1. MINLP Formulations

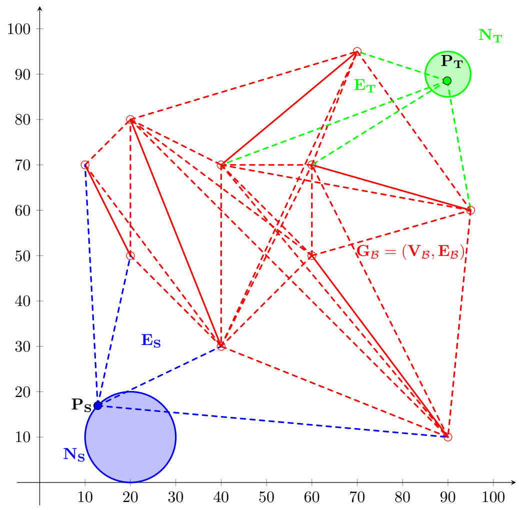
In this section it is introduced a Mixed Integer Non-Linear Programming formulation for the problems described in Section ??. Firstly, the formulation for the H-SPP-S is described and then the model H-TSP-S is presented as an extension of the previous problem.

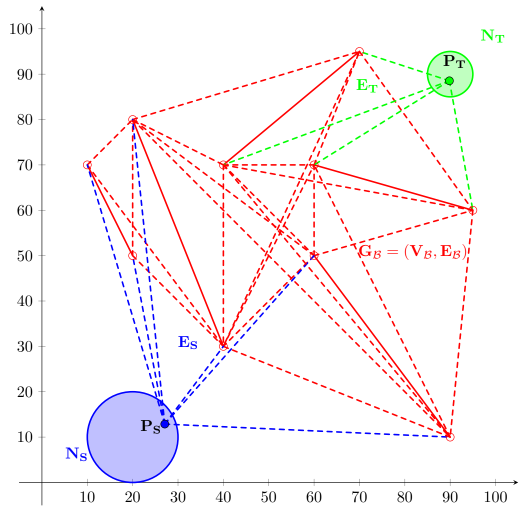
1.1. A formulation for the H-SPP-S

The main idea of the H-SPP-S is to solve a shortest path problem in the undirected graph induced by the endpoints of the barriers and the neighborhoods. Here, it is necessary to define the following sets:

- $V_S = \{P_S\}$: set composed by the point selected in the source neighborhood N_S.
- $V_{\mathcal{B}} = \{P_B^1, P_B^2 : B = \overline{P_B^1 P_B^2} \in \mathcal{B}\}$: set of vertices that form the barriers of the problem.
- $V_T = \{P_T\}$: set composed by the point selected in the target neighborhood N_T.
- $E_S = \{(P_S, P_B^i) : P_B^i \in V_{\mathcal{B}}$ and $\overline{P_S P_B^i} \cap B'' = \emptyset, \forall B'' \in \mathcal{B},\ i = 1, 2\}$: set of edges formed by the line segments that join the point selected in the source neighborhood and every endpoint in the barriers and do not cross any barrier in $\mathcal{B}$.
- $E_{\mathcal{B}} = \{(P_B^i, P_{B'}^j) : P_B^i, P_{B'}^j \in V_{\mathcal{B}}$ and $\overline{P_B^i P_{B'}^j} \cap B'' = \emptyset,\ \forall B'' \in \mathcal{B},\ i, j = 1, 2\}$: set of edges formed by the line segments that join two vertices of $V_{\mathcal{B}}$ and do not cross any barrier in $\mathcal{B}$.
- $E_T = \{(P_T, P_B^i) : P_B^i \in V_{\mathcal{B}}$ and $\overline{P_T P_B^i} \cap B'' = \emptyset, \forall B'' \in \mathcal{B},\ i = 1, 2\}$: set of edges formed by the line segments that join the point selected in the target neighborhood and every endpoint in the barriers and do not cross any barrier in $\mathcal{B}$. Definimos tambien la arista que pueda unir a los dos entornos o asumimos que no hay un camino que los una? Depende del caso que consideremos el problema es convexo o no.

