# The Hampered Traveling Salesman Problem with Neighbourhoods

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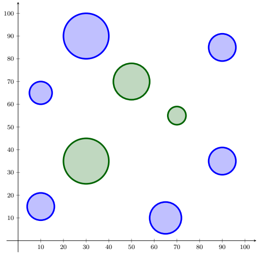
# Problem Motivation

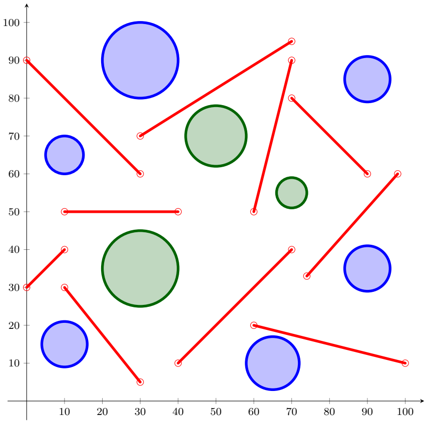
The starting point of this work is the $k$-Median Problem with Neighbourhoods (KMPN):




- $\mathcal{S}$: Set of neighbourhoods that describe the possible sources where a facility can be allocated. It is assumed, wlog, that one facility can be allocated to each source at most once.
- $\mathcal{T}$: Set of neighbourhoods that represent the targets that a facility must serve. It is assumed, wlog, that each target is served when it has been assigned to a facility.

# Problem Motivation

Barriers that can not be crossed are taken into account in the KMPN:



# Problem Assumptions

We denote by $\mathcal{B}$ the set of line segments that model the barriers, and we adopt the assumptions that are listed below.

**A1** The line segments of $\mathcal{B}$ are located in general position, i.e., the endpoints of these segments are not aligned. Although it is possible to model the other case, one can always slightly modify one of the endpoints so that the segments are in general position.

**A2** The line segments of $\mathcal{B}$ are open sets, that is, it is possible that the drone visits endpoints of segments, but entering in its interior is not allowed. Observe that without loss of generality, we can always slightly enlarge these segments to make them open.

**A3** If there are two overlapping barriers, we assume that there is only one barrier given by the union of them.

This problem is called the **Hampered $k$-Median Problem with Neighbourhoods** (H-KMPN).

The goal of the H-KMPN is to find a subset of $k$ points, denoted by $P_S$, in the source set $\mathcal{S}$, at most one in each neighbourhood, and one point in each target set $\mathcal{T}$, denoted by $P_T$, that minimise the weighted length of the path joining each target point with its associated source point and the weighted link distance without crossing any barrier of $\mathcal{B}$ assuming A1-A3.

# Problem Assumptions

We denote by $\mathcal{B}$ the set of line segments that model the barriers, and we adopt the assumptions that are listed below.

In addition, a special for the H-KMPN is studied by assuming, in addition, that:

**A4** There is no rectilinear path joining two neighbourhoods without crossing an obstacle.

This problem is called the **Hampered $k$-Median Problem with Hidden Neighbourhoods** (H-KMPHN).

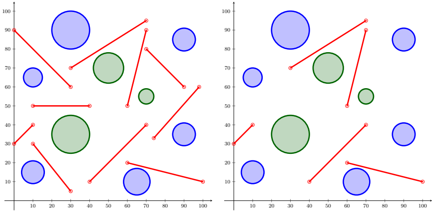



Figure: Problem data of the KMPHN and KMPN

# Notation

- $P$ and $Q$ are referred to as generic points that, at the same time, are identified with their coordinates $P = (P_x, P_y)$ and $Q = (Q_x, Q_y)$, respectively.
- Given two points $P^1$ and $P^2$, the line segment joining $P^1$ and $P^2$ is denoted by $\overline{P^1P^2}$. It can be parameterized as follows:

$$\overline{P^1P^2} = \{P \in \mathbb{R}^2 : P = \lambda P^1 + (1-\lambda)P^2,\ \lambda \in [0,1]\}.$$

- Given two points $P^1$ and $P^2$, the edge whose vertices are $P^1$ and $P^2$ is denoted by $(P^1, P^2)$.
- Given two points $P^1$ and $P^2$, the vector pointing from $P^1$ to $P^2$ is denoted by $\overrightarrow{P^1P^2}$. It is computed as $\overrightarrow{P^1P^2} = P^2 - P^1$.
- Given three points $P^1$, $P^2$ and $P^3$, $\det(P^1|P^2P^3)$ denotes the following determinant:

$$\det(P^1|P^2P^3) = \det\left( \overrightarrow{P^1P^2} \;\middle|\; \overrightarrow{P^1P^3} \right) := \det\begin{pmatrix} P_x^2 - P_x^1 & P_x^3 - P_x^1 \\ P_y^2 - P_y^1 & P_y^3 - P_y^1 \end{pmatrix}.$$

  The sign of $\det(P^1|P^2P^3)$ gives the orientation of the point $P^1$ with respect to the line segment $\overline{P^2P^3}$. Note that $\det(P^1|P^2P^3) \neq 0$ by **A1**.

