First price Sealed-bid Procurement Auction

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1 Introduction

Recent work in mechanism design has advanced the worst–case analysis of resource–allocation mechanisms through *price of anarchy* (PoA) guarantees. These results bound the welfare attained in (Bayes–Nash) equilibrium without requiring an explicit solution of the equilibria. For example, in the standard (selling) first–price auction, recent results show that any equilibrium achieves at least $1 - 1/e^2$ of the optimal social welfare (Jin & Lu, 2023).

By contrast, analogous guarantees for the *reverse* setting—procurement mechanisms in which agents submit a single offer and the lowest compliant bid wins—are largely missing. This project addresses that gap for first–price procurement auctions.

Classical PoA bounds for selling auctions often rely on the smoothness framework (Roughgarden, 2015; Syrgkanis & Tardos, 2013), which decomposes welfare into bidders' utilities plus the mechanism's revenue and then derives a lower bound on their sum via a deviation argument. In procurement, however, "revenue" becomes the buyer's expenditure with the opposite sign, so this decomposition fails to yield a useful bound.

Following (Hoy et al., 2018), we instead analyze a different pair of quantities: (i) the welfare contributed by *rightful winners*—agents who win and have the lowest cost—and (ii) the welfare from *improper allocations*—agents who win despite not having the lowest cost. Under independent private costs and monotone bidding, we show that the expected cost of improper allocations can be upper bounded by the mechanism's expected payments. This yields a clean trade-off between rightful-winner welfare and payments, from which we derive welfare guarantees for first–price procurement auctions.

2 Technical Preliminaries

We now lay the formal groundwork for our result. This paper analyzes the *single-item sealed-bid first-price procurement auction*. In such an auction a single item is sold to n agents. Each agent i simultaneously submits a bid b_i to the auctioneer. The agent i^* with the lowest bid wins the item, and pays their bid b_{i^*}. All other agents pay nothing and win nothing. Let $\tilde{x}_i(b)$ denote the indicator for whether agent i is allocated under bid profile b, and let $\tilde{p}_i(b)$ denote the payments received by agent i under that same bid profile. Each agent evaluates their allocation and payment using the linear utility function $\tilde{u}_i(b) = (b_i - c_i)\tilde{x}_i(b)$, where c_i is agent i's true cost for providing the product/service.

We consider a Bayesian environment, in which each agent i's cost is drawn independently from a distribution with CDF F_i and density f_i. Note that we do not require agents' cost distributions to be identical. We assume agents' costs are private and independent, but that the prior distributions are common knowledge.

We adopt the standard solution concept of Bayes–Nash equilibrium (BNE). Informally, a BNE is a strategy mapping $b_i(\cdot)$ from costs to bids for every agent such that each agent's bid given their cost maximizes their expected utility given the strategies of other bidders. Formally, given a profile $b(\cdot)$ of bidding strategies for each agent, define the interim allocation probability of agent i bidding b to be

$$\tilde{x}_i(b) = \mathbb{E}_{c_{-i}}\left[\tilde{x}_i\big(b,\, b_{-i}(c_{-i})\big)\right].$$

Similarly, define the interim expected payments of agent i to be

$$\tilde{p}_i(b) = \mathbb{E}_{c_{-i}}\left[\tilde{p}_i\big(b,\, b_{-i}(c_{-i})\big)\right].$$

Define the interim expected utility $\tilde{u}_i(b)$ similarly. A profile of bidding strategies $b(\cdot)$ is a BNE if for every agent i with cost c_i, the following best response inequality holds for every alternate bid b:

$$\tilde{u}_i\big(b_i(c_i)\big) \;\geq\; \tilde{u}_i(b).$$

In what follows, we argue assuming agents are bidding according to an arbitrary BNE profile of bidding strategies. Since the strategies map costs to bids and bids are mapped to allocation and payments, we will often consider allocations, payments, and utilities as a function of cost, taking the bid functions as implicit. Formally, we will let

$$x_i(c_i) = \mathbb{E}_{c_{-i}}\left[\tilde{x}_i\big(b_i(c_i),\, b_{-i}(c_{-i})\big)\right], \quad p_i(c_i) = \mathbb{E}_{c_{-i}}\left[\tilde{p}_i\big(b_i(c_i),\, b_{-i}(c_{-i})\big)\right], \quad u_i(c_i) = \mathbb{E}_{c_{-i}}\left[\tilde{u}_i\big(b_i(c_i),\, b_{-i}(c_{-i})\big)\right].$$

Note that we use tildes when the argument to the function is a bid, and omit the tildes when an argument to the function is the cost instead.

