assignment06

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1 Straight-line Fit

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1.1 Least Squares Problem

Suppose we have a tall matrix A, so $Ax = b$ is *over-determined*. For most b, there is no x that satisfies $Ax = b$.

The *least squares problem* is: choosing x to minimize $||Ax - b||^2$.
We call the solution *least squares approxmimata solution* of $Ax = b$, and notate it as $\hat{x}$.

1.2 Least squares data fitting

If we have a scalar y and an n-vector x that are related by model

$$y \approx f(x)$$

x is the *independent variable*,
y is the *outcome*,
$f : R^n \to R$ gives the relation between x and y.

Often x is a feature vector, and y is something we want to predict.
We do not know the true relationship f between x and y. So what we want to do is to find a *approximation* of f.

We define a *linear in the parameters* model:

$$\hat{f}(x) = \theta_1 f_1(x) + \cdots + \theta_p f_p(x)$$

$f_i : R^n \to R$: are *basis fuctions* that we choose.
θ_i: are *model parameters* that we choose.
$\hat{y}^{(i)} = \hat{f}(x^{(i)})$ is prediction of $y^(i)$.

Thus, our goal becomes to choose model parameters θ_i to minimize *residuals* $r^i = y^{(i)} - \hat{y}^{(i)}$. This can be formulated and solved as a **least squares problem**.

If we define
$y^d = (y^{(1)}, \cdots, y^{(N)})$ is vecor of outcomes,

$\hat{y}^d = (\hat{y}^{(1)}, \cdots, \hat{y}^{(N)})$ is vector of predictions,
$r^d = (r^{(1)}, \cdots, r^{(N)})$ is vector of residuals.
have $N \times p$ matrix A with elements $A_{ij} = fj(x^{(i)})$, so $\hat{y}^d = A\theta$.
$||r^d||^2 = ||y^d - \hat{y}^d||^2 = ||y^d - A\theta||^2$

1.3 Straight-line Fit

Use straight-line to fit the data set, we know general line relation $f(x) = ax + b$.
Since the *linear in the parameters model* is

$$\hat{f}(x) = \theta_1 f_1(x) + \cdots + \theta_p f_p(x)$$

Here, we have $p = 2$ with $f_1(x) = 1$, $f_2(x) = x$.
Model becomes to $\hat{f}(x) = \theta_1 + \theta_2 x$ which matches general line relation.

Let's try a exercise.

1.3.1 First, generate some random numbers as *outcomes*.

In [4]:
```
import numpy as np
import matplotlib.pyplot as plt

num      = 201
std      = 20
a        = 2
b        = 10

# x  : x-coordinate data
# y1 : (noisy) y-coordinate data
# y2 : (clean) y-coordinate data
# y = f(x) = a * x + b

n        = np.random.rand(num)
nn       = n - np.mean(n)
x        = np.linspace(-100,100,num)
y1       = a * x + nn * std + b
y2       = a * x + b
```

1.3.2 Second, set model

As we said above, we have model $\hat{f}(x) = \theta_1 + \theta_2 x$. Change it to matrix form, we have

$$\hat{f}(x) = A\theta$$

which, $A = \begin{bmatrix} 1 & x^{(1)} \\ \vdots & \vdots \\ 1 & x^{(N)} \end{bmatrix}$, $\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$

In [15]:
```
A = np.vstack([np.ones(len(x)), x]).T
```

1.3.3 Third, define object funtion

By *least square* problem, we have object function $||r^d||^2 = ||y^d - \hat{y}^d||^2 = ||y^d - A\theta||^2$

```
In [51]: def computeCost(A, outcome, theta):
             inner = np.power((outcome - (np.inner(A, theta))), 2)
             return np.sum(inner)
```

1.3.4 Fourth, compute solution θ

To compute θ we have two method.