# Problem Description H-KMPHN

- $V_{\mathcal{S}} = \{P_S : S \in \mathcal{S}\}$. Set of points selected in the sources of $\mathcal{S}$.
- $V_{\mathcal{B}} = \{P_B^1, P_B^2 : B = \overline{P_B^1 P_B^2} \in \mathcal{B}\}$. Set of vertices that come from the endpoints of barriers in the problem.
- $V_{\mathcal{T}} = \{P_T : T \in \mathcal{T}\}$. Set of the points selected in the targets of $\mathcal{T}$.
- $E_{\mathcal{S}} = \{(P_S, P_B^i) : P_S \in V_{\mathcal{S}}, P_B^i \in V_{\mathcal{B}}$ and $\overline{P_S P_B^i} \cap B'' = \emptyset, \forall B'' \in \mathcal{B},\ i = 1, 2\}$. Set of edges formed by the line segments that join the point selected in any source neighbourhood $S \in \mathcal{S}$ and every endpoint in the barriers that do not cross any other barrier in $\mathcal{B}$.
- $E_{\mathcal{B}} = \{(P_B^i, P_{B'}^j) : P_B^i, P_{B'}^j \in V_{\mathcal{B}}$ and $\overline{P_B^i P_{B'}^j} \cap B'' = \emptyset,\ \forall B'' \in \mathcal{B},\ i, j = 1, 2\}$. Set of edges formed by the line segments that join two vertices of $V_{\mathcal{B}}$ and do not cross any other barrier in $\mathcal{B}$. $E_{\mathcal{B}}^{int}$ denotes the edges represented by the linear barriers.
- $E_{\mathcal{T}} = \{(P_B^i, P_T) : P_B^i \in V_{\mathcal{B}}, P_T \in V_{\mathcal{T}}$ and $\overline{P_B^i P_T} \cap B'' = \emptyset, \forall B'' \in \mathcal{B}, i = 1, 2\}$. Set of edges formed by the line segments that join the selected point in any target neighbourhood $T \in \mathcal{T}$ and every endpoint on the barriers that do not cross any other barrier in $\mathcal{B}$.

We define the graph $G_{\text{KMPHN}} = (V_{\text{KMPHN}} = V_{\mathcal{N}} \cup V_{\mathcal{B}}, E_{\text{KMPHN}} = E_{\mathcal{S}} \cup E_{\mathcal{B}} \cup E_{\mathcal{T}})$ induced by barriers and neighbourhoods.

# Problem Description H-KMPN

By taking the same approach, the graph induced for the relaxed version H-KMPN can be described as $G_{\text{KMPN}} = (V_{\text{KMPN}}, E_{\text{KMPN}})$, $V_{\text{KMPN}} = V_{\text{KMPHN}}$ and $E_{\text{KMPN}} = E_{\text{KMPHN}} \cup E_{\mathcal{ST}}$, where:

- $E_{\mathcal{ST}} = \{(P_S, P_T) : P_S \in V_{\mathcal{S}}, P_T \in V_{\mathcal{T}}$ and $\overline{P_S P_T} \cap B'' = \emptyset, \forall B'' \in \mathcal{B},\ i = 1, 2\}$. Set of edges formed by the line segments that join the point selected in any source neighbourhood $S \in \mathcal{S}$ and the point selected in any target neighbourhood $T \in \mathcal{T}$ that do not cross any other barrier in $\mathcal{B}$.

# Variables

Table: Summary of decision variables used in the mathematical programming model

| **Binary Decision Variables** | |
|---|---|
| **Name** | **Description** |
| $\alpha(P\|QQ')$ | 1, if the determinant $\det(P\|QQ')$ is positive,<br>0, otherwise. |
| $\beta(PP'\|QQ')$ | 1, if the determinants $\det(P\|QQ')$ and $\det(P'\|QQ')$ have the same sign,<br>0, otherwise. |
| $\gamma(PP'\|QQ')$ | 1, if the determinants $\det(P\|QQ')$ and $\det(P'\|QQ')$ are both positive,<br>0, otherwise. |
| $\delta(PP'\|QQ')$ | 1, if the line segments $\overline{PP'}$ and $\overline{QQ'}$ do not intersect,<br>0, otherwise. |
| $\varepsilon(PP')$ | 1, if the line segment $\overline{PP'}$ does not cross any barrier,<br>0, otherwise. |
| $f(PQ\|ST)$ | 1, if edge $(P, Q)$ is traversed in the path joining $S$ and $T$,<br>0, otherwise. |
| $y(S)$ | 1, if a facility is allocated in the source $S$ in the solution of the model,<br>0, otherwise. |
| $x(ST)$ | 1, if source $S$ and target $T$ are joined by a path in the solution of the model,<br>0, otherwise. |
| **Continuous Decision Variables** | |
| **Name** | **Description** |
| $P_N$ | Coordinates representing the point selected in the neighbourhood $N \in \mathcal{S} \cup \mathcal{T}$. |
| $d(PQ)$ | Euclidean distance between the points $P$ and $Q$. |

# Conic Constraints

### $d$-Constraints

We introduce the non-negative continuous variable $d(PQ)$ that represents the distance between $P$ and $Q$:

$$\|P - Q\| \leq d(PQ), \quad \forall (P, Q) \in E_X,$$

where $E_X$ is the set of edges $E_{\text{KMPHN}}$ or $E_{\text{KMPN}}$.

