1 For the first question, it's sufficient to prove that the Lebesgue integral is liner.

For two functionf_1, f_2which is Lebesgue integrable,there exist two sequence of simple function f_{1n}, f_{2n} , such that

$$\lim_{n\to\infty}\int_\Omega (f_{1n}+f_{2n})\mathrm{d}\mu = \int_\Omega (f_1+f_2)\mathrm{d}\mu$$

It's easy to show that the integrate is liner for simple function,so the integrate is liner for any Lebesgue integral function.

2 first prove that $\|A_g\| \le \|g\|_{L^{p'}}$

$\forall f \in L^p(\Omega)$

$|A_g f| = |\int_\Omega fg\mathrm{d}\mu| \le \int_\Omega |fg|\mathrm{d}\mu = \|fg\|_{L^1}$ using the Holder inequality , we can get

$|A_g f| \le \|f\|_{L^p}\|g\|_{L^{p'}} \Rightarrow$

$\|A_g\| = sup\dfrac{|A_g f|}{\|f\|_{L^p}} \le \|g\|_{L^{p'}}$

then prove that $\|A_g\| \ge \|g\|_{L^{p'}}$

using the lemma given in the question,there exist a function f_g , such that

$A_g(f_g) = \|f_g\|_{L^p}\|g\|_{L^{p'}} \Rightarrow$

$\|A_g\| = sup\dfrac{A_g f}{\|f\|_{L^p}} \ge \dfrac{|A_g f_g|}{\|f_g\|_{L^p}} = \|g\|_{L^{p'}}$ so we have:

$$\|A_g\| = \|g\|_{L^{p'}}$$