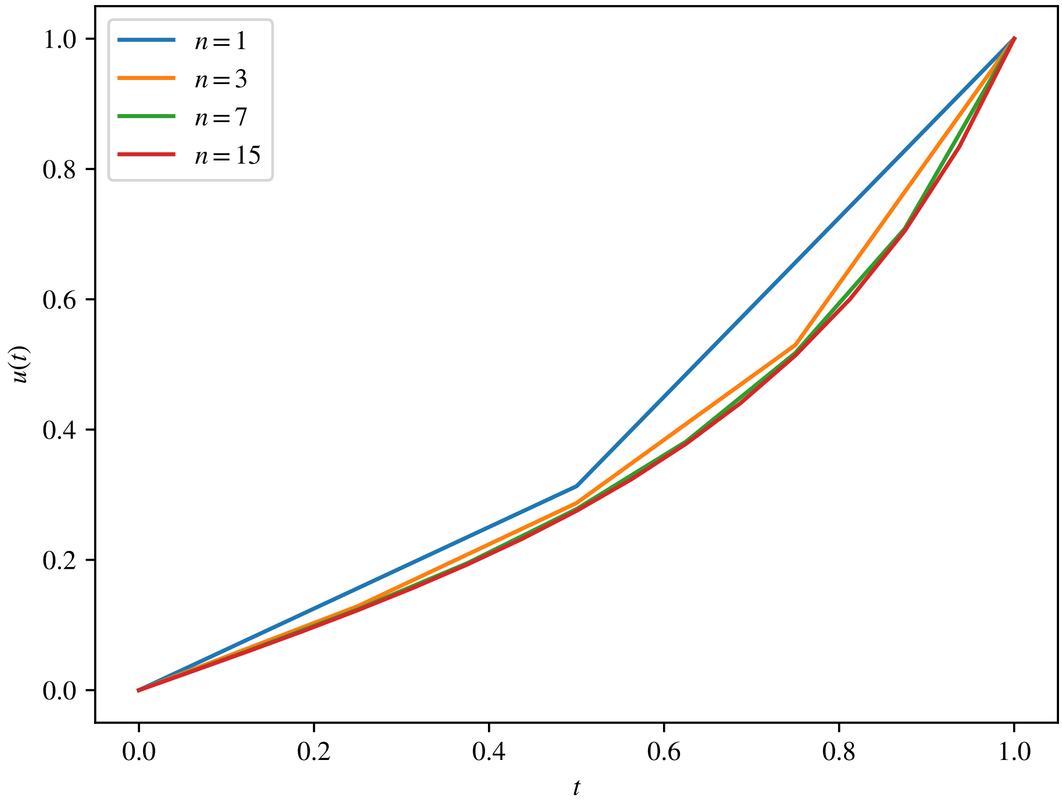

## b

## Core Code

```
# 牛顿迭代法
for _ in range(max_iter):
    # u_old = u.copy()
    F = np.zeros(n)
    J = np.zeros((n, n))

    for i in range(1, n + 1):
        F[i - 1] = (u[i + 1] - 2 * u[i] + u[i - 1]) / h**2 - (
            10 * u[i] ** 3 + 3 * u[i] + t_values[i] ** 2
        )
        J[i - 1, i - 1] = -2 / h**2 - 30 * u[i] ** 2 - 3
        if i > 1:
            J[i - 1, i - 2] = 1 / h**2
        if i < n:
            J[i - 1, i] = 1 / h**2

    # 求解线性方程组 J * delta_u = -F
    delta_u = np.linalg.solve(J, -F)

    # 更新解
    u[1 : n + 1] += delta_u
```

```python
    # 检查收敛性
    if np.linalg.norm(delta_u) < tol:
        break
```