### $N$-Constraints

We are assuming that the neighbourhoods are second-order cone (SOC) representable, they can be expressed by means of the constraints:

$$P_N \in N \iff \|A_N^i P_N + b_N^i\| \leq (c_N^i)^T P_N + d_N^i, \quad i = 1, \ldots, nc_N,$$

where $A_N^i, b_N^i, c_N^i$ and $d_N^i$ are parameters of the constraint $i$ and $nc_N$ denotes the number of constraints that appear in the block associated with the neighbourhood $N$.

# Checking whether a segment is an edge of the induced graph

**Remark**

*Let* $\overline{PQ}$ *and* $B = \overline{P_B^1 P_B^2} \in \mathcal{B}$ *be two different line segments. If*

$$\text{sign}\left(\det(P|P_B^1 P_B^2)\right) = \text{sign}\left(\det(Q|P_B^1 P_B^2)\right)$$

*or*

$$\text{sign}\left(\det(P_B^1|PQ)\right) = \text{sign}\left(\det(P_B^2|PQ)\right),$$

*then* $\overline{PQ}$ *and* $B$ *do not intersect.*

# Sign of the determinants

Let $P, Q \in V_X$, where $V_X$ denotes the set of vertices $V_{\text{KMPHN}}$ or $V_{\text{KMPN}}$. Let $B = \overline{P_B^1 P_B^2} \in \mathcal{B}$.

## $\alpha$-Constraints

We introduce the binary variable $\alpha$, that assumes the value one if the determinant is positive and zero, otherwise.

$$\begin{aligned}
\left[1-\alpha(P|P_B^1P_B^2)\right] L(P|P_B^1P_B^2) \leq \det(P|P_B^1P_B^2) \leq U(P|P_B^1P_B^2)\, \alpha(P|P_B^1P_B^2),\\
\left[1-\alpha(Q|P_B^1P_B^2)\right] L(Q|P_B^1P_B^2) \leq \det(Q|P_B^1P_B^2) \leq U(Q|P_B^1P_B^2)\, \alpha(Q|P_B^1P_B^2),\\
\left[1-\alpha(P_B^1|PQ)\right] L(P_B^1|PQ) \leq \det(P_B^1|PQ) \leq U(P_B^1|PQ)\, \alpha(P_B^1|PQ),\\
\left[1-\alpha(P_B^2|PQ)\right] L(P_B^2|PQ) \leq \det(P_B^2|PQ) \leq U(P_B^2|PQ)\, \alpha(P_B^2|PQ),
\end{aligned}$$

where $L$ and $U$ are lower and upper bounds for the value of corresponding determinants, respectively.

# Coincidence of sign in one pair of determinants

Let $P, Q \in V_X$, where $V_X$ denotes the set of vertices $V_{\text{KMPHN}}$ or $V_{\text{KMPN}}$. Let $B = \overline{P_B^1 P_B^2} \in \mathcal{B}$.

## $\beta$-Constraints

We introduce the binary variable $\beta$, that is one if the corresponding pair has the same sign, and zero otherwise.

$$\beta(PQ|P_B^1P_B^2) = \alpha(P|P_B^1P_B^2)\alpha(Q|P_B^1P_B^2) + \left[1 - \alpha(P|P_B^1P_B^2)\right]\left[1 - \alpha(Q|P_B^1P_B^2)\right],$$
$$\beta(P_B^1P_B^2|PQ) = \alpha(P_B^1|PQ)\alpha(P_B^2|PQ) + \left[1 - \alpha(P_B^1|PQ)\right]\left[1 - \alpha(P_B^2|PQ)\right].$$

This condition can be equivalently written using an auxiliary binary variable $\gamma$ that models the product of the $\alpha$ variables:

$$\beta(PQ|P_B^1P_B^2) = 2\gamma(PQ|P_B^1P_B^2) - \alpha(P|P_B^1P_B^2) - \alpha(Q|P_B^1P_B^2) + 1,$$
$$\beta(P_B^1P_B^2|PQ) = 2\gamma(P_B^1P_B^2|PQ) - \alpha(P_B^1|PQ) - \alpha(P_B^2|PQ) + 1.$$

# Coincidence of sign in any of two pairs of determinants

Let $P, Q \in V_X$, where $V_X$ denotes the set of vertices $V_{\text{KMPHN}}$ or $V_{\text{KMPN}}$. Let $B = \overline{P_B^1 P_B^2} \in \mathcal{B}$.

## $\delta$-Constraints

We need to check whether there exists any coincidence of the sign of determinants, so we define the binary variable $\delta$ that is one if segments do not intersect and zero, otherwise.

$$\frac{1}{2}\left[\beta(PQ|P_B^1P_B^2) + \beta(P_B^1P_B^2|PQ)\right] \leq \delta(PQ|P_B^1P_B^2) \leq \beta(PQ|P_B^1P_B^2)+\beta(P_B^1P_B^2|PQ).$$

# Extending these conditions to set $\mathcal{B}$

Finally, we need to check that

$$\overline{PQ} \cap B'' = \emptyset, \quad \forall B'' \in \mathcal{B}, \quad \Longleftrightarrow \quad \delta(PQ|P^1_{B''}, P^2_{B''}) = 1, \quad \forall B'' \in \mathcal{B}.$$

