Los nombres deberían ser H-SPPN-S   H-TSPN-S
para hacer referencia a neighboors.

1. Description of the Problem *second order cone representable*

In this section, it is described the two problems that are considered in this paper: the Hampered Shortest Path Problem with Segments H-SPP-S and the Hampered Traveling Salesman Problem with Segments H-TSP-S .

In H-SPP-S , we have a source neighborhood $N_S \subset \mathbb{R}^2$ and a target neighborhood $N_T \subset \mathbb{R}^2$, that we assume to be convex sets and a set $\mathcal{B}$ of line segments that plays the role of barriers that the drone cannot cross. In this model we state the following assumptions:

A1 The line segments of $\mathcal{B}$ are located in general position, i.e., the endpoints of these segments are not aligned. Although it is possible to model the most general case, it is beyond of the scope of this paper.

A2 The line segments of $\mathcal{B}$ are opened, that is, it is possible that the drone visits the endpoints of the segments, but it cannot be located in the interior points of them. Otherwise, we can take a small quantity to enlarge these segments to make them opened.

A3 If there are two barriers that have a common portion of them, it is only considered the smallest line segment that contains both barriers.

A4 There is not a rectilinear path to go from N_S to N_T. Otherwise, the problem becomes straightforward and the solution is the minimum distance between both neighborhoods.

The aim of the H-SPP-S is to find the best pair of points $(P_S, P_T) \in N_S \times N_T$ in the source and target neighborhoods that minimize the length of the path that joins both points without crossing any barrier of $\mathcal{B}$ by assuming **A1-A4**.

The H-TSP-S is an extension of the H-SPP-S where a neighborhood set $\mathcal{N}$ is considered that plays the role of source and target neighborhoods in the H-SPP-S . The aim of the H-TSP-S is to seek the shortest tour that visits each neighborhood $N \in \mathcal{N}$ exactly once without crossing any barrier $B \in \mathcal{B}$ by assuming again **A1-A4**.

The Figure 1 shows an example of the problem that are being solved. In the left picture, the blue neighborhood represents the source, the green one, representing the H-SPP-S , the target and the red line segments show the barriers that the drone can not cross. In the right picture, an instance of the H-TSP-S is shown, where the neighborhood are balls and the barriers are, again, the red line segments.
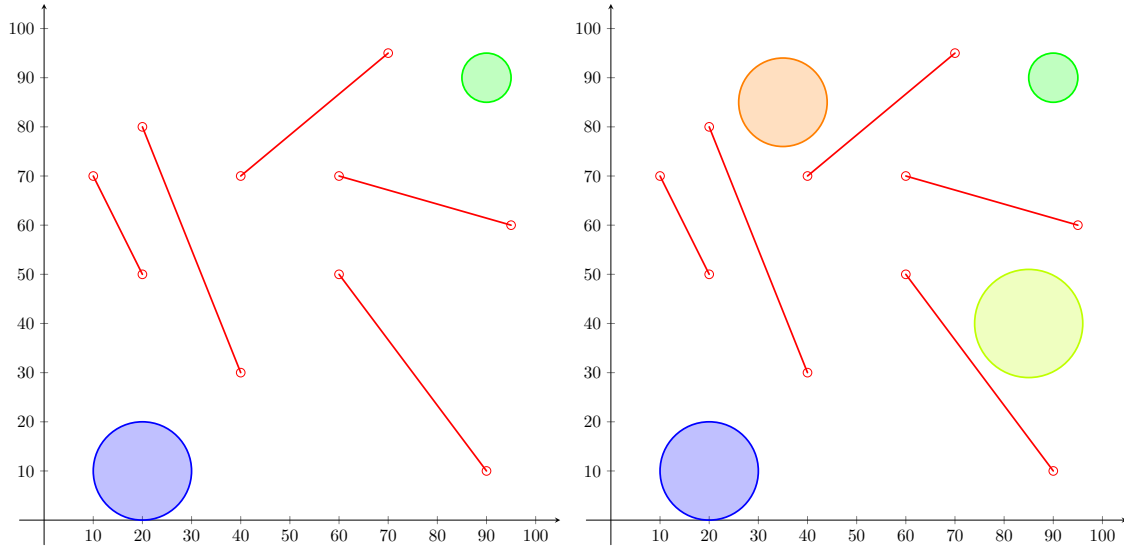

Figure 1: Problem data of the H-SPP-S and H-TSP-S

2. MINLP Formulations

In this section it is introduced a Mixed Integer Non-Linear Programming formulation for the problems described in Section **??**. Firstly, we set the constraints that check if two segments have crossing points or not. Then, we give the formulation for the H-SPP-S is described and then the model H-TSP-S is presented as an extension of the previous problem.

2.1. Checking if two segments intersect

Let $\overline{PP'}$ and $\overline{QQ'}$ be two line segments that we want to check if they intersect. The following well-known computational geometry result can be used to analysis their relative position:

Remark 1. *Let $\overline{PP'}$ and $\overline{QQ'}$ be two different line segments. Let also denote $\det(P|\overline{QQ'}) = \det\left(\ \overrightarrow{PQ} \ \middle| \ \overrightarrow{PQ'} \ \right)$ the determinant whose arguments are P, Q and Q'. If*

$$\mathrm{sign}\left(\det(P|\overline{QQ'})\right) = \mathrm{sign}\left(\det(P'|\overline{QQ'})\right) \quad \text{or} \quad \mathrm{sign}\left(\det(Q|\overline{PP'})\right) = \mathrm{sign}\left(\det(Q'|\overline{PP'})\right),$$

then $\overline{PP'}$ and $\overline{QQ'}$ do not intersect.

