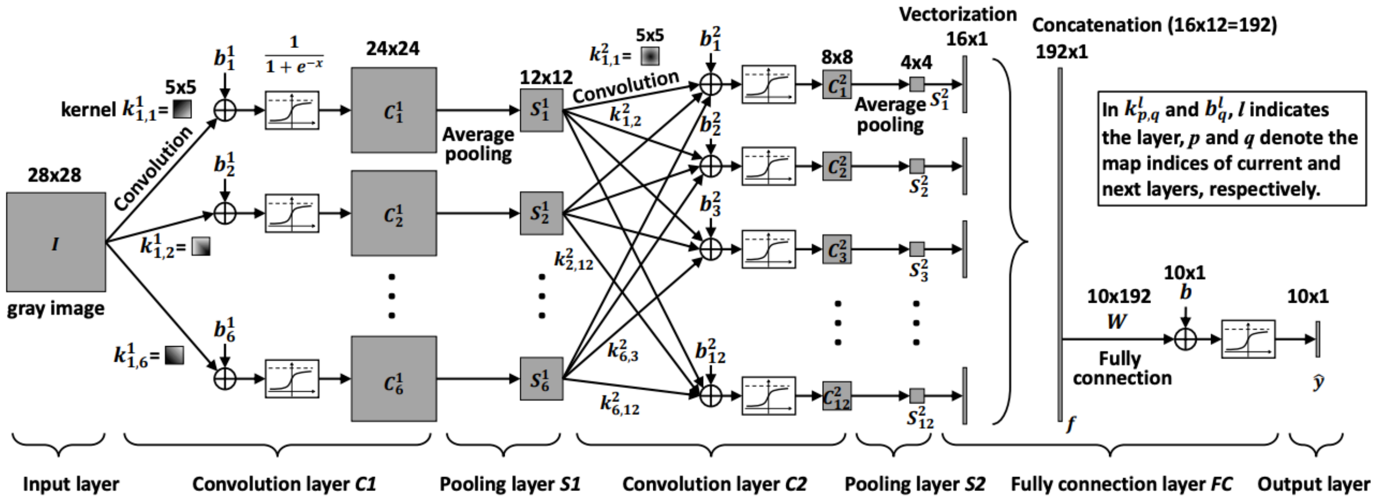
Matlab DeepLearnToolbox



Convolution layer C1

$$k_{1,p}^1 = (5,5) \quad (p = 1,2,\dots,6)$$
$$b_p^1 = (1,1) \quad (p = 1,2,\dots,6)$$
$$C_p^1 = \sigma\left(I * k_{1,p}^1 + b_p^1\right) \quad (p = 1,2,\dots,6)$$
$$C_p^1(i,j) = \sigma\left(\sum_{u=-2}^{2}\sum_{u=-2}^{2} I(i-u,j-v)\cdot k_{1,p}^1(u,v) + b_p^1\right) = (24,24)$$
$$\sigma(x) = \frac{1}{1+e^{-x}}$$

Pooling Layer S1

$$S_p^1(i,j) = \frac{1}{4}\sum_{u=0}^{1}\sum_{v=0}^{1} C_p^1(2i-u,2j-v) = (12,12) \quad (i,j = 1,2,\dots,12)$$

The indexing $2i-u, 2j-v$ accesses the 2×2 grids that are pooled.

Convolution layer C2

$$k_{p,q}^2 = (5,5) \quad (q = 1,2,\dots,12)$$
$$b_q^2 = (1,1) \quad (q = 1,2,\dots,12)$$
$$C_q^2 = \sigma\left(\sum_{p=1}^{6} S_p^1 * k_{p,q}^2 + b_q^2\right) \quad (q = 1,2,\dots,12)$$
$$C_q^2(i,j) = \sigma\left(\sum_{p=1}^{6}\sum_{u=-2}^{2}\sum_{u=-2}^{2} S_p^1(i-u,j-v)\cdot k_{p,q}^2(u,v) + b_q^2\right) = (8,8)$$

Pooling Layer S2

$$S_q^2(i,j) = \frac{1}{4}\sum_{u=0}^{1}\sum_{v=0}^{1} C_q^2(2i-u,2j-v) = (4,4) \quad (i,j = 1,2,\dots,4)$$

Vectorisation and concatenation

Each S_q^2 is a 4×4 matrix, and there are 12 such matrices on the S2 layer.