At this point, we can define the graph $G = (V, E)$ induced by the barriers and the neighborhoods, where $V = V_S \cup V_{\mathcal{B}} \cup V_T$ and $E = E_S \cup E_{\mathcal{B}} \cup E_T$. It is interesting to note that this graph can be split into two parts: a fixed graph $G_{\mathcal{B}} = (V_{\mathcal{B}}, E_{\mathcal{B}})$ and the sets V_S, E_S, V_T and E_T that depend on where the points P_S and P_T are located as shown in Figure. The figures show how the graph G is generated. The blue dashed line segments represent the edges of E_S, the green ones, the edges of E_T and the red dashed lines, the edges of $E_{\mathcal{B}}$. A special case that can be remarked occurs when the neighborhoods are points. In that case, the induced graph is completely fixed and it is only necessary to find which edges are included by keeping in mind that there can not have crossings.





The following computational geometry result is useful to compute the edges in $E_S, E_{\mathcal{B}}$ and E_T:

Remark 1. *Let* $\overline{P_S P_B^i}$ *and* $\overline{P_{B'}^1 P_{B'}^2}$ *be two different line segments. Let also denote* $\det(P_S | P_{B'}^1 P_{B'}^2) = \det\left(\overrightarrow{P_S P_{B'}^1} \;\middle|\; \overrightarrow{P_S P_{B'}^2} \right)$ *the determinant whose arguments are* P_S, $P_{B'}^1$ *and* $P_{B'}^2$. *If*

$$\operatorname{sign}\left(\det(P_S | P_{B'}^1 P_{B'}^2)\right) = \operatorname{sign}\left(\det(P_B^i | P_{B'}^1 P_{B'}^2)\right) \quad \text{or} \quad \operatorname{sign}\left(\det(P_{B'}^1 | P_S P_B^i)\right) = \operatorname{sign}\left(\det(P_{B'}^2 | P_S P_B^i)\right),$$

then $\overline{P_S P_B^i}$ *and* $\overline{P_{B'}^1 P_{B'}^2}$ *do not intersect.*

Since E_S and E_T are not fixed, the determinants in Remark 1 also depend on the location of P_S and P_T. Hence, it is essential to model the previous constraint by using binary variables. We only focus on the case of E_S but the same rationale is used for E_T.

Let $B \in \mathcal{B}$ be a barrier and P_B^i an endpoint of B. Hence, the edge (P_S, P_B^i) belongs to E_S if

$$\overline{P_S P_B^i} \cap B' = \emptyset, \quad \forall B' \in \mathcal{B},$$

or, by Remark 1, if

$$\text{sign}\left(\det(P_S|P_{B'}^1 P_{B'}^2)\right) = \text{sign}\left(\det(P_B^i|P_{B'}^1 P_{B'}^2)\right) \quad \text{or} \quad \text{sign}\left(\det(P_{B'}^1|P_S P_B^i)\right) = \text{sign}\left(\det(P_{B'}^2|P_S P_B^i)\right), \quad \forall B' \in \mathcal{B}.$$

To model the sign of a determinant defined by the points P_S, $P_{B'}^1$ and $P_{B'}^2$, we introduce the following binary variable:

- $\alpha(P_S|P_{B'}^1 P_{B'}^2)$, that is one if $\det(P_S|P_{B'}^1 P_{B'}^2)$ is positive and zero, otherwise.

Note that the case when the determinant is null is not considered , because segments are located in general position.

It is possible to represent the sign condition by including the following constraints:

$$\left[1 - \alpha(P_S|P_{B'}^1 P_{B'}^2)\right] L(P_S|P_{B'}^1 P_{B'}^2) \leq \det(P_S|P_{B'}^1 P_{B'}^2) \leq U(P_S|P_{B'}^1 P_{B'}^2)\, \alpha(P_S|P_{B'}^1 P_{B'}^2), \tag{α-C}$$

where $L(P_S|P_{B'}^1 P_{B'}^2)$ and $U(P_S|P_{B'}^1 P_{B'}^2)$ are a lower and a upper bound for the determinant, respectively. If the determinant is positive, then $\alpha(P_S|P_{B'}^1 P_{B'}^2)$ must be one to make the second inequality feasible. The analagous case happens if the determinant is not positive.

Now, to check if two determinants $\det(P_S|P_{B'}^1 P_{B'}^2)$ and $\det(P_B^i|P_{B'}^1 P_{B'}^2)$ have the same sign, it is required to introduce the binary variables:

- $\beta(P_S P_B^i|P_{B'}^1 P_{B'}^2)$, that is one if $\det(P_S|P_{B'}^1 P_{B'}^2)$ and $\det(P_B^i|P_{B'}^1 P_{B'}^2)$ have the same sign, zero otherwise.

