assignment12

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1 Polynomial Fit with Regularizations

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Samilar with assignment11, this is also a *multi-objective least squares* problem.

This time we want to find a optimal set of model parameters that provide the least square approximate slolution:

$$E(\theta;\lambda) = ||A\theta - y||_2^2 + \lambda||\theta||_2^2$$

By stacking we can change our weighted sum least squares to a standard least squares form

$$\left\| \begin{bmatrix} \sqrt{\lambda_1}A_1 \\ \vdots \\ \sqrt{\lambda_k}A_k \end{bmatrix} x - \begin{bmatrix} \sqrt{\lambda_1}b_1 \\ \vdots \\ \sqrt{\lambda_k}b_k \end{bmatrix} \right\|^2 = \left\| \tilde{A}x - \tilde{b} \right\|^2$$

So, our cost function can be expressed as

$$\left\| \begin{bmatrix} A \\ \sqrt{\lambda}I \end{bmatrix} \theta - \begin{bmatrix} y \\ 0 \end{bmatrix} \right\|^2 = \left\| \tilde{A}x - \tilde{b} \right\|^2$$

We define our feature function as

$$\hat{f}(x) = \theta_0 + \theta_1 x + \cdots + \theta_p x^p$$

Our matrix A becomes to:

$$\begin{bmatrix} 1 & x^{(1)} & \cdots & (x^1)^p \\ 1 & x^{(2)} & \cdots & (x^2)^p \\ \vdots & \vdots & & \vdots \\ 1 & x^{(N)} & \cdots & (x^N)^p \end{bmatrix}$$

Have all this infromations, we can compute the parameters θ.

1.1 Implementation

1.1.1 Generate data set

In [1]:
```
import numpy as np
import math
import matplotlib.pyplot as plt

num      = 1001
std      = 5

# x  : x-coordinate data
# y1 : (clean) y-coordinate data
# y2 : (noisy) y-coordinate data

def fun(x):

    # f = np.sin(x) * (1 / (1 + np.exp(-x)))
    f = np.abs(x) * np.sin(x)

    return f

n        = np.random.rand(num)
nn       = n - np.mean(n)
x        = np.linspace(-10,10,num)
y1       = fun(x)                          # clean points
y2       = y1 + nn * std                 # noisy points

plt.plot(x, y1, 'b.', x, y2, 'k.')
plt.show()
```

```
<Figure size 640x480 with 1 Axes>
```

1.2 Build cost function

Assume our input data set x is a n-vector, our parameter is a p-vector, then A is a $n \times p$ matrix and I is a p-identity matrix. We combine them into a $\tilde{A}$ matrix.

In [2]:
```
def create_tilde_matrix(n, p, weight, data):
    """Build tilde matrix A in cost function

    Arguments:
        n: number of input data
        p: degree of polynomial
        weight: weight of secondary objective
        data: input data
    """
```

```
    I = np.identity(p+1)
    I = math.sqrt(weight) * I

    feature_matrix = build_feature_matix(data, p)

    A = np.vstack((feature_matrix, I))

    return A


def build_feature_matix(data, p):
    """Build basic function matrix

    data: input data set
    p: degree of polynomial
    """

    matrix = []

    for xi in data:
        poly = [xi**dg for dg in range(p+1)]
        matrix.append(poly)

    return np.array(matrix)
```

1.2.1 Generate $\tilde{b}$

$\tilde{b}$ has same length with $\tilde{A}$: $n + p$

```
In [3]: def create_tilde_b(n, p, data_noisy):
            """Create b vector in cost functions

            Arguments:
                n: number of input data
                p: degree of polynomial
                data_noisy: noisy data
            Return:
                macthing b vector
            """

            length = n + p + 1  # add 1 for degree 0

            b = np.zeros(length)
            for i in range(len(data_noisy)):
                b[i] = data_noisy[i]

            return b
```

1.2.2 Compute θ

```
In [4]: def compute_params(A, y):
            return np.linalg.lstsq(A, y, rcond=None)[0]
```

1.2.3 Fitting function

Let's combine all the part together to generate our approximate values.

```
In [5]: def fitting(data_input, output_noisy, p, weight):
            """Compute approximate values with given poly degree

            Arguments:
                data_input: input data
                output_noisy: noisy measurements
                p: degree of polynomial
                weight: weight of secondary objective
            Return:

            """

            n = len(data_input)

            A = create_tilde_matrix(n, p, weight, data_input)
            b = create_tilde_b(n, p, output_noisy)

            # compute approximation
            params = compute_params(A, b)
            approx = np.inner(np.array([data_input**dg for dg in range(p+1)]).T, params)

            # compute error with approximation and clear data
            basic_A = build_feature_matix(data_input, p)
            error = error_function(A, params, b)

            return approx, error


        def error_function(A, theta, b):
            cost_func = np.inner(A, theta) - b

            return np.linalg.norm(cost_func)**2
```

1.3 Check fitting function

Now let's try our fitting function to see how p and λ affect the result.

1.3.1 Fix λ while change p

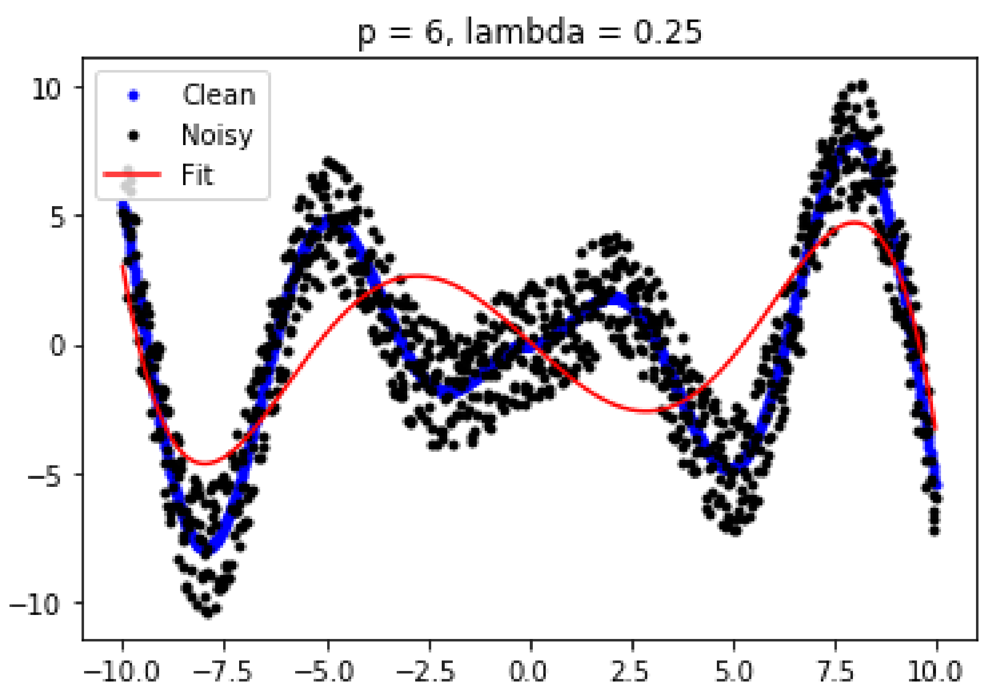
First use a fixed λ and chose p from $6 - 15$ to see the result.

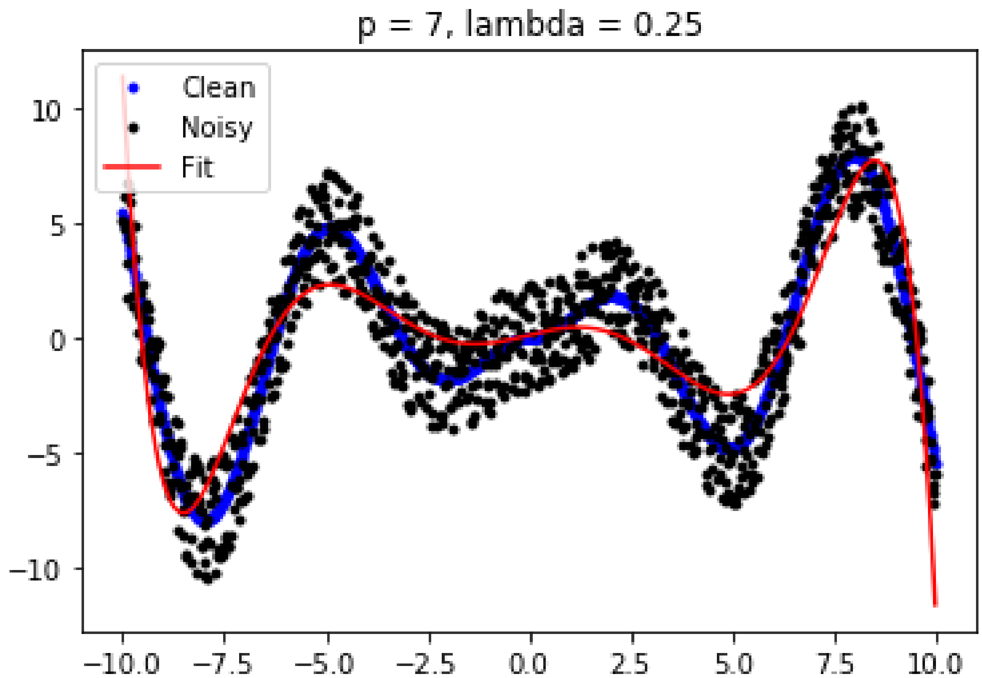
To convenient plotting, we define a plot function here

