第一次检测

基科 32 曾柯又 2013012266

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我承诺我将独立完成本次开卷测验

1 可令 $r_n = n$

由中心极限定理 $\sqrt{n}(\hat{\mu_n} - \mu) \xrightarrow{d} n(0, \sigma^2)$

设 $g(x) = \cos(x)$, 有 $g'(0) = 0$ $g''(0) = -1$

由二阶 delta 方法

$$n(\cos(\hat{\mu_n}) - \cos(\mu)) \xrightarrow{d} -\frac{\sigma^2}{2}\chi_1^2$$

满足要求

2 (a) 先求出 $X_{(n+1)}, X_{(2n)}$ 的联合概率分布，为简化记号，记 $U = X_{(n+1)}$, $V = X_{(2n)}$, 有:

$$f_{U,V}(u,v) = \frac{(2n+1)!}{n!(2n-2)!} f_X(u) f_X(v) F_X^n(u)[F_X(v) - F_X(u)]^{n-2}[1 - F_X(v)]$$

$F_X(u) = \frac{1}{\theta}$, $F_X(u) = \frac{u}{\theta}$ $\quad 0 \le u \le v \le \theta$ 可得：

$$\begin{aligned} f_{U,V}(u,v) &= \frac{(2n+1)!}{n!(2n-2)!}(\frac{1}{\theta})^2(\frac{u}{\theta})^n(\frac{v-u}{\theta})^{n-2}(1 - \frac{v}{\theta}) \\ &= \frac{(2n+1)!}{n!(2n-2)!\theta^{2n+1}} u^n (v-u)^{n-2}(\theta - v) \qquad 0 \le u \le v \le \theta \end{aligned}$$

设 $S = V + U$, $T = V - U$, 则：

$$\begin{aligned} f_{S,T}(s,t) &= f_{U,V}(\frac{s-t}{2}, \frac{s+t}{2})\Big|\frac{\partial(u,v)}{\partial(s,t)}\Big| \\ &= \frac{(2n+1)!}{n!(n-2)!\theta^{2n+1}}(\frac{s-t}{2})^n t^{n-2}(\theta - \frac{s+t}{2}) \times \frac{1}{2} \end{aligned}$$

s, t 的取值范围为：$s \in [0, \theta]$ 时，$t \in [0, s]$; $s \in [\theta, 2\theta]$ 时，$t \in [0, 2\theta - s]$, 因此，当 $s \in [0, \theta]$ 时：

$$
\begin{aligned}
f_S(s) &= \int_0^s f_{S,T}(s,t)\mathrm{d}t \\
&= \int_0^s \frac{(2n+1)!s^{2n-1}}{n!(n-2)!\theta^{2n+2}2^{n+1}}(1-\frac{t}{s})^n(\frac{t}{s})^{n-2}(\frac{2\theta}{s}-1-\frac{t}{s})\mathrm{d}t \\
&= \int_0^1 \frac{(2n+1)!s^{2n}}{n!(n-2)!\theta^{2n+1}2^{n+2}}(1-t)^n t^{n-2}(\frac{2\theta}{s}-1-t)\mathrm{d}t \\
&= \frac{(2n+1)}{2^{n+2}\theta}(4n(\frac{s}{\theta})^{2n-1}-(3n-1)(\frac{s}{\theta})^{2n})
\end{aligned}
$$