On the following, we are going to model this condition by introducing the necessary binary variables that check the sign of the determinants, the equality of signs and the disjunctive condition.

To model the sign of a determinant defined by the points P, Q and Q', we introduce the binary variable $\alpha(P|\overline{QQ'})$, that is one if $\det(P|\overline{QQ'})$ is positive and zero, otherwise. Note that the case when the determinant is null is not considered , because segments are located in general position.

It is possible to represent the sign condition by including the following constraints:

$$\left[1 - \alpha(P|\overline{QQ'})\right] L(P|\overline{QQ'}) \leq \det(P|\overline{QQ'}) \leq U(P|\overline{QQ'})\,\alpha(P|\overline{QQ'}), \tag{α-C}$$

where $L(P|\overline{QQ'})$ and $U(P|\overline{QQ'})$ are a lower and a upper bound for the determinant, respectively. If the determinant is positive, then $\alpha(P|\overline{QQ'})$ must be one to make the second inequality feasible. The analogous case happens if the determinant is not positive.

Now, to check if two determinants $\det(P|\overline{QQ'})$ and $\det(P'|\overline{QQ'})$ have the same sign, it is required to introduce the binary variable $\beta(\overline{PP'}|\overline{QQ'})$, that is one if $\det(P|\overline{QQ'})$ and $\det(P'|\overline{QQ'})$ have the same sign, zero otherwise.

Hence, the β variable can be represented by the equivalence constraint of the α variables

$$\beta(\overline{PP'}|\overline{QQ'}) = \alpha(P|\overline{QQ'})\alpha(P'|\overline{QQ'}) + \left[1 - \alpha(P|\overline{QQ'})\right]\left[1 - \alpha(P'|\overline{QQ'})\right], \tag{β-C}$$
$$\beta(\overline{PP'}|\overline{QQ'}) = 2\gamma(\overline{PP'}|\overline{QQ'}) - \alpha(P|\overline{QQ'}) - \alpha(P'|\overline{QQ'}) + 1,$$

where $\gamma(\overline{PP'}|\overline{QQ'})$ is the auxiliary binary variable that models the product of the α variables that can be linearized by using the following constraints:

$$\gamma(\overline{PP'}|\overline{QQ'}) \leq \alpha(P|\overline{QQ'}), \tag{γ-C}$$
$$\gamma(\overline{PP'}|\overline{QQ'}) \leq \alpha(P'|\overline{QQ'}),$$
$$\gamma(\overline{PP'}|\overline{QQ'}) \geq \alpha(P|\overline{QQ'}) + \alpha(P'|\overline{QQ'}) - 1.$$

Finally, we need to check if there exists any coincidence in the sign of the determinants, so we define the binary variable $\delta(\overline{PP'}|\overline{QQ'})$ that is one if $\overline{PP'}$ and $\overline{QQ'}$ do not intersect and zero, otherwise. This condition can be modelled by using these disjunctive constraints:

$$\frac{1}{2}\left[\beta(\overline{PP'}|\overline{QQ'}) + \beta(\overline{QQ'}|\overline{PP'})\right] \leq \delta(\overline{PP'}|\overline{QQ'}) \leq 2\left[\beta(\overline{PP'}|\overline{QQ'}) + \beta(\overline{QQ'}|\overline{PP'})\right]. \tag{δ-C}$$

If there exists a sign coincidence, then $\delta(\overline{PP'}|\overline{QQ'})$ is one to satisfy the first constraint, and the second one is always fullfilled. However, if the sign of the determinants is not the same, then the second constraint is active and $\delta(\overline{PP'}|\overline{QQ'})$ is null.

2.2. Conic constraints in the models

In both problems, there exist two second-order cone constraints that model the distance between a pair of points P and Q and the representation of the neighborhoods where the points can be selected.

For the former case, we introduce the non-negative continuous variable $d(PQ)$ that represents the distance between P and Q:

$$\|P - Q\| \leq d(PQ), \quad \forall(P,Q) \in E. \tag{d-C}$$

For the latter case, since we are assuming that the neighborhoods are second order cone (SOC) representable, they can be expressed by means of the constraints:

$$P_N \in N \iff \|A_N^i P_N + b_N^i\| \le (c_N^i)^T P_N + d_N^i, \quad i = 1, \dots, nc_N, \tag{N-C}$$

where A_N^i, b_N^i, c_N^i and d_N^i are parameters of the constraint i and nc_N denotes the number of constraints that appear in the block associated to the neighborhood N.

It is remarkable that these inequalities can model the special case of linear constraints (for $A_N^i, b_N^i \equiv 0$), ellipsoids and hyperbolic constraints (see [?] for more information).

Puede ser muy interesante el caso 3D pero quizás en otro trabajo, no?

2.3. A formulation for the H-SPP-S

The main idea of the H-SPP-S is to solve a shortest path problem in the undirected graph induced by the endpoints of the barriers and the neighborhoods. Here, it is necessary to define the following sets:

- $V_S = \{P_S\}$: set composed by the point selected in the source neighborhood N_S.

- $V_{\mathcal{B}} = \{P_B^1, P_B^2 : B = \overline{P_B^1 P_B^2} \in \mathcal{B}\}$: set of vertices that form the barriers of the problem.

- $V_T = \{P_T\}$: set composed by the point selected in the target neighborhood N_T.

- $E_S = \{(P_S, P_B^i) : P_B^i \in V_{\mathcal{B}} \text{ and } \overline{P_S P_B^i} \cap B'' = \emptyset, \forall B'' \in \mathcal{B}, i = 1, 2\}$: set of edges formed by the line segments that join the point selected in the source neighborhood and every endpoint in the barriers and do not cross any barrier in $\mathcal{B}$.