1. Use derivative
2. Use inverse matrix

Here we use inverse matrix to get θ, have $\hat{\theta} = (A^T A)^{-1} A^T y^d$
Bring our A into formular, we have
Gram matrix $A^T A = \begin{bmatrix} N & 1^T x^d \\ 1^T x^d & (x^d)^T x^d \end{bmatrix}$
so, $(A^T A)^{-1} = \frac{1}{N(x^d)^T x^d - (1^T x^d)^2} \begin{bmatrix} (x^d)^T x^d & -1^T x^d \\ -1^T x^d & N \end{bmatrix}$
$A^T y = \begin{bmatrix} 1^T y^d \\ (x^d)^T y^d \end{bmatrix}$
Multiplying the scalar term by N^2, and dividing the matrix and vector terms by N, we can change the expression to

$$\frac{1}{rms(x^d)^2 - avg(x^d)^2} \begin{bmatrix} rms(x^d)^2 & -avg(x^d) \\ -avg(x^d) & 1 \end{bmatrix} \begin{bmatrix} avg(y^d) \\ (x^d)^T y^d / N \end{bmatrix}$$

Change them into more simply terms, we have
$\hat{\theta}_2 = \frac{std(y_d)}{std(x^d)}\rho$, $\hat{\theta}_1 = avg(y^d) - \hat{\theta}_2 avg(x^d)$
where, $\rho = \frac{(x^d - avg(x^d)1)^T (y^d - avg(y^d)1)}{N std(x^d) std(y^d)}$
Finaly, we get the approximate $\hat{f}(x) = avg(y^d) + \rho \frac{std(y^d)}{std(x^d)}(x - avg(x^d))$

```
In [30]: def rho(var, outcome):
             # function for computing rho in formular

             numerator = (np.inner((var - np.full(len(var), np.average(var))).T,
                                   (outcome - np.full(len(outcome),
                                                      np.average(outcome)))))

             denominator = len(var) * np.std(var) * np.std(outcome)

             return numerator/denominator


         # trans list to numpy array
         y = np.array(y1)
```

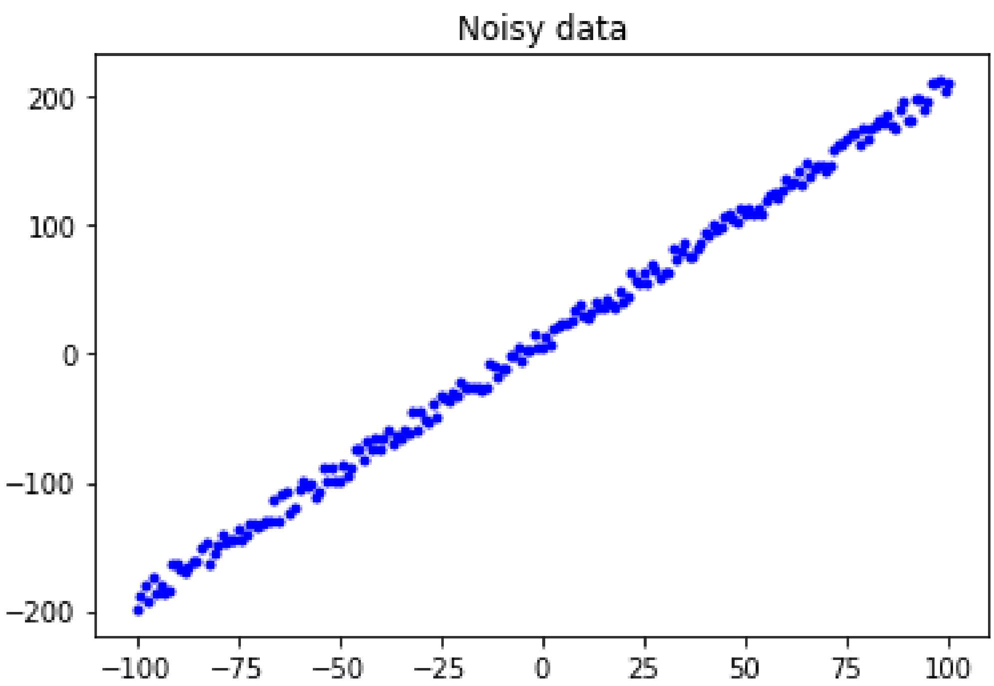
```
y_avg = np.average(y)
ce = rho(x, y) * np.std(y) / np.std(x)
x_avg = np.average(x)

# final result
f_x = y_avg + ce*(x - x_avg)
```

1.3.5 Plot result

Let's plot the result to see if the result matches well.