Hence, the β variable can be represented by the equivalence constraint of the α variables

$$\begin{aligned} \beta(P_S P_B^i|P_{B'}^1 P_{B'}^2) &= \alpha(P_S|P_{B'}^1 P_{B'}^2)\alpha(P_B^i|P_{B'}^1 P_{B'}^2) + \left[1 - \alpha(P_S|P_{B'}^1 P_{B'}^2)\right]\left[1 - \alpha(P_B^i|P_{B'}^1 P_{B'}^2)\right], \\ \beta(P_S P_B^i|P_{B'}^1 P_{B'}^2) &= 2\gamma(P_S P_B^i|P_{B'}^1 P_{B'}^2) - \alpha(P_S|P_{B'}^1 P_{B'}^2) - \alpha(P_B^i|P_{B'}^1 P_{B'}^2) + 1, \end{aligned} \tag{β-C}$$

where $\gamma(P_S P_B^i|P_{B'}^1 P_{B'}^2)$ is the auxiliary binary variable that models the product of the α variables that can be linearized by using the following constraints:

$$\begin{aligned} \gamma(P_S P_B^i|P_{B'}^1 P_{B'}^2) &\leq \alpha(P_S|P_{B'}^1 P_{B'}^2), \\ \gamma(P_S P_B^i|P_{B'}^1 P_{B'}^2) &\leq \alpha(P_B^i|P_{B'}^1 P_{B'}^2), \\ \gamma(P_S P_B^i|P_{B'}^1 P_{B'}^2) &\geq \alpha(P_S|P_{B'}^1 P_{B'}^2) + \alpha(P_B^i|P_{B'}^1 P_{B'}^2) - 1. \end{aligned} \tag{γ-C}$$

Finally, we need to check if there exists any coincidence in the sign of the determinants, so we define the binary variable $\delta(P_S P_B^i|P_{B'}^1 P_{B'}^2)$ that is one if $\overline{P_S P_B^i}$ and $\overline{P_{B'}^1 P_{B'}^2}$ do not intersect and zero, otherwise. This condition can be modelled by using these disjunctive constraints:

$$\frac{1}{2}\left[\beta(P_S P_B^i|P_{B'}^1 P_{B'}^2) + \beta(P_{B'}^1 P_{B'}^2|P_S P_B^i)\right] \leq \delta(P_S P_B^i|P_{B'}^1 P_{B'}^2) \leq 2\left[\beta(P_S P_B^i|P_{B'}^1 P_{B'}^2) + \beta(P_{B'}^1 P_{B'}^2|P_S P_B^i)\right]. \tag{δ-C}$$

If there exists a sign coincidence, then $\delta(P_S P_B^i|P_{B'}^1 P_{B'}^2)$ is one to satisfy the first constraint, and the second one is always fullfilled. However, if the sign of the determinants is not the same, then the second constraint is active and $\delta(P_S P_B^i|P_{B'}^1 P_{B'}^2)$ is null. Finally, $(P_S, P_B^i) \in E_S$ if

$$\delta(P_S P_B^i|P_{B'}^1 P_{B'}^2) = 1, \quad \forall B' \in \mathcal{B}.$$

Hence, if we denote by $\varepsilon(P_S P_B^i)$ the binary variable that is one when $(P_S, P_B^i) \in E_S$, this variable can be represented by means of the following inequalities:

$$\left[\sum_{B' \in \mathcal{B}} \delta(P_S P_B^i|P_{B'}^1 P_{B'}^2) - |\mathcal{B}|\right] + 1 \leq \varepsilon(P_S P_B^i) \leq \frac{1}{|\mathcal{B}|} \sum_{B' \in \mathcal{B}} \delta(P_S P_B^i|P_{B'}^1 P_{B'}^2). \tag{ε-C}$$

If there is a barrier $B' \in \mathcal{B}$ that intersects the segment $\overline{P_S P_B^i}$, then $\delta(P_S P_B^i | P_{B'}^1 P_{B'}^2)$ is zero and the second inequality enforces $\varepsilon(P_S P_B^i)$ to be zero because the right side is fractional and the first inequality is non-positive. Nonetheless, if there is not any barrier that intersects the segment, then $\varepsilon(P_S P_B^i)$ is equals to one, because the left side of the first inequality is one and the right side of the second inequality too.

