# Assignment 3

Name: 陈子蔚 SID: 12332440

## 1

Data points $(-1,1),(0,0),(1,1)$.

### a

Since there are 3 data points, $\phi_1(t)=1,\phi_2(t)=x,\phi_3(t)=t^2,\theta=[a,b,c]^\top$.

$$\Phi_{i,j}=\phi_j(i) \tag{26}$$

$$\Phi=\begin{bmatrix}1 & -1 & 1\\ 1 & 0 & 0\\ 1 & 1 & 1\end{bmatrix} \tag{27}$$

$$\Phi\theta=y\Leftrightarrow\begin{bmatrix}1 & -1 & 1\\ 1 & 0 & 0\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}\Leftrightarrow\begin{bmatrix}a\\ b\\ c\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix} \tag{28}$$

Thus, $p(t)=\phi(t)^\top\theta=t^2$.

### b

As for Lagrange basis,

$$\phi_i(t)=\prod_{0\le j\le n\wedge i\ne j}\frac{t-t_j}{t_i-t_j} \tag{29}$$

$$\phi_1(t)=\frac{t-0}{-1-0}\frac{t-1}{-1-1}=\frac{t(t-1)}{2} \tag{30}$$

$$\phi_2(t)=\frac{t+1}{0+1}\frac{t-1}{0-1}=-t^2+1 \tag{31}$$

$$\phi_3(t)=\frac{t+1}{1+1}\frac{t-0}{1-0}=\frac{t(t+1)}{2} \tag{32}$$

$$p(t)=\phi(t)^\top y=t^2 \tag{33}$$

### c

As for Newton basis, $\theta=[a,b,c]^\top$,

$$\phi_i(t)=\prod_{0\le j\le n\wedge i\ne j}(t-t_j) \tag{34}$$

$$\phi_1(t)=1 \tag{35}$$

$$\phi_2(t)=t+1 \tag{36}$$

$$\phi_3(t)=t(t+1) \tag{37}$$

$$\theta_1=y_1=1 \tag{38}$$

$$\theta_{i+1}=\frac{y_{i+1}-p_i(t_{i+1})}{\phi_{i+1}(t_{i+1})} \tag{39}$$

Thus, $\theta=[1,-1,1]^\top, p(t)=\phi(t)^\top\theta=t^2$.

## d

$p(t)=t^2$ is always obtained.

## e

As for linear interpolating,

$$p(t)=\begin{cases}-t, -1\le t<0\\ t, 0\le t\le 1\end{cases} \tag{40}$$

## f

As for cubic natural spline interpolation,

$$p(t)=\begin{cases}p_1(t)=\alpha_1+\alpha_2 t+\alpha_3 t^2+\alpha_4 t^3, -1\le t<0\\ p_2(t)=\beta_1+\beta_2 t+\beta_3 t^2+\beta_4 t^3, 0\le t\le 1\end{cases} \tag{41}$$

Interpolate the data at the endpoints,

$$p_1(-1)=\alpha_1-\alpha_2+\alpha_3-\alpha_4=1 \tag{42}$$
$$p_1(0)=\alpha_1=0 \tag{43}$$
$$p_2(0)=\beta_1=0 \tag{44}$$
$$p_2(1)=\beta_1+\beta_2+\beta_3+\beta_4=1 \tag{45}$$

The first derivative of the interpolating function is continuous at $t=0$,

$$\frac{\partial p_1(t)}{\partial t}_{|t=0}=\alpha_2=\beta_2=\frac{\partial p_2(t)}{\partial t}_{|t=0} \tag{46}$$

The second derivative of the interpolating function is continuous at $t=0$,

$$\frac{\partial^2 p_1(t)}{\partial t^2}_{|t=0}=2\alpha_3=2\beta_3=\frac{\partial^2 p_2(t)}{\partial t^2}_{|t=0} \tag{47}$$

A natural spline has second derivative equal to zero at the endpoints,

$$\frac{\partial^2 p_1(t)}{\partial t^2}_{|t=-1}=2\alpha_3-6\alpha_4=0 \tag{48}$$
$$\frac{\partial^2 p_2(t)}{\partial t^2}_{|t=1}=2\beta_3+6\beta_4=0 \tag{49}$$

