Q1

(a) Using monomial basis, linear system is

$$Ax = \begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y$$

For these three specific points, linear system is

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

whose solution is $x = [0, 0, 1]$, so interpolating polynomial is

$$p_2(t) = t^2$$

(b) Lagrange basis

$$l_1(t) = \frac{(t-0)(t-1)}{(-1-0)(-1-1)} = \frac{t(t-1)}{2}$$

$$l_2(t) = \frac{(t+1)(t-1)}{(0+1)(0-1)} = \frac{(t+1)(t-1)}{-1}$$

$$l_3(t) = \frac{(t+1)t}{(1+1)(1-0)} = \frac{(t+1)t}{2}$$

so interpolating polynomial is

$$y = \sum_{i=1}^{3} y_i l_i(t) = \frac{t(t-1)}{2} + 0 + \frac{(t+1)t}{2} = t^2.$$

(c) Using Newton basis, linear system is

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & t_2 - t_1 & 0 \\ 1 & t_3 - t_1 & (t_3 - t_1)(t_3 - t_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

The solution is $x = [1, -1, 1]$, so interpolating polynomial is

$$y = 1 - (t+1) + (t+1)t = t^2.$$

(d) It shows the three representations give the same polynomial.

(e) In subinterval [-1,1], $p_1(t) = -t$

In subinterval [0,1], $p_1(t) = t$

Thus, $y = \begin{cases} -t & t \le 0 \\ t & t > 0 \end{cases}$

(f) We have three known data points $(t_1, y_1) = (-1,1), (t_2, y_2) = (0,0), (t_3, y_3) = (1,1)$

Assume the two cubic polynomials

$$p_1(t) = a_1 + a_2 t + a_3 t^2 + a_4 t^3$$
$$p_2(t) = b_1 + b_2 t + b_3 t^2 + b_4 t^3$$

Interpolating three given data gives 4 equations

$$y_1 = a_1 + a_2 t_1 + a_3 t_1^2 + a_4 t_1^3$$
$$y_2 = a_1 + a_2 t_2 + a_3 t_2^2 + a_4 t_2^3$$
$$y_3 = b_1 + b_2 t_2 + b_3 t_2^2 + b_4 t_2^3$$
$$\mathrm{y}_4 = b_1 + b_2 t_3 + b_3 t_3^2 + b_4 t_3^3$$

According to condition of the first derivative being continuous, we have

$$a_2 + 2a_3 t_2 + 3a_4 t_2^2 = b_2 + 2b_3 t_2 + 3b_4 t_2^2$$

According to condition of the second derivative being continuous, we have

$$2a_3 + 6a_4 t_2 = 2b_3 + 6b_4 t_2$$

For natural spline has second derivatives equal to zero at endpoints, we get two equations

$$2a_3 + 6a_4 t_1 = 0$$
$$2b_3 + 6b_4 t_3 = 0$$

To summary, we get the total equations of the problem

$$y_1 = a_1 + a_2 t_1 + a_3 t_1^2 + a_4 t_1^3$$
$$y_2 = a_1 + a_2 t_2 + a_3 t_2^2 + a_4 t_2^3$$
$$y_3 = b_1 + b_2 t_2 + b_3 t_2^2 + b_4 t_2^3$$
$$y_4 = b_1 + b_2 t_3 + b_3 t_3^2 + b_4 t_3^3$$
$$a_2 + 2a_3 t_2 + 3a_4 t_2^2 = b_2 + 2b_3 t_2 + 3b_4 t_2^2$$
$$2a_3 + 6a_4 t_2 = 2b_3 + 6b_4 t_2$$
$$2a_3 + 6a_4 t_1 = 0$$
$$2b_3 + 6b_4 t_3 = 0$$

By solving the linear system above, we can get

$$(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4) = (0, 0, 1.5, 0.5, 0, 0, 1.5, -0.5)$$

Therefore, the interpolation

$$p_1(t) = 1.5t^2 + 0.5t^3 \quad t \in [-1, 0]$$
$$p_2(t) = 1.5t^2 - 0.5t^3 \quad t \in [0, 1]$$

Q2

Weight function w(t) = 1 on interval [−1,1], so

$$\int_{-1}^{1} P_1(x) P_2(x) \mathrm{d}x = \int_{-1}^{1} x(3x^2 - 1) \mathrm{d}x = \frac{3}{4}x^4 - \frac{1}{2}x^2 \Big|_{-1}^{1} = 0$$

Thus, the two Legendre polynomials are orthogonal to each other.

