Kingdom of Saudi Arabia
Ministry of Education
College of Computer
Department of Computer Science

المملكة العربية السعودية
وزارة التعليم
كلية الحاسب
قسم علوم الحاسب

CS 348 - Optimization Techniques

Formulae / Cheat Sheet with Relevant Examples

$$ax^2 + bx + c = 0$$
$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Equation of a circle:

$$x_1^2 + x_2^2 = r^2$$

Center: $(0,0)$, Radius: r

$$(x_1 - a)^2 + (x_2 - b)^2 = r^2$$

Center: (a,b), Radius: r

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Finding the standard form of a quadratic function:

$$f(x)=ax_1^{\ 2}+bx_2^{\ 2}+cx_3^{\ 2}+dx_1x_2+ex_2x_3+fx_1x_3+gx_1+hx_2+ix_3+j$$

$$Q=\begin{bmatrix}2a & d & f\\ d & 2b & e\\ f & e & 2c\end{bmatrix}, c=\begin{bmatrix}g\\ h\\ i\end{bmatrix}, k=j \text{ (constant)}$$

$x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ $x^T=[x_1\ x_2\ x_3]$

$$f(x)=\frac{1}{2}x^TQx+c^Tx+K$$

$$f(x)=\overset{a}{9}x_1^2+\overset{b}{7}x_2^2+\overset{c}{3}x_3^2\overset{d}{-2}x_1x_2+\overset{f}{4}x_1x_3\overset{h}{-6}x_2+\overset{j}{6}$$

[2 Marks]

Express $f(x)$ in terms of standard quadratic form.

Sol:

$$f(x)=\frac{1}{2}x^TQx+c^Tx+K$$

$a=9$, $b=7$, $c=3$, $d=-2$, $e=0$, $f=4$,

$g=0$, $h=-6$, $i=0$, $j=6$

$$Q=\begin{bmatrix}18 & -2 & 4\\ -2 & 14 & 0\\ 4 & 0 & 6\end{bmatrix}$$

$$c=\begin{bmatrix}0\\ -6\\ 0\end{bmatrix}, \quad K=6$$

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Consider the following minimization problem **[5 marks]**

$f: \quad \Re^n \longrightarrow \Re^p$ $f(x) = x_1 + x_2 - 5$ objective الهدف	$g: \quad \Re^r \longrightarrow \Re^q$ $g(x) = x_1 - x_2 - 2$ القيود Constrains	$h: \quad \Re^s \longrightarrow \Re^m$ $h(x) = \begin{bmatrix} -x_1 \\ -x_2 - 3 \end{bmatrix}$

i. Find the values of n, p, r, q, s, m. **[0.5 Mark]**

input, output

$f: R^n \longrightarrow R^p$ $g: R^r \longrightarrow R^q$ $h: R^s \longrightarrow R^m$

$n = 2, \quad p = 1$ $r = 2, \quad q = 1$ $s = 2, m = 2$

ii. Write the problem in standard form. **[1.5 Marks]**

$$f(x) = c^T X + k$$

x_1, x_2 متغيرات

$$c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad k = -5$$

$f: \quad \Re^n \longrightarrow \Re^p$

$f(x) = x_1 + x_2 - 5$

$$g(x) = 0$$

$$x_1 - x_2 - 2 = 0$$

$$x_1 - x_2 = 2$$

$$A X = b$$

$$A = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad b = 2$$

$g: \quad \Re^r \longrightarrow \Re^q$

$g(x) = x_1 - x_2 - 2$

$$h(x) \le 0$$

القيد الأول $\quad -x_1 \le 0 \Rightarrow$

$$-x_2 - 3 \le 0 \Rightarrow -x_2 \le 3$$ القيد الثاني

$h: \quad \Re^s \longrightarrow \Re^m$

$h(x) = \begin{bmatrix} -x_1 \\ -x_2 - 3 \end{bmatrix}$

$$c X \le d$$

القيد الأول

$$c = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad d = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

