基科 32 曾柯又 2013012266

**5.23** $f_U(u) = I(u \in [0,1]) \quad F_U(u) = u\ (0 \le u \le 1)$

$$f(z|X=x) = \frac{x!}{(x-1)!} f_U(z)(1-F_U(z))^{x-1} = x(1-z)^{x-1}\ (0 \le z \le 1)$$

$$\begin{aligned} f_Z(z) &= \sum_{x=1}^{\infty} f(z|X=x)P(X=x) \\ &= \sum_{x=1}^{\infty} x(1-z)^{x-1}\frac{c}{x!} \\ &= \sum_{x=1}^{\infty} \frac{c(1-z)^{x-1}}{(x-1)!} \\ &= ce^{1-z} = \frac{e^{1-z}}{e-1}\ (0 \le z \le 1) \end{aligned}$$

**5.24**

$$\begin{aligned} f_{(\frac{X_{(1)}}{X_{(n)}}, X_{(n)})}(t_1,t_2) &= f_{(X_{(1)},X_{(n)})}(t_1t_2,t_2)\left|\frac{\partial(X_{(1)}/X_{(n)}, X_{(n)})}{\partial(X_{(1)},X_{(n)})}\right|^{-1} \\ &= \frac{n!}{1!1!(n-2)!} f_X(t_1t_2)f_X(t_2)(F_X(t_2)-F_X(t_1t_2))^{n-2}t_2 \\ &= \frac{n(n-1)}{\theta^2}(\frac{t_2}{\theta}-\frac{t_1t_2}{\theta})^{n-2}t^2 \\ &= \frac{n(n-1)}{\theta^n}t_2^{n-1}(1-t_1)^{n-2} \quad t_1 \in (0,1)\ t_2 \in (0,\theta) \end{aligned}$$

由此易判断 $\frac{X_{(1)}}{X_{(n)}}$ 与$X_{(n)}$ 独立。

**5.25**

$$\begin{aligned} &f_{(\frac{X_{(1)}}{X_{(2)}},\ldots,\frac{X_{(n-1)}}{X_{(n)}},X_{(n)})}(t_1,\ldots,t_n) \\ &= f_{(X_{(1)},\ldots,X_{(n)})}(t_1\cdots t_n, t_2\cdots t_n,\ldots,t_n)\left|\frac{\partial(\frac{X_{(1)}}{X_{(2)}},\ldots,\frac{X_{(n-1)}}{X_{(n)}},X_{(n)})}{\partial(X_{(1)},\ldots,X_{(n)})}\right|^{-1} \end{aligned}$$

$$= n!(\frac{a}{\theta^a})^n t_1^{a-1} t_2^{2(a-1)} t_n^{n(a-1)} \begin{vmatrix} \frac{1}{X_{(2)}} & -\frac{X_{(1)}}{X_{(2)}^2} & 0 & \cdots \\ 0 & \frac{1}{X_{(3)}} & -\frac{X_{(2)}}{X_{(3)}^2} & \cdots \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix}^{-1}$$

$$= n!(\frac{a}{\theta^a})^n t_1^{a-1} t_2^{2a-1} \cdots t_n^{na-1}$$

$t_i \in (0,1)$ $i = 1, \ldots, n-1$ $t_n \in (0, \theta)$

该式表明 $\frac{X_{(1)}}{X_{(2)}}, \ldots, \frac{X_{(n-1)}}{X_{(n)}}, X_{(n)}$ 互为独立变量

$$\begin{aligned} f_{X_{(n)}} &= \frac{n!}{(n-1)!} f_X(t_n) F_X(t_n)^{n-1} \\ &= n \frac{a}{\theta^a} t_n^{a-1} (\frac{t_n^a}{\theta^a})^{n-1} \\ &= \frac{na}{\theta^{na}} t_n^{na-1} \quad t_n \in (0, \theta) \end{aligned}$$

对任意 $k$

$$\begin{aligned} f_{\frac{X_{(k)}}{X_{(k+1)}}}(t_k) &= \int \cdots \int f(t_1, \ldots, t_n) \mathrm{d}t_1 \cdots \mathrm{d}t_{k-1} \mathrm{d}t_{k+1} \ldots \mathrm{d}t_n \\ &= \frac{t_k^{ka-1}}{\int_0^1 t_k^{ka-1} \mathrm{d}t_k} \\ &= ka t_k^{ka-1} \end{aligned}$$

**5.26** (a) 每个样本在 $u$ 之前的概率 $F_X(u)$, 在 $u, v$ 之间的概率为 $F_X(v) - F_X(u)$, 在 $v$ 之后的概率为 $1 - F_X(v)$, 因此 $(U, V, n-U-V)$ 服从尝试次数为 $n$, 元概率为 $(F_X(u), F_X(v) - F_X(u), 1 - F_X(v))$ 的多项分布, 即

$$f_{(U,V,n-U-V)}(n_1, n_2, n_3) = \frac{n!}{n_1! n_2! n_3!} F_X(u)^{n_1} [F_X(v) - F_X(u)]^{n_2} [1 - F_X(v)]^{n_3}$$