**$\varepsilon$-Constraints**

We denote by $\varepsilon(PQ)$ the binary variable that is one if the previous condition is satisfied for all $B'' \in \mathcal{B}$ and zero otherwise.

$$\left[\sum_{B'' \in \mathcal{B}} \delta(PQ|P^1_{B''}, P^2_{B''}) - |\mathcal{B}|\right] + 1 \leq \varepsilon(PQ) \leq \frac{1}{|\mathcal{B}|} \sum_{B'' \in \mathcal{B}} \delta(PQ|P^1_{B''}, P^2_{B''}).$$

Based on the above description, we can identify the set of actual edges of graph $G$ by means of the $\varepsilon$ variables as follows:

$$E_X = \{(P, Q) : P, Q \in V_X \wedge \varepsilon(PQ) = 1, P \neq Q\}, \quad X \in \{KMPHN, KMPN\}.$$

# Formulation for the H-KMPN

The rationale of the formulation for the H-KMPN is to consider the $k$-Median with geodesic distances. Firstly, it is necessary to define the binary variables inherited from the $k$-median:

- $y(S)$, that assumes value one if the source neighbourhood $S \in \mathcal{S}$ is selected.
- $x(ST)$, that is one if the target neighbourhood $T \in \mathcal{T}$ is assigned to the selected source $S \in \mathcal{S}$.

## $k$-Median Constraints

The block of constraints inherited from the $k$-Median are the following:

$$\sum_{S \in \mathcal{S}} y(S) = k,$$

$$x(ST) \leq y(S), \quad \forall S \in \mathcal{S}, \quad \forall T \in \mathcal{T},$$

$$\sum_{S \in \mathcal{S}} x(ST) = 1, \quad \forall T \in \mathcal{T}.$$

# Formulation for the H-KMPN

For each edge $(P, Q) \in E_X$, a binary variable $f(PQ|ST)$ is defined, and takes the value of one when edge $(P, Q)$ is traversed in the path to go from the source $S$ to the target $T$.

## $f$-Constraints

The inequalities

$$f(PQ|ST) \leq \varepsilon(PQ), \forall S \in \mathcal{S}, \forall T \in \mathcal{T},$$

assure that the drone can go from $P$ to $Q$ only if the segment $\overline{PQ}$ does not cross any barrier.

# Formulation for the H-KMPN

For each edge $(P, Q) \in E_X$, a binary variable $f(PQ|ST)$ is defined, and takes the value of one when edge $(P, Q)$ is traversed in the path to go from the source $S$ to the target $T$.

## Flow conservation Constraints

For all $S \in \mathcal{S}$ and $T \in \mathcal{T}$,

$$\sum_{\{Q\in V_X:(P,Q)\in E_X\}} f(PQ|ST) - \sum_{\{Q\in V_X:(Q,P)\in E_X\}} f(PQ|ST) = \begin{cases} x(ST), & \text{if } P \in S, \\ 0, & \text{if } P \in V_{\mathcal{B}}, \\ -x(ST), & \text{if } P \in T, \end{cases}$$

model the flow conservation for each $P \in V_X$.

# Formulation for the H-KMPN

Hence, we can adjust the multi-commodity flow formulation to the induced graph $G_{\text{KMPHN}}$ as follows:

$$\text{minimize} \quad \sum_{(P,Q)\in E_{\text{KMPHN}}} d(PQ)y(PQ) \tag{H-TSPHN}$$

subject to

$$\sum_{\{Q:(P_N,Q)\in E_{\mathcal{N}}\}} y(P_N Q) \geq 1, \qquad \forall P_N \in V_{\mathcal{N}},$$

$$\sum_{\{Q:(P,Q)\in E_{\text{KMPHN}}\}} y(PQ) = \sum_{\{Q:(Q,P)\in E_{\text{KMPHN}}\}} y(QP), \qquad \forall P \in V_{\text{KMPHN}},$$

$$\sum_{\{Q:(Q,P_N)\in E_{\mathcal{N}}\}} g(QP_N) - \sum_{\{Q:(P_N,Q)\in E_{\mathcal{N}}\}} g(P_N Q) = 1, \qquad \forall P_N \in V_{\mathcal{N}} \setminus \{P_{N_1}\},$$

$$\sum_{\{Q:(Q,P_B^i)\in E_{\text{KMPHN}}\}} g(QP_B^i) - \sum_{\{Q:(P_B^i,Q)\in E_{\text{KMPHN}}\}} g(P_B^i Q) = 0, \quad \forall P_B^i \in V_{\mathcal{B}},$$

$$g(PQ) \leq (|\mathcal{N}| - 1)y(PQ), \qquad \forall (P, Q) \in E_{\text{KMPHN}},$$

$$(\alpha - C), (\beta - C), (\delta - C), \quad \forall P, Q \in V_{\text{KMPHN}}, \quad \forall P_B^1, P_B^2 \in V_{\mathcal{B}},$$

$$(\varepsilon - C), (y - C), (d - C), \quad \forall P, Q \in V_{\text{KMPHN}},$$

$$(N - C) \quad \forall P_N \in V_{\mathcal{N}}.$$

# Formulation for the H-KMPN

Hence, we can adjust the single-commodity flow formulation to the induced graph $G_{\mathrm{TSP}}$ as follows:

