1

$$
\begin{aligned}
< u_N, v > &= \sum_{k=1}^{N} < \mathcal{L}(e_k)e_k, v > \\
&= \sum_{k=1}^{N} \mathcal{L}(e_k) < e_k, v > \\
&= \mathcal{L}(\sum_{k=1}^{N} < e_k, v > e_k) \\
&= \mathcal{L}(v)
\end{aligned}
$$

2 it's obvious that $u_N \in V_N$

so $\mathcal{L}(u_N) = \|u_N\|^2$

and $\mathcal{L}(u_N) =< \mathcal{L}, u_N >\leq \|\mathcal{L}\|\|u_N\|$

$\Rightarrow \|u_N\| \leq \|\mathcal{L}\|$

3 $\because \|u_N\|^2 = \sum_{k=1}^{N} \mathcal{L}^2(e_k) \leq \|\mathcal{L}\|^2$

so we have $\sum_{k=1}^{\infty} |\mathcal{L}(e_k)|^2 \leq \|\mathcal{L}\|^2$

using the lemma in week5 PDF we can know that there exist $u = \sum_{k=1}^{\infty} \mathcal{L}(e_k)e_k$

which prove the convergence of $\sum_{k=1}^{\infty} \mathcal{L}(e_k)e_k$

$\because u = \sum_{k=1}^{\infty} < u, e_k > e_k$

$\|u - u\| = |\mathcal{L}(e_k) - < u, e_k > |^2 = 0$

$\therefore < u, e_k >= \mathcal{L}(e_k) \; \forall k = 1 \ldots$

$\therefore \forall v \in \mathcal{H} v = \sum_{k=1}^{\infty} < v, e_k > e_k$

$$\mathcal{L}(v) = \mathcal{L}(\sum_{k=1}^{\infty} < v, e_k > e_k) = \sum_{k=1}^{\infty} < v, e_k > \mathcal{L}(e_k)$$

$$=< v, \sum_{k=1}^{\infty} \mathcal{L}(e_k) e_k >=< u, v >$$

then prove the uniqueness of u if we have au^* $s.t$ $\forall v$

$$< u^*, v >= \mathcal{L}(v)$$

then $< u^*, v >= \mathcal{L}(e_k)$ $\forall k \geq 0$

$$u^* = \sum_{k=1}^{\infty} < u^*, e_k > e_k = \sum_{k=1}^{\infty} \mathcal{L}(e_k) e_k = u$$