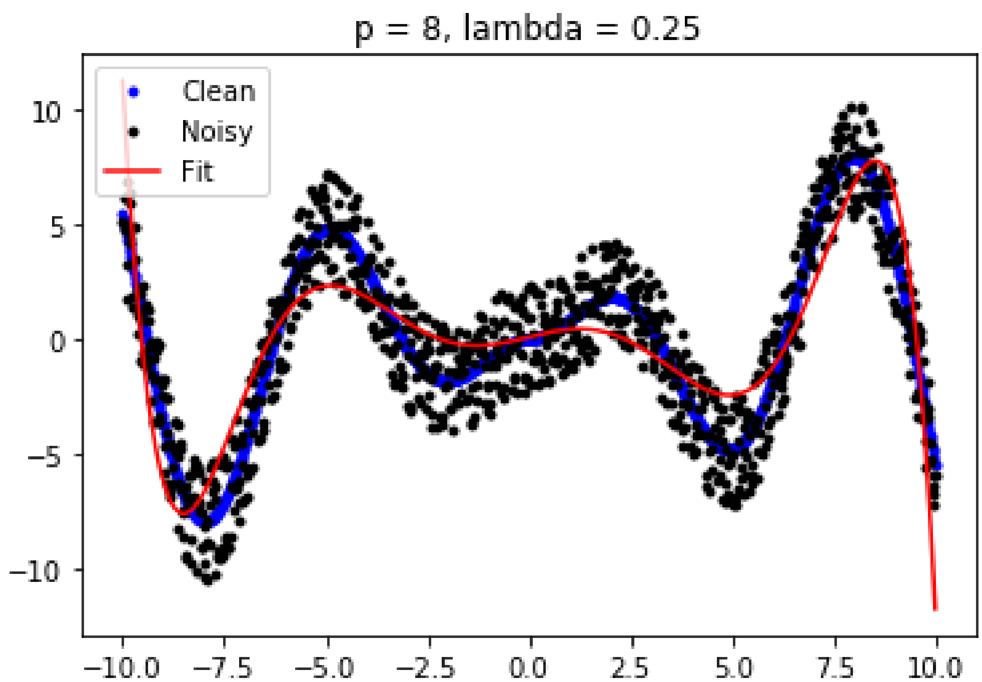
In [6]:
```
def plot_vary_p(x, y_clean, y_noisy, p, weight):

    y, error = fitting(x, y_noisy, p, weight)

    # Plot part
    plt.title("p = {}, lambda = {}".format(p, weight))
    plt.plot(x, y_clean, 'b.', x, y_noisy, 'k.', x, y, 'r')
    plt.legend(['Clean', 'Noisy', 'Fit'], loc='upper left')
    plt.show()

    return error
```

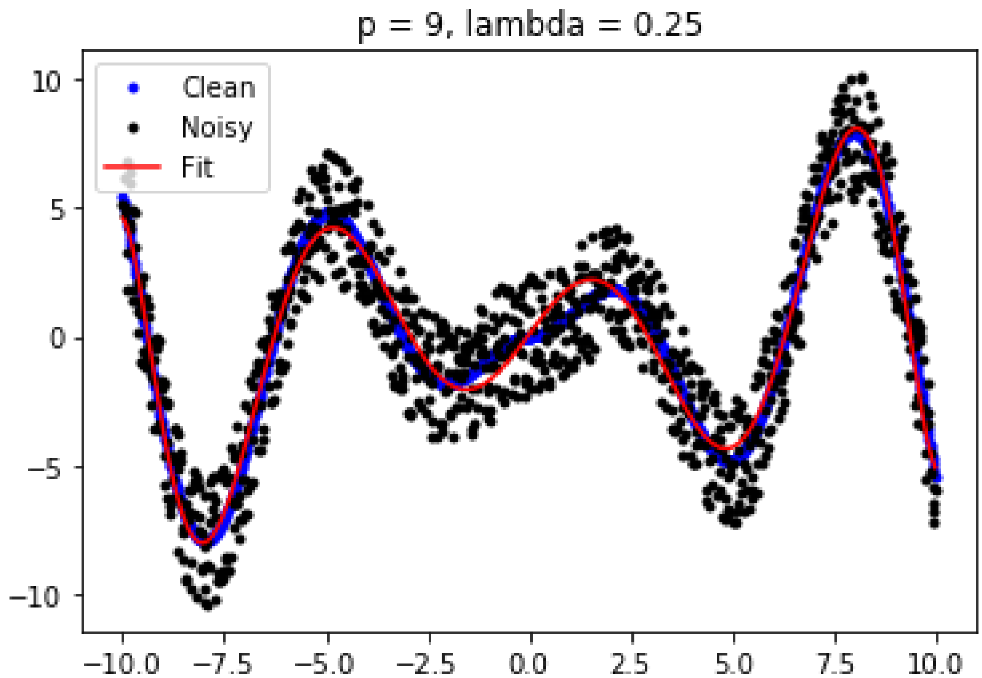
Here use define our λ to 0.25.

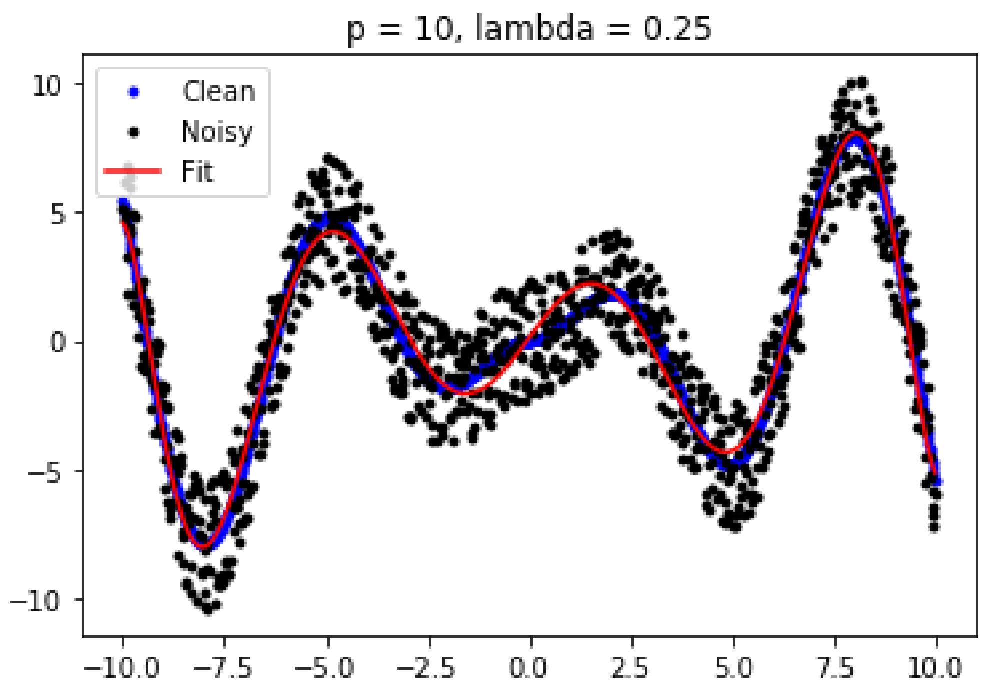
In [7]:
```
error_histo_p = []

for p in range(6, 16):
    error_histo_p.append(plot_vary_p(x, y1, y2, p, 0.25))
```

