基科 32 曾柯又 2013012266

7.44 先证明 $\bar{X}^2-\frac{1}{n}$ 是一个无偏估计

$\because\ \bar{X}\sim n(\theta,\dfrac{1}{n})$

$\therefore\ \sqrt{n}(\bar{X}-\theta)\sim(0,1)\Rightarrow n(\bar{X}-\theta)^2\sim\chi_1^2$

$\therefore nE(\bar{X}-\theta)^2=nE(\bar{X}^2-\theta^2)=1$

$\therefore E(\bar{X}^2-\dfrac{1}{n})=\theta^2$ 由指数分布族的性质可知 $\bar{X}$ 是 θ 的完全统计量，且由前面的例子可知 $\bar{X}$ 是充分统计量，因此 $\bar{X}^2-\frac{1}{n}$ 是 θ 的最佳无偏估计。

$$\begin{aligned}
Var(\bar{X}^2-\frac{1}{n}) &= E(\bar{X}^2-\frac{1}{n}-\theta^2)^2\\
&= E[(\bar{X}-\theta)(\bar{X}+\theta)-\frac{1}{n}]^2\\
&= E(\bar{X}+\theta)^2(\bar{X}-\theta)^2-\frac{2}{n}E(\bar{X}-\theta)(\bar{X}+\theta)+\frac{1}{n^2}\\
&= \frac{1}{n}E[2(\bar{X}+\theta)(\bar{X}-\theta)+(\bar{X}+\theta)^2]-\frac{2}{n^2}+\frac{1}{n^2}\\
&= \frac{1}{n}(\frac{2}{n}+E(\bar{X}^2+2\bar{X}\theta+\theta^2))-\frac{2}{n^2}+\frac{1}{n^2}\\
&= \frac{1}{n}(4\theta^2+\frac{3}{n})-\frac{2}{n^2}+\frac{1}{n^2}\\
&= \frac{4\theta^2}{n}+\frac{2}{n^2}
\end{aligned}$$

而由 Cramer-Rao 不等式给出的下界为:

$$\begin{aligned}
\frac{(2\theta)^2}{-nE(\frac{\partial^2}{\partial\theta^2}\log f)} &= \frac{(2\theta)^2}{-nE[\frac{\partial^2}{\partial\theta^2}(-\frac{1}{2}\log(2\pi)-\frac{1}{2}(x-\theta)^2)]}\\
&= \frac{4\theta^2}{n}
\end{aligned}$$

即 $Var(\bar{X}-\frac{1}{n^2})>\frac{4\theta^2}{n}$

7.45

(a)

$$
\begin{aligned}
MSE(aS^2) &= E(aS^2-\sigma^2)^2 \\
&= E(a(S^2-\sigma^2)+(a-1)\sigma^2)^2 \\
&= E[a^2(S^2-\sigma^2)^2+(a-1)^2\sigma^4+2a(S^2-\sigma^2)(a-1)\sigma^2] \\
&= a^2VarS^2+(a-1)^2\sigma^2
\end{aligned}
$$

(b) 首先容易证明 $S^2=\frac{1}{2n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}(X_i-X_j)^2$

记 $Z_i=X_i-\mu$, 有$EZ_i=0$, $EZ_i^2=\sigma^2$, $EZ_i^4=\kappa$

有 $S^2=\frac{1}{2n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}(Z_i-Z_j)^2$

$S^4=(\frac{1}{2n(n-1)})^2\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}(Z_i-Z_j)^2(Z_k-Z_l)^2$