- $E_{\mathcal{B}} = \{(P_B^i, P_{B'}^j) : P_B^i, P_{B'}^j \in V_{\mathcal{B}} \text{ and } \overline{P_B^i P_{B'}^j} \cap B'' = \emptyset, \forall B'' \in \mathcal{B}, i, j = 1, 2\}$: set of edges formed by the line segments that join two vertices of $V_{\mathcal{B}}$ and do not cross any barrier in $\mathcal{B}$.

- $E_T = \{(P_T, P_B^i) : P_B^i \in V_{\mathcal{B}} \text{ and } \overline{P_T P_B^i} \cap B'' = \emptyset, \forall B'' \in \mathcal{B}, i = 1, 2\}$: set of edges formed by the line segments that join the point selected in the target neighborhood and every endpoint in the barriers and do not cross any barrier in $\mathcal{B}$. Definimos tambien la arista que pueda unir a los dos entornos o asumimos que no hay un camino que los una? Depende del caso que consideremos el problema es convexo o no.

At this point, we can define the graph $G = (V, E)$ induced by the barriers and the neighborhoods, where $V = V_S \cup V_{\mathcal{B}} \cup V_T$ and $E = E_S \cup E_{\mathcal{B}} \cup E_T$. It is interesting to note that this graph can be split into two parts: a fixed graph $G_{\mathcal{B}} = (V_{\mathcal{B}}, E_{\mathcal{B}})$ whose edges can be computed by using the Remark 1 and the sets V_S, E_S, V_T and E_T that depend on where the points P_S and P_T are located as shown in Figure. The figures show how the graph G is generated. The blue dashed line segments represent the edges of E_S, the green ones, the edges of E_T and the red dashed lines, the edges of $E_{\mathcal{B}}$. A special case that can be remarked occurs when the neighborhoods are points. In that case, the induced graph is completely fixed and it is only necessary to find which edges are included by keeping in mind that there can not have crossings. This idea is exploited in the Subsection 2.3.1.

Me estoy dando cuenta que nos pueden decir que para qué tenemos en cuenta todo el conjunto si realmente se pueden coger los puntos de la frontera mas cerca a cada uno de los puntos por donde puede salir de las barreras. Es cierto que solo ocurre con el SPP. En el caso del TSP esto no es cierto.

Since E_S and E_T are not fixed, the determinants in Remark 1 also depend on the location of P_S and P_T. Hence, it is essential to model the previous constraint by using binary variables. We only focus on the case of E_S but the same rationale is used for E_T.

Let $B \in \mathcal{B}$ be a barrier and P_B^i an endpoint of B. Hence, the edge (P_S, P_B^i) belongs to E_S if
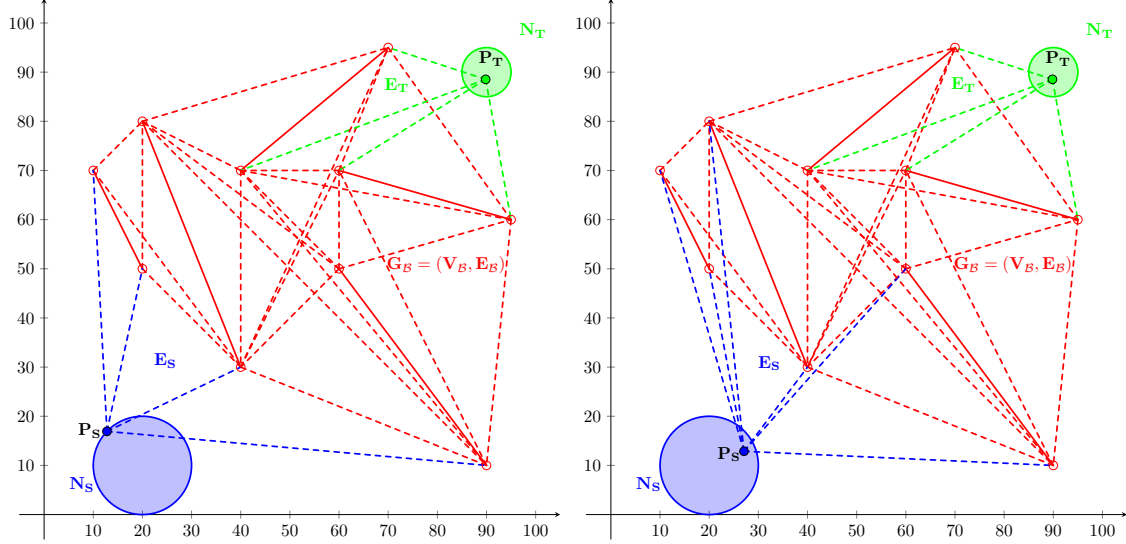
$$\overline{P_S P_B^i} \cap B'' = \emptyset, \quad \forall B'' \in \mathcal{B},$$

or, by the preceding subsection if

$$\delta(\overline{P_S P_B^i} | \overline{P_{B''}^1 P_{B''}^2}) = 1, \quad \forall B'' \in \mathcal{B}.$$

Hence, if we denote by $\varepsilon(P_S P_B^i)$ the binary variable that is one when $(P_S, P_B^i) \in E_S$, this variable can be represented by means of the following inequalities:

$$\left[\sum_{B'' \in \mathcal{B}} \delta(\overline{P_S P_B^i} | \overline{P_{B''}^1 P_{B''}^2}) - |\mathcal{B}| \right] + 1 \le \varepsilon(P_S P_B^i) \le \frac{1}{|\mathcal{B}|} \sum_{B'' \in \mathcal{B}} \delta(\overline{P_S P_B^i} | \overline{P_{B''}^1 P_{B''}^2}). \tag{ε-C}$$

If there is a barrier $B'' \in \mathcal{B}$ that intersects the segment $\overline{P_S P_B^i}$, then $\delta(\overline{P_S P_B^i} | \overline{P_{B''}^1 P_{B''}^2})$ is zero and the second inequality enforces $\varepsilon(P_S P_B^i)$ to be zero because the right side is fractional and the first inequality is non-positive. Nonetheless, if there is not any barrier that intersects the segment, then $\varepsilon(P_S P_B^i)$ is equals to one, because the left side of the first inequality is one and the right side of the second inequality too.