We study the objective of utilitarian social welfare. The social welfare of a BNE is the expected cost of the winner. In other words,

$$\mathrm{WELF}\big(b(\cdot)\big) = \sum_i c_i\, x_i(c_i).$$

As our benchmark, we use the efficient (first-best) cost—the expected minimum cost across bidders: $\mathrm{OPT}_{\mathrm{cost}} \coloneqq \mathbb{E}_{\mathbf{c}}[\min_i c_i]$. This is the welfare of the mechanism which always allocates the agent with the lowest cost. We state performance relative to the cost benchmark by lower-bounding the efficiency through upper-bounding the WELF.

$$\mathrm{Eff}\big(b(\cdot)\big) \coloneqq \frac{\mathrm{OPT}_{\mathrm{cost}}}{\mathrm{WELF}\big(b(\cdot)\big)}.$$

Finally, we note that it will be useful to consider allocation probabilities, expected payments, and expected utility in smaller probability spaces, conditioning, for example, on agent i having the lowest cost with cost c_i. Given such an event $\mathcal{E}$, we will use the shorthand

$$\tilde{x}_i(b \mid \mathcal{E}) = \mathbb{E}_{c_{-i}}\left[\tilde{x}_i\big(b,\, b_{-i}(c_{-i})\big) \,\big|\, \mathcal{E}\right]$$

to denote the allocation probability of agent i given a bid of b conditioned on this event, and so on for payments and utilities.

3 Finding a lower bound on the Efficiency

In order to find a lower bound on the PoA, we need first to prove some Lemmas that would be later useful.

3.1 Lemma

Given any Bayes–Nash equilibrium of the first-price procurement auction, let i^* be the random variable $i^* = \arg\min_i b_i(c_i)$ (breaking ties arbitrarily). The expected welfare in any Bayes–Nash equilibrium of the first-price auction can be written as

$$\mathrm{WELF} = \mathbb{E}[c_i\, \mathbf{1}\{i \text{ wins}\}\mathbf{1}\{i \text{ has the minimum cost}\}] + \mathbb{E}\left[\sum_{j \neq i} c_j\, \mathbf{1}\{j \text{ wins}\}\right],$$

Expressing this in integral form yields the following:

$$\sum_{i=1}^{n} \int f_i(c_i)\, \Pr\big[\mathcal{E}_i(c_i)\big]\big(c_i\, x_i(c_i \mid \mathcal{E}_i(c_i)) + \mathbb{E}[c_{i^*} \mid \mathcal{E}_i(c_i),\, i^* \neq i] \Pr[i^* \neq i \mid \mathcal{E}_i(c_i)]\big)\, dc_i.$$

Given a cost profile $\mathbf{c}$, let $\tau_i(c_{-i})$ denote agent i's *threshold bid*. That is, $\tau_i(c_{-i})$ is the bid of the lowest bidder other than i; agent i wins iff $b_i(c_i) \leq \tau_i(c_{-i})$ (modulo tiebreaking). When i loses the auction, the winner pays their bid, which is $\tau_i(c_{-i})$ by definition. Upper bounding agent i's threshold bid will then translate to revenue and hence welfare. We produce such an upper bound with the following lemma:

3.2 Lemma

Let $\tau_i(\varrho, c_i)$ be the threshold bid with quantile ϱ in the distribution of $\tau_i(c_{-i})$ conditioned on $\mathcal{E}_i(c_i)$. That is, $\tau_i(\varrho, c_i)$ is the cost such that

$$\Pr\big[\tau_i(c_{-i}) \geq \tau_i(\varrho, c_i) \,\big|\, \mathcal{E}_i(c_i)\big] = \varrho.$$

Then, as long as $\tau_i(\varrho, c_i) \leq b_i(c_i)$,

$$\tau_i(\varrho, c_i) \;\leq\; c_i + \frac{u_i(c_i \mid \mathcal{E}_i(c_i))}{\varrho}. \tag{1}$$