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iii. Draw the feasible set. **[1 Mark]**

$g(x) = 0$

$x_1 - x_2 - 2 = 0$

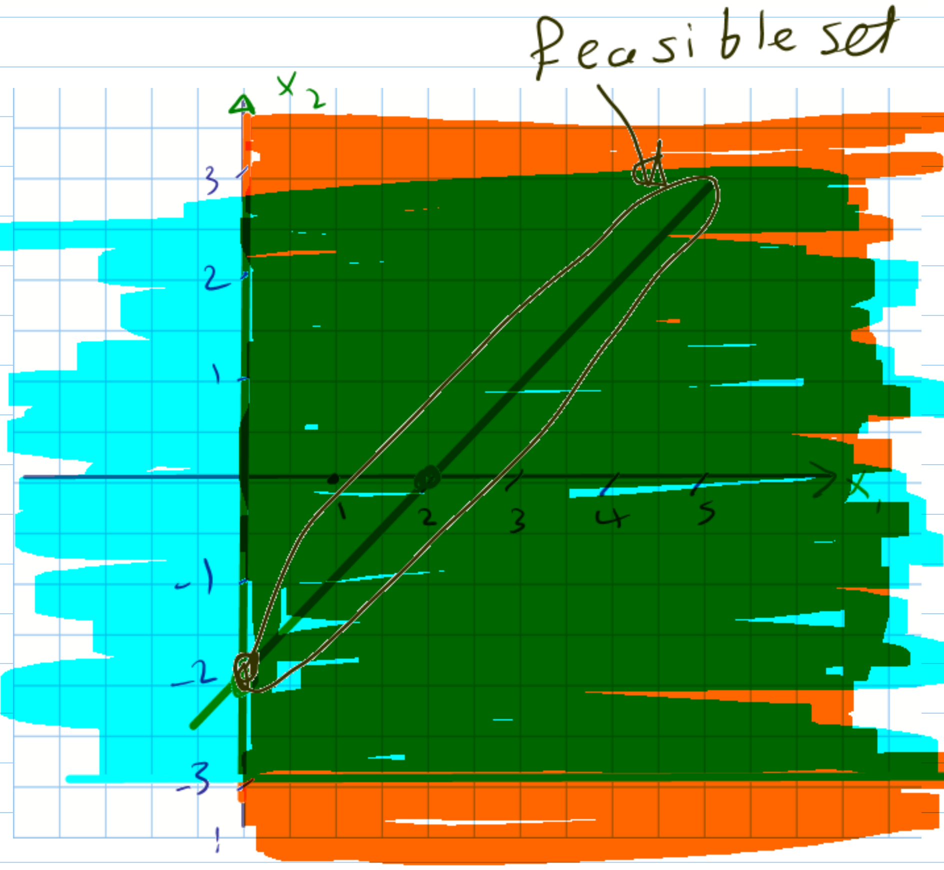
$\boxed{x_1 - x_2 = 2}$

x_1	0	2
x_2	-2	0

$h(x)$

$-x_1 \leq 0 \quad = x_1 \geq 0$

$\Rightarrow -x_2 \leq 3 \quad x_2 \geq -3$



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iv. Draw the contour sets $C_f(\tilde{f})$ for values of objective function $\tilde{f} = -2, -1, 0, 1$. **[1 Mark]**

