基科 32 曾柯又 2013012266

5.44

(a) $\mu = EX_i = p\ \sigma^2 = VarX_i = p(1-p)$

$\therefore\ \dfrac{\sqrt{n}(Y_n - p)}{\sqrt{p(1-p)}} \xrightarrow{\mathcal{D}} n(0,1)$

$\Rightarrow\ \sqrt{n}(Y_n - p) \xrightarrow{\mathcal{D}} n(0, p(1-p))$

(b) 令$g(x) = x(1-x)$ 当$p \neq \dfrac{1}{2}$ 时$g'(x) \neq 0$

$\sqrt{n}(g(Y_n) - g(p)) \xrightarrow{\mathcal{D}} g'(p)n(0, p(1-p))$

即$\sqrt{n}(g(Y_n) - g(p)) \xrightarrow{\mathcal{D}} n(0, (1-2p)^2 p(1-p))$

(c) $Y_n(1-Y_n) = (\dfrac{1}{2})^2 - (Y_n - \dfrac{1}{2})^2$

$4n[\dfrac{1}{4} - Y_n(1-Y_n)] = (2\sqrt{n}(Y_n - \dfrac{1}{2}))^2$

由 CMT

$2\sqrt{n}(Y_n - \dfrac{1}{2}) \xrightarrow{\mathcal{D}} X\ \Rightarrow\ (2\sqrt{n}(Y_n - \dfrac{1}{2}))^2 \xrightarrow{\mathcal{D}} X^2$

$X \sim n(0,1),\ X^2 \sim \chi_1^2$

$\Rightarrow n[Y_n(1-Y_n) - \dfrac{1}{4}] \to -\dfrac{1}{4}\chi_1^2$

5.50 $X_1 = cos(2\pi U_1)\sqrt{-2\log U_2}\quad X_2 = sin(2\pi U_1)\sqrt{-2\log U_2}$

$\Rightarrow U_2 = e^{-\frac{1}{2}(X_1^2 + X_2^2)}\quad U_1 = \dfrac{1}{2\pi}\arctan(\dfrac{X_2}{X_1})$

$f_{X_1,X_2}(x_1,x_2) = f_{U_1,U_2}(u_1,u_2)\left|\dfrac{\partial(U_1,U_2)}{\partial(X_1,X_2)}\right|$

$$= \begin{vmatrix} \frac{1}{2\pi}\frac{-x_2}{x_1^2+x_2^2} & \frac{1}{2\pi}\frac{x_1}{x_1^2+x_2^2} \\ -x_1 e^{-\frac{1}{2}(x_1^2+x_2^2)} & -x_2 e^{-\frac{1}{2}(x_1^2+x_2^2)} \end{vmatrix} = \frac{1}{2\pi}e^{-\frac{1}{2}(x_1^2+x_2^2)}$$

即 X_1, X_2 是独立的 $n(0,1)$ 变量。

5.52

(a) 一种方法仿照书中的

令 $Y = i$ 当$U \in (\sum_{k=0}^{i-1} \binom{8}{k} (\frac{2}{3})^k (\frac{1}{3})^{8-k}, \sum_{k=0}^{i} \binom{8}{k} (\frac{2}{3})^k (\frac{1}{3})^{8-k})$ $i = 0, \cdots 8$

其中$U \sim \text{uniform}(0,1)$

还可以通过产生 8 个 0,1 分布，求和得到二项分布, 即设:

$$f(U) = \begin{cases} 1 & 0 \le U < \frac{2}{3} \\ 0 & \frac{2}{3} \le U < 1 \end{cases}$$

再令 $Y = \sum_{i=1}^{8} f(U_i)$ 则

$Y \sim B(8, \frac{2}{3})$

(b) $P(X = x) = \frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}}$ $N = 10,\, M = 8,\, K = 4$

$P(X = 2) = \dfrac{2}{15},\ P(X = 3) = \dfrac{8}{15},\ P(X = 4) = \dfrac{1}{3}$

$$Y = \begin{cases} 2 & 0 \le U < \frac{4}{30} \\ 3 & \frac{2}{15} \le U < \frac{2}{3} \\ 4 & \frac{2}{3} \le U \le 1 \end{cases}$$

