2.17 (a) $\int_0^m f(x)\mathrm{d}x = m^3 = \frac{1}{2} \Rightarrow m = (\frac{1}{2})^{\frac{1}{3}}$

(b) $f(-x) = f(x)$ 因此 $\int_{-\infty}^0 f(x)\mathrm{d}x = \int_0^\infty f(x)\mathrm{d}x = \frac{1}{2}$ 因此 $m = 0$.

2.18 $E|x-a| = \int_{-\infty}^\infty |x-a|f(x)\mathrm{d}x = \int_0^\infty (x-a)f(x)\mathrm{d}x + \int_{-\infty}^0 (a-x)f(x)\mathrm{d}x$

$$\begin{aligned} E|x-a| - E|x-m| &= 2\int_a^m xf(x)\mathrm{d}x + \int_{-\infty}^a af(x)\mathrm{d}x - \int_a^\infty af(x)\mathrm{d}x \\ &= 2\int_a^m xf\mathrm{d}x + a\left(1 - 2\int_a^\infty f(x)\mathrm{d}x\right) \\ &= 2\int_a^m xf\mathrm{d}x + a\left(\int_m^\infty f(x)\mathrm{d}x - 2\int_a^\infty f(x)\mathrm{d}x\right) \\ &= 2\int_a^m (m-a)f(x)\mathrm{d}x \quad or \quad 2\int_m^a (a-m)f(x)\mathrm{d}x \end{aligned}$$

因为 $\int_a^m (m-a)f(x)\mathrm{d}x \geq 0$

所以 $E|x-a| - E|x-m| \geq 0 \Rightarrow E|x-a| \geq E|x-m|$

即 $\min_a E|x-a| = E|x-m|$.

2.27 (a) $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$

(b)

$$f(x) = \begin{cases} \frac{1}{\pi}\sin^2(x) & \text{if } x \in [-\pi, \pi] \\ 0 & \text{others} \end{cases}$$

(c) 假众数在 b 处且 $b \neq a$, 不妨假设 $a < b$, 由题意可知，对 $a < x < b, f(a) < f(x) < f(b)$, 因为 a 是对称点，所以存在点 $x' < a$ 满足 $a - x' = x - a$, 有 $f(x') = f(x)$ 此时，$f(x') > f(a)$ 且 $x' < a < b$ 与 b 是 $f(x)$ 的众数矛盾。故 $a = b$，即 a 是 $f(x)$ 的众数。

(d) $x = 0$

3.13 (a)$P\left(X=x|\lambda\right)=\dfrac{e^{-\lambda}\lambda^{x}}{x!}$ $\qquad P(X=0)=e^{-\lambda}$

所以 $\quad P\left(X\geq 0\right)=1-e^{-\lambda}$

$P(X_T=x)=\dfrac{e^{-\lambda}}{1-e^{-\lambda}}\dfrac{\lambda^{x}}{x!}$

所以 $EX_T=\displaystyle\sum_{x=1}^{\infty}xP(X_T=x)=\frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!}=\frac{\lambda}{1-e^{-\lambda}}$

$$
\begin{aligned}
VarX_T &= EX_T^2-(EX_T)^2\\
&= \sum_{x=1}^{\infty}x^2\frac{e^{-\lambda}\lambda^{x}}{(1-e^{-\lambda})x!}-(EX_T)^2\\
&= \frac{\lambda(\lambda+1)}{1-e^{-\lambda}}-\left(\frac{\lambda}{1-e^{-\lambda}}\right)^2\\
&= \frac{\lambda}{1-e^{-\lambda}}(1-\frac{\lambda e^{-\lambda}}{1-e^{-\lambda}})
\end{aligned}
$$

(b) $P(X=x)=\dbinom{r+x-1}{x}p^r(1-p)^x$, $P(X>0)=1-P(X=0)=1-p^r$

$P(X_T=x)=\dbinom{r+x-1}{x}\dfrac{p^r}{1-p^r}(1-p)^x$

$EX_T=\displaystyle\sum_{x=1}^{\infty}x\binom{r+x-1}{x}\frac{p^r}{1-p^r}(1-p)^x=\frac{EX}{1-p^r}=r\frac{1-p}{p(1-p^r)}$

$$
\begin{aligned}
VarX_T &= EX_T^2-(EX_T)^2=\frac{EX^2}{1-p^r}-(EX_T)^2\\
&= \frac{VarX+(EX)^2}{1-p^r}-(EX_T)^2\\
&= \frac{r(1-p)+r^2(1-p)^2}{p^2(1-p^r)}-[\frac{r(1-p)}{p(1-p^r)}]^2
\end{aligned}
$$