根据 i,j,k,l 的不同取值，平均值不同，先分情况讨论

有 $n(n-1)(n-2)(n-3)$ 对满足 i,j,k,l 互不相同

$E(Z_1-Z_2)^2(Z_3-Z_4)^2=(2\sigma^2)^2=4\sigma^4$

有 $2n(n-1)$ 对满足 $i\neq j$ 且$i=k$, $j=l$或$i=l$, $j=k$

$E(Z_1-Z_2)^4=E(Z_1^4-4Z_1^3Z_2+6Z_1^2Z_2^2-4Z_1Z_2^3+Z_2^4)=2\kappa+6\sigma^4$

有 $4n(n-1)(n-2)$ 对满足 i,j,k,l 中仅有一对相等且 $i\neq j$, $k\neq l$

$E(Z_1-Z_2)^2(Z_1-Z_3)^2=E(Z_1^2+Z_2^2-2Z_1Z_2)(Z_1^2+Z_3^2-2Z_1Z_3)=\kappa+3\sigma^4$

因此

$$
\begin{aligned}
ES^4 &= \left[\frac{1}{2n(n-1)}\right]^2[4n(n-1)(n-2)(n-3)\sigma^4+2n(n-1))(2\kappa+6\sigma^4) \\
&+4n(n-1)(n-2)(\kappa+3\sigma^4)] \\
&= \left[\frac{1}{2n(n-1)}\right]^2\left[4n(n-1)^2\kappa+4\sigma^4n(n-1)(n^2-2n+3)\right]
\end{aligned}
$$

$$\begin{aligned}
VarS^2 &= ES^4-(ES^2)^2\\
&= \frac{\kappa}{n}+\frac{n^2-2n+3}{n(n-1)}\sigma^4-\sigma^4\\
&= \frac{\kappa}{n}+\frac{(3-n)}{n(n-1)}\sigma^4\\
&= \frac{1}{n}(\kappa-\frac{n-3}{n-1}\sigma^4)
\end{aligned}$$

(c) 若 $X\sim n(\mu,\sigma^2)$

$E(X-\mu)^4 = E(X-\mu)^3(X-\mu)$

$= \sigma^2 E3(X-\mu)^2$

$= 3\sigma^4$

$\therefore \kappa = 3\sigma^4 \Rightarrow Var(S^2)=\frac{1}{n}(\kappa-\frac{n-3}{n-1}\sigma^4)=\frac{2\sigma^4}{n-1}$

$MSE(aS^2)=a^2VarS^2+(a-1)^2\sigma^4=a^2\frac{2\sigma^4}{n-1}+(a-1)^2\sigma^4$

对a求导可得$a=\frac{n-1}{n+1}$，而该二次函数的极值为极小值，故 MSE 取极小值对应的aS^2为$\frac{n-1}{n+1}S^2$

(d) $MSE(aS^2)=a^2\frac{1}{n}(\kappa-\frac{n-3}{n-1}\sigma^4)+(a-1)^2\sigma^4$ 对 a 求导可得

$a\frac{1}{n}(\frac{\kappa}{\sigma^4}-\frac{n-3}{n-1})+(a-1)=0$

$\Rightarrow a=\frac{n-1}{(n+1)+\frac{(n-1)(\frac{\kappa}{\sigma^4}-3)}{n}}$

由于 $Var(S^2)>0 \Rightarrow Var(S^2)+\sigma^4>0$

a 对应二次函数的 2 次幂系数大于 0, 故该极值恰对应极小值

书中的公式应该有错

(e) 若 $\kappa > 3\sigma^4$, $\dfrac{(n-1)(\frac{\kappa}{\sigma^4}-3)}{n} > 0 \Rightarrow a < \dfrac{n-1}{n+1}$

若$\kappa < 3\sigma^4$, $\dfrac{(n-1)(\frac{\kappa}{\sigma^4}-3)}{n} < 0 \Rightarrow a > \dfrac{n-1}{n+1}$

另一方面 $VarS^2 > 0 \Rightarrow \kappa > \dfrac{n-3}{n-1}\sigma^4$

$\therefore \dfrac{(n-1)(\frac{\kappa}{\sigma^4}-3)}{n} > -\dfrac{2}{n-1} > -2$

$a = \dfrac{n-1}{(n+1)+\frac{(n-1)(\frac{\kappa}{\sigma^4}-3)}{n}} < \dfrac{n-1}{(n+1)+-2} = 1$