Solution

$$f(x) = x_1 + x_2 - 5$$

$\tilde{f} = -2$

$x_1 + x_2 - 5 = -2$

$x_1 + x_2 = 3$

x_1	0	3
x_2	3	0

$\tilde{f} = -1$

$x_1 + x_2 - 5 = -1$

$x_1 + x_2 = 4$

x_1	0	4
x_2	4	0

$\tilde{f} = 0$

$x_1 + x_2 - 5 = 0$

$x_1 + x_2 = 5$

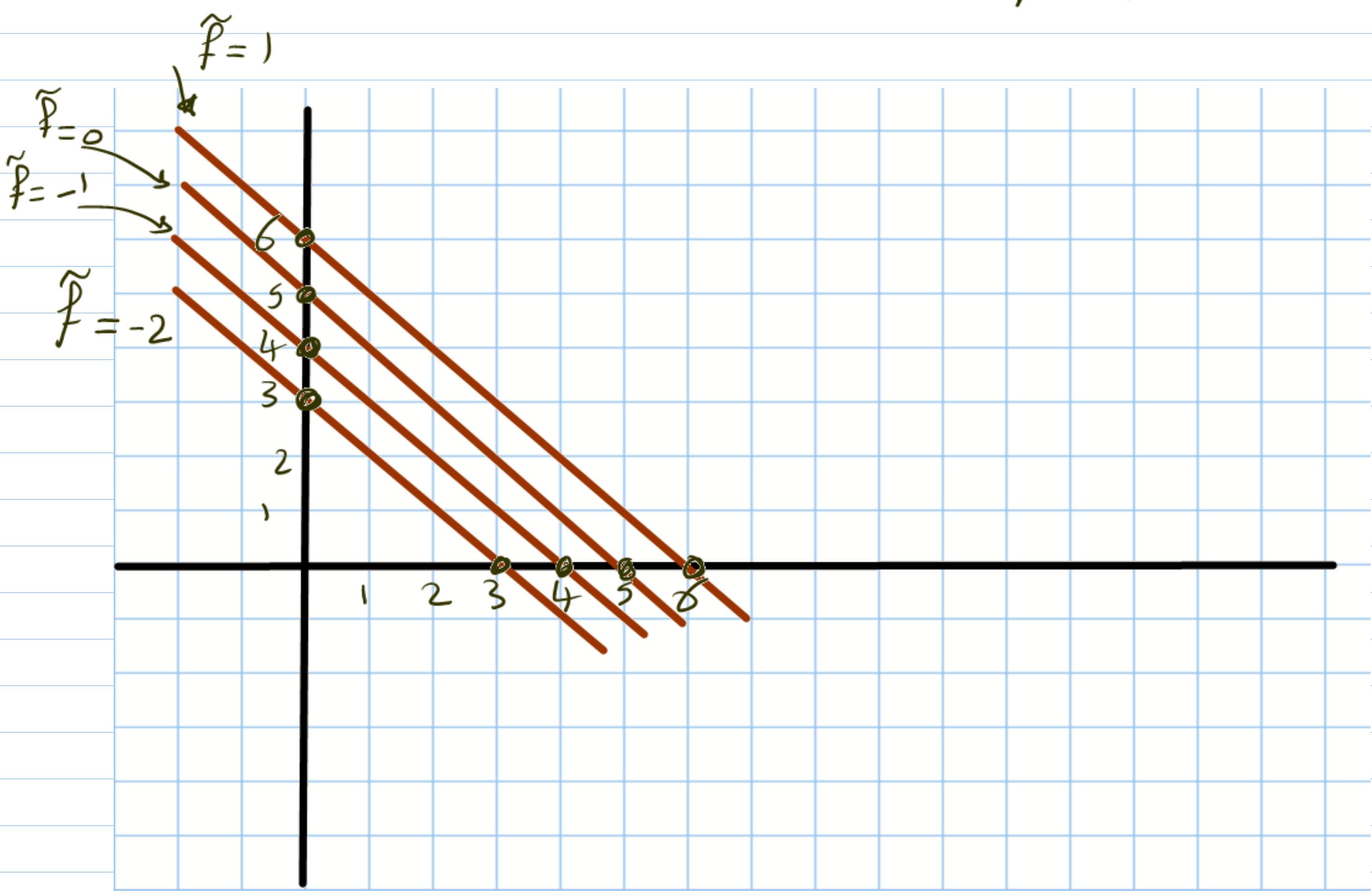
x_1	0	5
x_2	5	0

$\tilde{f} = 1$

$x_1 + x_2 - 5 = 1$

$x_1 + x_2 = 6$

x_1	0	6
x_2	6	0



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v. Find the *constrained* minimizer. **[1 Mark]**