$$f = \text{concatenate}\left(\text{asColumnVector}\left(\left\{S_q^2\right\}_{q=1,2,\dots,12}\right)\right) = F\left(\left\{S_q^2\right\}_{q=1,2,\dots,12}\right)$$
$$\left\{S_q^2\right\}_{q=1,2,\dots,12} = F^{-1}(f)$$

FC layer

$$W = (10,192)$$
$$b = (10,1)$$
$$\hat{y} = \sigma(W \times f + b) = a$$

$$\hat{y}_i = \sigma\left(\sum_{j=1}^{192} W_{i,j} f_j + b_i\right) = \sigma(Z_i)$$

$$L = \frac{1}{2}\sum_{i=1}^{10} (\hat{y}_i - y_i)^2$$

Backpropagation

We update the parameters in reverse order. $W, b, k_{p,q}^2, b_q^2, k_{1,p}^1, b_p^1$.

Backpropagation, part I

Backpropagation 3a, $\Delta W = (10, 192)$

$$\Delta W_{i,j} = \frac{\partial L}{\partial W_{i,j}} = \frac{\partial L}{\partial \hat{y}_i}\frac{\partial \hat{y}_i}{\partial W_{i,j}} = (\hat{y}_i - y_i)\frac{\partial}{\partial W_{i,j}}\sigma\left(\sum_{j=1}^{192} W_{i,j} f_j + b_i\right)$$

$$= (\hat{y}_i - y_i)\sigma'(Z_i)\frac{\partial}{\partial W_{i,j}}\left(\sum_{j=1}^{192} W_{i,j} f_j + b_i\right) = (\hat{y}_i - y_i)\sigma(Z_i)\big(1 - \sigma(Z_i)\big)f_j$$

$$= (\hat{y}_i - y_i)\hat{y}_i(1 - \hat{y}_i)f_j = (10, 1)$$

Denote $\Delta\hat{y}_i = (\hat{y}_i - y_i)\hat{y}_i(1 - \hat{y}_i)$

$$\Delta W_{i,j} = \Delta\hat{y}_i f_j$$

$$\Delta W = \Delta\hat{y} \times f^T \quad (10, 192) = (10, 1) \times (1, 192)$$

Backpropagation 3b, $\Delta b = (10, 1)$

$$\Delta b_i = \frac{\partial L}{\partial b_i} = \frac{\partial L}{\partial \hat{y}_i}\frac{\partial \hat{y}_i}{\partial b_i} = (\hat{y}_i - y_i)\frac{\partial}{\partial b_i}\sigma\left(\sum_{j=1}^{192} W_{i,j} f_j + b_i\right)$$

$$= (\hat{y}_i - y_i)\sigma'(Z_i)\frac{\partial}{\partial b_i}\left(\sum_{j=1}^{192} W_{i,j} f_j + b_i\right) = (\hat{y}_i - y_i)\sigma(Z_i)\big(1 - \sigma(Z_i)\big)$$

$$= (\hat{y}_i - y_i)\hat{y}_i(1 - \hat{y}_i) = \Delta\hat{y}_i$$

$$\Delta b = \Delta\hat{y} \quad (10, 1) = (10, 1)$$

Backpropagation, part II

Backpropagation 2a, $\Delta S_q^2 = (4, 4)$

$$\Delta f_j = \frac{\partial L}{\partial f} = \sum_{i=1}^{10}\frac{\partial L}{\partial \hat{y}_i}\frac{\partial \hat{y}_i}{\partial f_j} = \sum_{i=1}^{10}\left\{(\hat{y}_i - y_i)\frac{\partial}{\partial f_j}\sigma\left(\sum_{j=1}^{192} W_{i,j} f_j + b_i\right)\right\}$$

$$= \sum_{i=1}^{10}(\hat{y}_i - y_i)\hat{y}_i(1 - \hat{y}_i)W_{i,j} = \sum_{i=1}^{10}\Delta\hat{y}_i W_{i,j}$$

$$\Delta f = W^T \times \Delta\hat{y} \quad (192, 1) = (192, 10) \times (10, 1)$$

Backpropagation 2b, $\Delta C_q^2 = (8, 8)$

We reshape Δf by

$$\left\{\Delta S_q^2\right\}_{q=1,2,\ldots,12} = F^{-1}(\Delta f)$$

which gives the error on the S2 layer made of twelve 4×4 maps. There are no parameters on the S2 layer so we do not need to take any derivatives. Perform upsampling to obtain the error on the C2 layer.