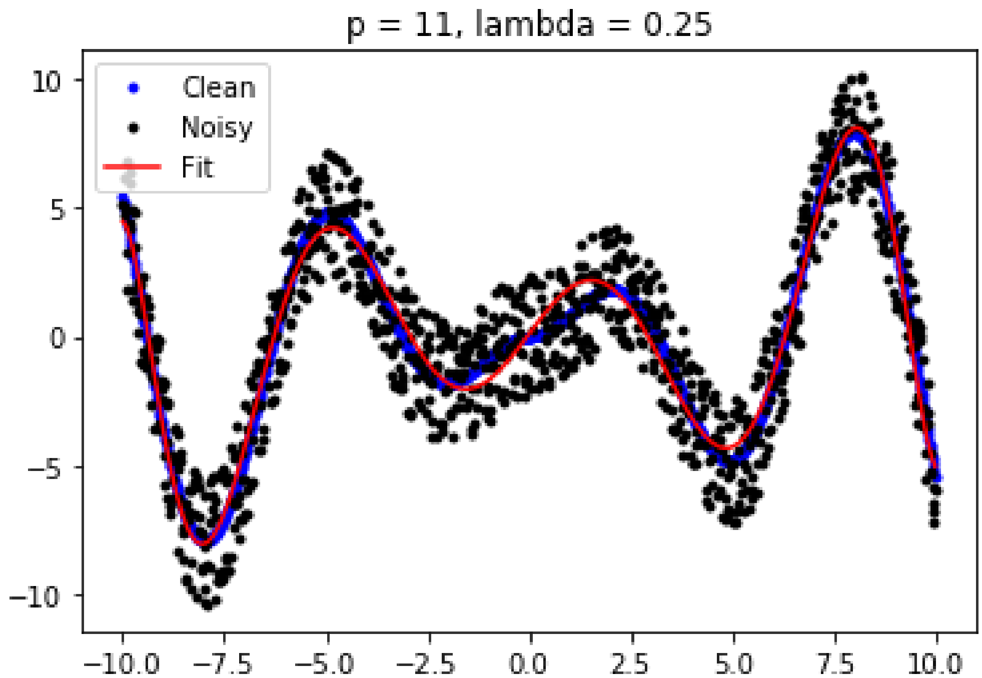


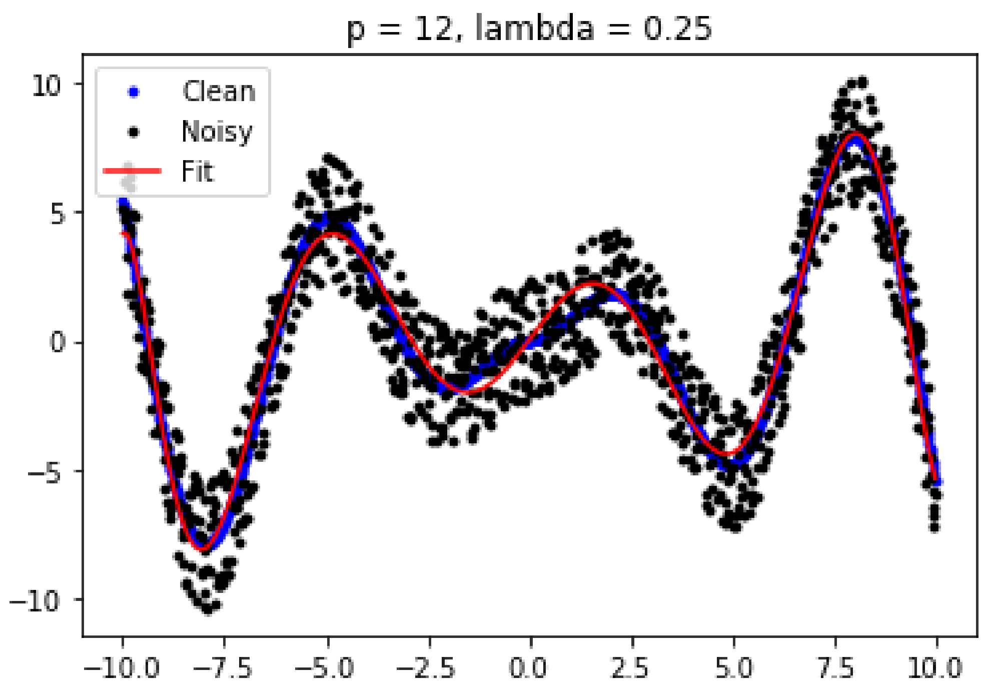

p = 7, lambda = 0.25
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0


p = 8, lambda = 0.25
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

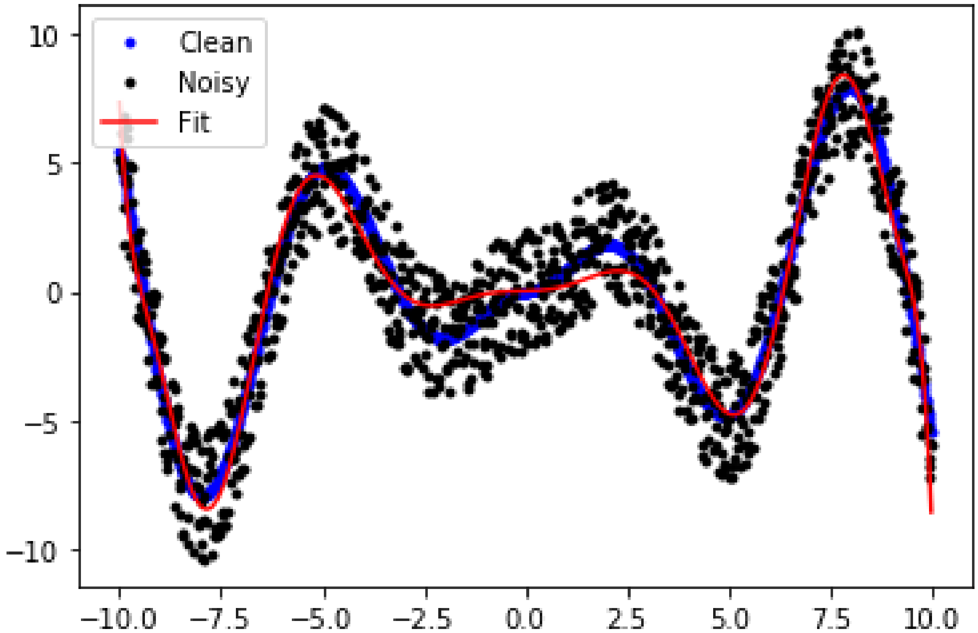

p = 9, lambda = 0.25
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0


p = 10, lambda = 0.25
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

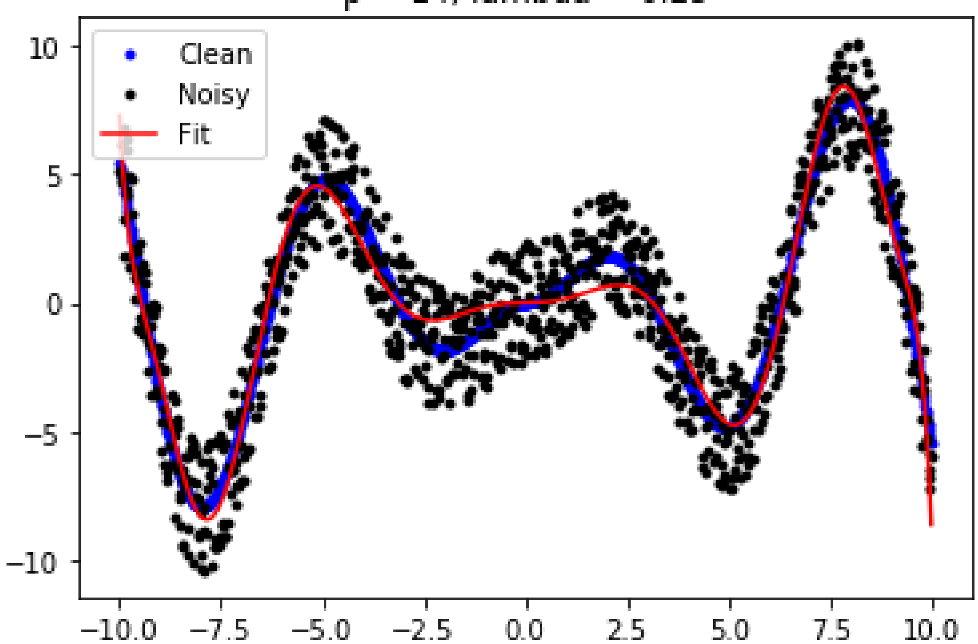

p = 11, lambda = 0.25
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

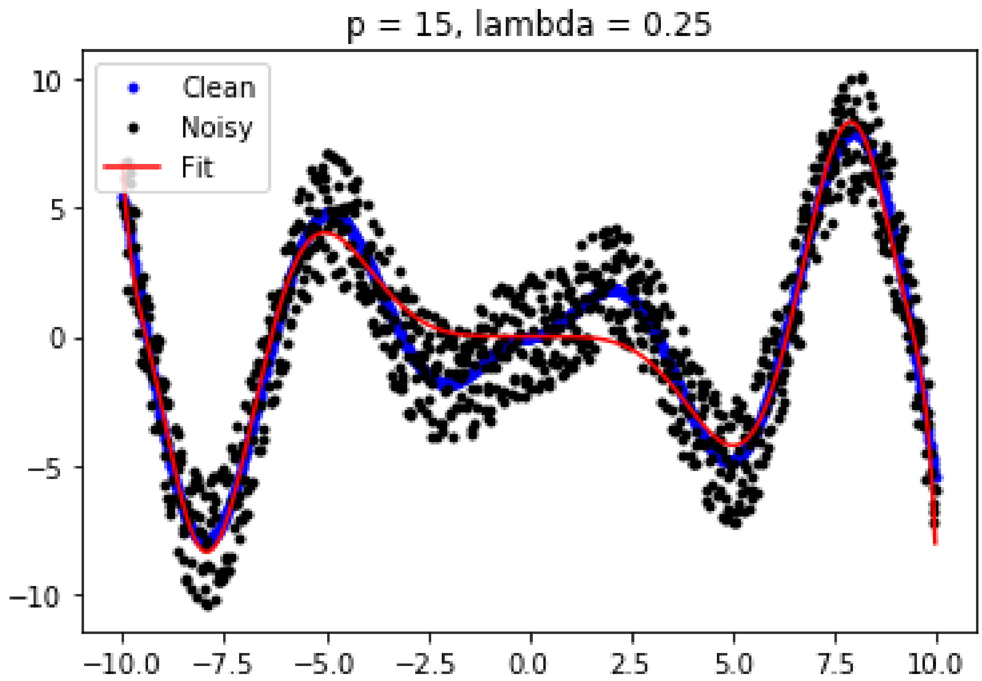

p = 12, lambda = 0.25
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

p = 13, lambda = 0.25

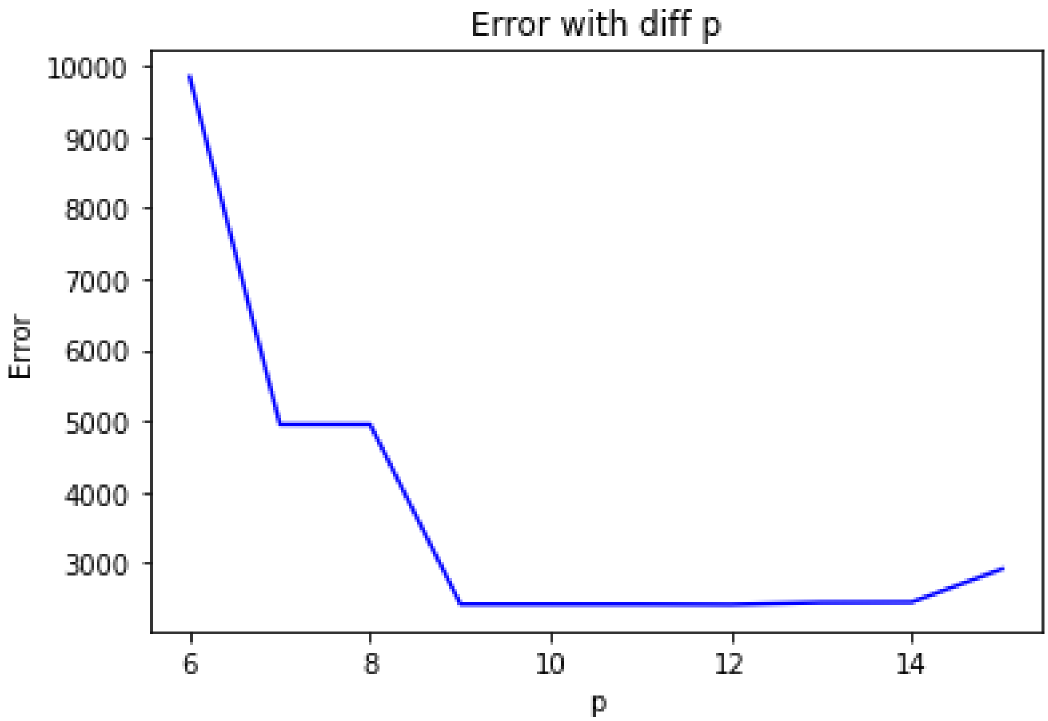


p = 14, lambda = 0.25






```
In [8]: p_axis = [p for p in range(6, 16)]

        plt.title("Error with diff p")
        plt.plot(p_axis, error_histo_p, 'b-')
        plt.xlabel('p')
        plt.ylabel('Error')
        plt.show()
```



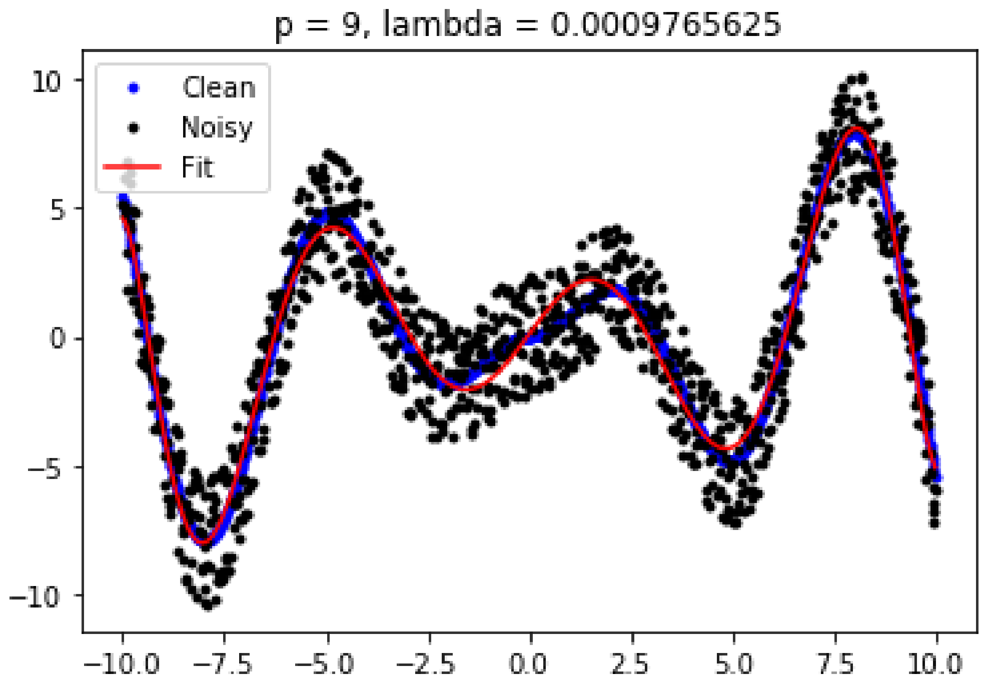
First along p increase error is dreacing, but after 12 error changes to increasing again.