3.24 (a) $f(x) = \dfrac{1}{\beta}e^{-x/\beta} \quad , \quad Y = X^{1/\gamma} = g(X) \quad , \quad g^{-1}(y) = y^{\gamma}$

$$f_Y(y) = f_X(g^{-1}(y))\frac{\mathrm{d}}{\mathrm{d}y}g^{-1}(y) = \frac{1}{\beta}e^{-y^{\gamma}/\beta}\gamma y^{\gamma-1}$$

$$\begin{aligned} EY &= \int_0^{\infty} \frac{\gamma}{\beta}y^{\gamma}e^{-y^{\gamma}/\beta}\mathrm{d}y = \int_0^{\infty} ye^{-y^{\gamma}/\beta}\mathrm{d}(y^{\gamma}/\beta) \\ &= \int_0^{\infty} (\beta t)^{1/\gamma}e^{-t}\mathrm{d}t = \beta^{1/\gamma}\Gamma(1+\frac{1}{\gamma}) \\ &= \frac{\beta^{1/\gamma}}{\gamma}\Gamma(\frac{1}{\gamma}) \end{aligned}$$

$$\begin{aligned} VarY &= EY^2 - (EY)^2 \\ &= \int_0^{\infty} y^2e^{-y^{\gamma}/\beta}\mathrm{d}(\frac{y^{\gamma}}{\beta}) - (EY)^2 \\ &= \int_0^{\infty} (\beta t)^{2/\gamma}e^{-t}\mathrm{d}t - (EY)^2 \\ &= \beta^{2/\gamma}[\Gamma(\frac{2}{\gamma}+1) - \Gamma^2(\frac{1}{\gamma}+1)] \end{aligned}$$

(b)$f_X(x) = \dfrac{1}{\beta}e^{-x/\beta} \quad , \quad Y = (2X/\beta)^{\frac{1}{2}} = g(x) \quad , \quad g^{-1}(y) = \dfrac{\beta y^2}{2}$

$$f_Y(y) = f_X(g^{-1}(y))\frac{\mathrm{d}}{\mathrm{d}y}g^{-1}(y) = ye^{-y^2/2}$$

$$EY = \int_0^{\infty} yf_Y(y)\mathrm{d}y = \int_0^{\infty} y^2e^{-\frac{y^2}{2}}\mathrm{d}y = \sqrt{\frac{\pi}{2}}$$

$$VarY = EY^2 - (EY)^2 = \int_0^{\infty} y^3e^{\frac{-y^2}{2}}\mathrm{d}y - \frac{\pi}{2} = \int_0^{\infty} 2te^{-t}\mathrm{d}t - \frac{\pi}{2}$$

$$= 2 - \frac{\pi}{2}$$

(c) $f_X(x) = \dfrac{1}{\Gamma(a)b^a}x^{a-1}e^{-x/b} \quad , \quad Y = \dfrac{1}{X}, g^{-1}(y) = \dfrac{1}{y}$

$$f_Y(y) = \frac{1}{\Gamma(a)b^ay^{a+1}}e^{-\frac{1}{by}}$$

$$\begin{aligned} EY &= \int_0^{\infty} yf_Y(y)\mathrm{d}y = \int_0^{\infty} \frac{1}{x}\frac{1}{\Gamma(a)b^a}x^{a-1}e^{-x/b}\mathrm{d}x \\ &= \int_0^{\infty} \frac{1}{\Gamma(a)b^a}x^{a-2}e^{-x/b}\mathrm{d}x = \frac{\Gamma(a-1)}{\Gamma(a)b} \\ &= \frac{1}{(a-1)b} \end{aligned}$$

$$
\begin{aligned}
EY^2 &= \int_0^\infty y^2 f_Y(y)\mathrm{d}y = \int_0^\infty \frac{1}{x^2\Gamma(a)b^a} x^{a-1}e^{-x/b}\mathrm{d}x \\
&= \int_0^\infty \frac{1}{\Gamma(a)b^a} x^{a-3}e^{-x/b}\mathrm{d}x = \frac{\Gamma(a-2)}{\Gamma(a)b^2} \\
&= \frac{1}{(a-1)(a-2)b^2}
\end{aligned}
$$