Now, we can define the path that the drone can follow by taking into account the edges of the induced graph. Let $y(PQ)$ be the binary variable that is one if the drone goes from P to Q. Then, the inequalities

$$y(P_S P_B^i) \leq \varepsilon(P_S P_B^i), \quad \forall P_B^i \in V_{\mathcal{B}}. \tag{y-C}$$

assure that the drone can go from $P_S \in V_S$ to a point of a barrier only if it does not cross any barrier.

Next, we need to introduce the continuous variable $d(PQ)$ that represents the distance between P and Q. To model this distance, we use the following second order cone constraints:

$$\|P - Q\| \leq d(PQ), \quad \forall (P, Q) \in E. \tag{d-C}$$

Finally, we assume that the sets N_S and N_T are second order cone (SOC) representable, that is, the sets can be represented by using second-order cone constraints:

$$P_S \in N_S \Longleftrightarrow \|A_S^i P_S + b_S^i\| \leq (c_S^i)^T P_S + d_S^i, \quad i = 1, \ldots, nc_S,$$

where A_S^i, b_S^i, c_S^i and d_S^i are parameters of the constraint i and nc_S denotes the number of constraints that appear in the block associated to the neighborhood N_S.

It is remarkable that these inequalities can model the special case of linear constraints (for $A_S^i, b_S^i \equiv 0$), ellipsoids and hyperbolic constraints (see [?] for more information).

At this time, we have all the necessary elements to give a MINLP formulation for the H-SPP-S as follows:

$$\begin{aligned}
\text{minimize} \quad & \sum_{(P,Q)\in E} d(PQ)y(PQ) \qquad\qquad \text{(H-SPP-S)}\\
\text{subject to} \quad & \sum_{\{Q:(P,Q)\in E\}} y(PQ) - \sum_{\{Q:(Q,P)\in E\}} y(QP) = \begin{cases} 1, & \text{if } P \in V_S, \\ 0, & \text{if } P \in V_{\mathcal{B}}, \\ -1, & \text{if } P \in V_T. \end{cases},\\
& (\alpha\text{-C}), (\beta\text{-C}), (\delta\text{-C}), (\varepsilon\text{-C}), (\text{y-C}), (\text{d-C}),\\
& P_S \in N_S,\ P_T \in N_T.
\end{aligned}$$

The objective function minimizes the length of the path followed by the drone in the edges of the induced graph. The first constraints are the flow conservation constraints, the second constraints represent the sets E_S and E_T and the third ones state that the point selected must be in their respective neighborhoods.

1.2. A formulation for the H-TSP-S

2. Strengthening the formulation of H-TSP-S

Incluir el resultado estructural que da una cota superior del numero de bolas que se puede generar

2.1. Preprocessing

In this subsection, we show a preprocessing result that allows to fix some variables by taking into account the relative position between the neighborhoods and the barriers. In particular, we are going to present an outcome that ensures that there are some barriers whose endpoints can not be incident in the edges of E_N and it is not necessary to include it in E_N.

The first issue that is solved is the one when the neighborhoods are segments. Let $N = \overline{P_N^1 P_N^2}$ be a line segment and $B = \overline{P_B^1 P_B^2} \in \mathcal{B}$ a barrier. Let also be

$$\begin{aligned}
\text{cone}(P, Q, R) &:= \{\mu_1 \overrightarrow{PQ} + \mu_2 \overrightarrow{PR} : \mu_1, \mu_2 \geq 0\},\\
\text{cone}(P, Q, R)^- &:= \{\mu_1 \overrightarrow{PQ} + \mu_2 \overrightarrow{PR} : \mu_1, \mu_2 \geq 0,\ \mu_1 + \mu_2 \leq 1\},\\
\text{cone}(P, Q, R)^+ &:= \{\mu_1 \overrightarrow{PQ} + \mu_2 \overrightarrow{PR} : \mu_1, \mu_2 \geq 0,\ \mu_1 + \mu_2 \geq 1\}.
\end{aligned}$$

Note that, $cone(P,Q,R)$ is the union of $\text{cone}(P,Q,R)^-$ and $\text{cone}(P,Q,R)^+$. It is also important to remark that $\text{cone}(P,Q,R)^+$ is the part of cone that crosses the barrier $\overline{QR}$ when we consider a segment whose endpoints are P and another point of this set, i.e.