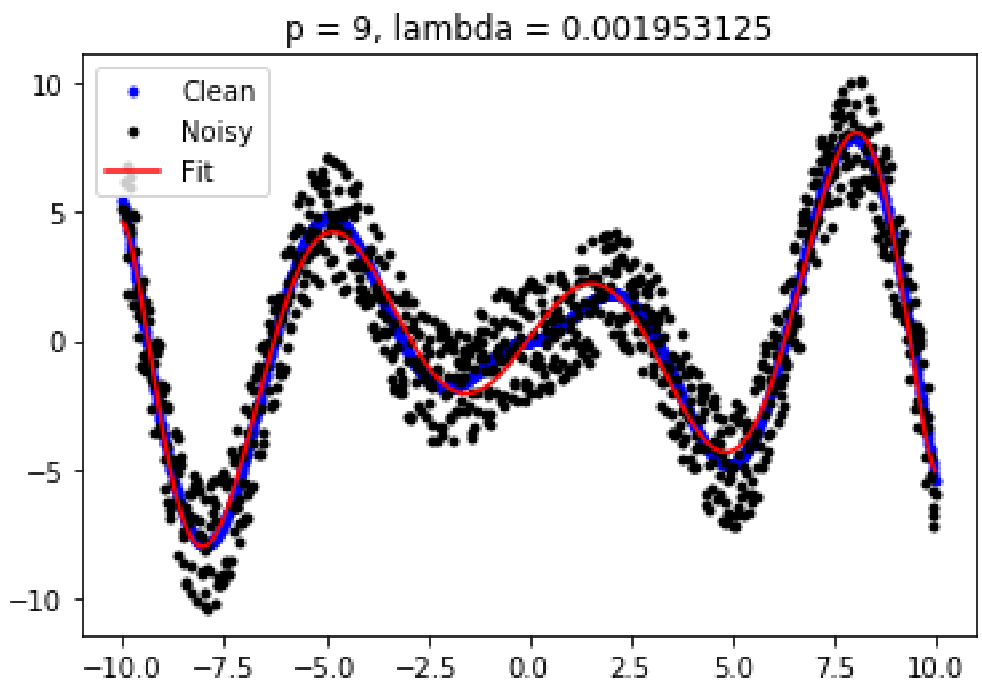
1.3.2 Fix p while vary λ

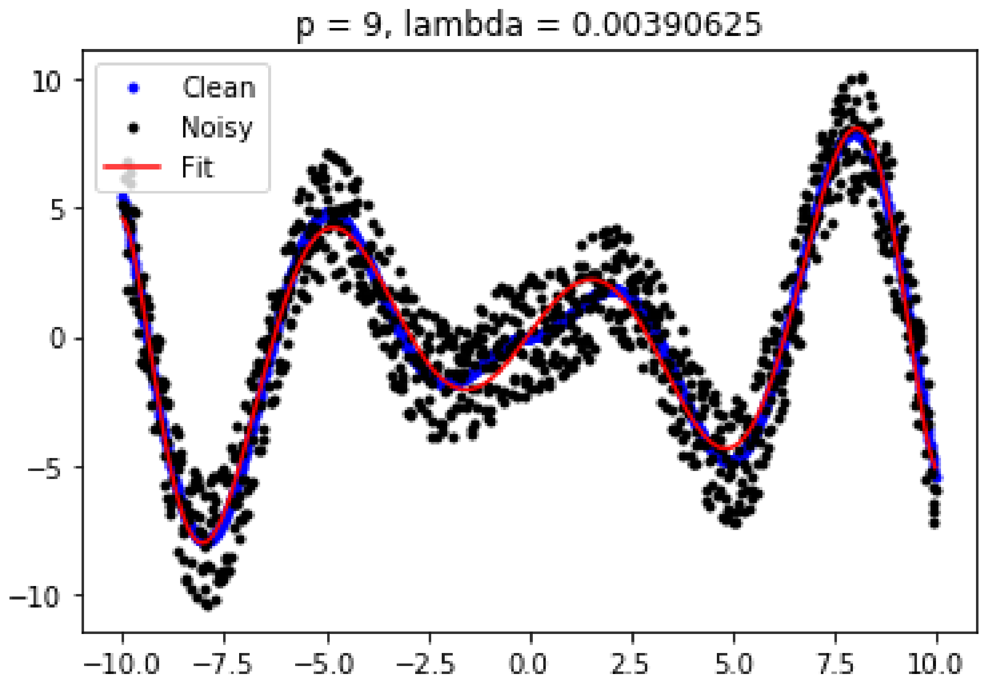
Now, let'us fix p and change λ to see how result changes.

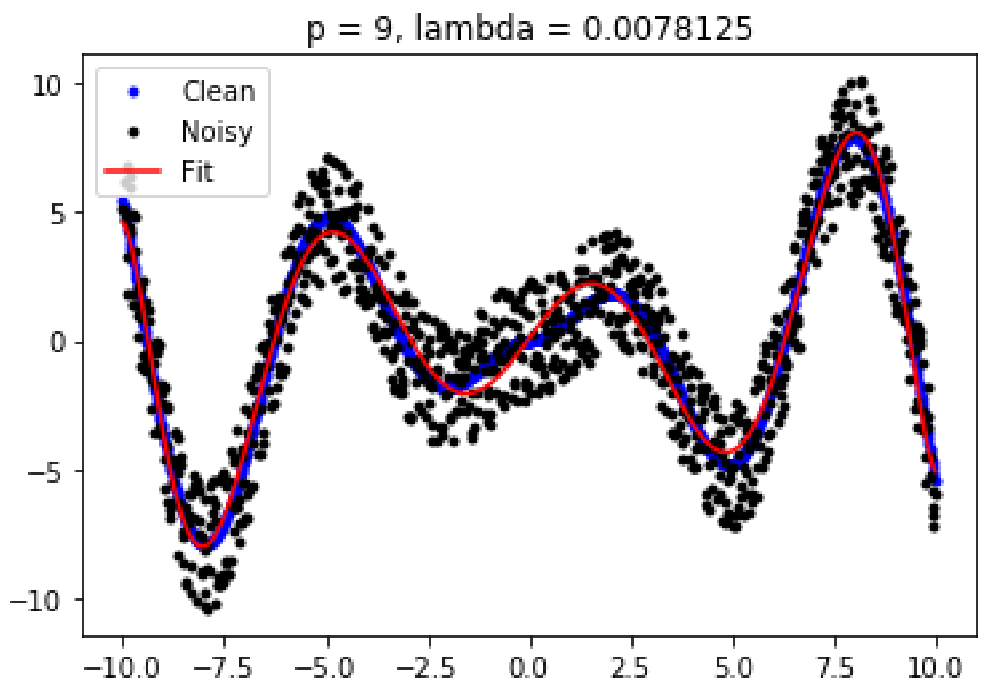
Since when $p = 9$ the error first time near the smallest value, we choose 9 be our p's value.

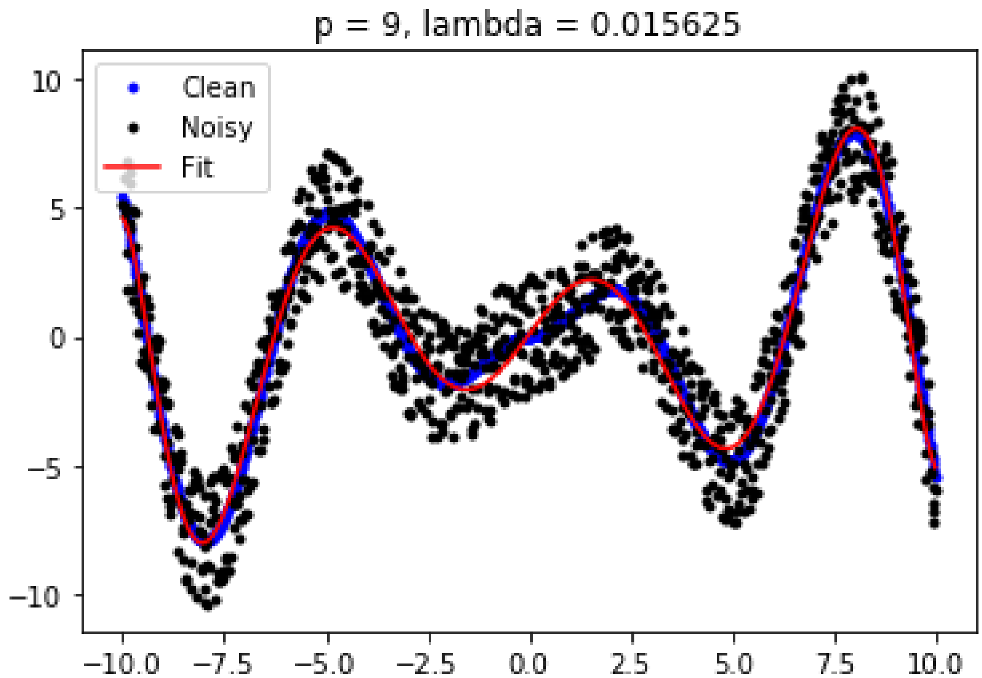
```
In [9]: lambdas = [2**n for n in range(-10, 11)]

        error_histo = []
        for lam in lambdas:
            error_histo.append(plot_vary_p(x, y1, y2, 9, lam))
```