minimizer

$x^* = [0 \quad -2]^T$

minimum

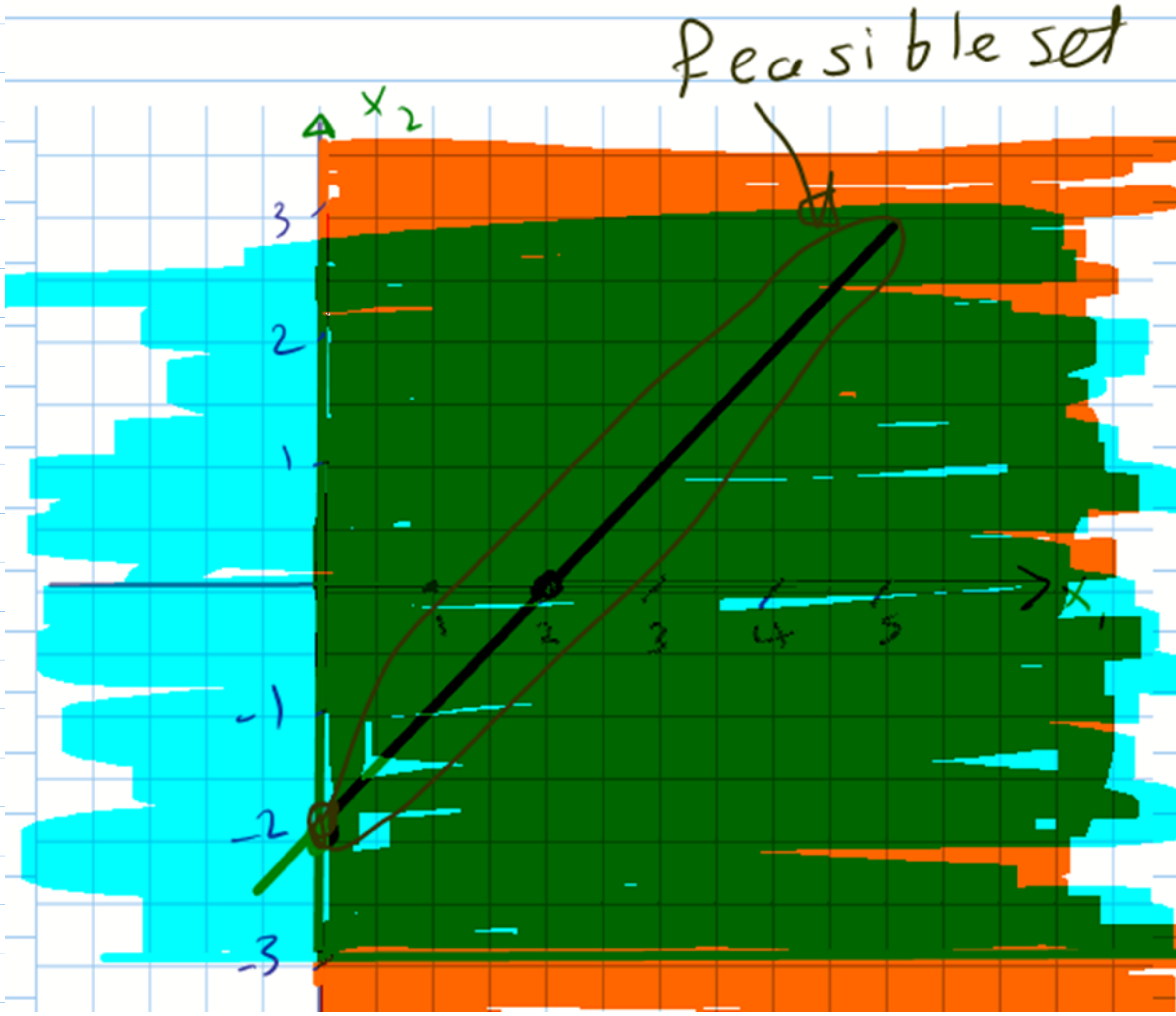
$f\begin{bmatrix} 0 \\ -2 \end{bmatrix}$

