

第一次检测

基科 32 曾柯又 2013012266

2014 年 11 月 16 日

我承诺我将独立完成本次开卷测验

1 可令 $r_n = n$

由中心极限定理 $\sqrt{n}(\hat{\mu}_n - \mu) \xrightarrow{d} N(0, \sigma^2)$

设 $g(x) = \cos(x)$, 有 $g'(0) = 0$ $g''(0) = -1$

由二阶 delta 方法

$$n(\cos(\hat{\mu}_n) - \cos(\mu)) \xrightarrow{d} -\frac{\sigma^2}{2}\chi_1^2$$

满足要求

2 (a) 先求出 $X_{(n+1)}, X_{(2n)}$ 的联合概率分布, 为简化记号, 记 $U = X_{(n+1)}$, $V = X_{(2n)}$, 有:

$$f_{U,V}(u, v) = \frac{(2n+1)!}{n!(2n-2)!} f_X(u) f_X(v) F_X^n(u) [F_X(v) - F_X(u)]^{n-2} [1 - F_X(v)]$$

$F_X(u) = \frac{1}{\theta}$, $F_X(u) = \frac{u}{\theta}$ $0 \leq u \leq v \leq \theta$ 可得:

$$\begin{aligned} f_{U,V}(u, v) &= \frac{(2n+1)!}{n!(2n-2)!} \left(\frac{1}{\theta}\right)^2 \left(\frac{u}{\theta}\right)^n \left(\frac{v-u}{\theta}\right)^{n-2} \left(1 - \frac{v}{\theta}\right) \\ &= \frac{(2n+1)!}{n!(2n-2)!\theta^{2n+1}} u^n (v-u)^{n-2} (\theta-v) \quad 0 \leq u \leq v \leq \theta \end{aligned}$$

设 $S = V + U$, $T = V - U$, 则:

$$\begin{aligned} f_{S,T}(s, t) &= f_{U,V}\left(\frac{s-t}{2}, \frac{s+t}{2}\right) \left| \frac{\partial(u, v)}{\partial(s, t)} \right| \\ &= \frac{(2n+1)!}{n!(n-2)!\theta^{2n+1}} \left(\frac{s-t}{2}\right)^n t^{n-2} \left(\theta - \frac{s+t}{2}\right) \times \frac{1}{2} \end{aligned}$$

s, t 的取值范围为: $s \in [0, \theta]$ 时, $t \in [0, s]$; $s \in [\theta, 2\theta]$ 时, $t \in [0, 2\theta - s]$, 因此, 当 $s \in [0, \theta]$ 时:

$$\begin{aligned} f_S(s) &= \int_0^s f_{S,T}(s, t) dt \\ &= \int_0^s \frac{(2n+1)! s^{2n-1}}{n!(n-2)! \theta^{2n+2} 2^{n+1}} \left(1 - \frac{t}{s}\right)^n \left(\frac{t}{s}\right)^{n-2} \left(\frac{2\theta}{s} - 1 - \frac{t}{s}\right) dt \\ &= \int_0^1 \frac{(2n+1)! s^{2n}}{n!(n-2)! \theta^{2n+1} 2^{n+2}} (1-t)^n t^{n-2} \left(\frac{2\theta}{s} - 1 - t\right) dt \\ &= \frac{(2n+1)}{2^{n+2}\theta} \left(4n\left(\frac{s}{\theta}\right)^{2n-1} - (3n-1)\left(\frac{s}{\theta}\right)^{2n}\right) \end{aligned}$$