$$\text{cone}(P,Q,R)^+ = \{P' : \overline{PP'} \cap \overline{QR} \neq \emptyset\}.$$

In the following proposition, we give a sufficient condition to not include the edge (P_N, P_B^i) in E_N:

Proposition 1. *Let $B' = \overline{P_{B'}^1 P_{B'}^2} \in \mathcal{B}$ and $\text{cone}(P_B^i, P_{B'}^1, P_{B'}^2)^+$ the conical hull generated by these points. If*

$$N \subset \bigcup_{B' \in \mathcal{B}} \text{cone}(P_B^i, P_{B'}^1, P_{B'}^2)^+,$$

then $(P_N, P_B^i) \notin E_N$.

Proof. If $P_N \in N$, then there exists a $B' \in \mathcal{B}$ such that $P_N \in \text{cone}(P_B^i, P_{B'}^1, P_{B'}^2)^+$. Therefore, $\overline{P_B^i P_N} \cap B' \neq \emptyset$ and $(P_N, P_B^i) \notin E_N$, as we claimed. □

We can check computationally the condition of the previous proposition by using the following procedure. Let r_N the straight line that contains the line segment N.

Initialization: Let P_B^i be the point whose edge (P_B^i, P_N) is going to check if $(P_N, P_B^i) \notin E_N$.
Set $flag = True$, $points = \{\}$.

for $B' \in \mathcal{B}$ **do**
- **for** $j \in \{1,2\}$ **do**
  - Compute the straight line $r(P_B^i, P_{B'}^j)$ that contains the points P_B^i and $P_{B'}^j$.
  - Intersect $r(P_B^i, P_{B'}^j)$ and r_N in the point $Q_{B'}^j$.
  - **if** $\|\overline{P_B^i Q_{B'}^j}\| \geq \|\overline{P_B^i P_{B'}^j}\|$ **then**
    - Include $Q_{B'}^j$ in $points$.

Estudiar resultados que eliminen algunas de las posibles aristas de E_S y E_T Hablar que las aristas $E_{\mathcal{B}}$ se pueden preprocesar porque los puntos están fijados

2.2. *Valid inequalities*

This subsection is devoted to show some results that adjust the big-M constants that appear in the previous formulation, specifically, in the (α-C), where the modelling of the sign requires to compute the lower and upper bounds L and U, respectively. We are going to expose that these bounds can be computed explicitly for the cases when the neighborhoods are ellipses and segments.

Let $\overline{P_{B'}^1 P_{B'}^2} = B' \in \mathcal{B}$ be a barrier and $P_S \in N_S$. Let $\det(P_B^i | P_{B'}^1 P_{B'}^2)$ also be the determinant whose values must be bounded. Hence, the solution of the following problem gives the lower bound of the determinant:

$$\overline{L} = \min_{P_S = (x,y) \in N_S} F(x,y) := \det(P_S | P_{B'}^1 P_{B'}^2) = \begin{vmatrix} P_{B'_x}^1 - x & P_{B'_x}^2 - x \\ P_{B'_y}^1 - y & P_{B'_y}^2 - y \end{vmatrix}. \tag{L-Problem}$$

2.2.1. *Lower and upper bounds when the neighborhoods are ellipsoids*

The first case that is considered is the one when N_S is an ellipse, that is, N_S is represented by the following inequality:

$$N_S = \{(x,y) \in \mathbb{R}^2 : ax^2 + by^2 + cxy + dx + ey + f \leq 0\} =$$

where a, b, c, d, e, f are coefficients of the ellipse. In an extended form, we need to find:

$$\text{minimize} \quad F(x,y) = \begin{vmatrix} P_{B'_x}^1 - x & P_{B'_x}^2 - x \\ P_{B'_y}^1 - y & P_{B'_y}^2 - y \end{vmatrix} = xP_{B'_y}^1 - xP_{B'_y}^2 + yP_{B'_x}^2 - yP_{B'_x}^1 + P_{B'_x}^1 P_{B'_y}^2 - P_{B'_y}^1 P_{B'_x}^2, \tag{L-Ellipse}$$

$$\text{subject to} \quad ax^2 + by^2 + cxy + dx + ey + f \leq 0.$$

Since we are minimizing a linear function in a convex set, we can conclude that the extreme points are located in the frontier, so we can use the Lagrangian function to compute these points.