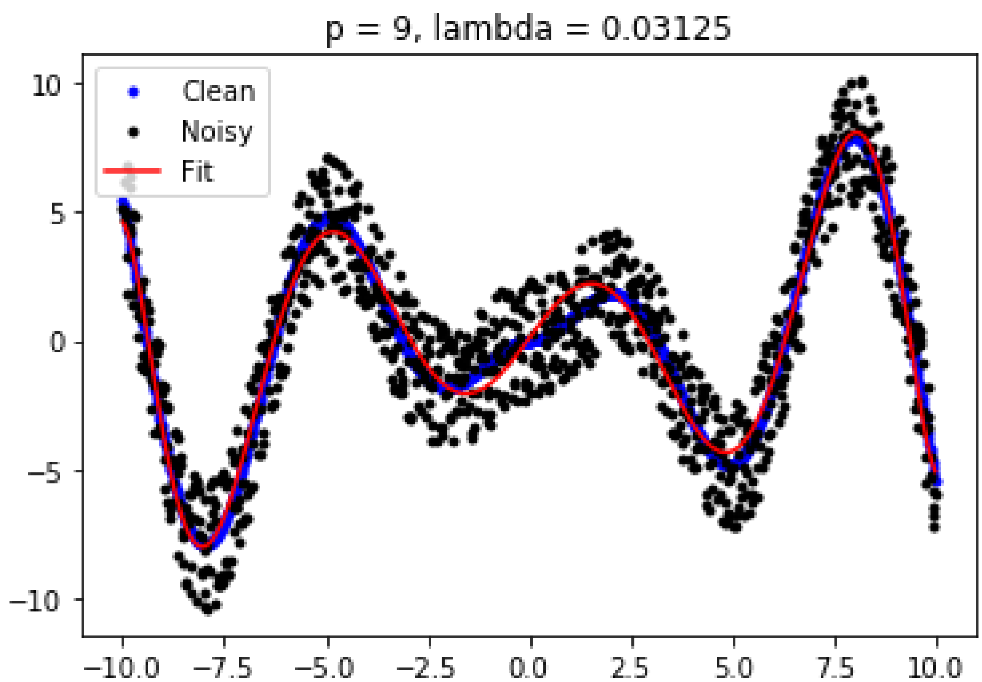

p = 9, lambda = 0.0009765625
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

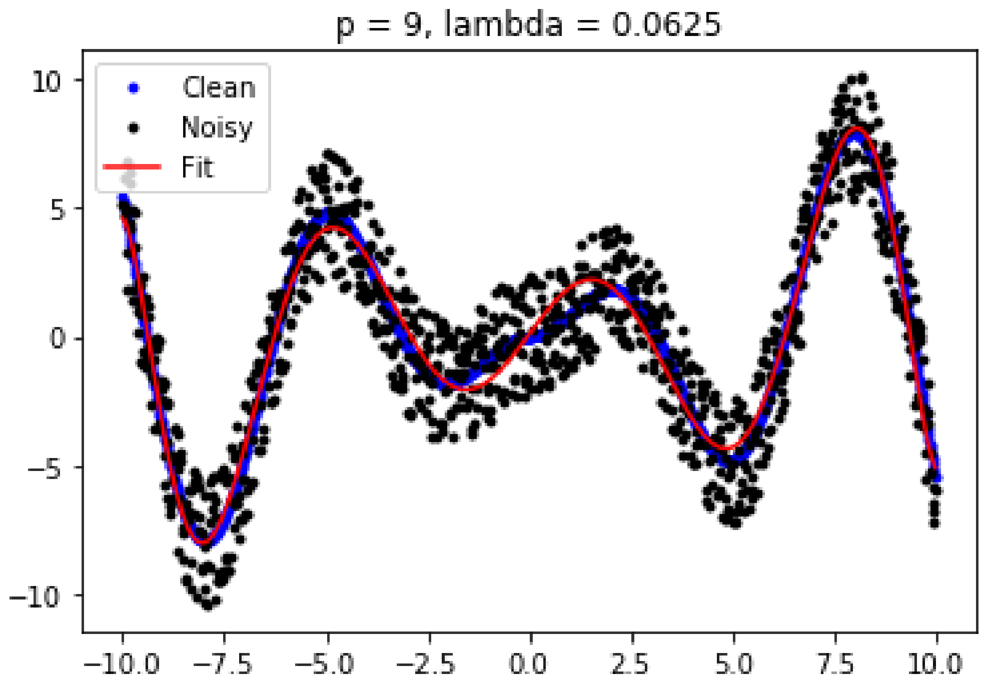

p = 9, lambda = 0.001953125
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

