基科 32 曾柯又 2013012266

5.11 $VarS = E(S-\bar{S})^2 \geq 0$

而 $VarS = ES^2 - (ES)^2 \Rightarrow ES \leq \sqrt{ES^2} = \sigma$

若 $ES^2 > 0$ 此时假设 $ES = \sigma$

记 $\mathcal{A} = \{X | f(X) > 0\}$

则 $VarS = 0$ 即$E(S-ES)^2 = 0$

$S - ES = 0$ a.e in $\mathcal{A} \Rightarrow S^2 = (ES)^2$ a.e in $\mathcal{A}$

由于每个随机变量是独立的，因此$VarX_n = 0 = \sigma^2$ 与条件矛盾

因此 $ES > \sigma$

5.14 (a) 记 $\widetilde{U}_i = \sum_{j=1}^{n} a_{ij}X_j = \sum_{j=1}^{n} a_{ij}\sigma_j Z_j + \sum_{j=1}^{n} a_{ij}\mu_j$

$\overset{记作}{=} \sum_{j=1}^{n} \widetilde{a}_{ij}Z_j + \widetilde{\mu}_j \overset{记作}{=} U_i + \widetilde{\mu}_i$

采用同样的记号$\widetilde{V}_r = \sum_{j=1}^{n} \widetilde{b}_{rj}Z_j + \widetilde{\mu}_r = V_r + \widetilde{\mu}_r$

$Cov(\widetilde{U}_i, \widetilde{V}_r) = 0$

$\Rightarrow Cov(U_i, V_r) = 0$

$\Rightarrow U_i$, V_r 独立

$\Rightarrow U_i + \widetilde{\mu}_i$, $V_r + \widetilde{\mu}_r$ 独立

$\Rightarrow \widetilde{U}_i, \widetilde{V}_r$独立

(b)$Cov(\widetilde{U}_i, \widetilde{V}_r) = Cov(U_i, V_r)$

$\because EU_i = 0$, $EV_r = 0$, $EZ_j^2 = 1$

$$\therefore Cov(\widetilde{U}_i, \widetilde{V}_r) = EU_iV_r = E\sum_{j,k}\widetilde{a}_{ij}\widetilde{b}_{rk}Z_jZ_k = \sum_{j=1}^{n}\widetilde{a}_{ij}\widetilde{b}_{rj}Z_j^2 = \sum_{j=1}^{n}\widetilde{a}_{ij}\widetilde{b}_{rj}$$

$$\therefore Cov(\widetilde{U}_i, \widetilde{V}_r) = \sum_{j=1}^{n}a_{ij}b_{rj}\sigma_j^2$$

5.15 (a)$\bar{X}_{n+1} = \sum_{i=1}^{n+1} = \frac{X_i}{n+1} = \frac{X_{n+1}+\sum_{i=1}^{n}X_i}{n+1} = \frac{X_{n+1}+n\bar{X}_n}{n+1}$

(b)

$$\begin{aligned} nS_{n+1}^2 &= \sum_{i=1}^{n+1}(X_i-\bar{X}_{n+1})^2 \\ &= \sum_{i=1}^{n+1}(X_i-\bar{X}_n+\bar{X}_n-\bar{X}_{n+1})^2 \\ &= \sum_{i=1}^{n+1}(X_i-\bar{X}_n)^2+\sum_{i=1}^{n+1}(\bar{X}_n-\bar{X}_{n+1})+\sum_{i=1}^{n+1}2(X_i-\bar{X}_n)(\bar{X}_n-\bar{X}_{n+1}) \\ &= nS_n^2+(X_{n+1}-\bar{X}_n)^2-(n+1)(\bar{X}_{n+1}-\bar{X}_n)^2 \end{aligned}$$

而 $\bar{X}_{n+1}-\bar{X}_n = \frac{X_{n+1}+n\bar{X}_n}{n+1}-\bar{X}_n = \frac{X_{n+1}-\bar{X}_n}{n+1}$

因此 $nS_{n+1}^2 = nS_n^2+(1-\frac{1}{n+1})(\bar{X}_n+1-\bar{X}_n)^2 = nS_n^2+\frac{n}{n+1}(\bar{X}_{n+1}-\bar{X}_n)^2$

这两个式子可以用于求与均值或方差相关的一些量关于 n 的递推关系。例如书中的关于 Theorem 5.3.1(c) $\frac{(n-1)S^2}{\sigma^2}\sim\chi_{n-1}^2$ 的证明，便是利用上式并使用数学归纳法证明的。其他还可以利用上面两式来得到关于 $\bar{X}_n, S_n^2$ 均值与方差的递推关系式，从而给出均值方差通项的另一种算法。