Q3

(a)

$$\int_{-1}^{1} \frac{T_i(t)T_j(t)}{\sqrt{1-t^2}} \mathrm{d}t = \int_{-1}^{1} \frac{\cos(i \cdot \arccos(t)) \cdot \cos(j \cdot \arccos(t))}{\sqrt{1-t^2}} \mathrm{d}t$$
$$= \int_{\pi}^{0} \frac{\cos(i \cdot x) \cdot \cos(j \cdot x)}{\sqrt{1-\cos^2(x)}} \mathrm{d}(\cos(x))$$
$$= -\int_{\pi}^{0} \cos(i \cdot x) \cdot \cos(j \cdot x) \mathrm{d}x$$
$$= -\frac{1}{2} \int_{\pi}^{0} [\cos((i+j) \cdot x) + \cos((i-j) \cdot x)] \mathrm{d}x$$

Since $i, j \in \mathbb{Z}^+, i, j > 0$, such that

$$\int_{\pi}^{0} \cos((i+j) \cdot x) \mathrm{d}x \equiv 0$$

So when $i \neq j$

$$\int_{-1}^{1} \frac{T_i(t)T_j(t)}{\sqrt{1-t^2}} \mathrm{d}t = -\frac{1}{2} \int_{\pi}^{0} [\cos((i+j) \cdot x) + \cos((i-j) \cdot x)] \mathrm{d}x = 0$$

And only when $i = j$

$$\int_{-1}^{1} \frac{T_i(t)T_j(t)}{\sqrt{1-t^2}} \mathrm{d}t = -\frac{1}{2} \int_{\pi}^{0} [\cos((i+j) \cdot x) + \cos((i-j) \cdot x)] \mathrm{d}x$$
$$= -\frac{1}{2} \int_{\pi}^{0} [\cos((i+j) \cdot x) + \cos(0 \cdot x)] \mathrm{d}x$$
$$= \frac{\pi}{2}$$

When $i = j = 0$

$$\int_{-1}^{1} \frac{T_i(t)T_j(t)}{\sqrt{1-t^2}} \mathrm{d}t = -\frac{1}{2} \int_{\pi}^{0} [\cos((i+j) \cdot x) + \cos((i-j) \cdot x)] \mathrm{d}x$$
$$= -\frac{1}{2} \int_{\pi}^{0} [\cos(0 \cdot x) + \cos(0 \cdot x)] \mathrm{d}x = \pi$$

Therefore, we have

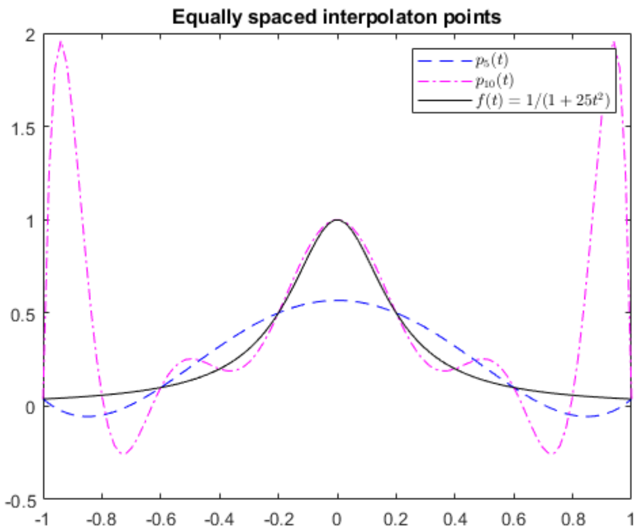
$$\int_{-1}^{1}\frac{T_i(t)T_j(t)}{\sqrt{1-t^2}}\mathrm{d}t=\frac{1}{2}\pi\delta_{ij}=\begin{cases}\pi, & i=j=0\\ \frac{1}{2}\pi, & i=j\neq 0\\ 0, & i\neq j\end{cases}$$