当 $s \in [\theta, 2\theta]$ 时：

$$
\begin{aligned}
f_S(s) &= \int_0^{2\theta-s} f_{S,T}(s,t)\mathrm{d}t \\
&= \int_0^{s'} \frac{(2n+1)!}{2^{n+2}n!(n-2)!\theta^{2n+1}}(2\theta-s'-t)^n t^{n-2}(s'-t)\mathrm{d}t \quad (s'=2\theta-s) \\
&= \int_0^1 \frac{(2n+1)!}{2^{n+2}n!(n-2)!\theta^{2n+1}s'^{2n}}(\frac{2\theta}{s'}-1-t)^n t^{n-2}(1-t)\mathrm{d}t \\
&= \frac{(2n+1)!}{2^{n+2}n!(n-2)!\theta^{2n+1}s'^{2n}}\sum_{k=0}^{n}\binom{n}{k}(-1)^k(\frac{2\theta}{s'}-1)^{n-k}\frac{1}{(n+k-1)(n+k)} \\
&= \frac{(2n+1)!}{2^{n+2}(n-2)!\theta^{2n+1}}\sum_{k=0}^{n}(-1)^k\frac{s^{n-k}}{(2\theta-s)^{3n-k}k!(n-k)!(n+k-1)(n+k)}
\end{aligned}
$$

3 可取 $r_n = \sqrt{n},\ \phi(x) = 2\arcsin(\sqrt{x})$

由中心极限定理:

$$\sqrt{n}(\hat{p_n}-p) \xrightarrow{d} n(0, p(1-p))$$

而 $\phi'(p) = \sqrt{\frac{1}{p(1-p)}}$, 且 $p \in (0,1)$, 故 $\phi'(p)^2 \times p(1-p) = 1$ 由 delta 方法：

$$\sqrt{n}(\phi(\hat{p_n})-\phi(p)) \xrightarrow{d} n(0,1)$$

满足要求

4 $P(n(1-X_{(n)}) \le t) = P(X_{(n)} > 1-\frac{t}{n}) = 1 - P(X_{(n)} < 1-\frac{t}{n})$

而 $P(X_{(n)} < 1-\frac{t}{n}) = \prod_{i=1}^{n} P(X_i < 1-\frac{t}{n}) = (1-\frac{t}{n})^n$

故 $P(n(1-X_{(n)}) \le t) = 1-(1-\frac{t}{n})^n$

$\Rightarrow \lim_{n\to\infty} P(n(1-X_{(n)}) \le t) = 1-e^{-t}$

即为指数分布 $n(1-X_{(n)}) \xrightarrow{d} X, \quad f_X(x)=e^{-x}$

5 由中心极限定理

$$\sqrt{n}(\bar{X}_n-\mu) \xrightarrow{d} n(0,\sigma^2)$$

记 $g(x)=x^2,\ g'(\mu)=2\mu,\ g''(\mu)=2$

若 $\mu\neq 0$, 由 delta 方法:

$$\sqrt{n}((\bar{X}_n)^2-\mu^2) \xrightarrow{d} n(0,4\sigma^2\mu^2)$$

即可取 $c_n=\sqrt{n},\ A=\mu^2$

若 $\mu=0$, 由二阶 delta 方法:

$$n((\bar{X}_n)^2-\mu^2) \xrightarrow{d} \sigma^2\chi_1^2$$

即取 $c_n=n,\ A=\mu^2$

6 (a) 第一部分 $\sqrt{n}(\bar{X}_n-\mu)-\frac{1}{\sqrt{n}}\sum_{i=1}^n(X_i-\mu)=0$

而

$$\begin{aligned}&\sqrt{n}(S_n^2-\sigma^2)-\frac{1}{\sqrt{n}}\sum_{i=1}^n[(X_i-\mu)^2-\sigma^2]\\&=\sqrt{n}(S_n^2-\sigma^2)-\frac{1}{\sqrt{n}}\sum_{i=1}^n[(X_i-\bar{X}_n+\bar{X}_n-\mu)^2-\sigma^2]\\&=-\sqrt{n}(\bar{X}_n-\mu)^2\end{aligned}$$

