Proof. 1. if $\|f\|_{L^p(\Omega)}$ or $\|g\|_{L^{p'}(\Omega)} = 0$

If $\|f\|_{L^p(\Omega)}$ or $\|g\|_{L^{p'}(\Omega)} = 0$. Then f or $g = 0$ a.e on Ω and $|fg| = 0$ a.e on Ω so $\|fg\|_{L^1(\Omega)} = 0$.

2.if $\|f\|_{L^p(\Omega)}$ or $\|g\|_{L^{p'}(\Omega)} = \infty$

Then it's easy to show that both sides of Hölder inequality is ∞

3.if $\|f\|_{L^p(\Omega)}$ and $\|g\|_{L^{p'}(\Omega)}\, in\, (0, \infty)$ and both p and $p' \in (1, \infty)$

Let $F(x) = \frac{f(x)}{\|f\|_{L^p(\Omega)}}$ and $G(x) = \frac{g(x)}{\|g\|_{L^{p'}(\Omega)}}$ we have

$$\|F\|_{L^p(\Omega)} = 1\ and\ \|G\|_{L^{p'}(\Omega)} = 1$$

use the inequality

$$|FG| \leq \frac{|F|^p}{p} + \frac{|G|^{p'}}{p'},$$

we got

$$\begin{aligned}
\int_\Omega |FG| \mathrm{d}\mu &\leq \int_\Omega \frac{|F|^p}{p} \mathrm{d}\mu + \int_\Omega \frac{|G|^{p'}}{p'} \mathrm{d}\mu \\
&= \frac{\|F\|_{L^p(\Omega)}}{p} + \frac{\|G\|_{L^{p'}(\Omega)}}{p'} \\
&= \frac{1}{p} + \frac{1}{p'} \\
&= 1
\end{aligned}$$

Then we obtained the Hölder inequality:

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)}$$

4. if $p = \infty, p' = 1\ or\ p = 1, p, = \infty$.

Let's suppose that $p = \infty, p' = 1$, then $|f(x)g(x)| \leq \|f\|_{L^\infty(\Omega)} |g(x)| \quad a.e\ in\ \Omega$

so we have

$$\int_\Omega |fg| \mathrm{d}\mu \leq \|f\|_{L^\infty(\Omega)} \int_\Omega |g| \mathrm{d}\mu$$

which is Hölder inequality:$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^\infty(\Omega)} \|g\|_{L^1(\Omega)}$ $\square$