且 $n_1 + n_2 + n_3 = n$

(b)

$$
\begin{aligned}
F_{(X_{(i)},X_{(j)})}(u,v) &= P(U \geq i, U+V \geq j) \\
&= P(i \leq U \leq j, U+V \geq j) + P(U \geq j) \\
&= \sum_{k=i}^{j-1} \sum_{m=j-k}^{n-k} P(U=k, V=m) + P(U \geq j) \\
&= \sum_{k=i}^{j-1} \sum_{m=j-k}^{n-k} \frac{n!}{k!m!(n-k-m)!} F_X(u)^k [F_X(v) - F_X(u)]^m \\
&\times [1 - F_X(v)]^{n-k-m} + P(U \geq j)
\end{aligned}
$$

(c) 为方便起见，记 $p_1 = F_X(u), p_2 = F_X(v) - F_X(u), p_3 = 1 - F_X(v)$

$$
\begin{aligned}
f_{(X_{(i)},X_{(j)})}(u,v) &= \frac{\partial^2 F(u,v)}{\partial u \partial v} \\
&= \frac{\partial^2}{\partial u \partial v} \sum_{k=i}^{j-1} \sum_{m=j-k}^{n-k} \frac{n!}{k!m!(n-k-m)!} p_1^k p_2^m p_3^{n-k-m} \\
&= \frac{\partial}{\partial u} \sum_{k=i}^{j-1} \frac{n!}{k!} f_X(v) p_1^k \Big( \sum_{m=j-k}^{n-k} \frac{1}{(m-1)!(n-k-m)!} p_2^{m-1} p_3^{n-k-m} \\
&- \sum_{m=j-k}^{n-k-1} \frac{1}{m!(n-k-m-1)} p_2^m p_3^{n-k-m-1} \Big)
\end{aligned}
$$

括号里的部分可化为

$$
\begin{aligned}
&\frac{1}{(j-k-1)!(n-j)!} p_2^{j-k-1} p_3^{n-j} + \sum_{m=j-k+1}^{n-k} \frac{1}{(m-1)!(n-k-m)!} p_2^{m-1} p_3^{n-k-m} \\
&- \sum_{m=j-k}^{n-k-1} \frac{1}{m!(n-k-m-1)} p_2^m p_3^{n-k-m-1} \\
&= \frac{1}{(j-k-1)!(n-j)!} p_2^{j-k-1} p_3^{n-j} \text{ 于是}
\end{aligned}
$$

$$
\begin{aligned}
f_{(X_{(i)},X_{(j)})}(u,v) &= \sum_{k=i}^{j-1} \frac{\partial}{\partial u} p_1^k f_X(v) \frac{n! p_2^{j-k-1} p_3^{n-j}}{(j-k-1)!(n-j)!} \\
&= \frac{n! f_X(v) f_X(u)}{(n-j)!} p_3^{n-j} \Big( \sum_{k=i}^{j-1} \frac{p_1^{k-1} p_2^{j-k-1}}{(k-1)!(j-k-1)!} \\
&- \sum_{k=i}^{j-2} \frac{p_1^k p_2^{j-k-2}}{k!(j-k-2)!} \Big)
\end{aligned}
$$

括号里的部分为 $\dfrac{p_1^{i-1}p_2^{j-i-1}}{(i-1)!(j-i-1)!}+\sum_{k=i}^{j-2}\dfrac{p_1^k p_2^{j-k-2}}{k!(j-k-2)!}-\sum_{k=i}^{j-2}\dfrac{p_1^k p_2^{j-k-2}}{k!(j-k-2)!}$

$=\dfrac{p_1^{i-1}p_2^{j-i-1}}{(i-1)!(j-i-1)!}$

最后得到

$$f_{(X_{(i)},X_{(j)})}(u,v)=\frac{n!}{(n-j)!(i-1)!(j-i-1)!}f_X(v)f_X(u)p_1^{i-1}p_2^{j-i-1}p_3^{n-j}$$

**5.27** (a)$f_{X_{(i)}|X_{(j)}}(u,v)=\dfrac{f_{(X_{(i)},X_{(j)})}(u,v)}{f_{X_{(j)}}(v)}$

对于$i<j$

$$f_{X_{(i)}|X_{(j)}}(u,v)=\frac{(j-1)!}{(i-1)!(j-i-1)!}f_X(u)\frac{F_X(u)^{i-1}[F_X(v)-F_X(u)]^{j-i-1}}{F_X(v)^{j-1}}$$

$(u<v)$

对于$i>j$

$$f_{(X_{(i)},X_{(j)})}(u,v)=\frac{n!}{(j-1)!(n-i)!(i-j-1)!}f_X(u)f_X(v)F_X(v)^{j-1}$$
$$\times[F_X(u)-F_X(v)]^{i-j-1}[1-F_X(u)]^{n-i}$$

