assignment11

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1 Image Denoising

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1.1 Multi-objective least squares

In some applications we have *multiple* objectives, all of which we would like to be small:

$$J_1 = ||A_1x - b_1||^2, \cdots, J_k = ||A_kx - b_k||^2$$

We seek a *single* $\hat{x}$ that gives a compromise, and makes them all small, tothe extent possible. We call this the *multi-objective* least squares problem.

We cannot get a single $\hat{x}$ which makes all the objectives be minimum at same time, so we have to come out a compromise plan. A standard method for finding a value of x that gives a compromise in making all the objectves small is to choose x to minimize a *weighted sum objective*:

$$J = \lambda_1 J_1 + \cdots + \lambda_k J_k = \lambda_1 ||A_1x - b_1||^2 + \cdots + \lambda_k ||A_kx - b_k||^2,$$

where λ are positive *weights*, that express our relative desire for the terms to be small.

1.1.1 Weighted sum least squares via stacking

We can minimize the weighted sum obejctive function by expressing it as a standard least squares problem, then we can solve it by the method we use before. Express J as the norm squared of a single vector:

$$J = \left\| \begin{bmatrix} \sqrt{\lambda_1}(A_1x - b_1) \\ \vdots \\ \sqrt{\lambda_k}(A_kx - b_k) \end{bmatrix} \right\|^2$$

so we have

$$J = \left\| \begin{bmatrix} \sqrt{\lambda_1}A_1 \\ \vdots \\ \sqrt{\lambda_k}A_k \end{bmatrix} x - \begin{bmatrix} \sqrt{\lambda_1}b_1 \\ \vdots \\ \sqrt{\lambda_k}b_k \end{bmatrix} \right\|^2 = \left\| \tilde{A}x - \tilde{b} \right\|^2$$

Now, we have reduced the problem of minimizing the weighted sum least squares objective to a standard lest squares problem.

Usually, we identify a *primary ojective* J_1 that we would like to be small. We also identify one or more *secondary objectives* that we would also like to be small. There are many possible sencondary objectives:

- $||x||^2$: x should be small.
- $||x - x^{prior}||^2$: x should be near x^{prior}.
- $||Dx||^2$, where D is the first difference matrix: x should be smooth.

1.2 Estination and inversion

In the broad application area of *estimation*, the goal is to estimate a set of n values (also called parameters), the entries of the n-vector x. We are given a set of m *measurements*, the entries of an m-vector y. They are related by

$$y = Ax + v$$

The m-vector v is the *measurement noise*, and is unknown but presumed to be small. The estimation problem is to make a sensible guess as to what x is, given y and prior knowledge about x.

Of course we cannot expect to find x exactly when the measurement noise is nonzero. This is called *approximate inversion*.

If we guess that x has the value $\hat{x}$, then we are implicitly making the guess that v has the value $y - A\hat{x}$. If we assume that v is small, then a sensible choive for $\hat{x}$ is the least squares approximate solution, which minimizes $||A\hat{x} - y||^2$. We will take this as our primary objective. And we choose secondary objectives by the informations we know about x.

1.3 Image Denoising

Image de-nosing is one of the *Inversion* problem. The vector x is an image, and the matrix A gives noise, so \$y = Ax + v \$ is a noisy image.

Our prior infromation about x is that it is smooth; neighboring pixels values are not very different from each other. So we choose $||Dx||^2$ be our secondary objective.

Because image is 2D, we form an estimate $\hat{x}$ image by minimizing a cost function of the form

$$||Ax - y||^2 + \lambda(||D_h x||^2 + ||D_v x||^2)$$

Here D_v and D_h are vertical and horizontal differencing operations.

1.3.1 Our specific denosing problem

In our problem, we do not use matrix A to add noisy on image, but just given noisy image y, so our function is

$$||x - y||^2 + \lambda(||D_h x||^2 + ||D_v x||^2)$$

Express it as a norm squared form, we have

$$\left\| \begin{bmatrix} I \\ \sqrt{\lambda} D_h \\ \sqrt{\lambda} D_v \end{bmatrix} x - \begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix} \right\|^2 = \|\tilde{A}x - \tilde{b}\|^2$$

Suppose the vector x has length MN then D_h is a $M(N-1) \times MN$ matrix, D_v be the $(M-1)N \times MN$ matrix, and I is a $M \times N$ identity matrix.