From (42) to (49),

$$[\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4]^\top=\left[0,0,\frac{3}{2},\frac{1}{2},0,0,\frac{3}{2},-\frac{1}{2}\right]^\top \tag{50}$$

Thus,

$$p(t)=\begin{cases}p_1(t)=\frac{1}{2}t^3+\frac{3}{2}t^2, -1\le t<0\\ p_2(t)=-\frac{1}{2}t^3+\frac{3}{2}t^2, 0\le t\le 1\end{cases} \tag{51}$$

## 2

Notice that $[-1,1]$ is a symmetric interval.

$$P_1(x)P_2(x)w(x) = (3x^3 - x)w(x) \tag{52}$$

is odd function when $w(x)$ is even.

That is, when $w(x)$ is even,

$$\int_{-1}^{1} P_1(x)P_2(x)w(x)\mathrm{d}x = 0 \tag{53}$$

## 3

$$T_k(x) = \cos(k\arccos(t)), t \in [-1,1] \tag{54}$$

### a

$$\begin{aligned}
\int_{-1}^{1} \frac{T_i(t)T_j(t)}{\sqrt{1-t^2}}\mathrm{d}t &= \int_{-\pi}^{0} \frac{T_i(\cos x)T_j(\cos x)}{\sqrt{1-\cos^2 x}}\mathrm{d}\cos x && (55)\\
&= \int_{-\pi}^{0} \frac{\cos ix \cos jx}{-\sin x}(-\sin x\mathrm{d}x) && (56)\\
&= \int_{-\pi}^{0} \cos ix \cos jx\mathrm{d}x && (57)
\end{aligned}$$

When $i \neq j$,

$$\int_{-\pi}^{0} \cos ix \cos jx\mathrm{d}x = \int_{-\pi}^{0} \frac{1}{2}[\cos{(i+j)t} + \cos{(i-j)t}]\mathrm{d}x = 0 \tag{58}$$

When $i = j$,

$$\begin{aligned}
\int_{-\pi}^{0} \cos ix \cos jx\mathrm{d}x &= \int_{-\pi}^{0} \cos^2 ix\mathrm{d}x && (59)\\
&= \int_{-\pi}^{0} \frac{1+\cos 2ix}{2}\mathrm{d}x && (60)\\
&= \frac{\pi}{2} && (61)
\end{aligned}$$

Thus,

$$\int_{-1}^{1} \frac{T_i(t)T_j(t)}{\sqrt{1-t^2}}\mathrm{d}t = \frac{\pi}{2}\delta_{i,j} \tag{62}$$

### b

$$\begin{aligned}
T_3(t) &= \cos{(3\arccos t)} = \cos 3x && (63)\\
&= \cos{(2x+x)} = \cos 2x \cos x - \sin 2x \sin x && (64)\\
&= (2\cos^2 x - 1)\cos x - 2\cos x \sin^2 x && (65)\\
&= 2\cos^3 x - \cos x - 2\cos x(1-\cos^2 x) && (66)\\
&= 4\cos^3 x - 3\cos x && (67)
\end{aligned}$$