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{(P,Q)\in E_{\mathrm{TSP}}} d(PQ)y(PQ) & \text{(H-TSPN)} \\
\text{subject to} & & \\
& \displaystyle\sum_{\{Q:(P_N,Q)\in E_{\mathcal{N}}\}} y(P_NQ) \geq 1, & \forall P_N \in V_{\mathcal{N}}, \\
& \displaystyle\sum_{\{Q:(P,Q)\in E_{\mathrm{TSP}}\}} y(PQ) = \sum_{\{Q:(Q,P)\in E_{\mathrm{TSP}}\}} y(QP), & \forall P \in V_{\mathrm{TSP}}, \\
& \displaystyle\sum_{\{Q:(Q,P_N)\in E_{\mathcal{N}}\}} g(QP_N) - \sum_{\{Q:(P_N,Q)\in E_{\mathcal{N}}\}} g(P_NQ) = 1, & \forall P_N \in V_{\mathcal{N}} \setminus \{P_{N_1}\}, \\
& \displaystyle\sum_{\{Q:(Q,P_B^i)\in E_{\mathrm{TSP}}\}} g(QP_B^i) - \sum_{\{Q:(P_B^i,Q)\in E_{\mathrm{TSP}}\}} g(P_B^iQ) = 0, & \forall P_B^i \in V_{\mathcal{B}}, \\
& g(PQ) \leq (|\mathcal{N}|-1)y(PQ), & \forall (P,Q) \in E_{\mathrm{TSP}}, \\
& (\alpha - C), (\beta - C), (\delta - C), \quad \forall P, Q \in V_{\mathrm{TSP}}, \quad \forall P_B^1, P_B^2 \in V_{\mathcal{B}}, & \\
& (\varepsilon - C), (y - C), (d - C), \quad \forall P, Q \in V_{\mathrm{TSP}}, & \\
& (N - C) \quad \forall P_N \in V_{\mathcal{N}}. &
\end{array}
$$

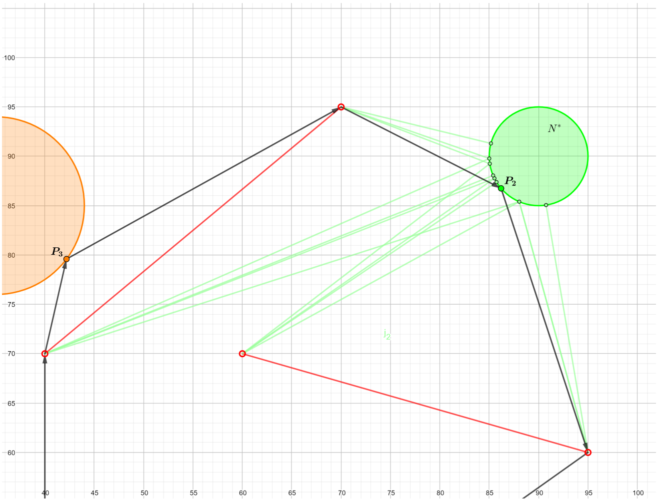
# Reformulating the H-KMPHN

## Proposition

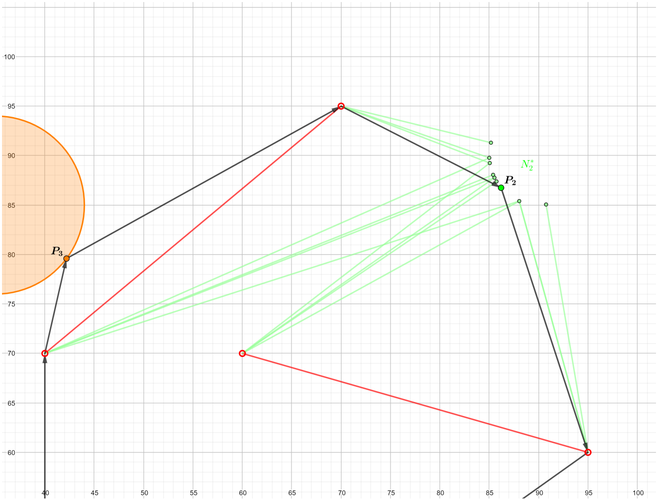
*Given a neighbourhood $N \in \mathcal{N}$, there exists a finite dominating set, $N^*$ of possible candidates to be in $N$. Moreover,*

$$N^* = \{P_N(P_B^i, P_{B'}^j) : \\ P_N(P_B^i, P_{B'}^j) = \arg\min_{P_N \in N} \|P_B^i - P_N\| + \|P_N - P_{B'}^j\| \text{ and } (P_B^i, P_N), (P_N, P_{B'}^j) \in E_{\mathcal{N}}\}.$$