可得

$$f_{X_{(i)}|X_{(j)}}(u,v)=\frac{(n-j)!}{(i-j-1)!(n-i)!}\frac{[F_X(u)-F_X(v)]^{i-j-1}[1-F_X(u)]^{n-i}}{[1-F_X(v)]^{n-j}}$$

$(v<u)$

(b) 由 Example 5.4.7

$$f_{R,V}(r,v)=\frac{n(n-1)r^{n-2}}{a^n},\quad 0<r<a,\quad r/2<v<a-r/2$$
$$f_R(r)=\frac{n(n-1)r^{n-2}(a-r)}{a^n},\quad 0<r<a$$
$$f_{V|R}(v,r)=\frac{f_{R,V}(r,v)}{f_R(r)}=\frac{1}{a-r},\quad r/2<v<a-r/2$$

**5.28** (a) 通过直观的办法，容易看出

$$f_{(X_{(i_1)},\ldots,X_{(i_l)})}(x_1,\cdots,x_l)$$
$$=\frac{n!}{(i_1-1)!(i_2-i_1-1)!\ldots(n-i_l)!}f_X(x_1)\cdots f_X(x_l)$$
$$\times F_X(x_1)^{i_1-1}[F_X(x_2)-F_X(x_1)]^{i_2-i_1-1}\cdots[1-F_X(x_l)]^{n-i_l}$$

且要求$x_1<x_2<\cdots<x_l$

cdf 为上式的积分, 为:

$$F_{(X_{(i_1)},\dots,X_{(i_l)})}(x_1,\cdots,x_l)=\int_{-\infty}^{x_1}\int_{x_1}^{x_2}\cdots\int_{x_n}^{\infty}f_{(X_{(i_1)},\dots,X_{(i_l)})}(u_1,\cdots,u_l)\mathrm{d}u_1\cdots\mathrm{d}u_l$$

(b) 由于无法讨论大小关系，先将 $(i_1,i_2,\dots,i_l,j_1,j_2,\dots,j_m)$ 排序，得到 $(k_1,k_2,\dots,k_{l+m})$, 并可由 (a) 得到分别关于角标 $k,j$ 的联合分布,

$$\begin{aligned}&f_{X_{(i_1)},\dots,X_{(i_l)}|X_{(j_1)},\dots,X_{(j_m)}}(u_{i_1},\cdots,u_{i_l},u_{j_1},\dots,u_{j_m})\\&\qquad=\frac{f_{(X_{(k_1)},\dots,X_{(k_{l+m})})}(u_{k_1},\cdots,u_{k_{m+l}})}{f_{(X_{(j_1)},\dots,X_{(j_m)})}(u_{j_1},\cdots,u_{j_m})}\end{aligned}$$

**课上作业** 设 $X\sim\chi_n^2$, $Y\sim\chi_m^2$, $Z=X+cY$, 则:

$f_Z(z)=\int_0^{z/c}f_X(z-cy)f_Y(y)\mathrm{d}y$

设$t=\frac{yc}{z}$则$y=\frac{tz}{c}$

$f_Z(z)=\int_0^1\frac{(z-tz)^{\frac{n}{2}-1}(\frac{tz}{c})^{\frac{m}{2}-1}e^{\frac{tz-z-tz/c}{2}}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})2^{\frac{m+n}{2}}}\mathrm{d}t$

记$p=\frac{m+n}{2}$

$f_Z(z)=\frac{z^{p-1}e^{-\frac{z}{2}}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})2^pc^{\frac{m}{2}}}\int_0^1(1-t)^{\frac{n}{2}-1}t^{\frac{m}{2}-1}e^{\frac{tz}{2}(1-\frac{1}{c})}\mathrm{d}t$

将$e^{\frac{tz}{2}(1-\frac{1}{c})}$展开，得:

$e^{\frac{tz}{2}(1-\frac{1}{c})}=\sum_{k=0}^{\infty}\frac{t^k(\frac{z}{2})^k}{k!}(1-\frac{1}{c})^k$

$$\begin{aligned}\therefore&\int_0^1(1-t)^{\frac{n}{2}-1}t^{\frac{m}{2}-1}e^{\frac{tz}{2}(1-\frac{1}{c})}\mathrm{d}t\\&=\sum_{k=0}^{\infty}\frac{t^k(\frac{z}{2})^k}{k!}(1-\frac{1}{c})^k\int_0^1(1-t)^{\frac{n}{2}-1}t^{\frac{m}{2}+k-1}\mathrm{d}t\\&=\sum_{k=0}^{\infty}\frac{t^k(\frac{z}{2})^k}{k!}(1-\frac{1}{c})^kB(\frac{m}{2}+k,\frac{n}{2})\end{aligned}$$

$$f_Z(z)=\sum_{k=0}^{\infty}\frac{\Gamma(\frac{m}{2}+k)e^{-\frac{z}{2}}z^{p+k-1}(1-\frac{1}{c})^k}{\Gamma(\frac{m}{2})\Gamma(p+k-1)2^{p+k-1}k!c^{\frac{m}{2}}}$$