Now we can use standard least mean square method to get our $\hat{x}$.

1.4 Implementation

1.4.1 Import packages & read input data

```
In [1]: import matplotlib.pyplot as plt
        import numpy as np
        import math
        from scipy import signal
        from skimage import io, color
        from skimage import exposure

        file_image        = 'cau-resized.jpg'

        im_color         = io.imread(file_image)
        im_gray          = color.rgb2gray(im_color)
        im          = (im_gray - np.mean(im_gray)) / np.std(im_gray)
        row, col     = im.shape
```

1.4.2 Function for generate noisy image

Here we add some noise on original image by normal distribution with mean 0 and standard deviation σ.

```
In [38]: def create_noisy_img(img, noise_std):
             """Add noise on image

             Arguments:
                 img(np.matrix): input clear image.
                 noise_std: noise standard deviation.
             Return:
                 noisy image
             """

             row, col = im.shape

             noise = np.random.normal(0, noise_std, (row, col))
             im_noise = im + noise

             return im_noise
```

1.4.3 Build cost function

As we talked above, to build cost function, we need a $M \times N$ identity matrix I, a $M(N-1) \times MN$ matrix D_h, and a $(M-1)N \times MN$ matrix D_v then combine them to a big matrix $\tilde{A}$.

In [3]:
```
def create_tilde_matrix(row, col, weight):
    """Build tilde matrix A in cost function

    Arguments:
        row: #row of image
        col: #column of image
        weight: weight of secondary objective
    Return:
        matrix row: row*(col-1) + (row-1)*col + row*col
               column: row * col
    """

    I = np.identity(row*col)
    Dx_weight = deri_h(row, col, weight)
    Dy_weight = deri_v(row, col, weight)

    A = np.vstack((I, Dx_weight, Dy_weight))

    return A


def deri_h(row, col, weight=1):
    dh = np.zeros((row*(col-1), row*col))

    for i in range(row*(col-1)):
        dh[i][i] = -1
        dh[i][i+row] = 1

    dh = math.sqrt(weight) * dh

    return dh


def deri_v(row, col, weight=1):
    m = np.zeros((row-1, row))
    for i in range(row-1):
        m[i][i] = -1
        m[i][i+1] = 1

    ident = np.identity(col)

    dv = np.kron(ident, m)
    dv = math.sqrt(weight) * dv
```

```
            return dv
```

Generate $\tilde{b}$

```
In [5]: def create_tilde_b(row, col, img):
            length = (row-1)*col + row*(col-1) + row*col

            b = np.zeros(length)
            img_vec = []

            for i in range(col):
                column = img[:, i]
                for j in range(row):
                    img_vec.append(column[j])

            for k in range(row*col):
                b[k] = img_vec[k]

            return b
```

Compute $\hat{x}$

```
In [39]: def compute_param(A, y):
             return np.linalg.lstsq(A, y, rcond=None)[0]
```

1.4.4 De-noising image

Since we have got all the components of the cost function, now we can compute the approximate inversion $\hat{x}$

```
In [47]: def denoising(row, col, weight, img_noisy):
             """de-noising image

             Arguments:
                 row: #row of image
                 col: #col of image
                 weight: weight for secondary objective
                 img_noisy: noisy image
             Return:
                 img_recon: de-noised image
                 error: error of this denoising
             """
             # tranform image to vector
             b = create_tilde_b(row, col, im_noise)

             # generate matrix A
             matrix_A = create_tilde_matrix(row, col, weight)
```

```
    # compute recon image
    img_recon = compute_param(matrix_A, b)

    # compute error
    cost_func = np.inner(matrix_A, img_recon) - b
    error = np.linalg.norm(cost_func)**2

    # transform to matrix shape
    img_recon = (img_recon.reshape((col, row))).T


    return img_recon, error
```

1.5 De-noising Images

Now, let's try our `denoising` function to do some image de-noising.