# Reformulating the H-KMPHN



# Reformulating the H-KMPHN



# Solution of the Example

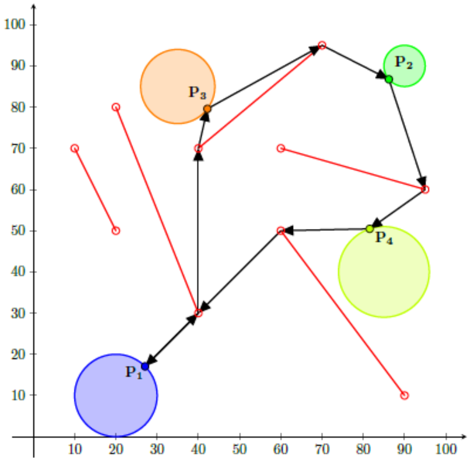



Figure: Solution for the TSPHN

# Preprocessing

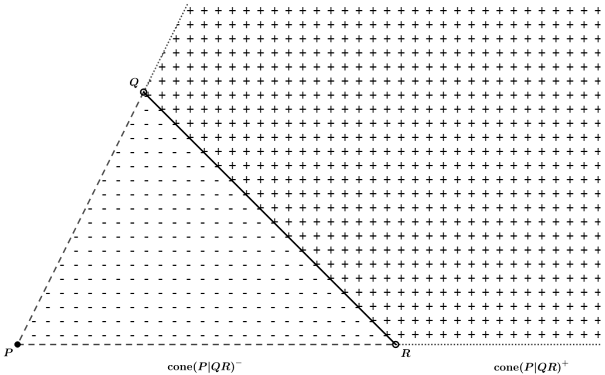



Figure: Representation of the cone generated by the point $P$ and the line segment $\overline{QR}$.

# Preprocessing

## Proposition

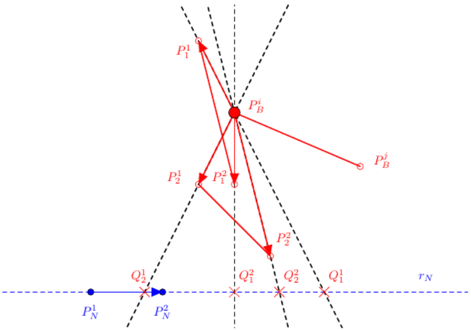
*Let* $B = \overline{P_B^1 P_B^2} \in \mathcal{B}$ *a barrier. If*

$$N \subset \bigcup_{B' \in \mathcal{B}} \mathrm{cone}(P_B^i | P_{B'}^1, P_{B'}^2)^+,$$

*then* $(P_N, P_B^i) \notin E_{\mathcal{N}}$.

## Proof.

If $P_N \in N$, then there exists a $B' \in \mathcal{B}$ such that $P_N \in \mathrm{cone}(P_B^i | P_{B'}^1, P_{B'}^2)^+$. Therefore, $\overline{P_B^i P_N} \cap B' \neq \emptyset$ and $(P_N, P_B^i) \notin E_{\mathcal{N}}$. $\square$




| | $Q_1^1$ | $P_N^1$ | $Q_2^1$ | $P_N^2$ | $Q_1^2$ | $Q_2^2$ |
|---|---|---|---|---|---|---|
| $\mu$ | $-\frac{5}{2}$ | $\times$ | $\frac{5}{2}$ | $\times$ | $\frac{5}{2}$ | $\frac{5}{4}$ |
| $\lambda$ | $M << 0$ | $0$ | $\frac{3}{4}$ | $1$ | $2$ | $\frac{9}{4}$ |

# Adjusting Big-Ms

Let $\overline{P^1_{B'}P^2_{B'}} = B' \in \mathcal{B}$ be a barrier, and $P_N \in N$. Let $\det(P_N|P^1_{B'}P^2_{B'})$ also be the determinant whose value must be bounded. Clearly, the solution of the following problem gives a lower bound of the determinant:

$$\overline{L} = \min_{P_N=(x,y)\in N} F(x,y) := \det(P_N|P^1_{B'}P^2_{B'}) = \begin{vmatrix} P^1_{B'_x} - x & P^2_{B'_x} - x \\ P^1_{B'_y} - y & P^2_{B'_y} - y \end{vmatrix}.$$

- If $N$ is a segment:

$$N = \{(x,y) \in \mathbb{R}^2 : (x,y) = \mu P^1_N + (1-\mu)P^2_N,\ 0 \leq \mu \leq 1\}.$$

  The function achieves its minimum and maximum at the extreme points of $N$, i.e., $P^1_N$ and $P^2_N$.

# Adjusting Big-Ms

- If $N$ is a ellipse:

$$N = \{(x,y) \in \mathbb{R}^2 : ax^2 + by^2 + cxy + dx + ey + f \leq 0\}.$$

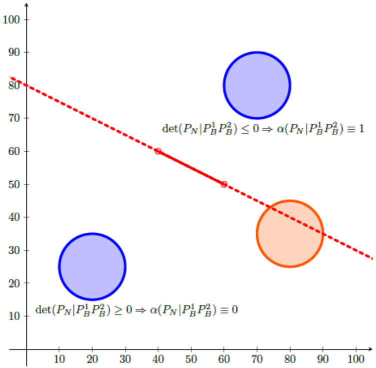
Solving the quadratic system:

$$\begin{cases} \left[(2a,c)\cdot \overrightarrow{P_{B'}^1,P_{B'}^2}\right]x + \left[(c,2b)\cdot \overrightarrow{P_{B'}^1,P_{B'}^2}\right]y + \left[(d,e)\cdot \overrightarrow{P_{B'}^1,P_{B'}^2}\right] & = 0, \\ ax^2 + by^2 + cxy + dx + ey + f & = 0, \end{cases}$$

they arise two solutions $x^{\pm}$ and $y^{\pm}$ that are evaluated in the objective function to obtain the lowest and highest value, respectively, according to $L(P_N|P_{B'}^1,P_{B'}^2)$ and $U(P_N|P_{B'}^1,P_{B'}^2)$, respectively.

