1. Description of the Problem

In this section, the framework of the two versions of the problem considered in the manuscript are analyzed: the Hampered k-Median Problem with Neighbourhoods and limited endurance H-KMPN and the Hampered k-Median Problem with Visible Neighbourhoods and limited endurance H-KMPVN. Since we have in mind their applications to the drone delivery problem with uniformly distributed demand in regions and inspection problems, at times, we will refer to the moving object as the *drone*.

1.1. Parameters and Assumptions of the Problem

The sets describing both versions of the problem are:

- $\mathcal{S}$: Set of neighbourhoods describing the possible sources where a facility can be allocated. It is assumed, wlog, that one facility can be allocated in each source at most.

- $\mathcal{T}$: Set of neighbourhoods representing the targets that must be served by a facility. It is assumed, wlog, that each target is served when it has been assigned to a facility.

- $\mathcal{B}$: Set of barriers that can not be crossed when a facility is joined with a target. The assumptions made for this set of line segments are the following:

  A1 The line segments of $\mathcal{B}$ are located in general position, i.e., the endpoints of these segments are not aligned. Although it is possible to model the most general case, one can always slightly modify one of the endpoints so that the segments are in general position.

  A2 The line segments of $\mathcal{B}$ are open sets, that is, it is possible that the drone visits endpoints of segments, but entering in its interior is not allowed. Observe that without loss of generality, we can always slightly enlarge these segments to make them open.

  A3 If there are two overlapping barriers, we assume that there is only one barrier given by the union of them.

  A4 There is no rectilinear path joining a pair of source-target neighbourhoods without crossing an obstacle.

The H-KMPVN is the relaxed version of the H-KMPN without imposing assumption **A4**. In this case, it is not required that the barriers separate neighbourhoods completely, i.e., when moving from one neighbourhood to another one it is possible to go following a straight line without crossing any barrier. Figure 1 shows an example of each version of the problem that is being considered. The left picture shows an instance of the H-KMPN, where green neighbourhoods represent possible sources to allocate the facilities, blue neighbourhoods represent targets to be assigned to the sources and the red line segments show the barriers that the drone cannot cross. The right picture illustrates an instance of the H-KMPVN where some sources and targets can be joined by a rectilinear path.
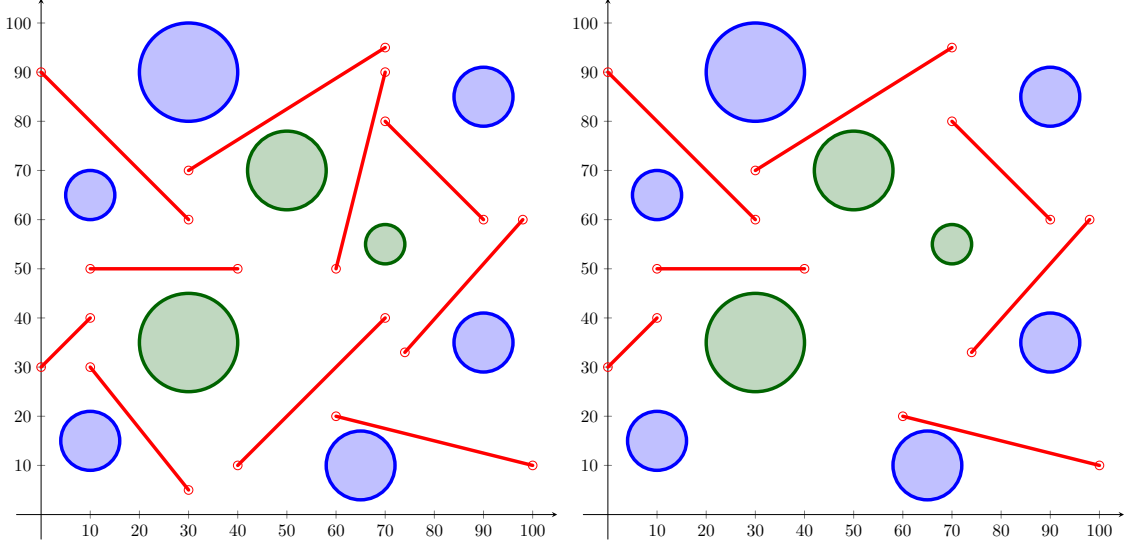
Figure 1: Problem data of the H-KMPN and H-KMPVN