而 $\dfrac{\sqrt{n}(\bar{X}_n-\mu)}{\sigma} \xrightarrow{d} n(0,1)$，由连续映射定理:

$$\frac{n(\bar{X}_n-\mu)^2}{\sigma^2} \xrightarrow{d} \chi_1^2$$

故：

$$\sqrt{n}(\bar{X}_n-\mu)^2 \xrightarrow{d} 0$$

即

$$\sqrt{n}(\bar{X}_n-\mu)^2 \xrightarrow{p} 0$$

因此

$$\sqrt{n}\begin{pmatrix}\bar{X}_n-\mu\\ S_n^2-\sigma^2\end{pmatrix}-\frac{1}{\sqrt{n}}\sum_{i=1}^n\begin{pmatrix}X_i-\mu\\ (X_i-\mu)^2-\sigma^2\end{pmatrix} \xrightarrow{p} 0$$

即

$$\sqrt{n}\begin{pmatrix}\bar{X}_n-\mu\\ S_n^2-\sigma^2\end{pmatrix}=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\begin{pmatrix}X_i-\mu\\ (X_i-\mu)^2-\sigma^2\end{pmatrix}+o_p(1)$$

(b) 记 $\mathcal{X}_i=\begin{pmatrix}X_i-\mu\\ (X_i-\mu)^2-\sigma^2\end{pmatrix}$，有 $E\mathcal{X}_i=0,\ \mathrm{cov}(\mathcal{X}_i)=\mathbf{\Sigma}$, 经过计算可得协方差矩阵 $\mathbf{\Sigma}$

$$\begin{pmatrix}E(X_i-\mu)^2 & E(X_i-\mu)((X_i-\mu)^2-\sigma^2)\\ E(X_i-\mu)((X_i-\mu)^2-\sigma^2) & E((X_i-\mu)^2-\sigma^2)^2\end{pmatrix}=\begin{pmatrix}\sigma^2 & \sigma^3\gamma_1\\ \sigma^3\gamma_1 & \sigma^4(\gamma_2+2)\end{pmatrix}$$

因此

$$\sqrt{n}(\bar{\mathcal{X}_n})\xrightarrow{d}\mathcal{N}_2(0,\mathbf{\Sigma})$$

而

$$\sqrt{n}\begin{pmatrix}\bar{X}_n-\mu\\ S_n^2-\sigma^2\end{pmatrix}=\sqrt{n}(\bar{\mathcal{X}_n})+o_p(1)$$

由 slusky 定理

$$\sqrt{n}\begin{pmatrix}\bar{X}_n-\mu\\ S_n^2-\sigma^2\end{pmatrix}\xrightarrow{d}\mathcal{N}_2(0,\mathbf{\Sigma})$$

(c) 记 $\boldsymbol{Y_n}=\begin{pmatrix}\bar{X}_n\\ S_n^2\end{pmatrix}$, $\boldsymbol{\theta}=\begin{pmatrix}\mu\\ \sigma^2\end{pmatrix}$, 则

$$\sqrt{n}(\boldsymbol{Y_n}-\boldsymbol{\theta})\xrightarrow{d}\mathcal{N}_2(0,\mathbf{\Sigma})$$

设函数 $g(x_1,x_2)=\dfrac{x_1}{\sqrt{x_2}}$,　$g'(\mu,\sigma^2)=(\dfrac{1}{\sigma},-\dfrac{\mu}{2\sigma^3})$,　$g'^T\mathbf{\Sigma}g'=1-\dfrac{\mu\gamma_1}{\sigma}+(\dfrac{\mu}{2\sigma})^2(\gamma_2+2)$, 则由多元函数的 delta 方法可得

$$\sqrt{n}(g(\boldsymbol{Y_n})-g(\boldsymbol{\theta}))\xrightarrow{d}n(0,\tau^2)$$

即

$$\sqrt{n}(\frac{\bar{X}_n}{S_n}-\frac{\mu}{\sigma})\xrightarrow{d}n(0,\tau^2)$$

其中 $\tau^2=1-\frac{\mu\gamma_1}{\sigma}+(\frac{\mu}{2\sigma})^2(\gamma_2+2)$