Now, we can define the path that the drone can follow by taking into account the edges of the induced graph. Let $y(PQ)$ be the binary variable that is one if the drone goes from P to Q. Then, the inequalities

$$y(P_S P_B^i) \le \varepsilon(P_S P_B^i), \quad \forall P_B^i \in V_\mathcal{B}. \tag{y-C}$$

assure that the drone can go from $P_S \in V_S$ to a point of a barrier only if it does not cross any barrier.

At this time, we have all the necessary elements to give a MINLP formulation for the H-SPP-S as follows:

$$\text{minimize} \quad \sum_{(P,Q) \in E} d(PQ) y(PQ) \tag{H-SPP-S}$$

$$\text{subject to} \quad \sum_{\{Q:(P,Q) \in E\}} y(PQ) - \sum_{\{Q:(Q,P) \in E\}} y(QP) = \left\{ \begin{array}{ll} 1, & \text{if } P \in V_S, \\ 0, & \text{if } P \in V_\mathcal{B}, \\ -1, & \text{if } P \in V_T. \end{array} \right. ,$$

$$(\alpha\text{-C}), (\beta\text{-C}), (\delta\text{-C}), (\gamma\text{-C}), (\varepsilon\text{-C}), (\text{y-C}),$$
$$(\text{d-C}), (\text{N-C}).$$

The objective function minimizes the length of the path followed by the drone in the edges of the induced graph. The first constraints are the flow conservation constraints, the second constraints represent the sets E_S and E_T and the third ones state that the point selected must be in their respective neighborhoods.

2.3.1. Reformulating the H-SPP-S

The problem exposed before can be reformulated by taking into account the following result:

Proposition 1. *There exists a finite dominant set (called N_S^*) in* H-SPP-S *that represents N_S.*

Proof. Note that the point selected in N_S in the optimal solution is the one that gives the minimum distance to the point of the barrier selected in the optimal solution. $\square$

Therefore, we can compute the set N_S^* by solving the convex problem for each endpoint of the barrier:

$$N_S^* = \{ P_S(P_B^i) = \underset{P \in N_S}{\arg\min} \| P_B^i - P \| : P_B^i \in V_\mathcal{B} \}.$$
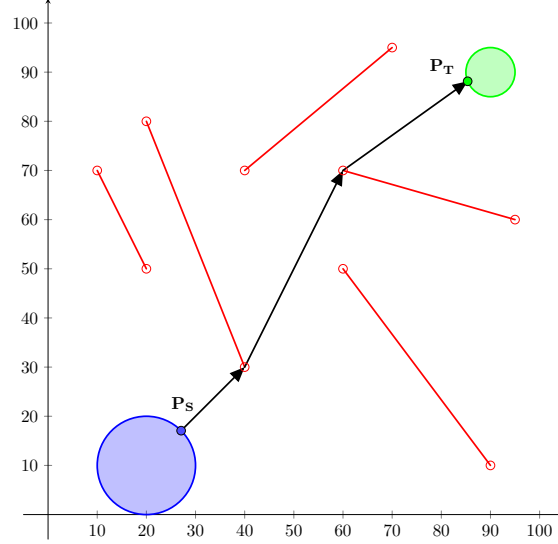
Figure 2: Solution for the instance of the H-SPP-S

Hence, the point selected P_S can be represented by these points as shown

$$P_S = \sum_{P_B^i \in V_{\mathcal{B}}} \mu_S(P_B^i) P_S(P_B^i), \qquad \text{(N*-C)}$$

$$1 = \sum_{P_B^i \in V_{\mathcal{B}}} \mu_S(P_B^i),$$

where $\mu_S(P_B^i)$ is a binary variable that attains one if $P_S(P_B^i)$ is selected to exit from N_S. The biggest advantage of this approach is that the whole graph G is fixed and the incident edges can be computed for each point $P_S(P_B^i)$, $P_B^i \in V_{\mathcal{B}}$ separately. Defining again the variable y^* for the edges in E, the new formulation for the H-SPP-S can be expressed as the following program:

$$\text{minimize} \quad \sum_{(P,Q) \in E} d(PQ) y(PQ) \qquad \text{(H-SPP-S*)}$$

$$\text{subject to} \quad \sum_{\{Q:(P,Q) \in E\}} y(PQ) - \sum_{\{Q:(Q,P) \in E\}} y(QP) = \begin{cases} 1, & \text{if } P \in V_S, \\ 0, & \text{if } P \in V_{\mathcal{B}}, \\ -1, & \text{if } P \in V_T. \end{cases}$$

$$\text{(d-C)}, \text{(N*-C)}.$$

2.4. A formulation for the H-TSP-S

In the H-TSP-S , the graph induced by the endpoints of the barriers and the neighborhoods is composed by the coming sets:

- $V_{\mathcal{N}} = \{P_N : N \in \mathcal{N}\}$: set of the points selected in the neighborhoods in $\mathcal{N}$.