1.2. Description of the Hampered k-Median Problem with Neighbourhoods

The goal of the H-KMPN is to find a subset of k points in the source set $\mathcal{S}$ and one point in each target set $\mathcal{T}$ that minimize the weighted length of the path, that is limited by the drone endurance, joining each target point with its associated source point and the weighted link distance without crossing any barrier of $\mathcal{B}$ assuming **A1-A4**. To state the model, we define the following sets:

- $V_{\mathcal{S}} = \{P_S : S \in \mathcal{S}\}$. Set of the points selected in the sources $\mathcal{S}$.

- $V_{\mathcal{B}} = \{P_B^1, P_B^2 : B = \overline{P_B^1 P_B^2} \in \mathcal{B}\}$. Set of vertices that come from the endpoints of barriers in the problem.

- $V_{\mathcal{T}} = \{P_T : T \in \mathcal{T}\}$. Set of the points selected in the targets $\mathcal{T}$.

- $E_{\mathcal{S}} = \{(P_S, P_B^i) : P_S \in V_{\mathcal{S}}, P_B^i \in V_{\mathcal{B}} \text{ and } \overline{P_S P_B^i} \cap B'' = \emptyset, \forall B'' \in \mathcal{B}, i = 1, 2\}$. Set of edges formed by the line segments that join the point selected in the source neighbourhood S and every endpoint in the barriers that do not cross any other barrier in $\mathcal{B}$.

- $E_{\mathcal{B}} = \{(P_B^i, P_{B'}^j) : P_B^i, P_{B'}^j \in V_{\mathcal{B}} \text{ and } \overline{P_B^i P_{B'}^j} \cap B'' = \emptyset, \forall B'' \in \mathcal{B}, i, j = 1, 2\}$. Set of edges formed by the line segments that join two vertices of $V_{\mathcal{B}}$ and do not cross any other barrier in $\mathcal{B}$.

- $E_{\mathcal{T}} = \{(P_B^i, P_T) : P_B^i \in V_{\mathcal{B}}, P_T \in V_{\mathcal{T}} \text{ and } \overline{P_B^i P_T} \cap B'' = \emptyset, \forall B'' \in \mathcal{B}, i = 1, 2\}$. Set of edges formed by the line segments that join the point selected in the target neighbourhood T and every endpoint in the barriers that do not cross any other barrier in $\mathcal{B}$.

The above sets allow us to define the graph $G_{\text{KMPN}} = (V_{\text{KMPN}}, E_{\text{KMPN}})$ induced by the barriers and neighbourhoods, where $V_{\text{KMPN}} = V_{\mathcal{S}} \cup V_{\mathcal{B}} \cup V_{\mathcal{T}}$ and $E_{\text{KMPN}} = E_{\mathcal{S}} \cup E_{\mathcal{B}} \cup E_{\mathcal{T}}$.

By taking the same approach, the graph induced for the relaxed version H-KMPVN can be described as $G_{\text{KMPVN}} = (V_{\text{KMPVN}}, E_{\text{KMPVN}})$, $V_{\text{KMPVN}} = V_{\text{KMPN}}$ and $E_{\text{KMPVN}} = E_{\text{KMPN}} \cup E_{\mathcal{ST}}$, where:

- $E_{\mathcal{ST}} = \{(P_S, P_T) : P_S \in V_{\mathcal{S}}, P_T \in V_{\mathcal{T}} \text{ and } \overline{P_S P_T} \cap B'' = \emptyset, \forall B'' \in \mathcal{B}, i = 1, 2\}$. Set of edges formed by the line segments that join the point selected in the source neighbourhood S and the point selected in the target neighbourhood T.