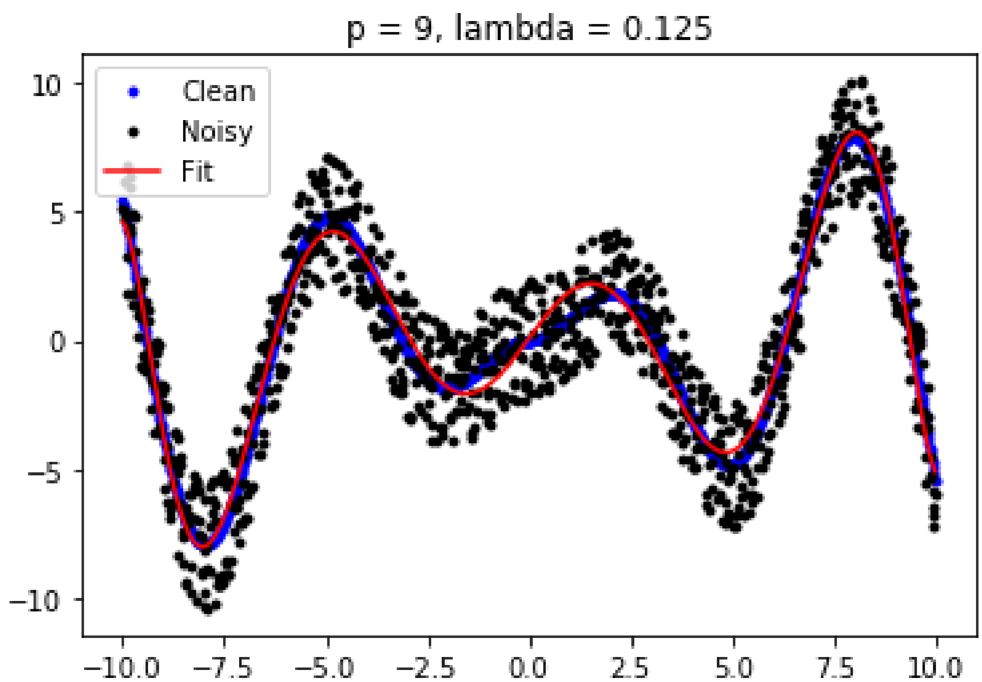

p = 9, lambda = 0.00390625
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

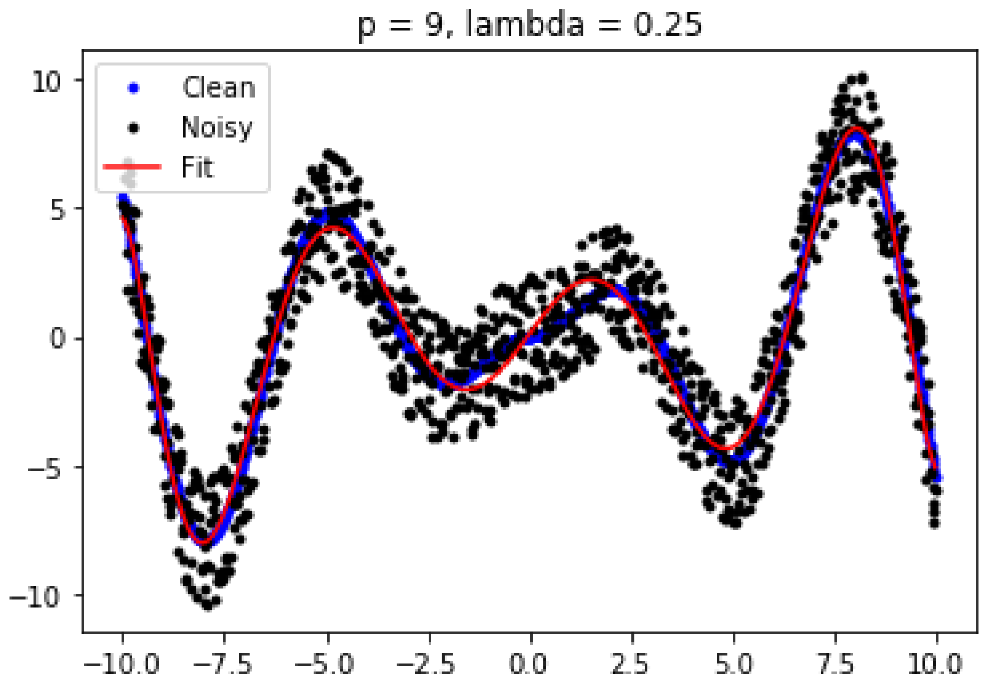

p = 9, lambda = 0.0078125
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

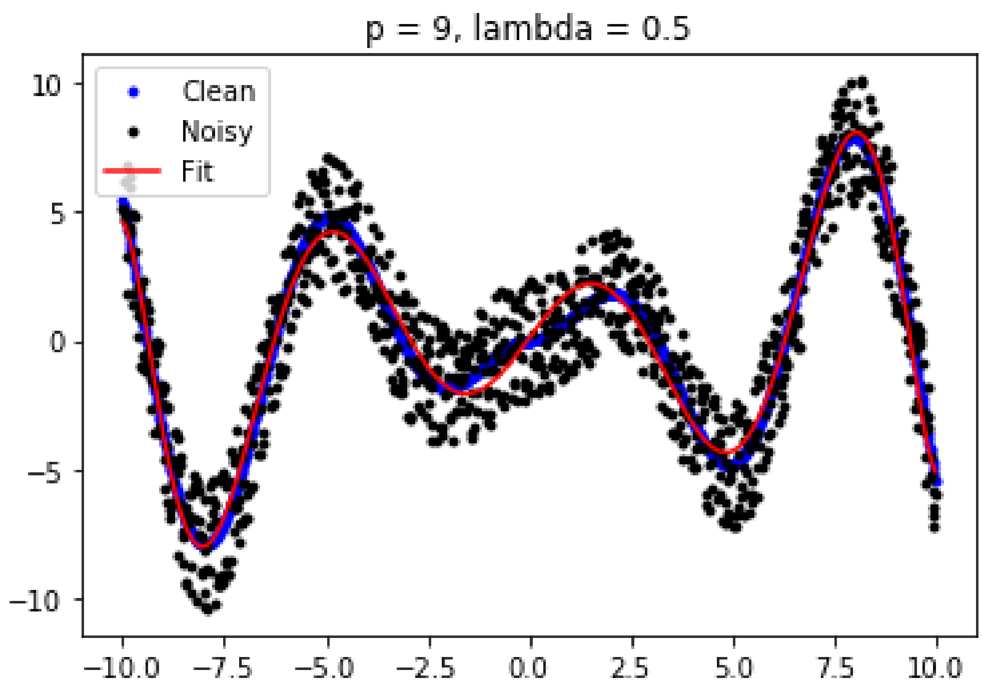

p = 9, lambda = 0.015625
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

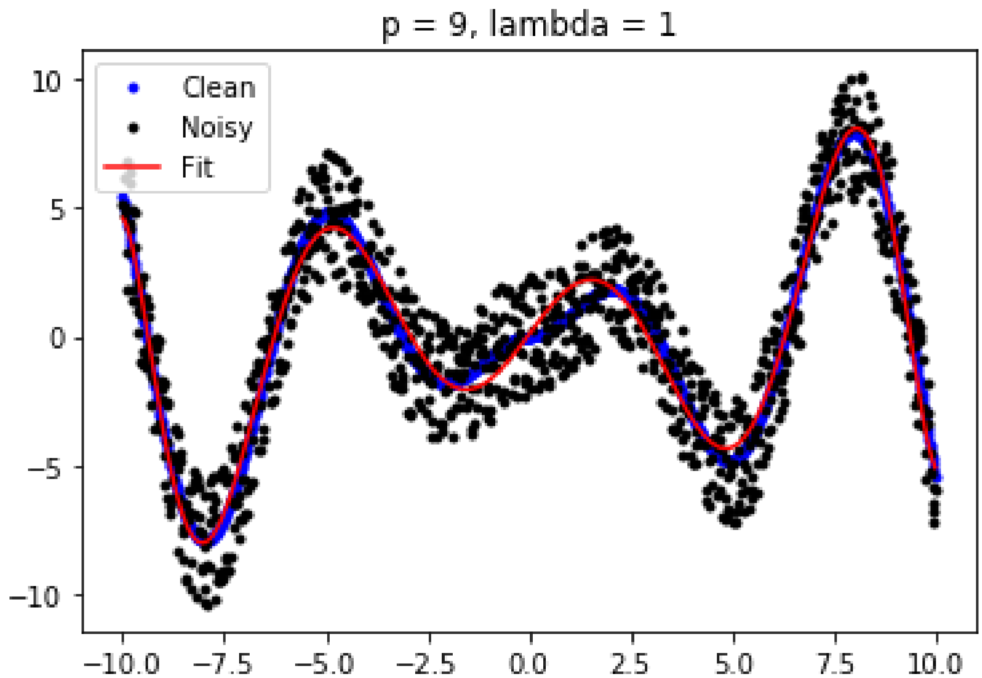

p = 9, lambda = 0.03125
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

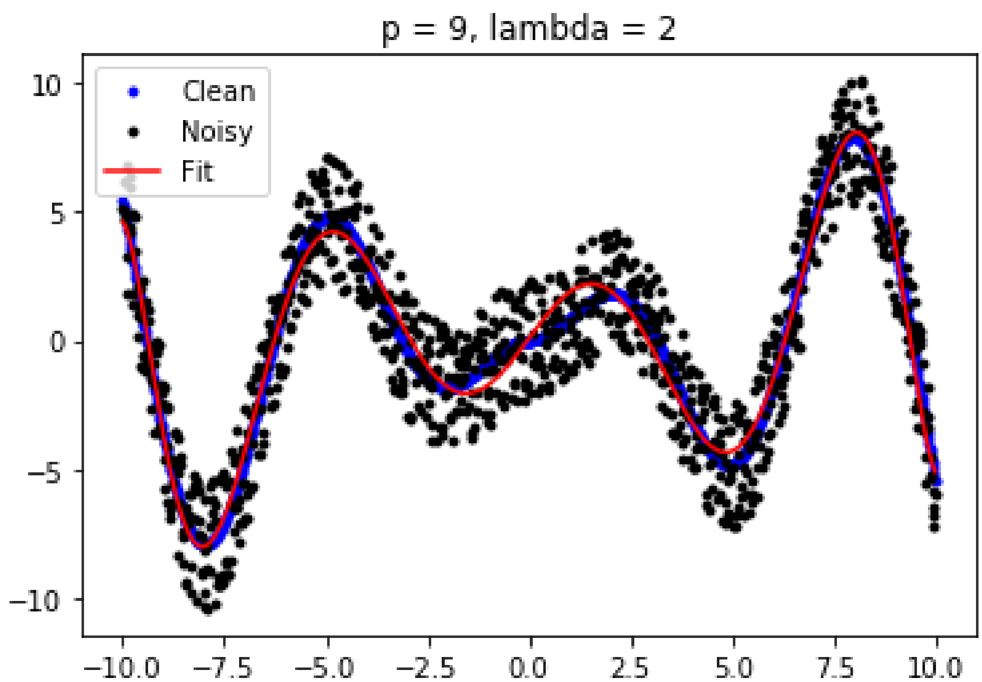

p = 9, lambda = 0.0625
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

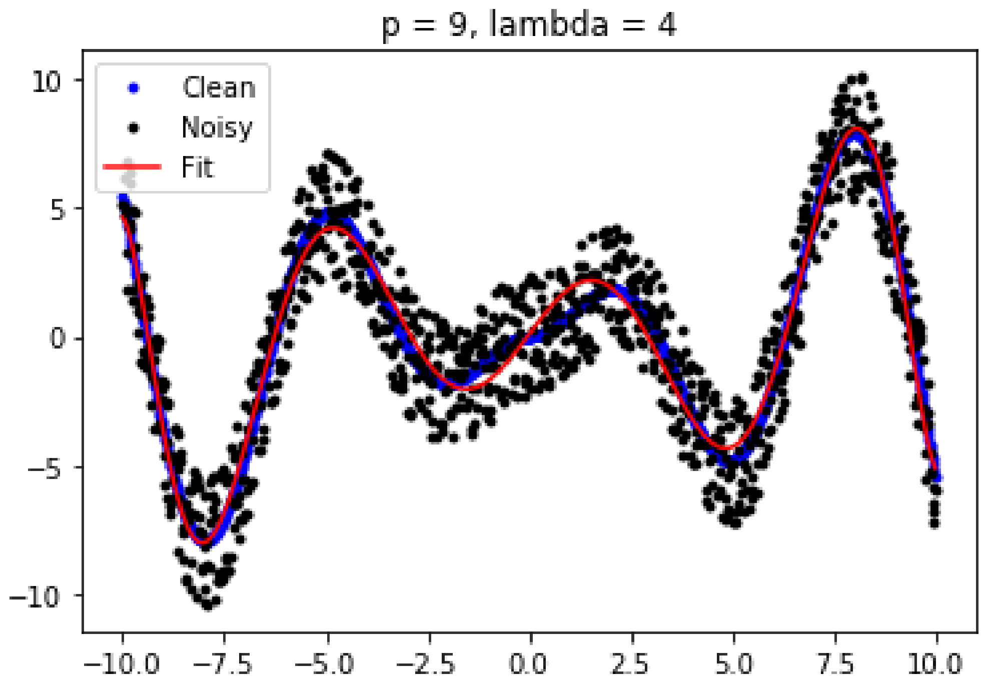

p = 9, lambda = 0.125
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

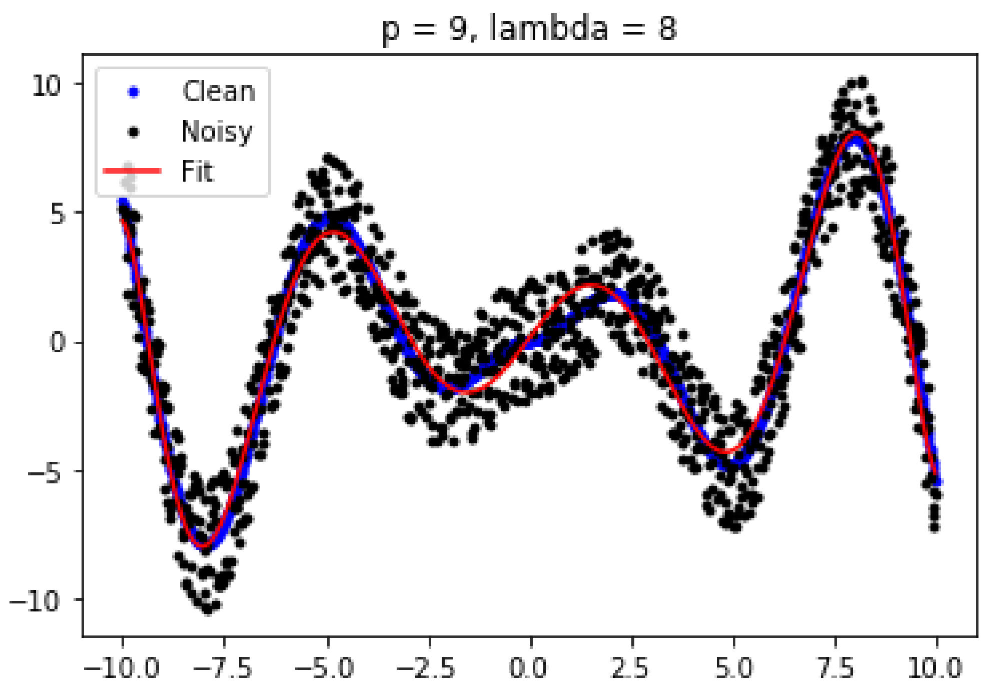

p = 9, lambda = 0.25
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

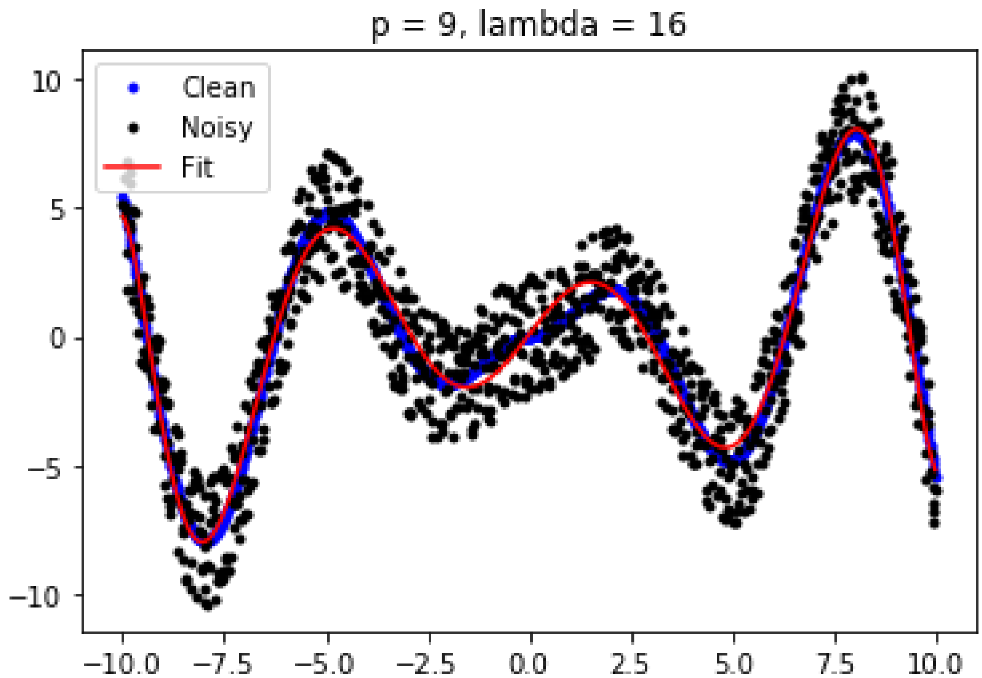