$= x_1 + x_2 - 5$

$= 0 + (-2) - 5$

$= -7$

minimum



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Function is in the standard form:

α $$f(x)=\frac{1}{2}x^T\begin{bmatrix} q_{11} & q_{12}=q_{21} \\ q_{21}=q_{12} & q_{22}\end{bmatrix}x+x^T\begin{bmatrix} a \\ b\end{bmatrix}+k$$

Hessian

$$\Rightarrow Q=\begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22}\end{bmatrix}, c=\begin{bmatrix} a \\ b\end{bmatrix}$$

اخليه نفسه

Function is NOT in the standard form:

α $$f(x)=x^T\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22}\end{bmatrix}x+x^T\begin{bmatrix} a \\ b\end{bmatrix}+k$$

Hessian

$$\Rightarrow Q=\begin{bmatrix} 2q_{11} & q_{12}+q_{21} \\ q_{12}+q_{21} & 2q_{22}\end{bmatrix}$$

For the given function $f: \Re^2 \longrightarrow \Re$, find the Hessian.

$$f(x)=x^T\begin{bmatrix} \overset{q_{11}}{4} & \overset{q_{12}}{-3} \\ \underset{q_{21}}{-3} & \underset{q_{22}}{8}\end{bmatrix}x+x^T\begin{bmatrix} 4 \\ -5\end{bmatrix}+10$$

Solution

Hessian $$\nabla^2=\begin{bmatrix} 2(4) & -3-3 \\ -3-3 & 2(8)\end{bmatrix}$$

$$\nabla^2=\begin{bmatrix} 8 & -6 \\ -6 & 16\end{bmatrix}$$

$$=\frac{1}{2}x^T\begin{bmatrix} 8 & -6 \\ -6 & 16\end{bmatrix}x+\begin{bmatrix} 4 \\ -5\end{bmatrix}x^T+10$$

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Determining if a function is convex (Using Sylvester Theorem):

Hessian

$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$$

$$\nabla^2 = \begin{vmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_2x_1} & f_{x_2x_2} \end{vmatrix}$$

$q_{11} > 0$

$|Q| = \det Q \geq 0$

$\Rightarrow q_{11}q_{22} - q_{21}q_{12} \geq 0$

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}$$

$q_{11} > 0$

$q_{11}q_{22} - q_{21}q_{12} > 0$

$|Q| = \det Q \geq 0$

$\Rightarrow q_{11}(q_{22}q_{33} - q_{32}q_{23}) - q_{12}(q_{21}q_{33} - q_{31}q_{23}) + q_{13}(q_{21}q_{32} - q_{31}q_{22}) \geq 0$

For what values of the parameters $\alpha, \beta \in \Re$ is the function

$$f(x_1, x_2) = 2x_1^2 - 2\beta x_1 x_2 + 2\alpha x_2^2,$$

a convex function? [3 Marks]

Soln

Hessian $\nabla^2 = Q = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_2x_1} & f_{x_2x_2} \end{bmatrix}$

عشان x_2 كالا خطية
متساوي صفر

على جنب

$f(x_1, x_2) = 3x_1^2 + 5x_2^2 + 7x_1^3 x_2^4 + 2x_1 x_2$

$f_{x_1} = 6x_1 + 21x_1^2 x_2^4 + 2x_2$

$f_{x_2} = -10x_2 + 28x_1^3 x_2^3 + 2x_1$

$f_{x_1x_1} = 6 + 44x_1 x_2^4$

$f_{x_2x_2} = -10 + 84x_1^3 x_2^2$

$f_{x_1x_2} = 84x_1^2 x_2^3 + 2$

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$$f(x_1,x_2) = 2x_1^2 - 2\beta x_1 x_2 + 2\alpha x_2^2,$$

