1 (1) $d(f,g)=\sum_{n=1}^{\infty}2^{-n}|<f-g,x_n>|=\sum_{n=1}^{\infty}2^{-n}|<g-f,x_n>|=d(g,f)$

(2) $d(f,f)=0$

when $d(f,g)=0$, $<f-g,x_n>=0$ for all x_n

Due to the definition of x_n ,$\forall x\in B$ $\exists\{x_{n_k}\}$ $s.t$ $x_{n_k}\to x$

we have $<f-g,x_{n_k}>\to<f-g,x>$, and from the continuity of inner product,we have $<f-g,x>=0$ .which means $f-g=0$

(3)

$$\because |<f-g,x_n>|=|<f-h,x_n>+<h-g,x_n>|\leq|<f-h,x_n>|+|<h-g,x_n>|$$

we have $d(f,g)\leq d(f,h)+d(g,h)$

(4) $<(f+h)-(g+h),x_n>=<f-g,x_n>$,so $d(f+h,g+h)=d(f,g)$

from those points all above, we can conclude that $d(f,g)$ is an inner product in E.

2 $\forall\epsilon>0$ define $U_\epsilon=(-\epsilon,\epsilon)$

then the set define by $\phi_x^{-1}(U_\epsilon)$ is a week* neighborhood of zero in E*.

define $S_\epsilon^*=\bigcup_{n\geq1}\phi_{x_n}^{-1}(U_\epsilon)$ then $\forall f\in S_\epsilon^*$ $|\phi_{x_n}(f)|<\epsilon\Rightarrow|<f,x_n>|<\epsilon$

$\therefore d(f,0)<\sum_{n=1}^{\infty}2^{-n}\epsilon=\epsilon$

$\therefore f\in S_\epsilon\Rightarrow S_\epsilon^*\subset B^*\cap S_\epsilon$

3 S* is a weak* neighborhood of zero in E*, then $\exists x\in E,$ and U a neighborhood of 0 in $\mathbb{R}$ $s.t$ $S^*=\phi_x^{-1}(U)$

$\because U$ is a neighborhood of zero

$\therefore \exists \epsilon > 0 \ s.t \ U_\epsilon = \Big\{ x \in \mathbb{R} \Big| |x| < \epsilon \Big\} \subset U$

$\forall f \in S(\epsilon) = \Big\{ f \in E^* \Big| d(f, 0) < \epsilon) \Big\}$

$\sum_{n=1}^{\infty} 2^{-n} | < f, x_n > | < \epsilon \Rightarrow 2^{-n} | < f, x_n > |$ for all n

we can find a $\{x_{n_k}\} \ s.t \ x_{n_k} \to x$

define $S = \bigcup_{k \geq 1} S(\frac{\epsilon}{x^{n_k}})$

then $\forall f \in S \quad 2^{-n_k} | < f, x_{n_k} > | < \frac{\epsilon}{2^{n_k}} \Rightarrow | < f, x_{n_k} > | < \epsilon$ for all n_k

so we have $< f, x >< \epsilon \Rightarrow |f(x)| < \epsilon$

$\therefore f(x) \in U_\epsilon \subset U \Rightarrow f \in \phi_x^{-1}(U) \Rightarrow S \subset S^*$