$$= 4t^3 - 3t \tag{68}$$

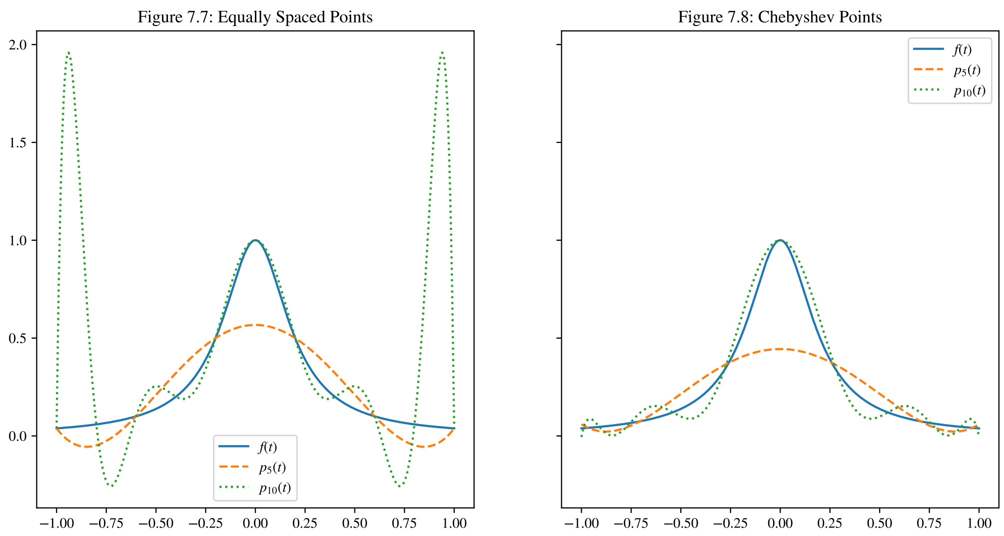
# 4

## Core Code

```python
def chebyshev_nodes(n, a, b):
    return 0.5 * (a + b) + 0.5 * (b - a) * np.cos(np.pi * (2 *
np.arange(1, n + 1) - 1) / (2 * n))

# Lagrange polynomial interpolation
def polynomial_interpolation(points, x_values):
    x_points, y_points = points
    polynomial = np.zeros_like(x_values)
    n = len(x_points)
    for i in range(n):
        term = y_points[i]
        for j in range(n):
            if i != j:
                term *= (x_values - x_points[j]) / (x_points[i] -
x_points[j])
        polynomial += term
    return polynomial
```

## Result




# 5

## a

$$M(f) = (b-a)f\left(\frac{a+b}{2}\right) = 1 \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{8} \tag{69}$$

$$T(f) = \frac{b-a}{2}(f(a) + f(b)) = \frac{1}{2}(0+1) = \frac{1}{2} \tag{70}$$

## b

$$E(f) = \frac{T(f) - M(f)}{3} = \frac{1}{8} \tag{71}$$

$$-2E(f) = -\frac{1}{4} \tag{72}$$

## c

$$S(f) = \frac{2M(f) + T(f)}{3} = \frac{1}{4} \tag{73}$$

## d

Yes, since when $I(f) = S(f)$, $\frac{\partial^3 R(f)}{\partial x^3} = 0$, which means the Simpson rule is of degree 3.

## e

$$\int_0^1 x^3 \mathrm{d}x = A_0 f(x_0) + A_1 f(x_1) \tag{74}$$

There are 4 parameters.

$$\int_0^1 1 = 1 = A_0 + A_1 \tag{75}$$

$$\int_0^1 x = \frac{1}{2} = A_0 x_0 + A_1 x_1 \tag{76}$$

$$\int_0^1 x^2 = \frac{1}{3} = A_0 x_0^2 + A_1 x_1^2 \tag{77}$$

$$\int_0^1 x^3 = \frac{1}{4} = A_0 x_0^3 + A_1 x_1^3 \tag{78}$$

The solution is $[A_0, A_1, x_0, x_1]^\top = [\frac{1}{2}, \frac{1}{2}, \frac{3-\sqrt{3}}{6}, \frac{3+\sqrt{3}}{6}]^\top$

$$G(f) = A_0 f(x_0) + A_1 f(x_1) = \frac{1}{4} \tag{79}$$

The result is exact since $G(f)$ is of degree $4 - 1 = 3$.

## f

$$M_2(f) = (0.5 - 0)f(0.25) + (1 - 0.5)f(0.75) = 0.21875 \tag{80}$$

## 6

$$f(x) = L_n(x) = \sum_{0 \le k \le n} f(x_k) l_k(x) \tag{81}$$

$$I(f) = \int_a^b L_n(x) \mathrm{d}x = \int_a^b \sum_{0 \le k \le n} f(x_k) l_k(x) \mathrm{d}x \tag{82}$$