2. MINLP Formulations

This section proposes a mixed-integer NonLinear Programming formulation for the problem described in Section 1. First of all, the conic programming representation of the neighbourhoods and distance is presented. Then, the constraints that check if a segment is included in the set of edges E_X with $X \in \{\text{KMPN}, \text{KMPVN}\}$ are set. Finally, the formulations for the H-KMPN and H-KMPVN are described.

First of all, we introduce the decision variables that represent the problem. These are summarized in Table 1.

Table 1: Summary of decision variables used in the mathematical programming model

Binary Decision Variables	
Name	**Description**
$\alpha(P|QQ')$	1, if the determinant $\det(P|QQ')$ is positive, 0, otherwise.
$\beta(PP'|QQ')$	1, if the determinants $\det(P|QQ')$ and $\det(P'|QQ')$ have the same sign, 0, otherwise.
$\gamma(PP'|QQ')$	1, if the determinants $\det(P|QQ')$ and $\det(P'|QQ')$ are both positive, 0, otherwise.
$\delta(PP'|QQ')$	1, if the line segments $\overline{PP'}$ and $\overline{QQ'}$ intersect, 0, otherwise.
$\varepsilon(PP')$	1, if the line segment $\overline{PP'}$ does not cross any barrier, 0, otherwise.
$f(PQ|ST)$	1, if edge (P, Q) is traversed in the path joining S and T, 0, otherwise.
$y(S)$	1, if a facility is allocated in the source S in the solution of the model, 0, otherwise.
$x(ST)$	1, if source S and target T are joined by a path in the solution of the model, 0, otherwise.
Continuous Decision Variables	
Name	**Description**
P_N	Coordinates representing the point selected in the neighbourhood $N \in \mathcal{S} \cup \mathcal{T}$.
$d(PQ)$	Euclidean distance between the points P and Q.

2.1. Conic programming constraints in the models

For problems in consideration, namely H-KMPN and H-KMPVN, there exist two typologies of second-order cone constraints. One models the distance between each pair of points P and Q in V_X, $X \in \{\text{KMPN}, \text{KMPVN}\}$, and the other, the representation of source and target neighbourhoods, where points are selected.

Firstly, we define the non-negative continuous variable $d(PQ)$ that represents the distance between P and Q:

$$\|P - Q\| \leq d(PQ), \quad \forall (P, Q) \in E_X, \qquad (d\text{-C})$$

where E_X is the set of edges E_{KMPN} or E_{KMPVN}.

Secondly, since we are assuming that the neighbourhoods are second-order cone (SOC) representable, they can be expressed by means of the constraints:

$$P_N \in N \iff \|A_N^i P_N + b_N^i\| \leq (c_N^i)^T P_N + d_N^i, \quad i = 1, \ldots, n(N), \qquad (N\text{-C})$$

where A_N^i, b_N^i, c_N^i and d_N^i are parameters of the constraint i and $n(N)$ denotes the number of constraints that appear in the block associated with the neighbourhood $N \in \mathcal{S} \cup \mathcal{T}$.

These inequalities can model the special case of linear constraints (for $A_N^i, b_N^i \equiv 0$), ellipsoids and hyperbolic constraints (see Lobo et al. (1998) and Boyd and Vandenberghe (2004) for more information).

2.2. Checking whether a segment is an edge of the induced graph

The goal of this subsection is to represent by linear constraints a test to check whether given two arbitrary vertices $P, Q \in V_X$, the edge $(P, Q) \in E_X$, with $X \in \{\text{KMPN}, \text{KMPVN}\}$, i.e., whether the line segment $\overline{PQ}$ does not intersect with any barrier of $\mathcal{B}$. Although this representation was firstly introduced in (nos citamos), it is again described in this paper to be self-content. The following well-known computational geometry result can be used to check if two line segments intersect.