- $V_{\mathcal{B}} = \{P_B^1, P_B^2 : B = \overline{P_B^1 P_B^2} \in \mathcal{B}\}$: set of vertices that form the barriers of the problem.

- $E_{\mathcal{N}} = \{(P_N, P_B^i) : P_B^i \in V_{\mathcal{B}} \text{ and } \overline{P_N P_B^i} \cap B'' = \emptyset, \forall B'' \in \mathcal{B}, i = 1, 2\}$: set of edges formed by the line segments that join the point selected in the neighborhood N and every endpoint in the barriers and do not cross any barrier in $\mathcal{B}$.

- $E_{\mathcal{B}} = \{(P_B^i, P_{B'}^j) : P_B^i, P_{B'}^j \in V_{\mathcal{B}} \text{ and } \overline{P_B^i P_{B'}^j} \cap B'' = \emptyset, \forall B'' \in \mathcal{B}, i, j = 1, 2\}$: set of edges formed by the line segments that join two vertices of $V_{\mathcal{B}}$ and do not cross any barrier in $\mathcal{B}$.

Following the same idea than before, we set $G = (V, E)$ induced by the barriers and the neighborhoords, where $V = V_{\mathcal{N}} \cup V_{\mathcal{B}}$ and $E = E_{\mathcal{N}} \cup E_{\mathcal{B}}$.

The idea of the formulation of the H-TSP-S is to considerate the variant called Steiner TSP (STSP) (references to this problem), where nodes in $V_{\mathcal{B}}$ does not have to be visited, but they can be visited more than once if desired, and edges may be traversed more that once if desired.

5

Note that, it is possible to convert any instance of the STSP into an instance of the standard TSP, by computing shortest paths between every pair of required nodes, when these nodes are fixed. In our problem, since the points in the neighborhoods are not fixed, this idea can not be applied to obtain the optimal solution for our problem, but it can produce a good approach to generate a good lower bound for the H-TSP-S .

We can assume wlog that the neighborhood N_1 is required and the drone departs the depot (that is assumed to be N_1) with $|\mathcal{N}|-1$ units of commodity that must be delivered to each required neighborhood. Then, for each edge $(P,Q) \in E$, we define the following variables:

- $y(PQ)$, binary variable that is equals to one if the drone goes from P to Q.

- $g(PQ)$, non-negative continuous variable that represents the amount of the commodity passing through the edge (P,Q).

Hence, we can the single-commodity flow formulation to the induced graph as shown:

$$\text{minimize} \quad \sum_{(P,Q)\in E} d(PQ)y(PQ) \hspace{4cm} \text{(H-TSP-S)}$$

$$\text{subject to} \quad \sum_{\{Q:(P_N,Q)\in E_\mathcal{N}\}} y(P_N Q) \geq 1, \hspace{2cm} \forall P_N \in V_\mathcal{N},$$

$$\sum_{\{Q:(P,Q)\in E\}} y(PQ) = \sum_{\{Q:(Q,P)\in E\}} y(QP), \quad \forall P \in V,$$

$$\sum_{\{Q:(Q,P_N)\in E_\mathcal{N}\}} g(QP_N) - \sum_{\{Q:(P_N,Q)\in E_\mathcal{N}\}} g(P_N Q) = 1, \hspace{1cm} \forall P_N \in V_\mathcal{N} \setminus \{P_{N_1}\},$$

$$\sum_{\{Q:(Q,P)\in E\}} g(QP) - \sum_{\{Q:(P,Q)\in E\}} g(PQ) = 0, \hspace{1cm} \forall P \in V_\mathcal{B},$$

$$g(PQ) \leq (|\mathcal{N}|-1)y(PQ), \hspace{1cm} \forall (P,Q) \in E,$$

$$(\alpha\text{-C}), (\beta\text{-C}), (\delta\text{-C}), (\gamma\text{-C}), (\varepsilon\text{-C}), (\text{y-C}),$$

$$(\text{d-C}), (\text{N-C}).$$

The first constraints impose that the drone departs from each neighborhood. The second constraints are the flow conservation constraints. The third inequalities ensure that one unit of commodity is delivered to each neighborhood where the the fourth ones conserve the amount of commodity for the points in the barriers. Finally, the last inequalities indicate that if any of the commodity passes along an edge, this edge must appears in the tour.

¿Solucion del problema del TSP propuesto en el dibujo?

3. Strengthening the formulations

Incluir el resultado estructural que da una cota superior del numero de bolas que se puede generar

3.1. Preprocessing

In this subsection, we show a preprocessing result that allows to fix some variables by taking into account the relative position between the neighborhoods and the barriers. In particular, we are going to present an outcome that ensures that there are some barriers whose endpoints can not be incident in the edges of $E_\mathcal{N}$ and it is not necessary to include it in $E_\mathcal{N}$.