当 $s \in [\theta, 2\theta]$ 时:

$$\begin{aligned} f_S(s) &= \int_0^{2\theta-s} f_{S,T}(s, t) dt \\ &= \int_0^{s'} \frac{(2n+1)!}{2^{n+2} n! (n-2)! \theta^{2n+1}} (2\theta - s' - t)^n t^{n-2} (s' - t) dt \quad (s' = 2\theta - s) \\ &= \int_0^1 \frac{(2n+1)!}{2^{n+2} n! (n-2)! \theta^{2n+1} s'^{2n}} \left(\frac{2\theta}{s'} - 1 - t\right)^n t^{n-2} (1-t) dt \\ &= \frac{(2n+1)!}{2^{n+2} n! (n-2)! \theta^{2n+1} s'^{2n}} \sum_{k=0}^n \binom{n}{k} (-1)^k \left(\frac{2\theta}{s'} - 1\right)^{n-k} \frac{1}{(n+k-1)(n+k)} \\ &= \frac{(2n+1)!}{2^{n+2} (n-2)! \theta^{2n+1}} \sum_{k=0}^n (-1)^k \frac{s^{n-k}}{(2\theta-s)^{3n-k} k! (n-k)! (n+k-1)(n+k)} \end{aligned}$$

3 可取 $r_n = \sqrt{n}$, $\phi(x) = 2 \arcsin(\sqrt{x})$

由中心极限定理:

$$\sqrt{n}(\hat{p}_n - p) \xrightarrow{d} n(0, p(1-p))$$

而 $\phi'(p) = \sqrt{\frac{1}{p(1-p)}}$, 且 $p \in (0, 1)$, 故 $\phi'(p)^2 \times p(1-p) = 1$ 由 delta 方法:

$$\sqrt{n}(\phi(\hat{p}_n) - \phi(p)) \xrightarrow{d} n(0, 1)$$

满足要求

4 $P(n(1 - X_{(n)}) \leq t) = P(X_{(n)} > 1 - \frac{t}{n}) = 1 - P(X_{(n)} < 1 - \frac{t}{n})$

而 $P(X_{(n)} < 1 - \frac{t}{n}) = \prod_{i=1}^n P(X_i < 1 - \frac{t}{n}) = \left(1 - \frac{t}{n}\right)^n$

故 $P(n(1 - X_{(n)}) \leq t) = 1 - \left(1 - \frac{t}{n}\right)^n$

$\Rightarrow \lim_{n \rightarrow \infty} P(n(1 - X_{(n)}) \leq t) = 1 - e^{-t}$

即为指数分布 $n(1 - X_{(n)}) \xrightarrow{d} X$, $f_X(x) = e^{-x}$

5 由中心极限定理

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} n(0, \sigma^2)$$

记 $g(x) = x^2$, $g'(\mu) = 2\mu$, $g''(\mu) = 2$

若 $\mu \neq 0$, 由 delta 方法:

$$\sqrt{n}((\bar{X}_n)^2 - \mu^2) \xrightarrow{d} n(0, 4\sigma^2\mu^2)$$

即可取 $c_n = \sqrt{n}$, $A = \mu^2$

若 $\mu = 0$, 由二阶 delta 方法:

$$n((\bar{X}_n)^2 - \mu^2) \xrightarrow{d} \sigma^2 \chi_1^2$$

即取 $c_n = n$, $A = \mu^2$

6 (a) 第一部分 $\sqrt{n}(\bar{X}_n - \mu) - \frac{1}{\sqrt{n}} \sum_{i=1}^n (X_i - \mu) = 0$

而

$$\begin{aligned} & \sqrt{n}(S_n^2 - \sigma^2) - \frac{1}{\sqrt{n}} \sum_{i=1}^n [(X_i - \mu)^2 - \sigma^2] \\ &= \sqrt{n}(S_n^2 - \sigma^2) - \frac{1}{\sqrt{n}} \sum_{i=1}^n [(X_i - \bar{X}_n + \bar{X}_n - \mu)^2 - \sigma^2] \\ &= -\sqrt{n}(\bar{X}_n - \mu)^2 \end{aligned}$$