Remark 1. Let $\overline{PQ}$ and $B = \overline{P_B^1 P_B^2} \in \mathcal{B}$ be two different line segments. If

$$\text{sign}\left(\det(P|P_B^1 P_B^2)\right) = \text{sign}\left(\det(Q|P_B^1 P_B^2)\right) \quad \text{or} \quad \text{sign}\left(\det(P_B^1|PQ)\right) = \text{sign}\left(\det(P_B^2|PQ)\right),$$

then $\overline{PQ}$ and B do not intersect.

Let $P, Q \in V_X$, where V_X can be the set of vertices V_{KMPN} or V_{KMPVN}. Let $P_B^1, P_B^2 \in V_{\mathcal{B}}$ also be points determining barrier $B \in \mathcal{B}$. To model the conditions of the Remark 1, the use of binary variables that verify the sign of determinants, the equality of signs, and the disjunctive condition is needed, since these determinants depend on the location of P and Q.

Firstly, the sign of each determinant in Remark 1 is modelled. The binary variable α is introduced and assumes the value one if the determinant is non-negative and zero, otherwise. Note that determinants can not be null, because the barriers are located in general position.

The following constraints represent the sign condition:

$$\left[1 - \alpha(P|P_B^1 P_B^2)\right] L(P|P_B^1 P_B^2) \leq \det(P|P_B^1 P_B^2) \leq U(P|P_B^1 P_B^2)\, \alpha(P|P_B^1 P_B^2), \qquad (\alpha\text{-C})$$

$$\left[1 - \alpha(Q|P_B^1 P_B^2)\right] L(Q|P_B^1 P_B^2) \leq \det(Q|P_B^1 P_B^2) \leq U(Q|P_B^1 P_B^2)\, \alpha(Q|P_B^1 P_B^2),$$

$$\left[1 - \alpha(P_B^1|PQ)\right] L(P_B^1|PQ) \leq \det(P_B^1|PQ) \leq U(P_B^1|PQ)\, \alpha(P_B^1|PQ),$$

$$\left[1 - \alpha(P_B^2|PQ)\right] L(P_B^2|PQ) \leq \det(P_B^2|PQ) \leq U(P_B^2|PQ)\, \alpha(P_B^2|PQ),$$

where L and U are lower and upper bounds for the value of corresponding determinants, respectively. These bounds are later tightened in Section 2.4.1. If a determinant is non-negative, then α must be one to make the second inequality feasible. Analogously, if the determinant is not positive, α must be zero to satisfy the correct condition.

Secondly, to check whether the sign of any pair

$$\det(P|P_B^1 P_B^2),\ \det(Q|P_B^1 P_B^2) \quad \text{or} \quad \det(P_B^1|PQ),\ \det(P_B^2|PQ) \qquad (1)$$

of determinants is the same, the binary variable β is defined, that is one if the corresponding pair has the same sign, and zero otherwise.

Hence, the correct value of β variable can be expressed by the following constraint of the α variables

$$\beta(PQ|P_B^1 P_B^2) = \alpha(P|P_B^1 P_B^2)\alpha(Q|P_B^1 P_B^2) + \left[1 - \alpha(P|P_B^1 P_B^2)\right]\left[1 - \alpha(Q|P_B^1 P_B^2)\right],$$

$$\beta(P_B^1 P_B^2|PQ) = \alpha(P_B^1|PQ)\alpha(P_B^2|PQ) + \left[1 - \alpha(P_B^1|PQ)\right]\left[1 - \alpha(P_B^2|PQ)\right].$$

This condition can be equivalently written by means of an auxiliary binary variable γ that models the product of the α variables:

$$\beta(PQ|P_B^1 P_B^2) = 2\gamma(PQ|P_B^1 P_B^2) - \alpha(P|P_B^1 P_B^2) - \alpha(Q|P_B^1 P_B^2) + 1, \qquad (\beta\text{-C})$$

$$\beta(P_B^1 P_B^2|PQ) = 2\gamma(P_B^1 P_B^2|PQ) - \alpha(P_B^1|PQ) - \alpha(P_B^2|PQ) + 1,$$