Let denote

$$\text{cone}(P,Q,R) := \{\mu_1 \overrightarrow{PQ} + \mu_2 \overrightarrow{PR} : \mu_1, \mu_2 \geq 0\},$$

$$\text{cone}(P,Q,R)^- := \{\mu_1 \overrightarrow{PQ} + \mu_2 \overrightarrow{PR} : \mu_1, \mu_2 \geq 0, \ \mu_1 + \mu_2 \leq 1\},$$

$$\text{cone}(P,Q,R)^+ := \{\mu_1 \overrightarrow{PQ} + \mu_2 \overrightarrow{PR} : \mu_1, \mu_2 \geq 0, \ \mu_1 + \mu_2 \geq 1\}.$$

Note that, $cone(P, Q, R)$ is the union of $\text{cone}(P, Q, R)^-$ and $\text{cone}(P, Q, R)^+$. It is also important to remark that $\text{cone}(P, Q, R)^+$ is the part of cone that crosses the barrier $\overline{QR}$ when we consider a segment whose endpoints are P and another point of this set, i.e.

$$\text{cone}(P, Q, R)^+ = \{P' : \overline{PP'} \cap \overline{QR} \neq \emptyset\}.$$

Let $B = \overline{P_B^1 P_B^2} \in \mathcal{B}$ a barrier. In the following proposition, we give a sufficient condition to not include the edge (P_N, P_B^i) in $E_{\mathcal{N}}$:

Proposition 2. *Let* $B' = \overline{P_{B'}^1 P_{B'}^2} \in \mathcal{B}$ *and* $\text{cone}(P_B^i, P_{B'}^1, P_{B'}^2)^+$ *the conical hull generated by these points. If*

$$N \subset \bigcup_{B' \in \mathcal{B}} \text{cone}(P_B^i, P_{B'}^1, P_{B'}^2)^+,$$

then $(P_N, P_B^i) \notin E_{\mathcal{N}}$.

Proof. If $P_N \in N$, then there exists a $B' \in \mathcal{B}$ such that $P_N \in \text{cone}(P_B^i, P_{B'}^1, P_{B'}^2)^+$. Therefore, $\overline{P_B^i P_N} \cap B' \neq \emptyset$ and $(P_N, P_B^i) \notin E_{\mathcal{N}}$, as we claimed. □

We can check computationally the condition of the previous proposition by using the following procedure. The first issue that is solved is the one when the neighborhoods are segments. Let $N = \overline{P_N^1 P_N^2}$ be a line segment and r_N the straight line that contains the line segment N that is represented as:

$$r_N : P_N^1 + \lambda \overrightarrow{P_N^1 P_N^2}, \qquad \lambda \in \mathbb{R}.$$

Algorithm 1: Checking computationally if $(P_N, P_B^i) \notin E_{\mathcal{N}}$ when N is a segment.

Initialization: Let P_B^i be the point whose edge (P_B^i, P_N) is going to check if $(P_N, P_B^i) \notin E_{\mathcal{N}}$. Set $points = \{P_N^1, P_N^2\}$, $lambdas = \{0, 1\}$.

1 **for** $B'' \in \mathcal{B}$ **do**

2   **for** $j \in \{1, 2\}$ **do**

3   Compute the straight line

$$r(P_B^i, P_{B''}^j) = P_B^i + \mu_{B''}^j \overrightarrow{P_B^i P_{B''}^j},$$

     that contains the points P_B^i and $P_{B''}^j$.

4   Intersect $r(P_B^i, P_{B''}^j)$ and r_N in the point $Q_{B''}^j$ and compute $\overline{\mu}_{B''}^j$ such that

$$Q_{B''}^j = P_B^i + \overline{\mu}_{B''}^j \overrightarrow{P_B^i P_{B''}^j}.$$

5   **if** $|\overline{\mu}_{B''}^j| \geq 1$ **then**

6   Compute $\lambda_{B''}^j$ such that
$$Q_{B''}^j = P_N^1 + \lambda_{B''}^j \overrightarrow{P_N^1 P_N^2}.$$

7   **if** $\overline{\mu}_{B''}^j \geq 1$ **then**

8   Include $\lambda_{B''}^j$ in $lambdas$.

9   **else**

10   **if** $\lambda_{B''}^j \geq 0$ **then**

11   Set $\lambda_{B''}^j = M << 0$ and include it in $lambdas$.

12   **else**

13   Set $\lambda_{B''}^j = M >> 0$ and include it in $lambdas$.

14 Order in non-decreasing order the set $lambdas$.

15 If it is verified that

$$\min\{\lambda_{B'}^1, \lambda_{B'}^2\} \leq 0 \leq \max\{\lambda_{B'}^1, \lambda_{B'}^2\}, \quad \text{for some } B' \in \mathcal{B},$$

$$\min\{\lambda_{B'}^1, \lambda_{B'}^2\} \leq 1 \leq \max\{\lambda_{B'}^1, \lambda_{B'}^2\}, \quad \text{for some } B' \in \mathcal{B},$$

$$\min\{\lambda_{B'}^1, \lambda_{B'}^2\} \leq \lambda_{B''}^j \leq \max\{\lambda_{B'}^1, \lambda_{B'}^2\}, \quad \text{for some } B' \in \mathcal{B} \setminus \{B''\}, \quad \forall \lambda_{B''}^j \in lambdas \setminus \{M\},$$

or

$$\min\{\lambda_{B'}^1, \lambda_{B'}^2\} \leq 0, 1 \leq \max\{\lambda_{B'}^1, \lambda_{B'}^2\}, \quad \text{for some } B' \in \mathcal{B},$$

then $(P_N, P_B^i) \notin E_{\mathcal{N}}$.

Note that this algorithm also allows us to decide if the drone can access to a point of a barrier from any point of the neighborhood N. It is enough to check in (15) that

$$0 \notin \left[\min\{\lambda_{B'}^1, \lambda_{B'}^2\}, \max\{\lambda_{B'}^1, \lambda_{B'}^2\}\right] \quad \text{and} \quad 1 \notin \left[\min\{\lambda_{B'}^1, \lambda_{B'}^2\}, \max\{\lambda_{B'}^1, \lambda_{B'}^2\}\right], \quad \forall B' \in \mathcal{B}.$$

For the case when N is an ellipse, the same rationale can be followed. The idea is to generate the largest line segment contained in the ellipse and repeat the procedure exposed in the Algorithm 1. Let F_1 and F_2 the focal points of N.