p = 9, lambda = 0.5
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

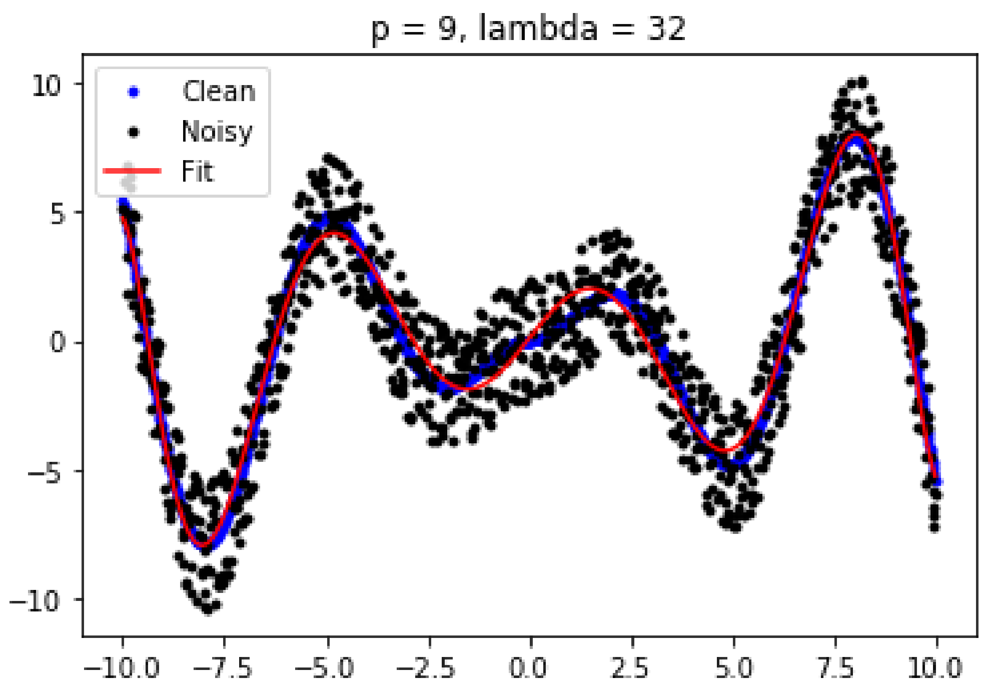

p = 9, lambda = 1
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

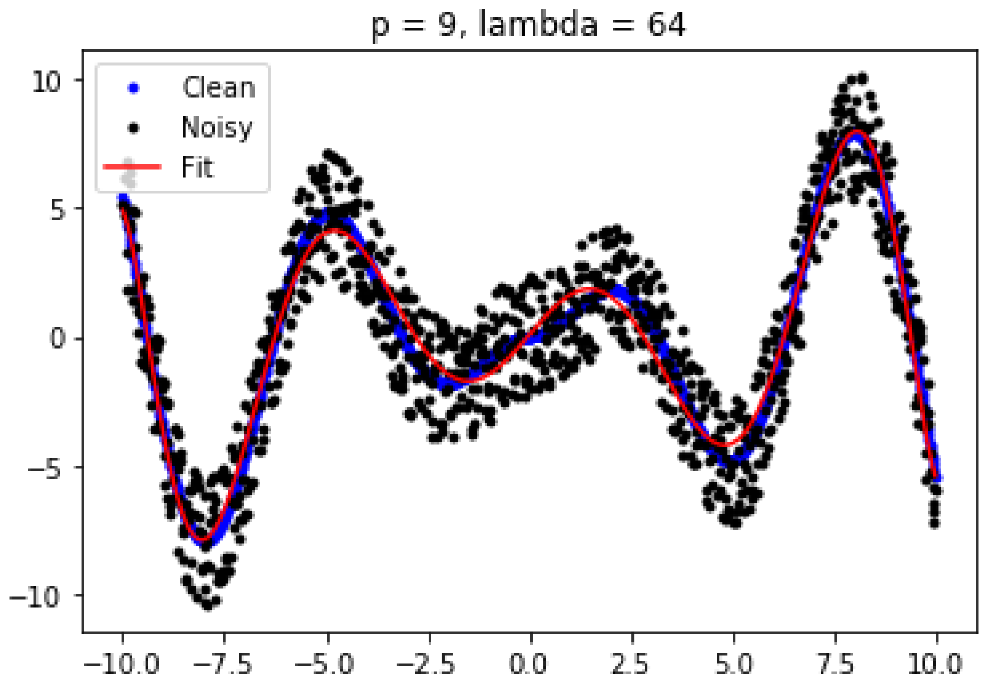

p = 9, lambda = 2
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

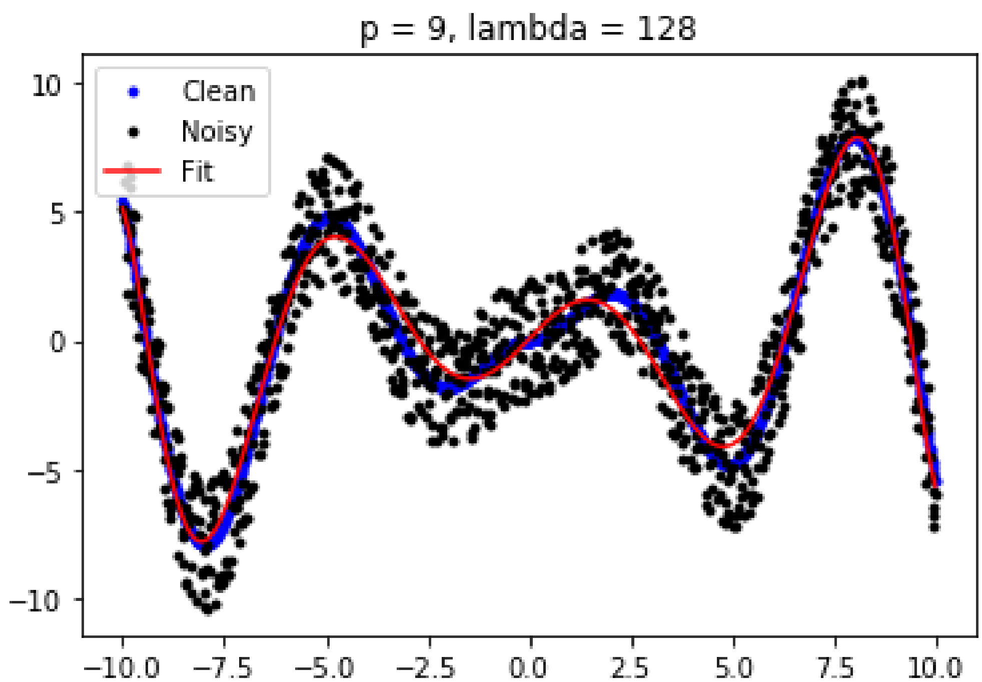

p = 9, lambda = 4
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

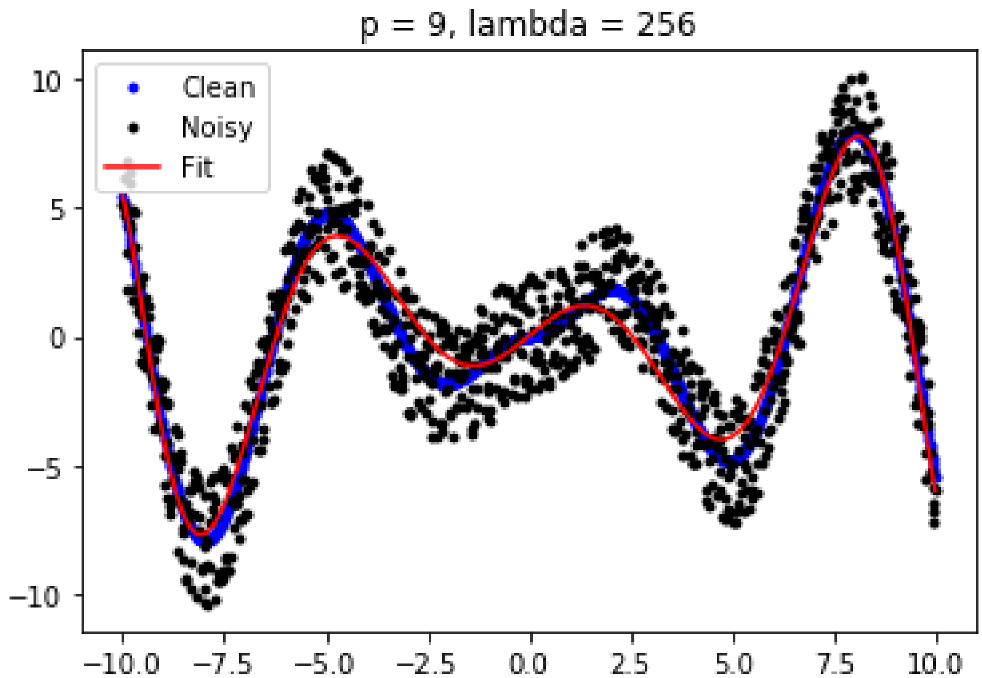

p = 9, lambda = 8
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

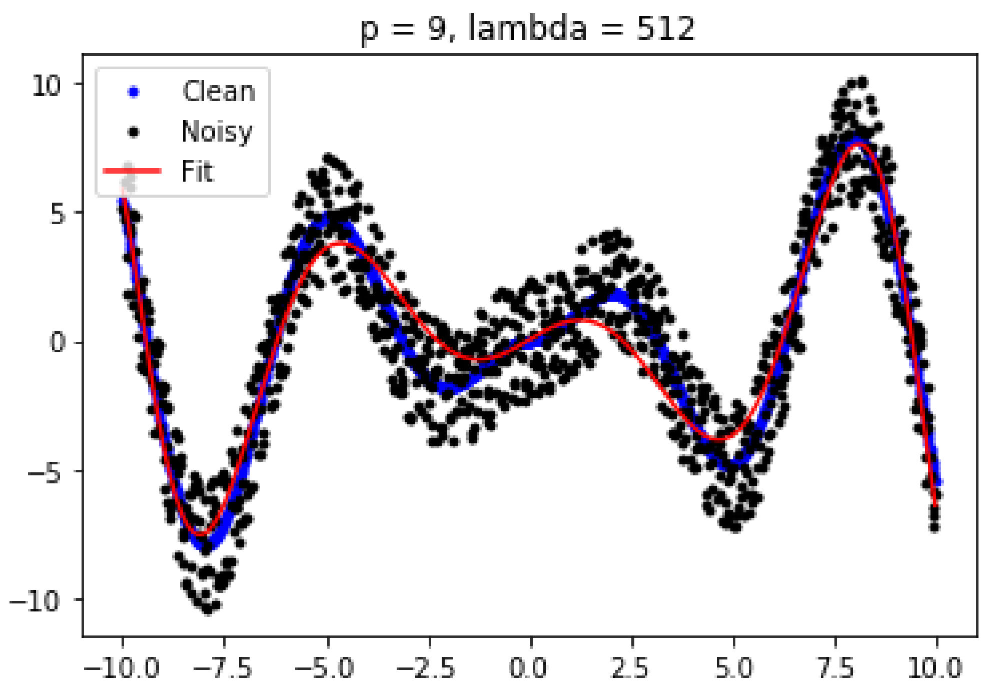

p = 9, lambda = 16
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

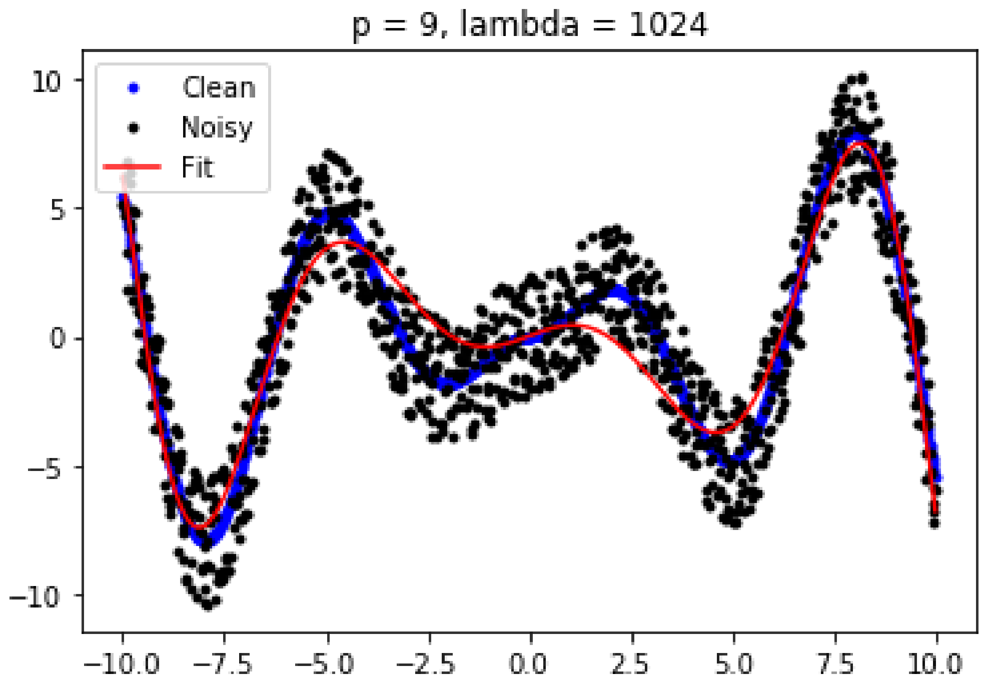

p = 9, lambda = 32
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0


p = 9, lambda = 64
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

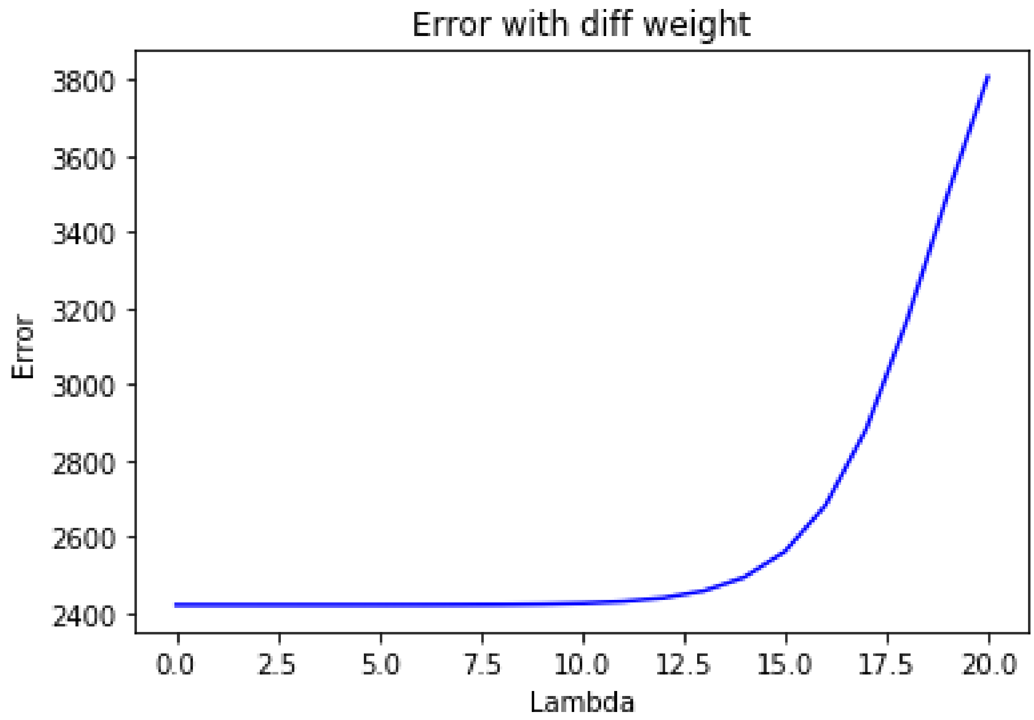

p = 9, lambda = 128