These γ variables can be linearized by using the following constraints:

$$\gamma(PQ|P_B^1 P_B^2) \leq \alpha(P|P_B^1 P_B^2), \qquad\qquad \gamma(P_B^1 P_B^2|PQ) \leq \alpha(P_B^1|PQ), \qquad (\gamma\text{-C})$$

$$\gamma(PQ|P_B^1 P_B^2) \leq \alpha(Q|P_B^1 P_B^2), \qquad\qquad \gamma(P_B^1 P_B^2|PQ) \leq \alpha(P_B^2|PQ),$$

$$\gamma(PQ|P_B^1 P_B^2) \geq \alpha(P|P_B^1 P_B^2) + \alpha(Q|P_B^1 P_B^2) - 1, \qquad \gamma(P_B^1 P_B^2|PQ) \geq \alpha(P_B^1|PQ) + \alpha(P_B^2|PQ) - 1.$$

Thirdly, verifying whether there exists any coincidence of the sign of determinants is required, so the binary variable δ is defined that assumes one if segments do not intersect and zero, otherwise. This condition can be modelled by adopting the following disjunctive constraints:

$$\frac{1}{2}\left[\beta(PQ|P_B^1 P_B^2) + \beta(P_B^1 P_B^2|PQ)\right] \leq \delta(PQ|P_B^1 P_B^2) \leq \beta(PQ|P_B^1 P_B^2) + \beta(P_B^1 P_B^2|PQ). \qquad (\delta\text{-C})$$

Indeed, the above restrictions state that if there exists a sign coincidence in any of the two pairs of determinants in (1), then δ is one to satisfy the left constraint, and the right one is always fulfilled. However, if none of the signs of any pairs of determinants is the same, then the second constraint is zero and δ must be null.

Finally, to check whether

$$\overline{PQ} \cap B'' = \emptyset, \quad \forall B'' \in \mathcal{B}, \quad \iff \quad \delta(PQ|P^1_{B''}P^2_{B''}) = 1, \quad \forall B'' \in \mathcal{B},$$

the binary variable $\varepsilon(PQ)$ is introduced, and it is one if this condition is verified for all $B'' \in \mathcal{B}$. This variable can be expressed as:

$$\left[\sum_{B'' \in \mathcal{B}} \delta(PQ|P^1_{B''}P^2_{B''}) - |\mathcal{B}| \right] + 1 \le \varepsilon(PQ) \le \frac{1}{|\mathcal{B}|} \sum_{B'' \in \mathcal{B}} \delta(PQ|P^1_{B''}P^2_{B''}). \qquad (\varepsilon\text{-C})$$

If there exists, at least, a barrier $B'' \in \mathcal{B}$ that intersects the segment $\overline{PQ}$, then $\delta(PQ|P^1_{B''}P^2_{B''})$ is zero and the second inequality enforces ε to be zero because the right hand side is fractional and the first inequality is non-positive. However, if no barrier intersects the segment $\overline{PQ}$, then ε is equals to one, because the left hand side of the first inequality is one and the right hand side of the second inequality too.

It is possible to identify the set of actual edges of graph G_X by using the ε variables based on the above description, as follows:

$$E_X = \{(P, Q) : P, Q \in V_X \wedge \varepsilon(PQ) = 1, P \ne Q\}, \quad X \in \{\text{KMPN}, \text{KMPVN}\}.$$

This representation of E_X with $X \in \{\text{KMPN}, \text{KMPVN}\}$ will be applied in the formulations that are presented in the following subsections.