Algorithm 2: Checking computationally if $(P_N, P_B^i) \notin E_{\mathcal{N}}$ when N is an ellipse.

Initialization: Let P_B^i be the point whose edge (P_B^i, P_N) is going to check if $(P_N, P_B^i) \notin E_{\mathcal{N}}$. Set $points = \{\}$, $lambdas = \{\}$.

1 Compute the straight line $r(F^1, F^2)$.

2 Intersect $r(F^1, F^2)$ and ∂N in the points P_N^1 and P_N^2.

3 Include P_N^1 and P_N^2 in $points$.

4 Apply the Algorithm 1.

Estudiar resultados que eliminen algunas de las posibles aristas de E_S y E_T Hablar que las aristas $E_{\mathcal{B}}$ se pueden preprocesar porque los puntos están fijados

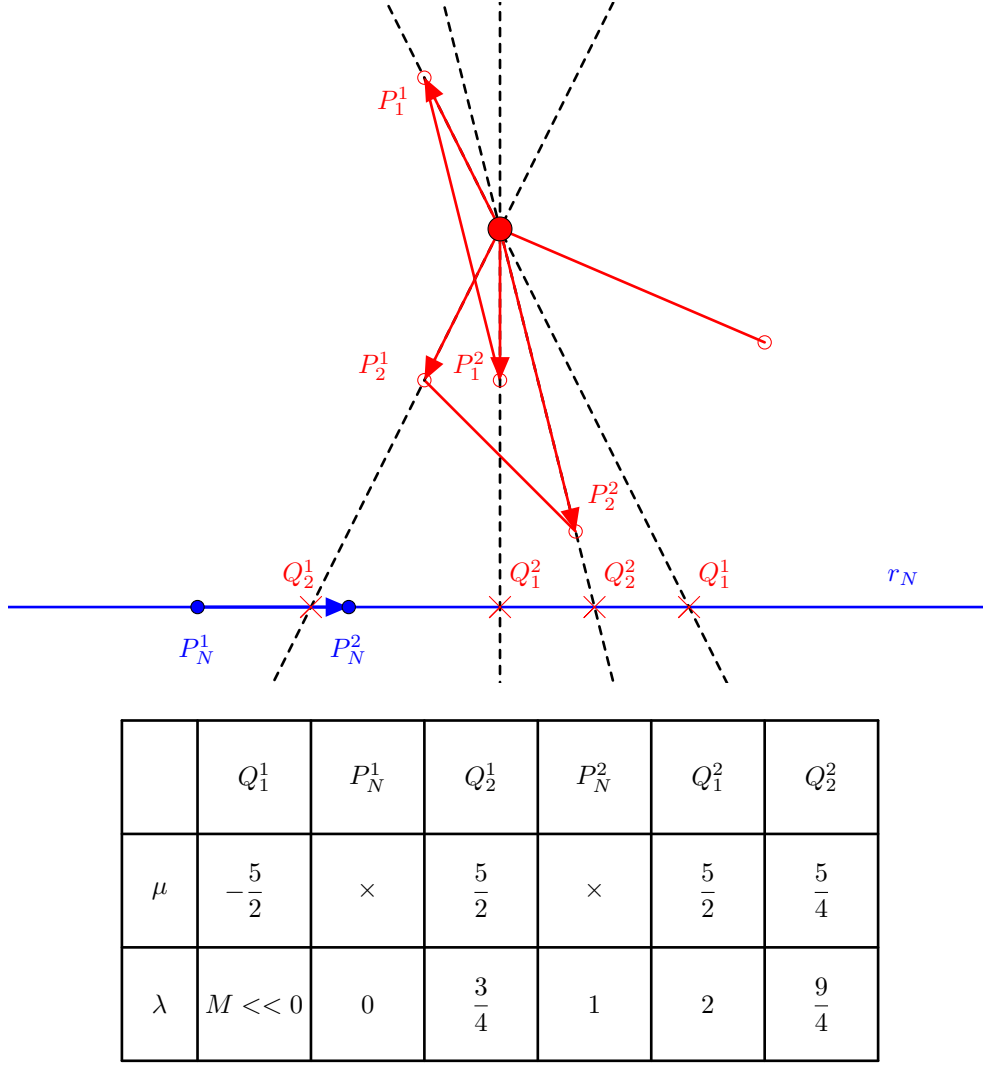|	Q_1^1	P_N^1	Q_2^1	P_N^2	Q_1^2	Q_2^2
μ	$-\dfrac{5}{2}$	$\times$	$\dfrac{5}{2}$	$\times$	$\dfrac{5}{2}$	$\dfrac{5}{4}$
λ	$M << 0$	0	$\dfrac{3}{4}$	1	2	$\dfrac{9}{4}$

Figure 3: Example of the Algorithm 1

3.2. Valid inequalities

This subsection is devoted to show some results that adjust the big-M constants that appear in the previous formulation, specifically, in the (α-C), where the modelling of the sign requires to compute the lower and upper bounds L and U, respectively. We are going to expose that these bounds can be computed explicitly for the cases when the neighborhoods are ellipses and segments.