$f' = 5x_1$
$f'' = 5$

Hessian $\nabla^2 = Q = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_2x_1} & f_{x_2x_2} \end{bmatrix}$

اتفاضل x_1 ← $f_{x_1} = 4x_1 - 2\beta x_2$

اتفاضل α_2 ← $f_{x_2} = -2\beta x_1 + 4\alpha x_2$

x_2

$f_{x_1x_1} = 4$, $f_{x_2x_2} = 4\alpha$, $f_{x_1x_2} = -2\beta$

$$Q = \begin{bmatrix} 4 & -2\beta \\ -2\beta & 4\alpha \end{bmatrix}$$

$Q_{11} = 4 > 0$ ✓

$$|Q| = \begin{vmatrix} 4 & -2\beta \\ -2\beta & 4\alpha \end{vmatrix}$$

$$= 16\alpha - 4\beta \geq 0$$

اتوقف هنا ← $\boxed{16\alpha \geq 4\beta}$

$4\alpha \geq \beta$ Convex $\alpha \geq \frac{\beta}{4}$

Determining if a function is convex

Hessian $Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$

$q_{11} > 0$

$|Q| = \det Q \geq 0$

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Length of a gradient:

$$\nabla f(x^0) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\|\nabla f(x^0)\| = \sqrt{a^2 + b^2 + c^2}$$

إمب محمد باملد

$f: \quad \Re^2 \longrightarrow \Re$

$f(x) = (x_1 + 4)^2 + (x_2 - 8)^2 - 2.5(x_1 - 4)(x_2 - 8)$

Starting at $x^{(0)} = [3 \quad 7]^T$,

Find the length of the gradient at $x^{(0)}$ [1 Mark]

$$\nabla f(x) = \begin{bmatrix} f_{x_1} \\ f_{x_2} \end{bmatrix} =$$

$$f_{x_1} = 2(x_1 + 4) - 2.5(x_2 - 8) = 2x_1 + 8 - 2.5x_2 + 20$$
$$= 2x_1 - 2.5x_2 + 28$$

$$f_{x_2} = 2(x_2 - 8) - 2.5(x_1 - 4) = 2x_2 - 16 - 2.5x_1 + 10$$
$$= 2x_2 - 2.5x_1 - 6$$

at

$$\nabla f(x) = \begin{bmatrix} 2x_1 - 2.5x_2 + 28 \\ 2x_2 - 2.5x_1 - 6 \end{bmatrix}$$

at Starting at $x^{(0)} = [3 \quad 7]^T$,

$$\nabla f(x^0) = \begin{bmatrix} 2(3) - 2.5(7) + 28 \\ 2(7) - 2.5(3) - 6 \end{bmatrix}$$

$$\nabla f(x) = \begin{bmatrix} 16.5 \\ 0.5 \end{bmatrix}$$

$$\|\nabla f(x^0)\| = \sqrt{(16.5)^2 + (0.5)^2}$$

$$= 16.51$$

طريقة اخرى لايجاد f_{x_2} ، f_{x_1}

$f(x) = (x_1 + 4)^2 + (x_2 - 8)^2 - 2.5(x_1 - 4)(x_2 - 8)$

$$f(x) = x_1^2 + 16 + 8x_1 + x_2^2 + 64 - 16x_2$$
$$-2.5[x_1 x_2 - 8x_1 - 4x_2 + 32]$$
$$= x_1^2 + x_2^2 + 8x_1 - 16x_2$$
$$+ \cancel{80} - 2.5x_1x_2 + 20x_1 + 10x_2$$
$$- \cancel{80}$$
$$= x_1^2 + x_2^2 - 2.5x_1x_2 + 28x_1$$
$$- 6x_2$$

$$f_{x_1} = 2x_1 - 2.5x_2 + 28$$

$$f_{x_2} = 2x_2 - 2.5x_1 - 6$$

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Direction of the Steepest Descent:

$\Delta x = -\nabla f(x^0)$

A skier is on a mountain with equation **1 Mark**

$$f(x) = 2 - 0.6x_1^2 - 0.9x_2^2 + 0.45x_3^2$$

where the function describes the height of the mountain. The skier is located at the ***point (2, 2, 2),*** and wants to ski downhill along the *steepest* possible path. In which direction (indicated by a ***vector*** (a, b) in the *xy*-plane) should the skier begin skiing?