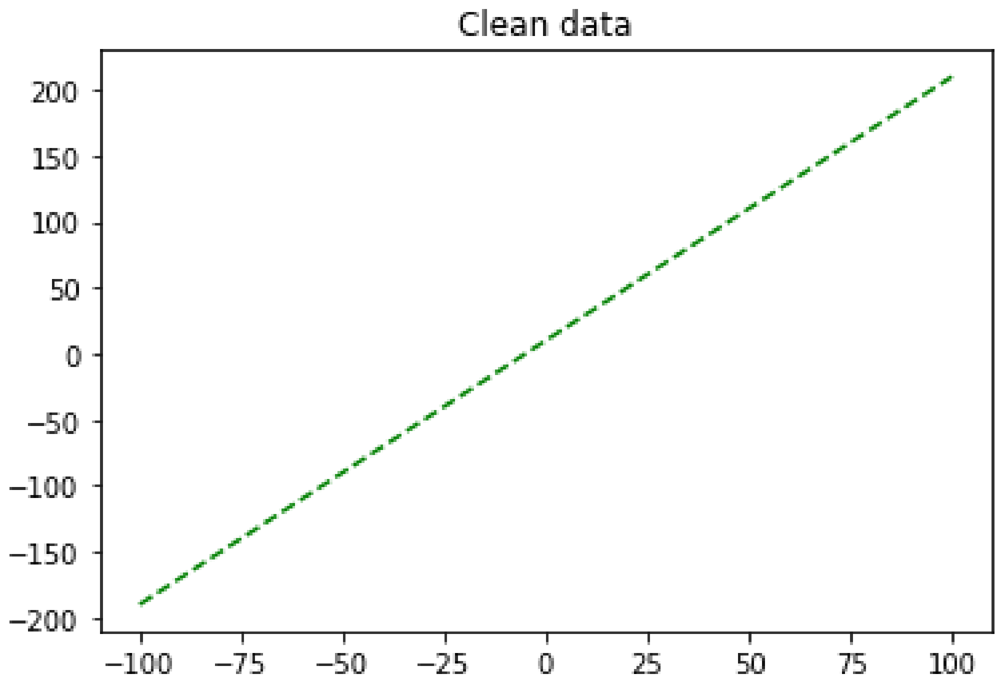
Nosiy data

```
In [26]: plt.title("Noisy data")
         plt.plot(x, y1, 'b.')
         plt.show()
```



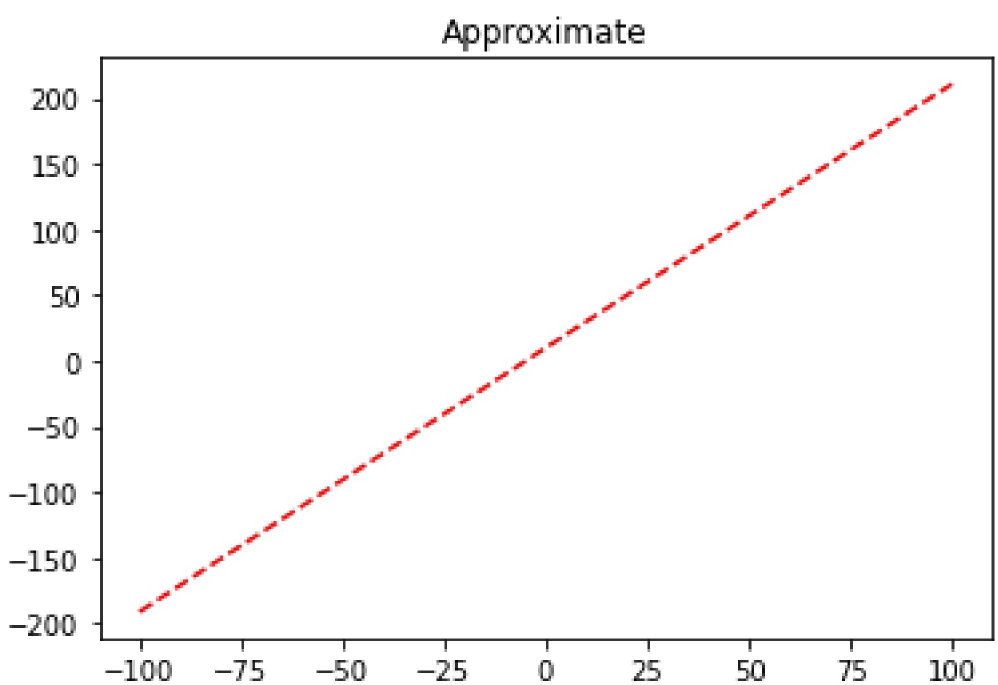
Clean data

```
In [27]: plt.title("Clean data")
         plt.plot(x, y2, 'g--')
         plt.show()
```