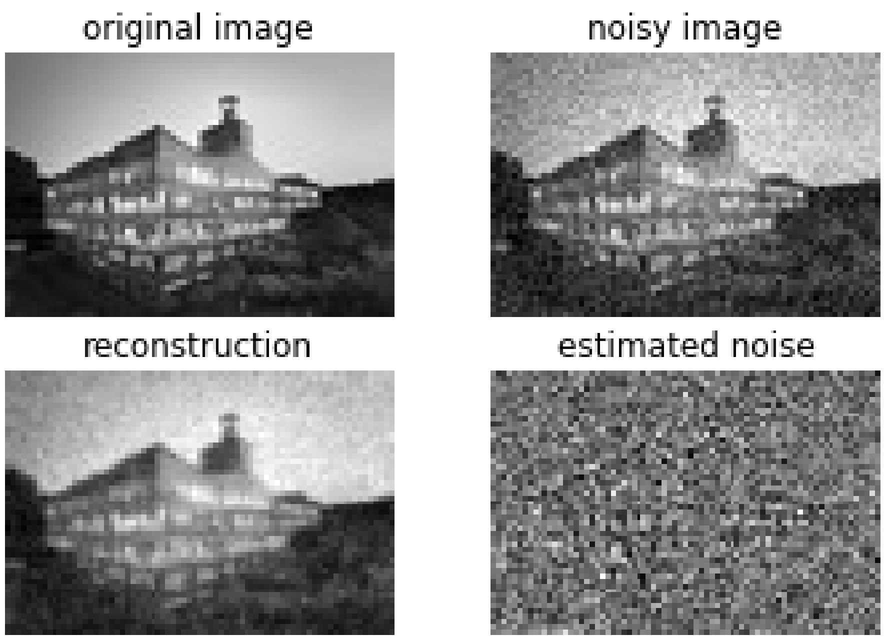
1.5.1 Try different standard deviation σ and regularization parameter λ

First create a function to convenient plotting.

```
In [54]: def plot_images(img, img_noisy, img_recon):
             noise_recon = img_noisy - img_recon

             p1 = plt.subplot(2,2,1)
             p1.set_title('original image')
             plt.imshow(img, cmap='gray')
             plt.axis('off')

             p2 = plt.subplot(2,2,2)
             p2.set_title('noisy image')
             plt.imshow(img_noisy, cmap='gray')
             plt.axis('off')

             p3 = plt.subplot(2,2,3)
             p3.set_title('reconstruction')
             plt.imshow(img_recon, cmap='gray')
             plt.axis('off')

             p4 = plt.subplot(2,2,4)
             p4.set_title('estimated noise')
             plt.imshow(noise_recon, cmap='gray')
             plt.axis('off')

             plt.show()
```

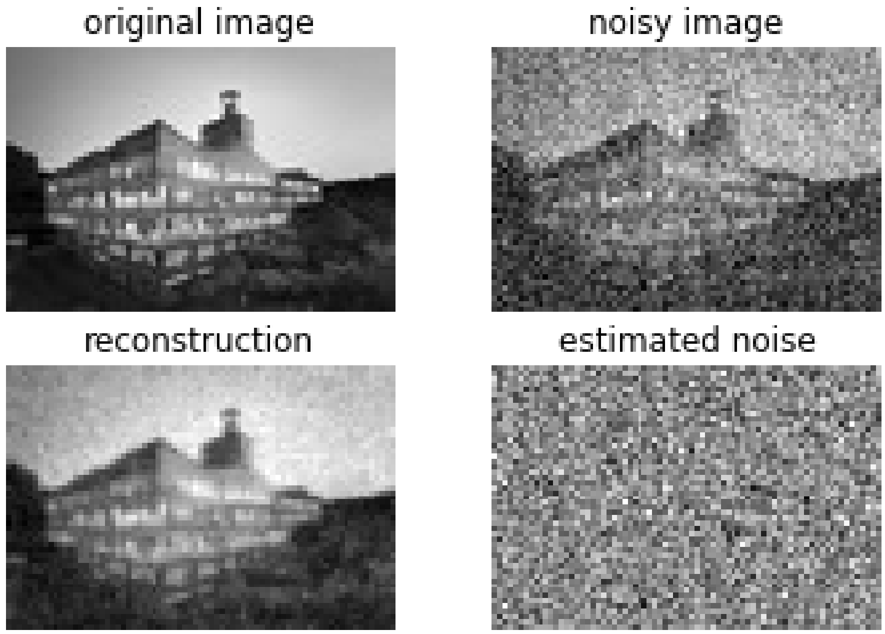
1. Try $\sigma = 0.2$, $\lambda = 0.25$

In [56]:
```
noise_std = 0.2
weight = 0.25

# generate noisy image
img_noisy = create_noisy_img(im, noise_std)

# denoising
img_recon, error = denoising(row, col, weight, img_noisy)

plot_images(im, img_noisy, img_recon)
```