(b) $T_3(t)=\cos(3\cdot\arccos(t))$ assume $x=\arccos(t)$ or $\cos(x)=t$

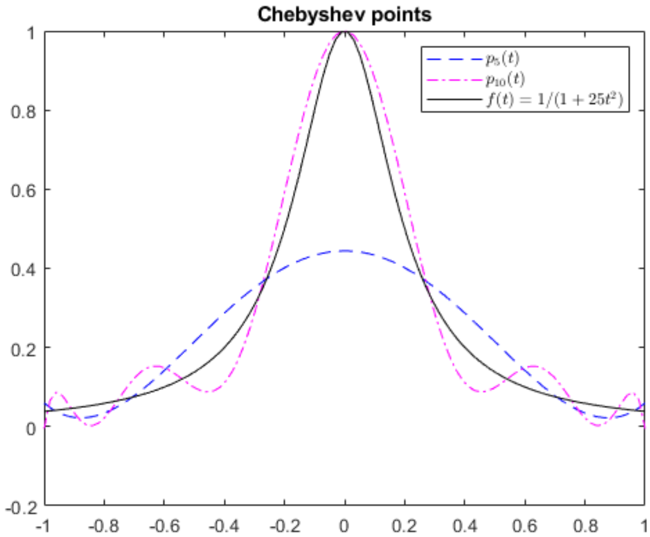
$$\begin{aligned}T_3(t)=T_3(x)&=\cos(3x)=\cos(x+2x)=\cos(x)\cos(2x)-\sin(x)\sin(2x)\\ &=\cos(x)(2\cos^2(x)-1)-\sin(x)\cdot 2\sin(x)\cos(x)\\ &=2\cos^3(x)-\cos(x)-2\cos(x)(1-\cos^2(x))\\ &=4\cos^3(x)-3\cos(x)\\ &=4t^3-3t\end{aligned}$$

Q4.

Equally spaced interpolation points:



Chebyshev points:



Q5

(a) Midpoint rule: $M(f)=(b-a)f(\frac{a+b}{2})=(b-a)(\frac{a+b}{2})^3=\frac{1}{8}$

trapezoid rule: $T(f)=\frac{b-a}{2}(f(a)+f(b))=\frac{1}{2}$

(b) $E(f)=\frac{T(f)-M(f)}{3}=\frac{1}{8}$

$M(f)_{error}=E(f)=\frac{1}{8}$

$T(f)_{error}=|2E(f)|=\frac{1}{4}$

(c) Simpson's rule

$S(f)=\frac{2}{3}M(f)+\frac{1}{3}T(f)=\frac{1}{4}$

(d) For Simpson's rule, $I(f)=S(f)-\frac{2}{3}F(f)+\cdots$. The error is $\frac{2}{3}F(f)$, where $\frac{f^{(4)}(\xi)}{1920}(b-a)^5$, for polynomials integrand with degrees lower than 4, the error is always zeros.

(e) Gaussian Quadrature

Two-point Gaussian rule on [-1,1] has form $\mathrm{G}_2(\mathrm{f})=f(-\frac{1}{\sqrt{3}})+f(\frac{1}{\sqrt{3}})$

By linear transformation $\mathrm{t}=\frac{\mathrm{x}+1}{2}$, we have

$$\int_0^1 x^3 dx = \frac{1}{2}\int_{-1}^{1}(\frac{1}{2}(t+1))^3 dt$$

$$= \frac{1}{2}(\frac{1}{2}(-\frac{1}{\sqrt{3}}+1))^3 + \frac{1}{2}(\frac{1}{2}(\frac{1}{\sqrt{3}}+1))^3 = \frac{1}{4}$$

Because two-point Gaussian quadrature has degree three(2n-1), the result is accurate.

(f) Composite midpoint rule

Mesh size is h = 0.5, such that $(a_1, b_1) = (0, 0.5), (a_2, b_2) = (0.5, 1)$

$$M(f) = M_1(f) + M_2(f) = (b_1 - a_1) f(\frac{a_1 + b_1}{2}) + (b_2 - a_2) f(\frac{a_2 + b_2}{2})$$

$$= 0.5 f(0.25) + 0.5 f(0.75) = \frac{7}{32}$$

Q6

The Lagrange interpolation of given function $f(x)$ at given nodes $x_1, x_2 \cdots x_n$ is

$$f(x) \approx \sum_{i=1}^{n} l_i(x) f(x_i)$$

where $l_i(x)$ is the Lagrange basis functions.