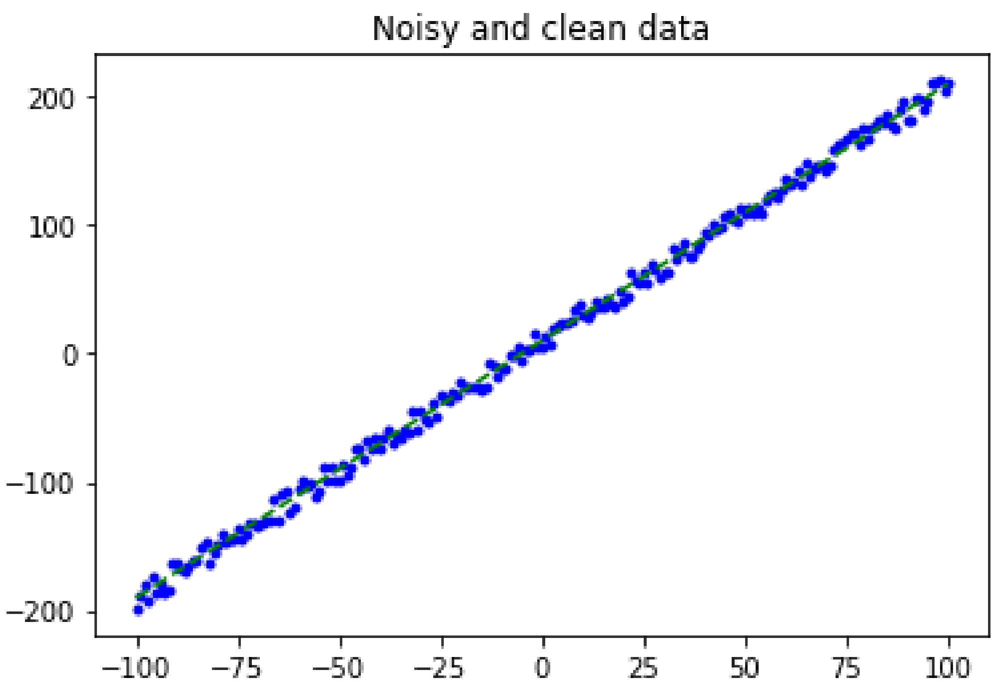
Approximate line

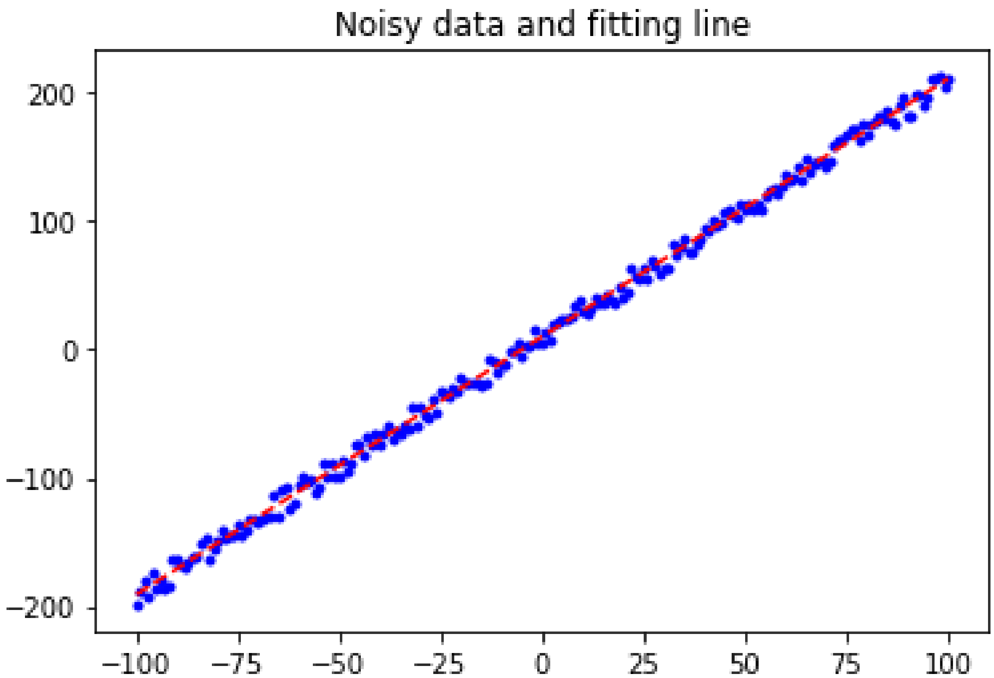
```
In [29]: plt.title("Approximate")
         plt.plot(x, f_x, 'r--')
         plt.show()
```