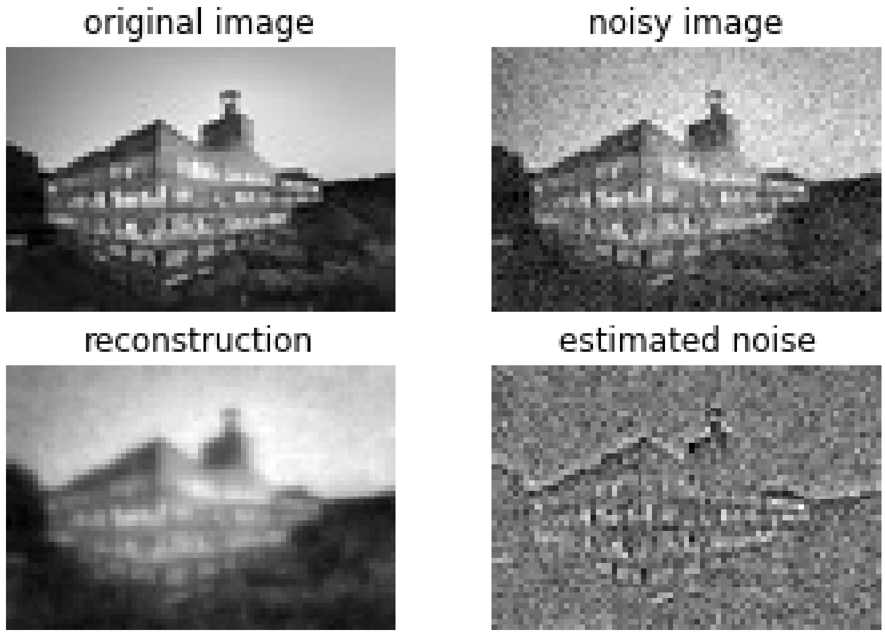
2. Try $\sigma = 0.5$, $\lambda = 0.25$

In [57]:
```
noise_std = 0.5
weight = 0.25

img_noisy = create_noisy_img(im, noise_std)
img_recon, error = denoising(row, col, weight, img_noisy)

plot_images(im, img_noisy, img_recon)
```



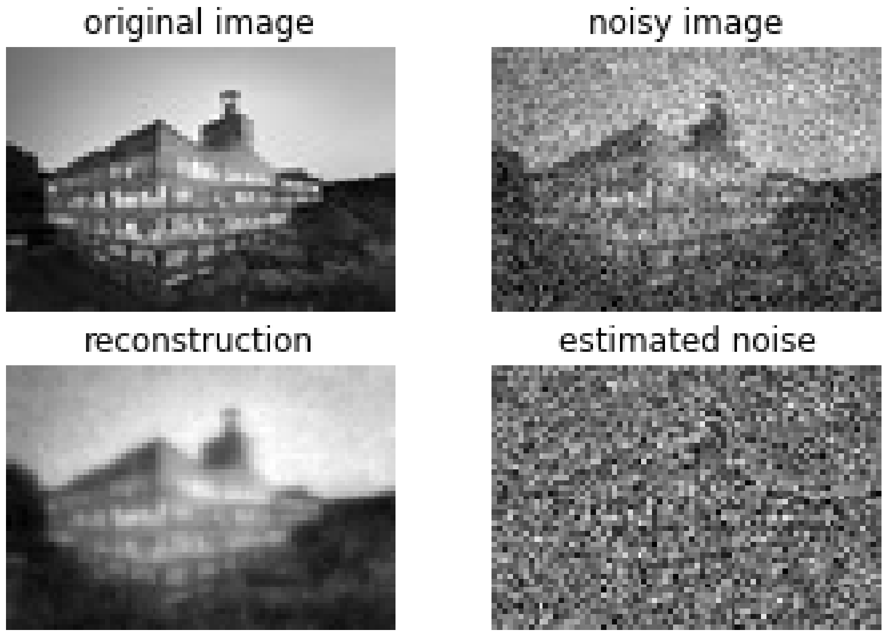
3. Try $\sigma = 0.2, \lambda = 1$

In [58]:
```
noise_std = 0.2
weight = 1

img_noisy = create_noisy_img(im, noise_std)
img_recon, error = denoising(row, col, weight, img_noisy)

plot_images(im, img_noisy, img_recon)
```



4. Try $\sigma = 0.5, \lambda = 1$

In [59]:
```
noise_std = 0.5
weight = 1

img_noisy = create_noisy_img(im, noise_std)
img_recon, error = denoising(row, col, weight, img_noisy)

plot_images(im, img_noisy, img_recon)
```



Since I do not get memory to compute bigger image, these may not very clear.