$$\Delta C_q^2(i, j) = \frac{1}{4}\Delta S_q^2\left(\left\lceil\frac{i}{2}\right\rceil, \left\lceil\frac{j}{2}\right\rceil\right) \quad (i, j = 1, 2, \ldots, 8)$$

Backpropagation 2c, $\Delta k_{p,q}^2 = (5, 5)$

$$\Delta k_{p,q}^2(u, v) = \frac{\partial L}{\partial k_{p,q}^2(u, v)} = \sum_{i=1}^{8}\sum_{j=1}^{8}\frac{\partial L}{\partial C_q^2(i, j)}\frac{\partial C_q^2(i, j)}{\partial k_{p,q}^2(u, v)}$$

$$= \sum_{i=1}^{8}\sum_{j=1}^{8}\left\{\Delta C_q^2(i,j)\frac{\partial}{\partial k_{p,q}^2(u,v)}\sigma\left(\sum_{p=1}^{6}\sum_{u=-2}^{2}\sum_{u=-2}^{2} S_p^1(i-u,j-v)\cdot k_{p,q}^2(u,v)+b_p^2\right)\right\}$$

$$= \sum_{i=1}^{8}\sum_{j=1}^{8}\Delta C_q^2(i,j)C_q^2(i,j)\left(1-C_q^2(i,j)\right)S_p^1(i-u,j-v)$$

Denote $\Delta C_{q,\sigma}^2(i,j) = \Delta C_q^2(i,j)C_q^2(i,j)\left(1-C_q^2(i,j)\right)$, which is actually the error before the sigmoid activation on the C2 layer.

$$C_{q,\sigma}^2(i,j) = \sum_{p=1}^{6}\sum_{u=-2}^{2}\sum_{u=-2}^{2} S_p^1(i-u,j-v)\cdot k_{p,q}^2(u,v)+b_p^2$$

$$\Delta C_{q,\sigma}^2(i,j) = \Delta C_q^2(i,j)C_q^2(i,j)\left(1-C_q^2(i,j)\right)$$

$$\Delta k_{p,q}^2(u,v) = \sum_{i=1}^{8}\sum_{j=1}^{8}\Delta C_{q,\sigma}^2(i,j)S_p^1(i-u,j-v)$$

Rotating S_p^1 by 180 degrees, we get $S_{p,180}^1(u-i,v-j) = S_p^1(i-u,j-v)$

$$\Delta k_{p,q}^2(u,v) = \sum_{i=1}^{8}\sum_{j=1}^{8} S_{p,180}^1(u-i,v-j)\Delta C_{q,\sigma}^2(i,j)$$

$$\Delta k_{p,q}^2 = S_{p,180}^1 * \Delta C_{q,\sigma}^2$$

Backpropagation 2d, $\Delta b_q^2 = (1,1)$

$$\Delta b_q^2 = \frac{\partial L}{\partial b_q^2} = \sum_{i=1}^{8}\sum_{j=1}^{8}\frac{\partial L}{\partial C_q^2(i,j)}\frac{\partial C_q^2(i,j)}{\partial b_q^2}$$

$$= \sum_{i=1}^{8}\sum_{j=1}^{8}\left\{\Delta C_q^2(i,j)\frac{\partial}{\partial b_q^2}\sigma\left(\sum_{p=1}^{6}\sum_{u=-2}^{2}\sum_{u=-2}^{2} S_p^1(i-u,j-v)\cdot k_{p,q}^2(u,v)+b_p^2\right)\right\}$$

$$= \sum_{i=1}^{8}\sum_{j=1}^{8}\Delta C_q^2(i,j)C_q^2(i,j)\left(1-C_q^2(i,j)\right) = \sum_{i=1}^{8}\sum_{j=1}^{8}\Delta C_{q,\sigma}^2(i,j)$$