$VarY = EY^2 - (EY)^2 = \frac{1}{(a-1)^2(a-2)b^2}$

(d)$f_X(x) = \dfrac{x}{\sqrt{\pi\beta}} x^{1/2}e^{-x/\beta}$, $Y = (x/\beta)^{1/2}$, $g^{-1}(y) = \beta y^2$

$f_Y(y) = \dfrac{4}{\sqrt{\pi}} y^2 e^{-y^2}$

$EY = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^\infty y^2 e^{-y^2}\mathrm{d}(y^2) = \dfrac{2}{\sqrt{\pi}}$

$EY^2 = \displaystyle\int_0^\infty \frac{2}{\sqrt{\pi}} y^3 e^{-y^2}\mathrm{d}(y^2) = \int_0^\infty \frac{2}{\sqrt{\pi}} t^{2/3}e^{-t}\mathrm{d}t = \frac{x}{\sqrt{\pi}}\Gamma(\frac{5}{2}) = \frac{3}{2}$

$VarY = EY^2 - (EY)^2 = \dfrac{9}{4} - \dfrac{4}{\pi}$

(e)$f_X(x) = e^{-x}$, $Y = \alpha - \gamma\log(X)$, $g^{-1}(y) = e^{\frac{\alpha-y}{\gamma}}$

$f_Y(y) = \dfrac{1}{\gamma}\exp(-e^{\frac{\alpha-y}{\gamma}})e^{\frac{\alpha-y}{\gamma}}$

$EY = \displaystyle\int_{-\infty}^\infty y f_Y(y)\mathrm{d}y = \int_0^\infty (\alpha - \gamma log(x))e^{-x}\mathrm{d}x = \alpha - \gamma\int_0^\infty \log(x)e^{-x}\mathrm{d}x$

因为 $\log(x) = \left.\dfrac{dx^t}{dt}\right|_{t=0}$

$\displaystyle\int_0^\infty \log(x)e^{-x}\mathrm{d}x = \left.\frac{\mathrm{d}}{\mathrm{d}t}(\int_0^\infty x^t e^{-x}\mathrm{d}x)\right|_{t=0} = \Gamma'(t+1)|_{t=0} = \Gamma'(1)$

并且 $\Gamma'(z) = \Gamma(z)\phi(z)$, $\phi(1) = -c$ (c 是欧拉常数) $\Rightarrow \Gamma'(1) = -c$

$EY = \alpha + \gamma c$

$EY^2 = \displaystyle\int_{-\infty}^\infty y^2 f_Y(y)\mathrm{d}y = \int_0^\infty (\alpha - \gamma\log(x))^2 e^{-x}\mathrm{d}x$

$\displaystyle\int_0^\infty \log^2(x)e^{-x}\mathrm{d}x = 2\left.\frac{d^2}{dt^2}(\int_0^\infty x^t e^{-x}\mathrm{d}x)\right|_{t=0} = \Gamma''(1)$

$\Gamma''(z) = \Gamma'(z)\phi(z) + \Gamma(z)\phi'(z)$

$\phi'(1) = \dfrac{\pi^2}{6} \Rightarrow \Gamma''(1) = c^2 + \dfrac{\pi^2}{6} \Rightarrow$

$EY^2 = \displaystyle\int_0^\infty (\alpha - \gamma\log(x))^2 e^{-x}\mathrm{d}x = \alpha^2 + 2\alpha\gamma c + \gamma^2(c^2 + \frac{\pi^2}{6})$(c 是欧拉常数)

$VarY = EY^2 - (EY)^2 = \frac{\gamma^2\pi^2}{6}$

3.25 $P(t \le T \le t+\delta) = P(t \le t+\delta) - P(T \le t) = F_T(t+\delta) - F_T(t)$

$h_T(t) = \lim_{\delta \to \infty} \frac{F_T(t+\delta) - F_T(t)}{\delta} \frac{1}{F_T(t)}$

$= \frac{f_T(t)}{1 - F_T(t)} = -\log\Big(1 - F_T(t)\Big)$

3.26 (a)$f_T(t) = \frac{1}{\beta^{-t/\beta}}$ $\quad , F_T(t) = 1 - e^{-t/\beta}$

$h_T(t) = \frac{1}{\beta}$

(b)$f_T(t) = \frac{\gamma}{\beta} t^{\gamma-1} e^{-t^\gamma/\beta} \quad , \quad F_T(t) = \int_0^t f_T(s)\mathrm{d}s = 1 - e^{-t^\gamma/\beta}$