Proof

Write $\tilde{x}_i(b)$ and $\tilde{x}_i(b \,|\, \mathcal{E}_i(c_i))$ for the (unconditional and conditional) probability that i wins when bidding b, and let $\tilde{u}_i(b)$ and $\tilde{u}_i(b \,|\, \mathcal{E}_i(c_i))$ be the corresponding expected utilities. Since this is a first-price procurement auction, $\tilde{u}_i(b) = (b - c_i)\tilde{x}_i(b)$ and, by independence across opponents,

$$\tilde{u}_i(b) = (b - c_i) \prod_{j \neq i} \big(1 - B_j(b)\big), \tag{2}$$

where B_j is the CDF of bids of opponent j.

For each j:

$$\Pr\big[b_j(c_j) \geq b \,\big|\, E_i(c_i)\big] = \min\left(\frac{1 - B_j(b)}{1 - F_j(c_i)}, 1\right),$$

Hence

$$\tilde{u}_i(b \,|\, E_i(c_i)) = (b - c_i) \prod_{j \neq i} \min\left(\frac{1 - B_j(b)}{1 - F_j(c_i)}, 1\right). \tag{3}$$

Condition on $\mathcal{E}_i(c_i) = \bigcap_{k \neq i}\{c_k \geq c_i\}$ and fix $j \neq i$. Since bids are (weakly) increasing in cost, let $b_j^{-1}(b) := \sup\{t : \, b_j(t) \leq b\}$ so that

$$\{b_j(c_j) \geq b\} \iff \{c_j \geq b_j^{-1}(b)\}.$$

Using conditional probability and independence across opponents,

$$\begin{aligned}
\Pr\big[b_j(c_j) \geq b \,\big|\, \mathcal{E}_i(c_i)\big] &= \frac{\Pr\big[b_j(c_j) \geq b, \; \mathcal{E}_i(c_i)\big]}{\Pr\big[\mathcal{E}_i(c_i)\big]} \\
&= \frac{\Pr\big[c_j \geq b_j^{-1}(b), \; c_j \geq c_i\big] \prod_{k \neq i,j} \Pr[c_k \geq c_i]}{\prod_{k \neq i} \Pr[c_k \geq c_i]} \\
&= \frac{\Pr\big[c_j \geq \max\{b_j^{-1}(b), c_i\}\big]}{\Pr[c_j \geq c_i]}.
\end{aligned}$$

Write F_j for the CDF of c_j. Then

$$\Pr\big[c_j \geq t\big] = 1 - F_j(t), \qquad \Pr[c_j \geq c_i] = 1 - F_j(c_i),$$

so

$$\Pr\big[b_j(c_j) \geq b \,\big|\, \mathcal{E}_i(c_i)\big] = \min\left(\frac{1 - F_j\big(b_j^{-1}(b)\big)}{1 - F_j(c_i)}, 1\right).$$

Since $B_j(b) = \Pr[b_j(c_j) \leq b] = F_j\big(b_j^{-1}(b)\big)$ is the bid CDF, this becomes

$$\Pr\big[b_j(c_j) \geq b \,\big|\, E_i(c_i)\big] = \min\left(\frac{1 - B_j(b)}{1 - F_j(c_i)}, 1\right).$$

Finally, conditional independence across j implies

$$\tilde{x}_i(b \,|\, \mathcal{E}_i(c_i)) = \prod_{j \neq i} \Pr\big[b_j(c_j) \geq b \,\big|\, \mathcal{E}_i(c_i)\big],$$

and therefore the conditional expected utility is

$$\tilde{u}_i(b \,|\, \mathcal{E}_i(c_i)) = (b - c_i) \prod_{j \neq i} \min\left(\frac{1 - B_j(b)}{1 - F_j(c_i)}, 1\right),$$

which is equation (3).

Define the ratio

$$R_i(b; c_i) \;:=\; \frac{\tilde{u}_i(b \,|\, E_i(c_i))}{\tilde{u}_i(b)} = \prod_{j \neq i} \min\left(\frac{1}{1 - B_j(b)}, \frac{1}{1 - F_j(c_i)}\right).$$

Because $B_j(b)$ is nondecreasing in b, each factor and hence $R_i(b; c_i)$ is nondecreasing in b.