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0


p = 9, lambda = 256
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0


p = 9, lambda = 512
Clean
Noisy
Fit
10
5
0
−5
−10
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0



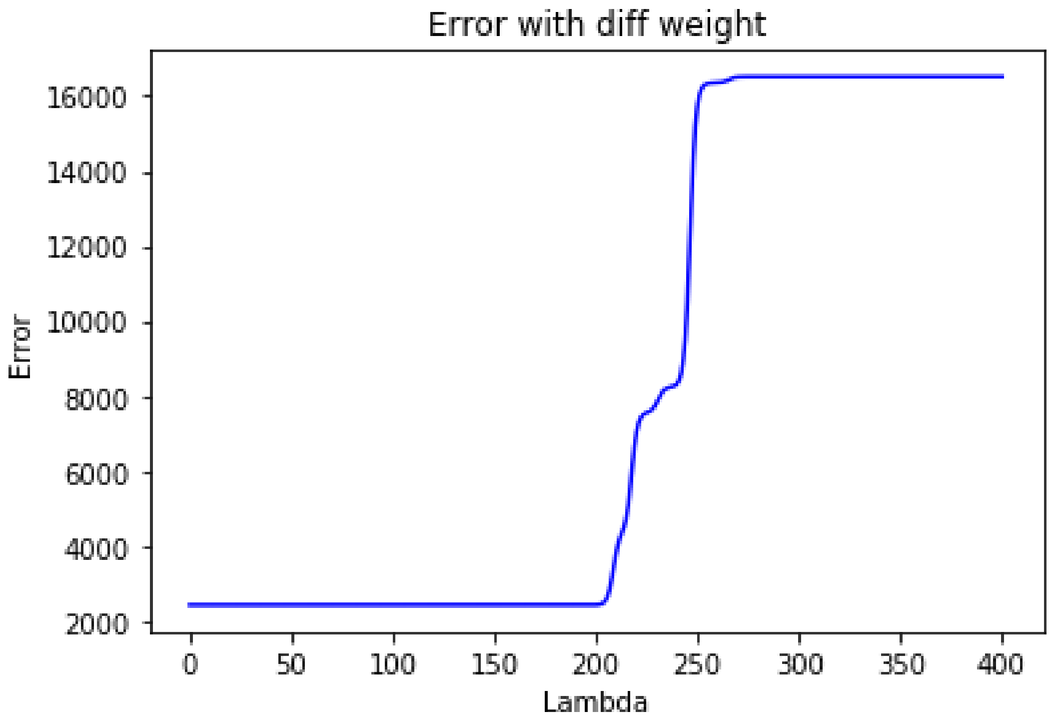

```
In [10]: plt.title("Error with diff weight")
         plt.plot(error_histo, 'b-')
         plt.xlabel('Lambda')
         plt.ylabel('Error')
         plt.show()
```



Extend the range of λ

In [11]:
```
lams = [2**n for n in range(-200, 201)]
errs = []

for lam in lams:
    _, error = fitting(x, y2, 9, lam)
    errs.append(error)
```

In [12]:
```
plt.title("Error with diff weight")
plt.plot(errs, 'b-')
plt.xlabel('Lambda')
plt.ylabel('Error')
plt.show()
```



When λ is small, the approximate result is close to input data. When λ becomes big, it focus on keep θ small, so error value start to grow again.