$h_T(t) = \frac{\gamma}{\beta} t^{\gamma-1}$

(c)$F_T(t) = \frac{1}{1 + e^{-(t-\mu)/\beta}} \quad , \quad f_T(t) = F_T'(t) = \frac{e^{-(t-\mu)/\beta}}{\beta(1 + e^{-(t-\mu)/\beta})^2}$

$h_T(t) = \frac{1}{\beta(1 + e^{-(t-\mu)/\beta})}$

3.28 (a) $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{(2\sigma^2)}} = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-\mu^2}{2\sigma^2}} e^{\frac{-x^2}{\sigma^2} + \frac{\mu x}{\sigma^2}}$

$h(x) = 1 \quad c = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-\mu^2}{2\sigma^2}} \quad \omega_1 = \frac{-1}{2\sigma^2} \quad t_1 = x^2 \quad \omega_2 = \frac{\mu}{\sigma^2} \quad t_2 = x$

(b) $f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} = \frac{1}{\Gamma(\alpha)\beta^\alpha} e^{(\alpha-1)\log x - x/\beta}$

$h(x) = 1 \ , \ c(\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \ , \ \omega_1 = \alpha - 1 \ , \ t_1 = \log x \ , \ \omega_2 = -\frac{1}{\beta} \ , \ t_2 = x$

(c)$f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} = \frac{1}{B(\alpha,\beta)} e^{(\alpha-1)\log x + (\beta-1)\log(1-x)}$

$h(x) = 1 \ , \ c(\alpha, \beta) = \frac{1}{B(\alpha,\beta)} \ , \ \omega_1 = \alpha - 1 \ , \ t_1 = \log x \ , \ \omega_2 = \beta - 1 \ ,$

$t_2 = \log(1-x)$

(d)$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-\lambda}}{x!} e^{\lambda \log x}$

$h(x) = \frac{1}{x!} \ , \ c(\lambda) = e^{-\lambda} \ , \ \omega_1 = \lambda \ , \ t_1 = \log x$

(e)$f(x) = \binom{r+x-1}{x} p^t (1-p)^x = \binom{r+x-1}{x} p^r e^{(1-p)\log x}$

$h(x)=1\binom{r+x-1}{x}$, $c(p)=p^r$, $\omega_1=1-p$, $t_1=\log x$

3.41 (a)$f(x|\mu)=\frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{(x-\mu)^2}{2\sigma^2}x)$

设$f_1(x)=f(x|\mu_1)$, $f_2(x)=f(x|\mu_2)$, 且$\mu_1<\mu_2$

$F_1(x)=\int_{-\infty}^{x}f_1(t)\mathrm{d}t \quad , \quad F_2(x)=\int_{-\infty}^{x}f_2(t)\mathrm{d}t$

因为$f_2(x)=f_1\big(x-(\mu_2-\mu_1)\big)$

所以$F_2(x)=\int_{-\infty}^{x}f_1\big(t-(\mu_2-\mu_1)\big)\mathrm{d}t=F_1(x)-\int_{x-(\mu_2-\mu_1)}f_1(t)\mathrm{d}t$

因为$f_1(t)>0$, 所以$F_2(x)<F_1(x)$(对所有的 x)

即$F(x|\mu)$关于 μ 随机递增

(b) $f(x|\beta)=\frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta}$

仍记$f_1(x)=f(x|\beta_1)$, $f_2(x)=f(x|\beta_2)$, 且$\beta_1<\beta_2$

$F_1(x)=\int_{-\infty}^{x}f_1(t)\mathrm{d}t \quad , \quad F_2(x)=\int_{-\infty}^{x}f_2(t)\mathrm{d}t$

因为$f_2(t)=\frac{\beta_1}{\beta_2}f_1(\frac{\beta_1}{\beta_2}t)$

所以$F_2(x)=\int_{0}^{x}f_1(\frac{\beta_1}{\beta_2}t)\frac{\beta_1}{\beta_2}\mathrm{d}t=\int_{0}^{\frac{\beta_1}{\beta_2}x}f_1(t)\mathrm{d}t$

而$\frac{\beta_1}{\beta_2}x<x$, $f_1(x)>0$(对$x\neq 0$,$x\neq\infty$)