Let $b^* = b_i(c_i)$ be bidder i's equilibrium bid. By optimality in BNE, $\tilde{u}_i(b^* \,|\, \mathcal{E}_i(c_i)) \geq \tilde{u}_i(b \,|\, \mathcal{E}_i(c_i))$ for all deviations b. In particular, for any $b \leq b^*$,

$$\tilde{u}_i(b^* \,|\, \mathcal{E}_i(c_i)) \;\geq\; \tilde{u}_i(b \,|\, \mathcal{E}_i(c_i)) = (b - c_i)\, \tilde{x}_i(b \,|\, \mathcal{E}_i(c_i)).$$

Take $b = \tau := \tau_i(\varrho, c_i)$ and assume $\tau \leq b^*$. By the definition of τ, $\tilde{x}_i(\tau \,|\, \mathcal{E}_i(c_i)) = \Pr[\tau_i(c_{-i}) \geq \tau \mid \mathcal{E}_i(c_i)] = \varrho$. Therefore,

$$u_i(c_i \,|\, E_i(c_i)) = \tilde{u}_i(b^* \,|\, E_i(c_i)) \;\geq\; (\tau - c_i)\, \varrho.$$

Rearranging yields (1).

3.3 Lemma

For any agent i with cost c_i, the following inequality holds:

$$\mathbb{E}[\, c_{i^*} \mid \mathcal{E}_i(c_i),\, i^* \neq i\,] \Pr[\, i^* \neq i \mid \mathcal{E}_i(c_i)\,] \;\leq\; c_i\Big(1 - x_i(c_i \mid \mathcal{E}_i(c_i))\Big) - u_i(c_i \mid \mathcal{E}_i(c_i)) \ln(x_i(c_i \mid \mathcal{E}_i(c_i)))\,.$$

Proof.

Consider a realization of $\mathbf{c}$ in which i does not win. This means that i's bid is more than its threshold bid $\tau_i(c_{-i})$, while the bid of the winner, i^*, is at most $\tau_i(c_{-i})$.

Since agents don't overbid, we have $c_{i^*} \leq \tau_i(c_{-i})$. We therefore have that

$$\mathbb{E}[c_{i^*} \mid \mathcal{E}_i(c_i),\, i^* \neq i] \Pr[i^* \neq i \mid \mathcal{E}_i(c_i)] \;\leq\; \mathbb{E}[\tau_i(c_{-i}) \mid \mathcal{E}_i(c_i),\, i^* \neq i] \Pr[i^* \neq i \mid \mathcal{E}_i(c_i)]\,. \tag{6}$$

To evaluate the quantity on the right-hand side of 6, we will integrate over the quantile space of agent i's threshold bid $\tau_i(c_{-i})$. We obtain the following sequence of inequalities:

$$
\begin{aligned}
&\mathbb{E}[\tau_i(c_{-i}) \,|\, \mathcal{E}_i(c_i),\, i^\star \neq i] \Pr[i^\star \neq i \,|\, \mathcal{E}_i(c_i)] \\
&\quad = \int_0^\infty \Pr[\tau_i(c_{-i})\, \mathbf{1}\{i^\star \neq i\} > y \,|\, \mathcal{E}_i(c_i)]\; dy \qquad \text{(tail–integral identity)} \\
&\quad = \int_0^\infty \Pr[\tau_i(c_{-i}) > y,\ \tau_i(c_{-i}) \le b_i(c_i) \,|\, \mathcal{E}_i(c_i)]\; dy \qquad (\text{since } i^\star \neq i \iff \tau_i(c_{-i}) \le b_i(c_i)) \\
&\quad = \int_0^\infty \Big(\Pr[\tau_i(c_{-i}) > y \,|\, \mathcal{E}_i(c_i)] - \Pr[\tau_i(c_{-i}) \ge b_i(c_i) \,|\, \mathcal{E}_i(c_i)] \Big)\, dy \\
&\quad = \int_0^1 \tau_i(q, c_i)\, dq - \int_0^{x_i(c_i \,|\, \mathcal{E}_i(c_i))} \tau_i(q, c_i)\, dq \qquad \text{(quantile–integral identity)} \\
&\quad = \int_{x_i(c_i \,|\, \mathcal{E}_i(c_i))}^1 \tau_i(q, c_i)\, dq \;=\; \int_{x_i(c_i \,|\, \mathcal{E}_i(c_i))}^1 \tau_i(\varrho, c_i)\, d\varrho \qquad (\text{rename } q \to \varrho) \\
&\quad \le \int_{x_i(c_i \,|\, \mathcal{E}_i(c_i))}^1 \left(c_i + \frac{u_i(c_i \,|\, \mathcal{E}_i(c_i))}{\varrho} \right) d\varrho \qquad \text{(by Lemma 3.2)} \\
&\quad = c_i \Big[\varrho\Big]_{x_i(c_i \,|\, \mathcal{E}_i(c_i))}^1 \;+\; u_i(c_i \,|\, \mathcal{E}_i(c_i)) \Big[\log \varrho\Big]_{x_i(c_i \,|\, \mathcal{E}_i(c_i))}^1 \\
&\quad = c_i(1 - x_i(c_i \,|\, \mathcal{E}_i(c_i))) \;-\; u_i(c_i \,|\, \mathcal{E}_i(c_i)) \log\Big(x_i(c_i \,|\, \mathcal{E}_i(c_i)) \Big).
\end{aligned}
$$