而 $\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} n(0, 1)$, 由连续映射定理:

$$\frac{n(\bar{X}_n - \mu)^2}{\sigma^2} \xrightarrow{d} \chi_1^2$$

故:

$$\sqrt{n}(\bar{X}_n - \mu)^2 \xrightarrow{d} 0$$

即

$$\sqrt{n}(\bar{X}_n - \mu)^2 \xrightarrow{p} 0$$

因此

$$\sqrt{n} \begin{pmatrix} \bar{X}_n - \mu \\ S_n^2 - \sigma^2 \end{pmatrix} - \frac{1}{\sqrt{n}} \sum_{i=1}^n \begin{pmatrix} X_i - \mu \\ (X_i - \mu)^2 - \sigma^2 \end{pmatrix} \xrightarrow{p} 0$$

即

$$\sqrt{n} \begin{pmatrix} \bar{X}_n - \mu \\ S_n^2 - \sigma^2 \end{pmatrix} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \begin{pmatrix} X_i - \mu \\ (X_i - \mu)^2 - \sigma^2 \end{pmatrix} + o_p(1)$$

(b) 记 $\mathcal{X}_i = \begin{pmatrix} X_i - \mu \\ (X_i - \mu)^2 - \sigma^2 \end{pmatrix}$, 有 $E\mathcal{X}_i = 0$, $\text{cov}(\mathcal{X}_i) = \mathbf{\Sigma}$, 经过计算可得

协方差矩阵 $\mathbf{\Sigma}$

$$\begin{pmatrix} E(X_i - \mu)^2 & E(X_i - \mu)((X_i - \mu)^2 - \sigma^2) \\ E(X_i - \mu)((X_i - \mu)^2 - \sigma^2) & E((X_i - \mu)^2 - \sigma^2)^2 \end{pmatrix} = \begin{pmatrix} \sigma^2 & \sigma^3\gamma_1 \\ \sigma^3\gamma_1 & \sigma^4(\gamma_2 + 2) \end{pmatrix}$$

因此

$$\sqrt{n}(\bar{\mathcal{X}}_n) \xrightarrow{d} \mathcal{N}_2(0, \mathbf{\Sigma})$$

而

$$\sqrt{n} \begin{pmatrix} \bar{X}_n - \mu \\ S_n^2 - \sigma^2 \end{pmatrix} = \sqrt{n}(\bar{\mathcal{X}}_n) + o_p(1)$$

由 slusky 定理

$$\sqrt{n} \begin{pmatrix} \bar{X}_n - \mu \\ S_n^2 - \sigma^2 \end{pmatrix} \xrightarrow{d} \mathcal{N}_2(0, \mathbf{\Sigma})$$

(c) 记 $\mathbf{Y}_n = \begin{pmatrix} \bar{X}_n \\ S_n^2 \end{pmatrix}$, $\boldsymbol{\theta} = \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix}$, 则

$$\sqrt{n}(\mathbf{Y}_n - \boldsymbol{\theta}) \xrightarrow{d} \mathcal{N}_2(0, \mathbf{\Sigma})$$

设函数 $g(x_1, x_2) = \frac{x_1}{\sqrt{x_2}}$, $g'(\mu, \sigma^2) = (\frac{1}{\sigma}, -\frac{\mu}{2\sigma^3})$, $g'^T \mathbf{\Sigma} g' = 1 - \frac{\mu\gamma_1}{\sigma} + (\frac{\mu}{2\sigma})^2(\gamma_2 + 2)$, 则由多元函数的 delta 方法可得

$$\sqrt{n}(g(\mathbf{Y}_n) - g(\boldsymbol{\theta})) \xrightarrow{d} n(0, \tau^2)$$

即

$$\sqrt{n}(\frac{\bar{X}_n}{S_n} - \frac{\mu}{\sigma}) \xrightarrow{d} n(0, \tau^2)$$

其中 $\tau^2 = 1 - \frac{\mu\gamma_1}{\sigma} + (\frac{\mu}{2\sigma})^2(\gamma_2 + 2)$