It is interesting to note that $E_{\mathcal{B}}$ is a fixed set whose edges can be computed by using the Remark 1. Then, ε variables can be prefixed in advance. However, edges in $E_X \setminus E_{\mathcal{B}}$ depend on the points selected in the neighbourhoods. A special case that can be highlighted happens when the set of neighbourhoods, $\mathcal{S}$ and $\mathcal{T}$, are represented by points. In that case, the induced graph is completely fixed and it is only necessary to find which edges are included by keeping in mind that the graph must be planar, i.e., without crossings. This idea is later exploited in Subsection **??**.

2.3. A formulation for the H-KMPN

The formulation of the H-KMPN is based on the structure of the well-known k-median problem where distances between each pair of source-target neighbourhoods are represented by the shortest path joining them without traversing any barrier.

Note that, although computing the shortest paths between every pair of neighbourhoods is possible, converting an instance of the H-KMPN into an instance of the standard k-median is not, since the points in neighbourhoods are not fixed. However, this simplification can be applied to produce an approximation to generate lower bounds for the H-KMPN.

Firstly, it is necessary to define the binary variables inherited from the k-median:

- $y(S)$, that assumes one if the source neighbourhood $S \in \mathcal{S}$ is selected.

- $x(ST)$, that is one if the target neighbourhood $T \in \mathcal{T}$ is assigned to the selected source $S \in \mathcal{S}$.

Secondly, adjusting a single-commodity flow formulation to ensure connectivity is required. The idea is that the model must deliver one unit of commodity from the selected source neighbourhood to each of

the required target neighbourhoods. Then, for each edge $(P, Q) \in E_{\text{KMPN}}$, the binary variable $f(PQ|ST)$ is defined, and takes the value of one when edge (P, Q) is traversed in the path to go from the source S to the target T.

Then, the inequalities

$$f(PQ) \leq |\mathcal{T}|\varepsilon(PQ), \qquad\qquad (f\text{-C})$$

are needed to assure that the drone can go from P to Q only if the segment $\overline{PQ}$ does not cross any barrier.

Hence, we can adjust the flow formulation to the induced graph G_{KMPN} as follows:

$$\text{minimize} \quad \alpha_E \sum_{(P,Q)\in E_{\text{KMPN}}} \sum_{S\in\mathcal{S}} \sum_{T\in\mathcal{T}} d(PQ)f(PQ|ST) + \frac{\alpha_L}{2} \sum_{(P,Q)\in E_{\text{KMPN}}} \sum_{S\in\mathcal{S}} \sum_{T\in\mathcal{T}} f(PQ|ST)$$

$$(\text{H-KMPN})$$

$$\text{subject to} \quad \sum_{S\in\mathcal{S}} y(S) = k,$$

$$x(ST) \leq y(S), \qquad\qquad \forall S \in$$

$$\sum_{S\in\mathcal{S}} x(ST) = 1, \qquad\qquad \forall T \in$$

$$\sum_{\{Q\in V_{\text{KMPN}}:(P,Q)\in E_{\text{KMPN}}\}} f(PQ|ST) - \sum_{\{Q\in V_{\text{KMPN}}:(Q,P)\in E_{\text{KMPN}}\}} f(PQ|ST) = \begin{cases} x(ST), & \text{if } P \in S, \\ 0, & \text{if } P \in V_{\mathcal{B}}, \quad \forall S \in \\ -x(ST), & \text{if } P \in T. \end{cases}$$

$$\sum_{(P,Q)\in E_{\text{KMPN}}} d(PQ)f(PQ|ST) \leq \text{endurance}\, x(ST), \qquad\qquad \forall S \in$$

$$(\alpha\text{-C}), (\beta\text{-C}), (\gamma\text{-C}), (\delta\text{-C}) \quad \forall P, Q \in V_{\text{KMPN}}, \quad \forall P_B^1, P_B^2 \in V_{\mathcal{B}},$$

$$(\varepsilon\text{-C}), (f\text{-C}), (d\text{-C}) \quad \forall P, Q \in V_{\text{KMPN}},$$

$$(N\text{-C}) \quad \forall P \in V_{\mathcal{S}} \cup V_{\mathcal{T}}.$$

Deberiamos quitar parte de las aristas en la distancia link?