### Result



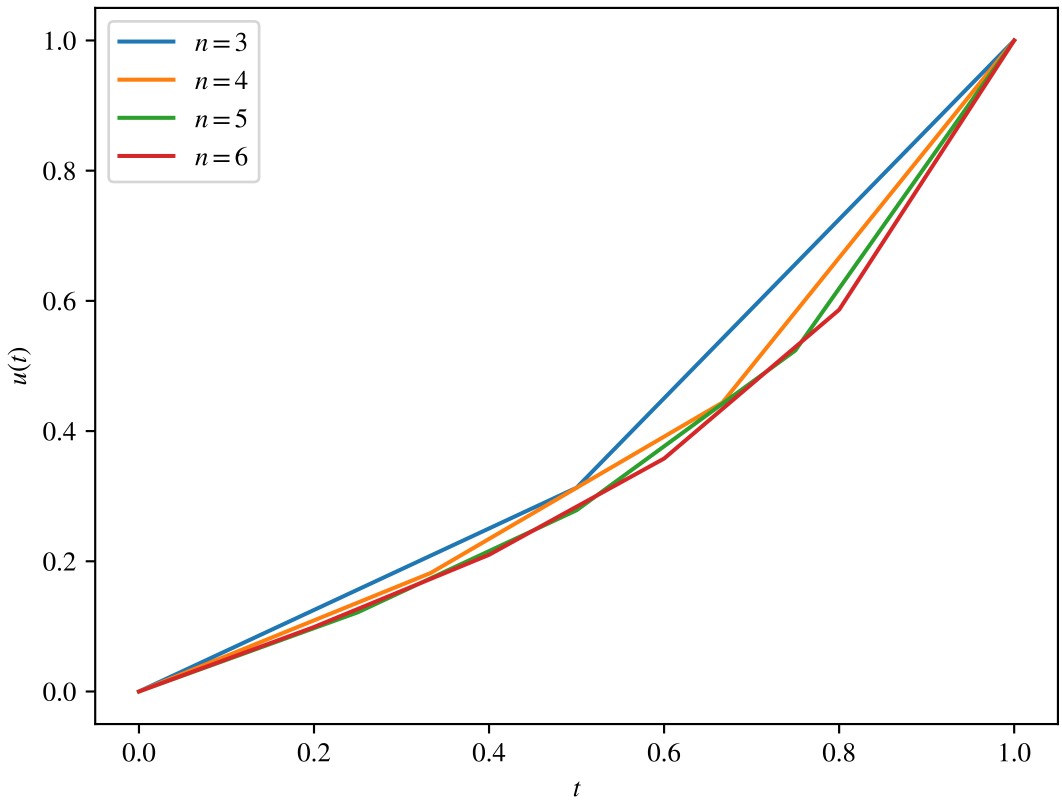

### c

### Core Code

```python
def func_u(t, x):
    _u = 0
    for i in range(n):
        _u += x[i] * t**i
    return _u

def func_ddu_dtt(t, x):
    _ddu_dtt = 0
    for i in range(2, n):
        _ddu_dtt += i * (i - 1) * x[i] * t ** (i - 2)
    return _ddu_dtt

def residuals(x):
    res = (
        [func_u(0, x) - 0]
        + [
```

```
            func_ddu_dtt(t, x)
            - 10 * func_u(t, x) ** 3
            - 3 * func_u(t, x)
            - t**2
            for t in t_values[1:-1]
        ]
        + [func_u(1, x) - 1]
    )
    return res

result = root(residuals, np.linspace(0, 1, n))
x = result.x
u = [func_u(t, x) for t in t_values]

return t_values, u
```

### Result




## 7

Let $h = 0.5$,

$$u_{xx} + u_{yy} = \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} \tag{42}$$

$$+ \frac{u(x,y+h) - 2u(x,y) + u(x,y-h)}{h^2} \tag{43}$$

$$= x + y \tag{44}$$

Thus,

$$\frac{u(1,0.5)-2u(0.5,0.5)+u(0,0.5)}{0.25}+\frac{u(0.5,1)-2u(0.5,0.5)+u(0.5,0)}{0.25}=0.5+0.5 \tag{45}$$

$$\Leftrightarrow 4(u(1,0.5)-2u(0.5,0.5)+u(0,0.5)+u(0.5,1)-2u(0.5,0.5)+u(0.5,0))=1 \tag{46}$$

From the figure,

$$u(1,0.5)=1, u(0,0.5)=0, u(0.5,1)=1, u(0.5,0)=0 \tag{47}$$

Thus,

$$4(2-4u(0.5,0.5))=1 \Leftrightarrow u(0.5,0.5)=\frac{7}{16}=0.4375 \tag{48}$$