# Variable fixing

When a neighbourhood $N$ is in the half-space generated by a barrier $B$, the sign of the determinant $\det(P_N|P_B^1 P_B^2)$ does not change for any point $P_N \in N$.




Therefore, a relevant number of variables $\alpha$ (hence $\beta$, $\gamma$, $\delta$ and $\varepsilon$) that model the sign of this determinant can be fixed 'a priori'.

# Data Generation

Assumptions **A1-A4** stated before are assumed to create the instances of experiments. In this case, w.l.o.g., the neighbourhoods generated are segments and circles.
The sketch of the procedure is:

1. Random sampling of the points in a square.
2. Generation of the bisectors that separate any pair of points.
3. Creation of the circles by assuming **A4**.

The line segments instances are generated by randomly selecting two diametrically opposite points in the boundary of the balls instances.

To generate instances for H-KMPN, it is only required to remove some bisectors for the instances generated before to ensure that it is possible to go directly from one to another neighbourhood.

# Configuration of the experiments

Formulations were coded in Python 3.9.2 and solved in Gurobi 9.1.2 on an AMD® Epyc 7402p 8-core processor. A time limit of 1 hour was set for the solver procedure.

- Five instances of randomly-sized neighbourhoods $|\mathcal{N}| \in \{5, 10, 20, 30, 50, 60, 65, 70, 75, 80, 100\}$ in $R = [0, 100] \times [0, 100]$
- Solving H-KMPN and H-KMPHN.
- Considering strengthening.

# Experimental Results (1)

Table: Computational results

| | | | | *Circles* | | | | | *Segments* | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert\mathcal{N}\rvert$ | $A_4$ | *Strengthening* | $\lvert\mathcal{B}\rvert$ | *#Found* | *Gap* | $Time_{model}$ | $Time_{prepro}$ | $Time_{total}$ | *#Found* | *Gap* | $Time_{model}$ | $Time_{prepro}$ | $Time_{total}$ |
| 5 | no | no | 4.8 | 5 | 0 | 157.24 | 0.29 | 157.53 | 5 | 0 | 42.59 | 0.91 | 43.5 |
| | | yes | 4.8 | 5 | 0 | 1.24 | 0.44 | 1.68 | 5 | 0 | 0.42 | 1.47 | 1.89 |
| | yes | no | 10.4 | 5 | 0.1 | 819.98 | 1.3 | 821.28 | 5 | 0 | 22.03 | 1.58 | 23.61 |
| | | yes | 10.4 | 5 | 0 | 0.61 | 1.16 | 1.77 | 5 | 0 | 0.38 | 1.57 | 1.95 |
| 10 | no | no | 9.2 | 5 | 0.17 | 2193.53 | 2.09 | 2195.62 | 5 | 0.42 | 2884.22 | 6.6 | 2890.82 |
| | | yes | 9.2 | 5 | 0 | 9.1 | 2.58 | 11.68 | 5 | 0 | 1.69 | 9.61 | 11.3 |
| | yes | no | 19.2 | 5 | 0.17 | 933.67 | 9.59 | 943.26 | 5 | 0.3 | 1448.73 | 11.8 | 1460.53 |
| | | yes | 19.2 | 5 | 0 | 2.56 | 6.36 | 8.92 | 5 | 0 | 1.53 | 9.24 | 10.77 |
| 20 | no | no | 17.6 | 5 | 0.17 | 3600 | 16.15 | 3616.15 | 5 | 0.2 | 3600 | 50.93 | 3650.93 |
| | | yes | 17.6 | 5 | 0 | 68.67 | 17.43 | 86.1 | 5 | 0 | 11.66 | 74.89 | 86.55 |
| | yes | no | 35.8 | 5 | 0.21 | 2349.26 | 87.7 | 2436.96 | 5 | 0 | 145.06 | 111.34 | 256.4 |
| | | yes | 35.8 | 5 | 0 | 42.45 | 43.49 | 85.94 | 5 | 0 | 8.05 | 64.01 | 72.06 |
| 30 | no | no | 28 | 4 | 0.5 | 3600 | 75.15 | 3675.15 | 5 | 0.4 | 3600 | 201.05 | 3801.05 |
| | | yes | 28 | 5 | 0 | 2034.53 | 65.23 | 2099.76 | 5 | 0 | 54.35 | 246.96 | 301.31 |
| | yes | no | 56.4 | 5 | 0.23 | 3027.02 | 512.03 | 3539.05 | 5 | 0.18 | 1039.08 | 672.46 | 1711.54 |
| | | yes | 56.4 | 5 | 0 | 147.98 | 179.95 | 327.93 | 5 | 0 | 96.72 | 270.56 | 367.28 |
| 50 | no | no | 44.2 | 3 | 0.87 | 3600 | 364.75 | 3964.75 | 4 | 0.3 | 3600 | 960.53 | 4560.53 |
| | | yes | 44.2 | 3 | 0 | 2485.76 | 297.99 | 2783.75 | 5 | 0 | 311.49 | 1043.15 | 1354.64 |
| | yes | no | 89 | 5 | 0.37 | 3600 | 4650.3 | 8250.3 | 5 | 0 | 1445.39 | 4654.95 | 6100.34 |
| | | yes | 89 | 5 | 0.1 | 3600 | 1213.85 | 4813.85 | 5 | 0 | 353.17 | 1292.12 | 1645.29 |