(c) 设 $f(U) = \begin{cases} 1 & 0 \le U < \frac{1}{3} \\ 0 & \frac{1}{3} \le U < 1 \end{cases}$

令 $X = f(U)$ U 为 $(0,1)$ 均匀分布随机变量

不断生成 X, 直到有 5 次 $X = 1$, 记下此时生成的 X 的次数 n, 令 $Y = n$,

则 Y 满足负二项分布。

5.57

(a) $EY_1 = EX_1 + EX_2 = \lambda_1 + \lambda_3$

$EY_2 = EX_2 + EX_3 = \lambda_2 + \lambda_3$

$EY_1Y_2 = E(X_1 + X_3)(X_2 + X_3) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + EX_3^2$

而$EX_3^2 = \sum_{x=1}^{\infty} x^2 \frac{e^{-\lambda_3}\lambda_3^x}{x!} = \lambda_3(1+\lambda_3)$

$\therefore\ Cov(Y_1, Y_2) = E((Y_1 - EY_1)(Y_2 - EY_2)) = EY_1Y_2 - EY_1E_Y2 = \lambda_3$

(b) $P(Z_1 = 1) = P(X_1 = 0, X_3 = 0) = e^{-(\lambda_1+\lambda_3)} = p_1$

$P(Z_2 = 1) = e^{-(\lambda_1+\lambda_3)} = p_2$

即$Z_i \sim \text{Bernouli}(p_i)$

$VarZ_i = p_i(1-p_i)$

$P(Z_1 = 1, Z_2 = 1) = P(X_1, X_2, X_3 = 0) = e^{-(\lambda_1+\lambda_2+\lambda_3)}$

$Cov(Z_1, Z_2) = EZ_1Z_2 - EZ_1EZ_2$

$= e^{-(\lambda_1+\lambda_2+\lambda_3)} - e^{-(\lambda_1+\lambda_3)}e^{-(\lambda_2+\lambda_3)} = p_1p_2(e^{\lambda_3} - 1)$

$\Rightarrow Corr(Z_1, Z_2) = \dfrac{p_1p_2(e^{\lambda_3} - 1)}{p_1(1-p_1)p_2(1-p_2)}$

(c) $e^{-\lambda_1} \leq 1$

$\Rightarrow p_1e^{\lambda_3} \leq 1$

$\Rightarrow p_1(e^{\lambda_3} - 1) \leq 1 - p_1$

$\Rightarrow \dfrac{p_1(e^{\lambda_3} - 1)}{\sqrt{1-p_1}} \leq \sqrt{1-p_1}$

$\Rightarrow Corr(Z_1, Z_2) \leq \sqrt{\dfrac{p_2(1-p_1)}{p_1(1-p_2)}}$

同理$e^{-\lambda_2} \leq 1 \Rightarrow \dfrac{p_2(e^{\lambda_3} - 1)}{\sqrt{1-p_2}} \leq \sqrt{1-p_2}$

$\Rightarrow Corr(Z_1, Z_2) = \sqrt{\dfrac{p_1(1-p_2)}{p_2(1-p_1)}}$

$\Rightarrow Corr(Z_1, Z_2) = min\sqrt{\dfrac{p_2(1-p_1)}{p_1(1-p_2)}}, \sqrt{\dfrac{p_1(1-p_2)}{p_2(1-p_1)}}$

5.60

(a) $\because\ a \le w \le b$

$$
\begin{aligned}
P(W = w) = P(X \le w, Y < f(X)) \\
\int_a^w \int_0^{f(x)} \mathrm{d}x\mathrm{d}y f_{X,Y}(x, y) \\
= \frac{1}{c(a-b)} \int_a^w f(x)\mathrm{d}x
\end{aligned}
$$

(b) 仿照书中的办法, 按下面步骤生成随机变量：

1. 生成分别服从 uniform(a, b) uniform$(0, c)$ 的独立随机变量 (X, Y) 。

2. 如果 $Y < f(X)$，则 $W = X$, 否则重新进行第一步。

则 W 的分布满足 $f(x)$。