$$= \sum_{0 \le k \le n} f(x_k) \int_a^b l_k(x) \mathrm{d}x \tag{83}$$

$$= \sum_{0 \le k \le n} w_k f(x_k) \tag{84}$$

## 7

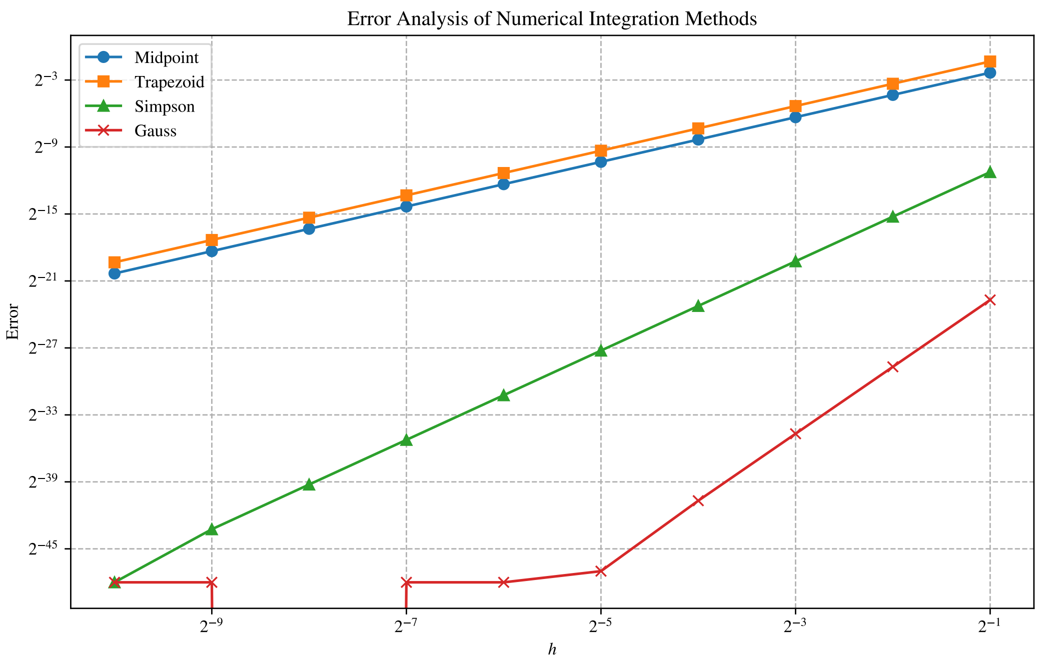
$$e = C \cdot h^n \Leftrightarrow \log e = \log C + n \log h \tag{85}$$

### Output

```
# (C, order)
0.7923224542232744 1.9992933732410973
1.587137649897742 1.9995965300241643
0.007343895022866783 4.041227723532598
9.372066198851523e-06 5.99563201077984
```

That is, the dominant term in error of composite midpoint or Trapezoid rules is $O(h^2)$, the dominant term in error of the composite Simpson rule is $O(h^4)$ and the dominant term in error of the composite 3-points Gauss rules is $O(h^6)$.

### Result



# 8

## 8.12

Average them,

$$\frac{\frac{f(x+h)-f(x)}{h}+\frac{f(x)-f(x-h)}{h}}{2}=\frac{f(x+h)-f(x-h)}{2h} \tag{86}$$

By Taylor series,

$$f(x+h)=f(x)+f'(x)h+\frac{1}{2}f''(x)h^2+O(h^3) \tag{87}$$
$$f(x-h)=f(x)-f'(x)h+\frac{1}{2}f''(x)h^2+O(h^3) \tag{88}$$

Thus,

$$\frac{f(x+h)-f(x-h)}{2h}=f'(x)+O(h^2) \tag{89}$$

, which is second-order accurate.

## 8.13

$$f(x+h)=f(x)+f'(x)h+\frac{1}{2}f''(x)h^2+O(h^3) \tag{90}$$
$$f(x+2h)=f(x)+2f'(x)h+2f''(x)h^2+O(h^3) \tag{91}$$

Consider the linear composition $af(x)+bf(x+h)+cf(x+2h)$,

$$\frac{b}{2}+2c=0 \tag{92}$$

Suppose $b=4, c=-1$, then $a=-b-c=-3$ and $b+2c=2$.

That is,

$$f'(x)=\frac{-3f(x)+4f(x+h)-f(x+2h)}{2h}+O(h^2) \tag{93}$$

, which is second-order accurate.