Backpropagation, part III

Backpropagation 3a, $\Delta S_p^1 = (12,12)$

$$\Delta S_p^1(i,j) = \frac{\partial L}{\partial S_p^1(i,j)} = \sum_{q=1}^{12}\sum_{u=-2}^{2}\sum_{u=-2}^{2}\frac{\partial L}{\partial C_{q,\sigma}^2(i+u,j+u)}\frac{\partial C_{q,\sigma}^2(i+u,j+u)}{\partial S_p^1(i,j)}$$

$$= \sum_{q=1}^{12}\sum_{u=-2}^{2}\sum_{u=-2}^{2}\left\{\Delta C_{q,\sigma}^2(i+u,j+v)\frac{\partial}{\partial S_p^1(i,j)}\left(\sum_{p=1}^{6}\sum_{u=-2}^{2}\sum_{u=-2}^{2} S_p^1(i,j)\cdot k_{p,q}^2(u,v)+b_q^2\right)\right\}$$

$$= \sum_{q=1}^{12}\sum_{u=-2}^{2}\sum_{u=-2}^{2}\Delta C_{q,\sigma}^2(i+u,j+v)k_{p,q}^2(u,v)$$

Backpropagation 3b, $\Delta C_p^1 = (24,24)$

Rotating $k_{p,q}^2$ by 180 degrees, we get $k_{p,q,180}^2(-u,-v) = k_{p,q}^2(u,v)$

$$\Delta S_p^1(i,j) = \sum_{q=1}^{12}\sum_{u=-2}^{2}\sum_{u=-2}^{2}\Delta C_{q,\sigma}^2\big(i-(-u),j-(-v)\big)k_{p,q,180}^2(-u,-v)$$

$$\Delta S_p^1(i,j) = \sum_{q=1}^{12}\Delta C_{q,\sigma}^2 * k_{p,q,180}^2$$

Perform upsampling to obtain the error on the C1 layer

$$\Delta C_p^1(i,j) = \frac{1}{4}\Delta S_p^1\left(\left\lceil\frac{i}{2}\right\rceil, \left\lceil\frac{j}{2}\right\rceil\right) \quad (i,j = 1,2,\dots,24)$$

Backpropagation 3c, $\Delta k_{1,p}^1 = (5,5)$

$$\Delta k_{1,p}^1(u,v) = \frac{\partial L}{\partial k_{1,p}^1(u,v)} = \sum_{i=1}^{24}\sum_{i=1}^{24}\frac{\partial L}{\partial C_p^1(i,j)}\frac{\partial C_p^1(i,j)}{\partial k_{1,p}^1(u,v)}$$

$$= \sum_{i=1}^{24}\sum_{i=1}^{24}\left\{\Delta C_p^1(i,j)\frac{\partial}{\partial k_{1,p}^1(u,v)}\sigma\left(\sum_{u=-2}^{2}\sum_{u=-2}^{2} I(i-u,j-v)\cdot k_{1,p}^1(u,v) + b_p^1\right)\right\}$$

$$= \sum_{i=1}^{24}\sum_{i=1}^{24}\Delta C_p^1(i,j) C_p^1(i,j)\left(1 - C_p^1(i,j)\right) I(i-u,j-v)$$

Denote $\Delta C_{p,\sigma}^1(i,j) = \Delta C_p^1(i,j) C_p^1(i,j)\left(1 - C_p^1(i,j)\right)$ and rotating I by 180 degrees

$$\Delta k_{1,p}^1(u,v) = \sum_{i=1}^{24}\sum_{i=1}^{24} I_{180}(u-i,v-j)\Delta C_{p,\sigma}^1(i,j)$$

$$\Delta k_{1,p}^1 = I_{180} * \Delta C_{p,\sigma}^1$$

Backpropagation 3d, $\Delta b_p^1 = (1,1)$

$$\Delta b_p^1 = \frac{\partial L}{\partial b_p^1} = \sum_{i=1}^{24}\sum_{i=1}^{24}\frac{\partial L}{\partial C_p^1(i,j)}\frac{\partial C_p^1(i,j)}{\partial b_p^1}$$

$$= \sum_{i=1}^{24}\sum_{i=1}^{24}\left\{\Delta C_{p,\sigma}^1(i,j)\frac{\partial}{\partial b_p^1}\sigma\left(\sum_{u=-2}^{2}\sum_{u=-2}^{2} I(i-u,j-v)\cdot k_{1,p}^1(u,v) + b_p^1\right)\right\}$$

$$= \sum_{i=1}^{24}\sum_{i=1}^{24}\Delta C_{p,\sigma}^1(i,j) C_{p,\sigma}^1(i,j)\left(1 - C_{p,\sigma}^1(i,j)\right) = \sum_{i=1}^{24}\sum_{i=1}^{24}\Delta C_{p,\sigma}^1(i,j)$$