The integral of $f(x)$ at interval [a, b] is

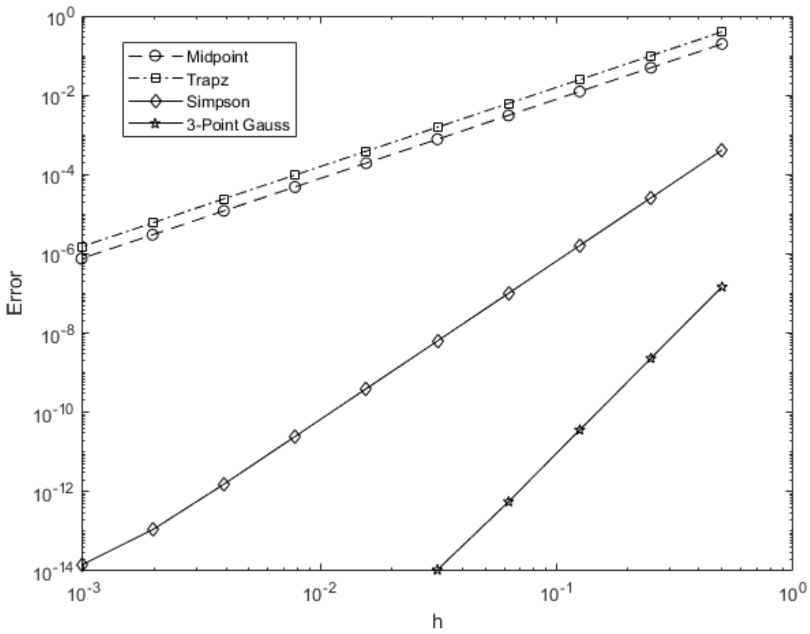
$$I(f) = \int_a^b f(x)\mathrm{d}x \approx \int_a^b \sum_{i=1}^{n} l_i(x) f(x_i)\mathrm{d}x$$

$$= \sum_{i=1}^{n} f(x_i) \int_a^b l_i(x)\mathrm{d}x = \sum_{i=1}^{n} f(x_i) w_i$$

where $w_i = \int_a^b l_i(x)\mathrm{d}x$

Q7

(a) – (d)



(e)

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Midpoint rule: C = 0.79232 h =1.9993
Trapezoid rule: C = 1.5871 h =1.9996
Simpson rule: C = 0.0054649 h =3.9238
3point Gaussian rule: C = 8.7158e-06 h =5.9442
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Remark: for 3 points Gaussian rule, when the mesh size lower than $\frac{1}{2^5}$,the error gets to machine precision. So we just use the points where mesh size is large than $\frac{1}{2^5}$ to fit curve.

Refer code HW3_7

Q8

8.12

Forward difference formula

$$f(x+h)=f(x)+f'(x)h+\frac{1}{2!}f''(x)h^2+\frac{1}{3!}f'''(x)h^3+\cdots$$

$$f_1'(x)\approx\frac{f(x+h)-f(x)}{h}-\frac{1}{2!}f''(x)h^1-\frac{1}{3!}f'''(x)h^2+\cdots$$

Backward difference formula

$$f(x-h)=f(x)-f'(x)h+\frac{1}{2}f''(x)h^2-\frac{1}{3!}f'''(x)h^3+\cdots$$

$$f_2'(x)\approx\frac{f(x)-f(x-h)}{h}+\frac{1}{2}f''(x)h-\frac{1}{3!}f'''(x)h^2+\cdots$$

The average of two first-order accurate approximations is

$$\overline{f}' = \frac{f_1'(x)+f_2'(x)}{2} = \frac{f(x+h)-f(x-h)}{2h} - \frac{1}{6}f'''(x)h^2 + \mathcal{O}(h^2)$$

To summary, the centered difference approximation is second-order accurate

8.13

Taylor series of f(x) at x

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2!}f''(x)h^2 + \frac{1}{3!}f'''(x)h^3 + \cdots$$

$$f(x+2h) = f(x) + f'(x)2h + \frac{1}{2!}f''(x)(2h)^2 + \frac{1}{3!}f'''(x)(2h)^3 + \cdots$$

use eq2 minus 4 times of eq1, we get

$$f(x+2h) - 4f(x+h) = -3f(x) - 2f'(x)h + 0 + \frac{2}{3}f'''(x)h^3$$

such that, the second-order accurate approximation is

$$f'(x) \approx \frac{4f(x+h) - 3f(x) - f(x+2h)}{2h}$$