3.4 Final Theorem

Plugging the lower bound of Lemma 3.3 into the welfare breakdown of Lemma 3.1 yields:

$$
\begin{aligned}
&\sum_{i=1}^n \int f_i(c_i)\, \Pr\big[\mathcal{E}_i(c_i)\big] \Big[c_i\, x_i(c_i \,|\, \mathcal{E}_i(c_i)) + \mathbb{E}[c_{i^\star} \,|\, \mathcal{E}_i(c_i),\, i^\star \neq i] \Pr[i^\star \neq i \,|\, \mathcal{E}_i(c_i)] \Big]\, dc_i \\
&\le \sum_{i=1}^n \int f_i(c_i)\, \Pr\big[\mathcal{E}_i(c_i)\big] \Big[c_i\, x_i(c_i \,|\, \mathcal{E}_i(c_i)) + c_i(1 - x_i(c_i \mid \mathcal{E}_i(c_i))) - u_i(c_i \mid \mathcal{E}_i(c_i)) \log x_i(c_i \mid \mathcal{E}_i(c_i)) \Big]\, dc_i \\
&= \sum_{i=1}^n \int f_i(c_i)\, \Pr\big[\mathcal{E}_i(c_i)\big] \Big[c_i - u_i(c_i \mid \mathcal{E}_i(c_i)) \log x_i(c_i \mid \mathcal{E}_i(c_i)) \Big]\, dc_i \\
&= \sum_{i=1}^n \int f_i(c_i)\, \Pr\big[\mathcal{E}_i(c_i)\big] \Big[c_i - \big(b_i - c_i\big)\, x_i\big(c_i \mid \mathcal{E}_i(c_i)\big)\, \log x_i\big(c_i \mid \mathcal{E}_i(c_i)\big) \Big]\, dc_i \\
&\le \sum_{i=1}^n \int f_i(c_i)\, \Pr\big[\mathcal{E}_i(c_i)\big] \Big[c_i + \big(b_i - c_i\big) \cdot e^{-1} \Big]\, dc_i \qquad \text{since } \max_{x \in (0,1]} \{-x \log x\} = e^{-1}. \\
&\le \sum_{i=1}^n \int f_i(c_i)\, \Pr\big[\mathcal{E}_i(c_i)\big]\, c_i \left[1 + \Big(\frac{\overline{c}}{\underline{c}} - 1 \Big) e^{-1} \right] dc_i \\
&\le \left[1 + \Big(\frac{\overline{c}}{\underline{c}} - 1 \Big) e^{-1} \right] \sum_{i=1}^n \int f_i(c_i)\, \Pr\big[\mathcal{E}_i(c_i)\big]\, c_i\, dc_i \\
&= \left[1 + \Big(\frac{\overline{c}}{\underline{c}} - 1 \Big) e^{-1} \right] \mathrm{OPT} \;=\; \left[1 + \Big(\frac{\overline{c}}{\underline{c}} - 1 \Big) e^{-1} \right] \mathbb{E}\Big[\min_j c_j \Big].
\end{aligned}
$$

Where the second inequality follows from Lemma 3.3 and the sixth follows from the following derivation under the assumption of i.i.d cost functions.