But we still can see from the above images, when σ bigger the noise is more, when λ get bigger the recon images are becoming smoother.

1.5.2 See how λ affects result

Let's fix σ and try different λs to see the result.

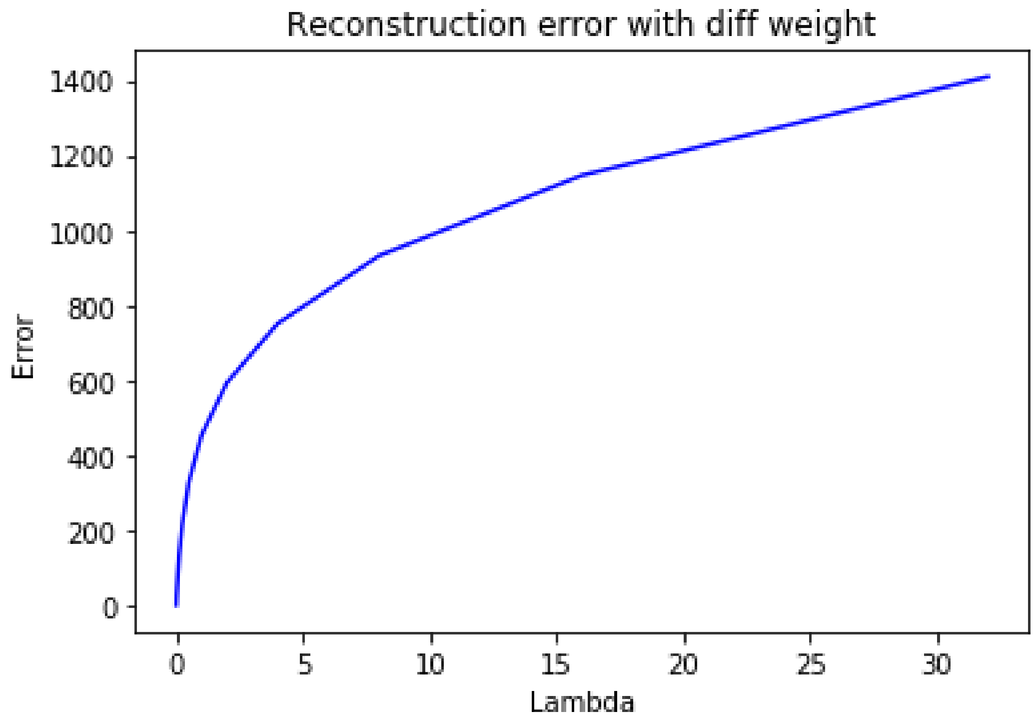
In [79]:
```
lambdas = [0.5**x for x in range(-5, 11)]
img_noisy = create_noisy_img(im, 0.5)

error_histo = []
imgs = []

for weight in lambdas:
    img_recon, error =denoising(row, col, weight, img_noisy)
    imgs.append(img_recon)
    error_histo.append(error)
```

In [95]:
```
plt.figure(figsize=(16, 16))
for i in range(len(error_histo)):
    p = plt.subplot(6,3,i+1)
    img_recon= imgs[i]
    p.set_title('weight = {}'.format(lambdas[i]))
    plt.imshow(img_recon, cmap='gray')
    plt.axis('off')
```

```
plt.show()
```

Let's see how the error of of reconstruction changes.
Error function is $E(u) = ||u - f||_2^2 + \lambda||\nabla u||_2^2$

In [98]:
```
plt.title("Reconstruction error with diff weight")
plt.plot(lambdas, error_histo, 'b-')
plt.xlabel('Lambda')
plt.ylabel('Error')
plt.show()
```



When λ becomes bigger, reconstructed images are more smooth, so the error between it with input noisy image also get bigger.