The objective function takes into account both the euclidean and link weighted distances to join the selected sources with their assigned targets. The first group of constraints imposes that a subset of k sources is selected in $\mathcal{S}$. The second constraints ensure that one target T is assigned to a source S only if it is selected. The third inequalities ensure that every target is assigned to one source. The fourth ones are the flow conservation constraints, where the units of commodity launched from the source must be the number of targets that are assigned to that source. The fifth group of inequalities ensures that the drone can reach, in endurance terms, the target T when it is launched from the source S. Constraints (α-C), (β-C), (γ-C), (δ-C), (ε-C), (f-C), (d-C), (N-C) enforce the variables of the problem to be well-defined.

To deal with the bilinear terms that appear in the objective function, the McCormick's envelope is used to linearize them by including variables $p(PQ|ST) \geq 0$ that represent the products and introducing

$$p(PQ|ST) \geq m(PQ)f(PQ|ST),$$

$$p(PQ|ST) \geq d(PQ) - M(PQ)(1 - f(PQ|ST)),$$

where $m(PQ)$ and $M(PQ)$ are, respectively, the lower and upper bounds of the distance variable $d(PQ)$.

Proposition 1. *The* H-KMPN *is NP-complete.*

Note that, once a point is fixed in each neighbourhood, the problem that results in the induced graph G_{KMPN} is the k-median with geodesic distances, that is NP-complete. Figure 2 shows the solutions of the problem data in Figure 1 for $k = 1, 2, 3$, respectively.
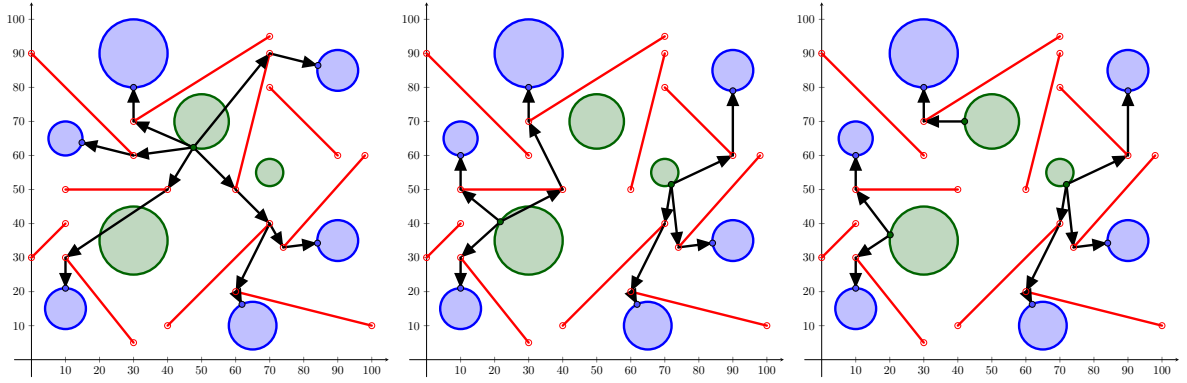

Figure 2: Optimal solution for the H-KMPN

2.4. Relaxing the assumptions of the problem: The H-KMPVN

In this subsection, we analyze the differences between the H-KMPN and the H-KMPVN, where there exists a rectilinear path joining a source neighbourhood with a target neighbourhood. The main difference lies in the description of the edges of the graph induced by the neighbourhoods and the endpoints of the barriers, as shown in Subsection 1.2.

The difference between the set of edges in the H-KMPN with respect to the graph in H-KMPVN is that, in the former case, the edges that join each pair of neighbourhoods must be considered. This fact leads to include product of continuous variables in the α constraints of the model that represent the determinants. These products make the problem to become non-convex.

2.4.1. Lower and upper bounds when the neighbourhoods are ellipses

References

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