Assume i.i.d. costs with CDF F (density f) on $[\underline{c}, \overline{c}]$, and a strictly increasing symmetric BNE with bid function $b(\cdot)$ (so $b_i(c_i) = b(c_i)$ for all i). Let $B_j(b)$ denote the CDF of opponent j's bid, as in the text. When agent i bids b, her win probability is

$$\tilde{x}_i(b) = \prod_{j \neq i} \bigl(1 - B_j(b)\bigr).$$

In the symmetric monotone equilibrium we have $B_j(b) = F\bigl(b^{-1}(b)\bigr)$, hence

$$\tilde{x}_i\bigl(b(c_i)\bigr) = \bigl(1 - F(c_i)\bigr)^{n-1}.$$

Her interim expected utility when bidding b with cost c_i is $\tilde{u}_i(b; c_i) = (b - c_i)\, \tilde{x}_i(b)$. FOC at $b = b(c_i)$ gives

$$0 = \frac{\partial \tilde{u}_i}{\partial b} = b'(c_i)\ \tilde{x}_i(b) + (b - c_i)\, \tilde{x}_i'(b) \quad \Longrightarrow \quad b'(c_i) = (n-1)\, \frac{f(c_i)}{1 - F(c_i)} \bigl(b(c_i) - c_i\bigr).$$

Let $m(c_i) := b(c_i) - c_i$. Then

$$m'(c_i) - (n-1)\, \frac{f(c_i)}{1 - F(c_i)}\, m(c_i) = -1. \tag{1}$$

Multiplying (1) by the integrating factor $\bigl(1 - F(c_i)\bigr)^{n-1}$ and integrating up to $\overline{c}$ (using the boundary condition $\lim_{x \uparrow \overline{c}} (1 - F(x))^{n-1} m(x) = 0$) yields the standard representation

$$\boxed{b(c_i) - c_i = \frac{\displaystyle\int_{c_i}^{\overline{c}} \bigl(1 - F(t)\bigr)^{n-1} dt}{\bigl(1 - F(c_i)\bigr)^{n-1}}} \tag{2}$$

Since $1 - F(t) \leq 1 - F(c_i)$ for all $t \geq c_i$,

$$0 \leq b(c_i) - c_i = \frac{\int_{c_i}^{\overline{c}} (1 - F(t))^{n-1} dt}{(1 - F(c_i))^{n-1}} \leq \frac{(\overline{c} - c_i)\, (1 - F(c_i))^{n-1}}{(1 - F(c_i))^{n-1}} = \overline{c} - c_i.$$

In particular, with bounded support and positive $\underline{c}$ we always have $b_i(c_i) \leq \overline{c} \leq \frac{\overline{c}}{\underline{c}} c_i$, since $c_i \in [\underline{c}, \overline{c}]$.

Reusing the auction theory argument. Consider a first–price procurement auction with n bidders. Each bidder i has a private cost $c_i \in [\underline{c}, \bar{c}]$ and submits a bid b_i.

Define a corresponding first–price *value* auction by setting, for each bidder i,

$$v_i := \bar{c} - c_i \in [0, \bar{c} - \underline{c}], \qquad \tilde{b}_i := \bar{c} - b_i.$$

The lowest bid in the procurement auction coincides with the highest bid in the value auction, and for each bidder i the utility is preserved:

$$u_i^{\text{proc}}(b, c) = (b_i - c_i)\,\mathbf{1}\{i \text{ wins in procurement}\} = (v_i - \tilde{b}_i)\,\mathbf{1}\{i \text{ wins in value}\} = u_i^{\text{val}}(\tilde{b}, v).$$

Thus every Bayes–Nash equilibrium of the procurement game corresponds to a Bayes–Nash equilibrium of the value game with the same allocation and utilities in the standard auction setting.

