基科 32 曾柯又 2013012266

7.53 考虑 $\phi_a = W + aU$

则$E_\theta \phi_a = E_\theta W + aE_\theta U$

$= \tau(\theta)$

故 $\forall\, a$, ϕ_a 为 $\tau(\theta)$ 的无偏估计量。

而$Var_\theta \phi_a = Var(W + aU)$

$= Var_\theta W + aCov_\theta(W,U) + a^2 Var_\theta U$

如果存在 θ_0 使得 $Cov_{\theta_0}(W,U) \neq 0$ 则可以取 a 使得 $aCov_{\theta_0}(W,U) + a^2 Var_{\theta_0}(U) < 0$

即若 $Cov_{\theta_0}(W,U) > 0$

取 $a \in (-\dfrac{Cov_{\theta_0}(W,U)}{Var_{\theta_0} W}, 0)$

若 $Cov_{\theta_0}(W,U) < 0$

取 $a \in (0, -\dfrac{Cov_{\theta_0}(W,U)}{Var_{\theta_0} W})$

此时 $Var_{\theta_0}\phi_a < Var_{\theta_0} W$, 即 W 不为最佳无偏估计

7.57

(a) $ET = \times P(\sum_{i=1}^{n} X_i > X_{n+1}|p) = h(p)$

(b) $X_1, \ldots X_{n+1}$ 的联合分布为:

$f(x_1, \cdots, x_{n+1}) = p^{\sum_{i=1}^{n+1} x_i}(1-p)^{n+1-\sum_{i=1}^{n+1} x_i}$

由因子分解定理 $\sum_{i=1}^{n+1} X_i$ 为一个充分统计量, 且由于伯努力分布为指数分布族且参数空间包含了 $\mathbb{R}$ 的开集，故 $T = \sum_{i=1}^{n+1} X_i$ 为完全充分统计量, 因此 $\phi(\sum_{i=1}^{n+1} X_i) = E(T|\sum_{i=1}^{n+1} X_i)$ 为最佳无偏估计。

记 $T_2 = \displaystyle\sum_{i=1}^{n+1} X_i$

$$P(\sum_{i=1}^{n} X_i > X_{n+1}|T_2 = t_2) = \frac{P(\sum_{i=1}^{n} X_i > X_{n+1}, T_2 = t_2)}{P(T_2 = t_2)}$$

$T_2 = 0$时, 显然$P(\sum_{i=1}^{n} X_i > X_{n+1}, T_2 = 0) = 0$

$\Rightarrow P(\sum_{i=1}^{n} X_i > X_{n+1}|T_2 = 0) = 0$

$T_2 = 1$时: $P(\sum_{i=1}^{n} X_i > X_{n+1}, T_2 = 1) = (1-p)\binom{n}{1}p(1-p)^{n-1}$

而$P(T_2 = 1) = \binom{n+1}{1}p(1-p)^n$

$\Rightarrow P(\sum_{i=1}^{n} X_i > X_{n+1}|T_2 = 1) = \dfrac{n}{n+1}$

$T_2 = 2$时$P(\sum_{i=1}^{n} X_i > X_{n+1}, T_2 = 2) = (1-p)\binom{n}{2}p^2(1-p)^{n-2}$

而$P(T_2 = 2) = \binom{n+1}{2}p^2(1-p)^{n-1}$

$\Rightarrow P(\sum_{i=1}^{n} X_i > X_{n+1}|T_2 = 2) = \dfrac{n-1}{n+1}$

$T_2 > 2$时，容易看出$P(\sum_{i=1}^{n} X_i > X_{n+1}|T_2) \equiv 1$

因此 $\phi(T_2) = \begin{cases} 0 & T_2 = 0 \\ \frac{n}{n+1} & T_2 = 1 \\ \frac{n-1}{n+1} & T_2 = 2 \\ 1 & T_2 > 2 \end{cases}$ 为最佳无偏估计

7.58

(a) $L(\theta|x) = (\frac{\theta}{2})^{|x|}(1-\theta)^{1-|x|}$

$\log L = |x|\log(\frac{\theta}{2}) + (1-|x|)\log(1-\theta)$

由$\frac{\partial}{\partial\theta}(\log L) = \frac{|x|}{\theta} - \frac{1-|x|}{1-\theta} = 0$

$\Rightarrow \theta = |x|$

且$\frac{|x|}{\theta} - \frac{1-|x|}{1-\theta}$单调递减, 故极值为极大值

因此$\hat{\theta} = |x|$为极大似然估计

(b) $ET(X) = 2P(X=1) = \theta$

(c) 由因子分解，$|x|$ 是 θ 的充分统计量, 考虑 $\phi(|x|) = E(T\big||x|)$

$\phi(0) = E(T|x=0) = 0$

$\phi(1) = E(T\big||x|=1) = 2P(x=1\big||x|=1) = 2\frac{P(x=1)}{|x|=1} = 2\frac{(\frac{\theta}{2})}{2\frac{\theta}{2}} = 1$ 即 $\phi(|x|) = |x|$ 是一个更好的 θ 的无偏估计量