$$F(x,y;\lambda) = xP^1_{B'_y} - xP^2_{B'_y} + yP^2_{B'_x} - yP^1_{B'_x} + P^1_{B'_x}P^2_{B'_y} - P^1_{B'_y}P^2_{B'_x} + \lambda(ax^2+by^2+cxy+dx+ey+f).$$

$$\nabla F(x,y;\lambda) = 0 \Longleftrightarrow \left\{ \begin{array}{rcll} \frac{\partial F}{\partial x} & = & P^1_{B'_y} - P^2_{B'_y} + 2ax\lambda + cy\lambda + d\lambda & = 0, \\ \frac{\partial F}{\partial y} & = & P^2_{B'_x} - P^1_{B'_x} + 2by\lambda + cx\lambda + e\lambda & = 0, \\ \frac{\partial F}{\partial \lambda} & = & ax^2+by^2+cxy+dx+ey+f & = 0. \end{array} \right.$$

From the first two equations we can obtain:

$$\lambda = \frac{P^2_{B'_y} - P^1_{B'_y}}{2ax+cy+d} = \frac{P^1_{B'_x} - P^2_{B'_x}}{2by+cx+e}.$$

From this equality, we can express y as a function of x:

$$\begin{aligned} (P^2_{B'_y} - P^1_{B'_y})(2by+cx+e) &= (P^1_{B'_x} - P^2_{B'_x})(2ax+cy+d), \\ y\left[2b(P^2_{B'_y} - P^1_{B'_y}) - c(P^1_{B'_x} - P^2_{B'_x})\right] &= \left[2a(P^1_{B'_x} - P^2_{B'_x}) - c(P^2_{B'_y} - P^1_{B'_y})\right]x + \left[d(P^1_{B'_x} - P^2_{B'_x}) - e(P^2_{B'_y} - P^1_{B'_y})\right], \\ y &= \left[\frac{2a(P^1_{B'_x} - P^2_{B'_x}) - c(P^2_{B'_y} - P^1_{B'_y})}{2b(P^2_{B'_y} - P^1_{B'_y}) - c(P^1_{B'_x} - P^2_{B'_x})}\right]x + \left[\frac{d(P^1_{B'_x} - P^2_{B'_x}) - e(P^2_{B'_y} - P^1_{B'_y})}{2b(P^2_{B'_y} - P^1_{B'_y}) - c(P^1_{B'_x} - P^2_{B'_x})}\right], \\ y &= mx+n. \end{aligned}$$

Finally, we can substitute y in the third equation to compute the value of x:

$$\begin{aligned} ax^2+by^2+cxy+dx+ey+f &= ax^2+b(mx+n)^2+cx(mx+n)+dx+e(mx+n)+f = \\ &= (a+bm^2+cm)x^2+(2bmn+cn+d+em)x+(n^2b+en+f) = 0. \end{aligned}$$

By using the standard form of the solution of a second grade equation:

$$\begin{aligned} x^{\pm} &= \frac{-(2bmn+cn+d+em) \pm \sqrt{(2bmn+cn+d+em)^2 - 4(a+bm^2+cm)(n^2b+en+f)}}{2(a+bm^2+cm)}, \\ y^{\pm} &= mx^{\pm}+n. \end{aligned}$$

Hence, to compute the lower and upper bound, we only need to evaluate $x^{\pm}$ and $y^{\pm}$ in the objective function and the lowest and highest value correspond to $L(P_S|P^1_{B'}P^2_{B'})$ and $U(P_S|P^1_{B'}P^2_{B'})$, respectively.

2.2.2. Lower and upper bounds when the neighborhoods are line segments

In this case, the segment whose endpoints are $P^1_{N_S}$ and $P^2_{N_S}$ can be expressed as the following convex set:

$$N_S = \{(x,y)\in\mathbb{R}^2 : (x,y) = \mu P^1_{N_S} + (1-\mu)P^2_{N_S}, 0 \le \mu \le 1\}.$$

Since we are optimizing a linear function in a compact set we can conclude that the objective function in (L-Problem) achieves its minimum and its maximum in the extreme points of N_S, that is, in $P^1_{N_S}$ and $P^2_{N_S}$.