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Jin and Lu (2023) proved that a first price auction is at least $1 - \frac{1}{e^2}$ efficient. Consequently, at any Bayes–Nash equilibrium of the value auction,

$$\mathbb{E}\big[v_j^{\text{eq}}\mathbf{1}\{j \text{ wins}\}\big] \;\geq\; \Big(1 - \tfrac{1}{e^2}\Big)\,\mathbb{E}\big[\max_i v_i\big].$$

Substitute $v_i = \bar{c} - c_i$. At equilibrium,

$$v_j^{\text{eq}} = \bar{c} - c_j^{\text{eq}}, \qquad \max_i v_i = \bar{c} - c^*, \qquad c^* = \min_i c_i$$

Hence

$$\mathbb{E}[\bar{c} - c^{\text{eq}}] \;\geq\; \Big(1 - \tfrac{1}{e^2}\Big)\,\mathbb{E}[\bar{c} - c^*].$$

Rearranging gives

$$\begin{aligned}
\mathbb{E}[c^{\text{eq}}] &\leq \bar{c} - \Big(1 - \tfrac{1}{e^2}\Big)\bar{c} + \Big(1 - \tfrac{1}{e^2}\Big)\mathbb{E}[c^*] \\
&= \Big(1 - \tfrac{1}{e^2}\Big)\mathbb{E}[c^*] + \tfrac{1}{e^2}\,\bar{c}, \\
&\implies \frac{\mathbb{E}[c^{\text{eq}}]}{\mathbb{E}[c^*]} \leq 1 + \frac{1}{e^2}(\frac{\bar{c}}{\underline{c}} - 1)
\end{aligned}$$

Where the multiplicative bound follows from $\mathbb{E}[c^*] \geq \underline{c}$.

What does not work

$$\mathbb{E}\big[\text{cost of } t\text{-threshold strategy}\big] \;\le\; (1-q(t))\cdot t - \sum_{i=1}^{n} \mathbb{E}\big[t-c_i \,\big|\, c_i \le t,\; c_j > t\,\forall j \ne i\big]\;\Pr[c_i \le t]\;\Pr[c_j > t\,\forall j \ne i] \tag{2'}$$

$$= \;(1-q(t))\cdot t \;-\; \sum_{i=1}^{n} \underbrace{\mathbb{E}\big[(t-c_i)^+\big]}_{=\;\mathbb{E}[\,t-c_i|c_i\le t\,]\Pr[c_i\le t]} \;\underbrace{\Pr[c_j > t\;\forall j \ne i]}_{\ge\; q(t)} \tag{3'}$$

$$\le\; (1-\bar{q}(t))\cdot t \;-\; \bar{q}(t)\sum_{i=1}^{n}\mathbb{E}\big[(t-c_i)^+\big]. \tag{4'}$$

Prophet (minimum) lower bound. For every t,

$$\begin{aligned}
\mathbb{E}\Big[\min_i c_i\Big] &= \mathbb{E}\Big[t + \min_i (c_i - t)\Big] \\
&= t + \mathbb{E}\Big[\min_i (c_i - t)\Big] \\
&= t - \mathbb{E}\Big[\max_i (t - c_i)\Big] \\
&\ge t - \mathbb{E}\Big[\max_i (t - c_i)^+\Big] \\
&\ge t - \sum_{i=1}^{n}\mathbb{E}\big[(t-c_i)^+\big] \qquad (5')
\end{aligned}$$

Combine (4') **and** (5')**.**

$$\mathbb{E}\big[\text{cost of } t\text{-threshold strategy}\big] \le\; q(t)\,\mathbb{E}\Big[\min_i c_i\Big] \;+\; \big(-2q(t)+1\big)\,t. \tag{3}$$

Choosing t to be a median of M so that $q(t) = \frac{1}{2}$. Then the t–term in (3) vanishes and

$$\mathbb{E}\big[\text{cost of } t\text{-threshold strategy}\big] \le\; \tfrac{1}{2}\,\mathbb{E}\Big[\min_i c_i\Big]$$

which is **impossible**.

References

Hoy, D., Taggart, S., & Wang, Z. (2018). A tighter welfare guarantee for first-price auctions. *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*, 132–137. https://doi.org/10.1145/3188745.3188944

Jin, Y., & Lu, P. (2023). First price auction is 1-1/e2 efficient. *J. ACM*, *70*(5). https://doi.org/10.1145/3617902

Roughgarden, T. (2015). Intrinsic robustness of the price of anarchy. *J. ACM*, *62*(5). https://doi.org/10.1145/2806883

Syrgkanis, V., & Tardos, E. (2013). Composable and efficient mechanisms. *Proceedings of the Forty-Fifth Annual ACM Symposium on Theory of Computing*, 211–220. https://doi.org/10.1145/2488608.2488635