Put noisy data with clean data and fitting line

```
In [34]: plt.title("Noisy and clean data")
         plt.plot(x, y1, 'b.', x, y2, 'g--')
         plt.show()

         plt.title("Noisy data and fitting line")
         plt.plot(x, y1, 'b.', x, f_x, 'r--')
         plt.show()
```



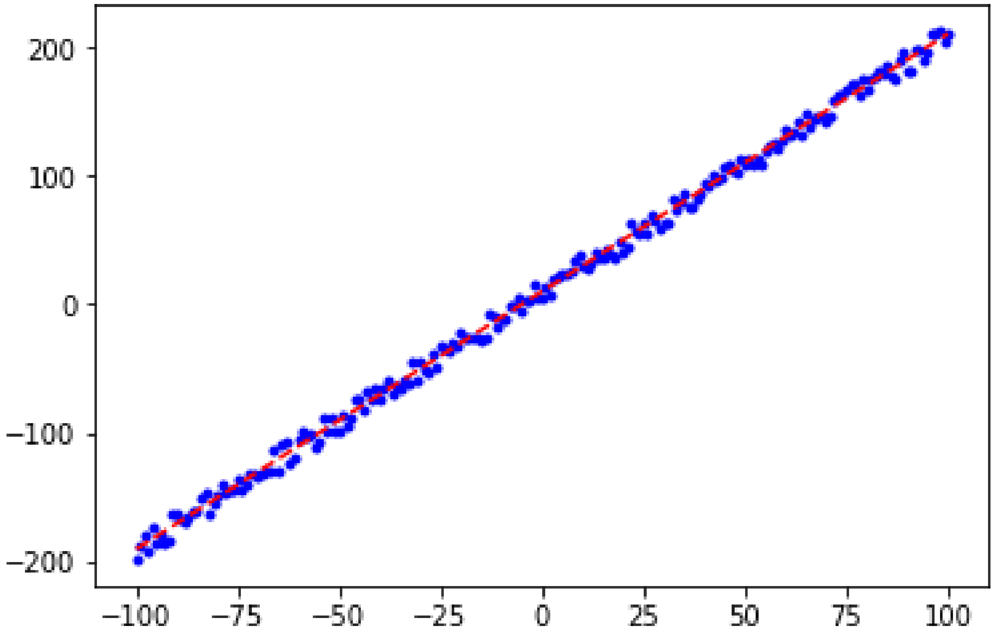


From above graph we can see, our approximate result is similar with the clean data.

Because `numpy` already have a function for computing *least-squares problem* to a linear matrix-- `numpy.linalg.lstsq`, we actually do not need to build our own one.

Let try `linalg.lstsq`

```
In [35]: the1, the2 = np.linalg.lstsq(A, y, rcond=None)[0]
         plt.plot(x, y1, 'b.', x, the1+the2*x, 'r--')
         plt.show()
```



The result is almost same as our formular.
If we compute the compare the square deviation, have

```
In [53]: theta_2 = rho(x, y) * np.std(y) / np.std(x)
         theta_1 = np.average(y) - theta_2*np.average(x)
         theta = np.array([theta_1, theta_2])

         cost_our = computeCost(A, y, theta)
         cost_np = computeCost(A, y, np.array([the1, the2]))

         print("Our formula cost:", cost_our)
         print("Numpy formula cost:", cost_np)
```

```
Our formula cost: 6477.384723477494
Numpy formula cost: 6477.384723477492
```