Soln

$$\nabla f(x) = \begin{bmatrix} f_{x_1} \\ f_{x_2} \\ f_{x_3} \end{bmatrix} = \begin{bmatrix} -1.2x_1 \\ -1.8x_2 \\ 0.9x_3 \end{bmatrix}$$

(2,2,2)

$$\nabla f(x^0) = \begin{pmatrix} -1.2(2) \\ -1.8(2) \\ 0.9(2) \end{pmatrix} = \begin{pmatrix} -2.4 \\ -3.6 \\ 1.8 \end{pmatrix}$$

direction

$$\Delta x = -\nabla f(x^0) = \begin{pmatrix} 2.4 \\ 3.6 \\ 1.8 \end{pmatrix}$$

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Finding the minimizer while solving simultaneous equations:

$\nabla f(x) = 0$

Finding the minmizer using Hessian:

$x^* = -Q^{-1}c \qquad \left(\text{since } \nabla f(x) = \frac{1}{2}2Qx + c = Qx + c\right)$

نقطة حرجة critical Point

تمثل وجود نهاية عظمى او صغرى

For the given function $f: \quad \Re^2 \longrightarrow \Re$, find the critical point.

$$f(x) = x^T \begin{bmatrix} 4 & -3 \\ -3 & 8 \end{bmatrix} x + x^T \begin{bmatrix} 4 \\ -5 \end{bmatrix} + 10$$

Can we classify the critical point as a *minimizer*, *maximizer*, or a *point of inflection*? Explain briefly.

$$f(x) = x^T \begin{bmatrix} \overset{q_{11}}{4} & \overset{q_{12}}{-3} \\ \underset{q_{21}}{-3} & \underset{q_{22}}{8} \end{bmatrix} x + x^T \begin{bmatrix} 4 \\ -5 \end{bmatrix} + 10$$

Solution

Hessian $\nabla^2 = \begin{bmatrix} 2(4) & -3-3 \\ -3-3 & 2(8) \end{bmatrix}$

$Q = \nabla^2 = \begin{bmatrix} 8 & -6 \\ -6 & 16 \end{bmatrix}$

$Q^{-1} = \frac{1}{|Q|} \begin{bmatrix} \text{بدل عناصر القطر الرئيسي وغير اشارات الثانوي} \end{bmatrix} = \frac{1}{128-36} \begin{bmatrix} 16 & 6 \\ 6 & 8 \end{bmatrix}$

$= \frac{1}{92} \begin{bmatrix} 16 & 6 \\ 6 & 8 \end{bmatrix}$

$x^* = -Q^{-1}c$

critical Point

$x^* = -\frac{1}{92} \begin{bmatrix} 16 & 6 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} 4 \\ -5 \end{bmatrix} = -\frac{1}{92} \begin{bmatrix} 16(4) + 6(-5) \\ 6(4) + 8(-5) \end{bmatrix}$

$x^* = -\frac{1}{92} \begin{bmatrix} 34 \\ -16 \end{bmatrix}$

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$$x^* = \begin{bmatrix} -34/92 \\ 16/92 \end{bmatrix} = \begin{bmatrix} -0.37 \\ 0.17 \end{bmatrix}$$ critical point

It can be regarded as a ***local minimizer*** since the determinant of the Hessian is ***positive definite*** (>0).

$$|Q| = \frac{1}{92} > 0$$

$$x^* = \begin{bmatrix} -0.37 \\ 0.17 \end{bmatrix}$$ minimizer

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