$F_2(x)\le F_1(x)$对所有 x , $F_2(x)<F_1(x)$对某些 x

即$f(x|\beta)$关于 β 随机递增

3.42 (a) 设标准函数为 $f_0(x)$

$f_1(x)=f_0(x-\mu_1) \quad f_2(x)=f_0(x-\mu_2)$, $mu_1<\mu_2$

$F_2(x)=\int_{0}^{x}f_2(t)\mathrm{d}t=\int_{0}^{x}f_1\big(t-(\mu_2-\mu_1)\big)\mathrm{d}t=F_1(x)-\int_{x-(\mu_2-\mu_1)}^{x}f_1(t)\mathrm{d}t$

因为$f_1(t)\ge 0$(对所有 x) , 且$f_1(t)\ge 0$(对某些 x)

$F_2(x)\le F_1(x)$(对所有 x) , 且$F_2(x)\le F_1(x)$(对某些 x)

$f_0(x)$的位置函数族关于位置随机递增

(b) 设标准函数为 $f_0(x)$，$x \in [0, \infty)$

$f_1(x) = \frac{1}{\sigma_1} f_0(\frac{x}{\sigma_1}) \quad f_2(x) = \frac{1}{\sigma_2} f_0(\frac{x}{\sigma_2})$，$\sigma_2 > \sigma_1$

$F2(x) = \int_0^x \frac{\sigma_1}{\sigma_2} f_1(\frac{\sigma_1}{\sigma_2} t) \mathrm{d}t = \int_0^{\frac{\sigma_1 x}{\sigma_2}} f_1(t) \mathrm{d}t = F_1(x) - \int_{\frac{\sigma_1 x}{\sigma_2}}^x f_1(t) \mathrm{d}t$

因为$f_1(t) \geq 0$(对所有 x)，且$f_1(t) \geq 0$(对某些 x)

$F_2(x) \leq F_1(x)$(对所有 x)，且$F_2(x) \leq F_1(x)$(对某些 x)

$f_0(x)$的尺度函数族关于尺度随机递增

Proof of Theorem 3.4.2. (1)首先$\int f(x|\theta)\mathrm{d}x = 1$ 并且有

$$\begin{aligned}\frac{\partial}{\partial\theta_j} f(x|\theta) &= h(x)\frac{\partial c(\theta)}{\partial\theta_j}\exp(\sum_i \omega_i t_i) + h(x)c(\theta)\exp(\sum_i \omega_i t_i)\Big(\sum_i \frac{\partial\omega_i t_i}{\partial\theta_j}\Big) \\ &= \frac{\partial \log c(\theta)}{\partial\theta_j} f(x|\theta) + f(x|\theta)\Big(\sum_i \frac{\partial\omega_i t_i}{\partial\theta_j}\Big)\end{aligned}$$

将上式两边积分 $\Rightarrow$

$$\frac{\partial \log c(\theta)}{\partial\theta_j} + \int f(x|\theta)\Big(\sum_i \frac{\partial\omega_i t_i}{\partial\theta_j}\Big)\mathrm{d}x = 0 \Rightarrow$$

$$E\Big(\sum_i \frac{\partial\omega_i t_i}{\partial\theta_j}\Big) = -\frac{\partial}{\partial\theta_j}\log c(\theta)$$

(2)$$\frac{\partial^2}{\partial\theta_j^2}\log c(\theta) = -\int \Big[f(x|\theta)(\sum_i \frac{\partial\omega_i t_i}{\partial\theta_j})^2 + f(x|\theta)(\frac{\partial^2\omega_i t_i}{\partial\theta_j^2}) - f(x|\theta)\Big(\frac{\partial}{\partial\theta_j}\log c(\theta)\Big)^2\Big]\mathrm{d}x$$

$$= -E\Big(\sum_i \frac{\partial\omega_i t_i}{\partial\theta_j}\Big)^2 - E\Big(\frac{\partial^2\omega_i t_i}{\partial\theta_j^2}\Big) + \Big(\frac{\partial}{\partial\theta_j}\log c(\theta)\Big)^2 \Rightarrow$$

$$Var\Big(\sum_i \frac{\partial\omega_i t_i}{\partial\theta_j}\Big) = E\Big(\sum_i \frac{\partial\omega_i t_i}{\partial\theta_j}\Big)^2 - \Big(E\sum_i \frac{\partial\omega_i t_i}{\partial\theta_j}\Big)^2$$

$$= E\Big(\sum_i \frac{\partial\omega_i t_i}{\partial\theta_j}\Big)^2 - \frac{\partial^2}{\partial\theta_j^2}\log c(\theta)$$

□