# Experimental Results (2)

Table: Computational results

| | | | | *Circles* | | | | | *Segments* | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert\mathcal{N}\rvert$ | $A_4$ | *Strengthening* | $\lvert\mathcal{B}\rvert$ | *#Found* | *Gap* | $Time_{model}$ | $Time_{prepro}$ | $Time_{total}$ | *#Found* | *Gap* | $Time_{model}$ | $Time_{prepro}$ | $Time_{total}$ |
| 60 | no | no | 50.2 | 0 | - | - | - | - | 5 | 0.53 | 3600 | 1213.08 | 4813.08 |
| | | yes | 50.2 | 3 | 0.22 | 3600 | 538.11 | 4138.11 | 5 | 0 | 1384.92 | 1103.43 | 2488.35 |
| | yes | no | 100.8 | 5 | 0.15 | 3600 | 8688.65 | 12288.65 | 5 | 0 | 1671.85 | 8726.63 | 10398.48 |
| | | yes | 100.8 | 5 | 0.13 | 3600 | 2179.26 | 5779.26 | 5 | 0.01 | 2903.27 | 2339.58 | 5242.85 |
| 65 | no | no | 52.8 | 0 | - | - | - | - | 4 | 0.75 | 3600 | 1561.46 | 5161.46 |
| | | yes | 52.8 | 2 | 0.35 | 3600 | 671.07 | 4271.07 | 5 | 0.02 | 3249.12 | 1343.12 | 4592.24 |
| | yes | no | 106 | 5 | 0.13 | 3600 | 11250.62 | 14850.62 | 5 | 0 | 1381.66 | 11269.25 | 12650.91 |
| | | yes | 106 | 5 | 0.17 | 3600 | 2843.48 | 6443.48 | 5 | 0.01 | 2877.66 | 3003.47 | 5881.13 |
| 70 | no | no | 57.6 | 0 | - | - | - | - | 4 | 0.85 | 3600 | 1977.3 | 5577.3 |
| | | yes | 57.6 | 3 | 0.12 | 3600 | 898.99 | 4498.99 | 5 | 0.04 | 3211.2 | 1754.51 | 4965.71 |
| | yes | no | 115.6 | 5 | 0.29 | 3600 | 17366.28 | 20966.28 | 5 | 0.04 | 2853.58 | 17433.28 | 20286.86 |
| | | yes | 115.6 | 5 | 0.32 | 3600 | 4106.95 | 7706.95 | 5 | 0.02 | 3203.33 | 4311.14 | 7514.47 |
| 75 | no | no | 63.2 | 0 | - | - | - | - | 3 | 0.74 | 3600 | 2976.6 | 6576.6 |
| | | yes | 63.2 | 0 | - | - | - | - | 5 | 0.24 | 3283.37 | 242407 | 5707,44 |
| | yes | no | 126.8 | 4 | 0.24 | 3600 | 26363.48 | 29963.48 | 5 | 0.01 | 1903.99 | 26153.87 | 28057.86 |
| | | yes | 126.8 | 3 | 0.23 | 3600 | 5382.14 | 8982.14 | 5 | 0.02 | 2458.75 | 6140.02 | 8598.77 |
| 80 | no | no | 64 | 0 | - | - | - | - | 1 | 0.83 | 3600 | 4205.41 | 7805.41 |
| | | yes | 64 | 0 | - | - | - | - | 5 | 0 | 1775.16 | 2858.51 | 4633.67 |
| | yes | no | 128.6 | 0 | - | - | - | - | 5 | 0.11 | 3471.06 | 29073.69 | 32544.75 |
| | | yes | 128.6 | 0 | - | - | - | - | 5 | 0 | 2701.21 | 7031.87 | 9733.08 |
| 100 | no | no | 81.6 | 0 | - | - | - | - | 0 | - | - | - | - |
| | | yes | 81.6 | 0 | - | - | - | - | 4 | 0 | 1761.08 | 6620.09 | 8381.17 |
| | yes | no | 163.2 | 0 | - | - | - | - | 5 | 0.48 | 3600 | 91153.55 | 94753.55 |
| | | yes | 163.2 | 0 | - | - | - | - | 5 | 0 | 2720.76 | 19429.1 | 22149.86 |

# Further research

- Study computationally non-disjoint neighbourhoods.
- Design some matheuristics to deal with larger instances.
- Consider non-convex neighbourhoods and/or nonlinear barriers.
- Propose alternative formulations that reduce the size of the problem.
- Extend to the third dimensional case.
- Study classical problems by including barriers.

# Acknowledgements

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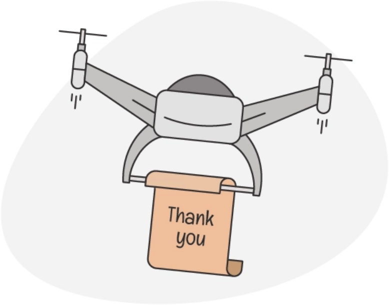
Thank
you