Let $\overline{P_{B'}^1 P_{B'}^2} = B' \in \mathcal{B}$ be a barrier and $P_N \in N$. Let $\det(P_N | \overline{P_{B'}^1 P_{B'}^2})$ also be the determinant whose values must be bounded. Hence, the solution of the following problem gives the lower bound of the determinant:

$$\overline{L} = \min_{P_N = (x,y) \in N} F(x,y) := \det(P_N | \overline{P_{B'}^1 P_{B'}^2}) = \begin{vmatrix} P_{B'_x}^1 - x & P_{B'_x}^2 - x \\ P_{B'_y}^1 - y & P_{B'_y}^2 - y \end{vmatrix}. \qquad \text{(L-Problem)}$$

3.2.1. Lower and upper bounds when the neighborhoods are ellipsoids

The first case that is considered is the one when N is an ellipse, that is, N is represented by the following inequality:

$$N = \{(x,y) \in \mathbb{R}^2 : ax^2 + by^2 + cxy + dx + ey + f \leq 0\} =$$

where a, b, c, d, e, f are coefficients of the ellipse. In an extended form, we need to find:

$$\text{minimize} \quad F(x,y) = \begin{vmatrix} P^1_{B'_x} - x & P^2_{B'_x} - x \\ P^1_{B'_y} - y & P^2_{B'_y} - y \end{vmatrix} = xP^1_{B'_y} - xP^2_{B'_y} + yP^2_{B'_x} - yP^1_{B'_x} + P^1_{B'_x}P^2_{B'_y} - P^1_{B'_y}P^2_{B'_x},$$

<div align="right">(L-Ellipse)</div>

$$\text{subject to} \quad ax^2 + by^2 + cxy + dx + ey + f \leq 0.$$

Since we are minimizing a linear function in a convex set, we can conclude that the extreme points are located in the frontier, so we can use the Lagrangian function to compute these points.

$$F(x,y;\lambda) = xP^1_{B'_y} - xP^2_{B'_y} + yP^2_{B'_x} - yP^1_{B'_x} + P^1_{B'_x}P^2_{B'_y} - P^1_{B'_y}P^2_{B'_x} + \lambda(ax^2 + by^2 + cxy + dx + ey + f).$$

$$\nabla F(x,y;\lambda) = 0 \Longleftrightarrow \begin{cases} \frac{\partial F}{\partial x} &= P^1_{B'_y} - P^2_{B'_y} + 2ax\lambda + cy\lambda + d\lambda &= 0, \\ \frac{\partial F}{\partial y} &= P^2_{B'_x} - P^1_{B'_x} + 2by\lambda + cx\lambda + e\lambda &= 0, \\ \frac{\partial F}{\partial \lambda} &= ax^2 + by^2 + cxy + dx + ey + f &= 0. \end{cases}$$

From the first two equations we can obtain:

$$\lambda = \frac{P^2_{B'_y} - P^1_{B'_y}}{2ax + cy + d} = \frac{P^1_{B'_x} - P^2_{B'_x}}{2by + cx + e}.$$

From this equality, we can express y as a function of x:

$$(P^2_{B'_y} - P^1_{B'_y})(2by + cx + e) = (P^1_{B'_x} - P^2_{B'_x})(2ax + cy + d),$$

$$y\left[2b(P^2_{B'_y} - P^1_{B'_y}) - c(P^1_{B'_x} - P^2_{B'_x})\right] = \left[2a(P^1_{B'_x} - P^2_{B'_x}) - c(P^2_{B'_y} - P^1_{B'_y})\right]x + \left[d(P^1_{B'_x} - P^2_{B'_x}) - e(P^2_{B'_y} - P^1_{B'_y})\right],$$

$$y = \left[\frac{2a(P^1_{B'_x} - P^2_{B'_x}) - c(P^2_{B'_y} - P^1_{B'_y})}{2b(P^2_{B'_y} - P^1_{B'_y}) - c(P^1_{B'_x} - P^2_{B'_x})}\right]x + \left[\frac{d(P^1_{B'_x} - P^2_{B'_x}) - e(P^2_{B'_y} - P^1_{B'_y})}{2b(P^2_{B'_y} - P^1_{B'_y}) - c(P^1_{B'_x} - P^2_{B'_x})}\right],$$

$$y = mx + n.$$

Finally, we can substitute y in the third equation to compute the value of x:

$$ax^2 + by^2 + cxy + dx + ey + f = ax^2 + b(mx + n)^2 + cx(mx + n) + dx + e(mx + n) + f =$$
$$= (a + bm^2 + cm)x^2 + (2bmn + cn + d + em)x + (n^2b + en + f) = 0.$$

By using the standard form of the solution of a second grade equation:

$$x^{\pm} = \frac{-(2bmn + cn + d + em) \pm \sqrt{(2bmn + cn + d + em)^2 - 4(a + bm^2 + cm)(n^2b + en + f)}}{2(a + bm^2 + cm)},$$

$$y^{\pm} = mx^{\pm} + n.$$

Hence, to compute the lower and upper bound, we only need to evaluate $x^{\pm}$ and $y^{\pm}$ in the objective function and the lowest and highest value correspond to $L(P_N|\overline{P^1_{B'}, P^2_{B'}})$ and $U(P_N|\overline{P^1_{B'}, P^2_{B'}})$, respectively.

3.2.2. Lower and upper bounds when the neighborhoods are line segments

In this case, the segment whose endpoints are $P^1_{N_S}$ and $P^2_{N_S}$ can be expressed as the following convex set:

$$N = \{(x,y) \in \mathbb{R}^2 : (x,y) = \mu P^1_N + (1-\mu)P^2_N, 0 \leq \mu \leq 1\}.$$

Since we are optimizing a linear function in a compact set we can conclude that the objective function in (L-Problem) achieves its minimum and its maximum in the extreme points of N_S, that is, in P^1_N and P^2_N.