PoA in First Price Procurement Auction

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February 17, 2026

Abstract

We study the efficiency of Bayes Nash equilibria in first price procurement (reverse) auctions through the lens of Price of Anarchy (PoA), defined as the ratio between the equilibrium expected social cost and the optimal expected social cost. We derive an explicit finite PoA bound under regularity assumptions on bidders' bid distributions, in particular a monotone hazard rate (MHR) condition. We then show that this guarantee is not universal by constructing a family of instances violating MHR for which the PoA becomes unbounded. Finally, for costs supported on a bounded interval $[\underline{c}, \bar{c}]$, we provide a reduction to the standard first price value auction and obtain a corresponding PoA bound for the bounded-support case.

1 Introduction

Recent work in mechanism design has advanced the worst–case analysis of resource–allocation mechanisms through *price of anarchy* (PoA) guarantees [1]. These results bound the welfare attained in (Bayes–Nash) equilibrium without requiring an explicit solution of the equilibria. For example, in the standard (selling) first–price auction, recent results show that any equilibrium achieves at least $1 - 1/e^2$ of the optimal social welfare [2].

By contrast, analogous guarantees for the *reverse* setting—procurement mechanisms in which agents submit a single offer and the lowest compliant bid wins—are largely missing. This project addresses that gap for first–price procurement auctions.

Classical PoA bounds for selling auctions often rely on the smoothness framework [3, 4], which decomposes welfare into bidders' utilities plus the mechanism's revenue and then derives a lower bound on their sum via a deviation argument. In procurement, however, "revenue" becomes the buyer's expenditure with the opposite sign, so this decomposition fails to yield a useful bound since this quantity could be driven to infinity in some equilibria.

Following [5], we instead analyze a different pair of quantities: (i) the welfare contributed by *rightful winners*—agents who win and have the lowest cost—and (ii) the welfare from *improper allocations*—agents who win despite not having the lowest cost. Under independent private costs and monotone bidding, we show that the expected cost of improper allocations can be upper bounded by the mechanism's expected payments only under a certain class of bid distributions with MHR. This yields a clean trade-off between rightful-winner welfare and payments, from which we derive welfare guarantees for first–price procurement auctions.

2 Technical Preliminaries

We now lay the formal groundwork for our result. This paper analyzes the *single-item sealed-bid first-price procurement auction.* In such an auction a single item is sold to n agents. Each agent i simultaneously submits a bid b_i to the auctioneer. The agent $i^\star$ with the lowest bid wins the item, and pays their bid $b_{i^\star}$. All other agents pay nothing and win nothing. Let $\tilde{x}_i(b)$ denote the probability for whether agent i is allocated under bid profile b, and let $\tilde{p}_i(b)$ denote the payments received by agent i under that same bid profile. Each agent evaluates their allocation and payment using the linear utility function $\tilde{u}_i(b) = (b_i - c_i)\tilde{x}_i(b)$, where c_i is agent i's true cost for providing the product/service.

We consider a Bayesian environment, in which each agent i's cost is drawn independently from a distribution with CDF F_i and density f_i. Note that we do not require agents' cost distributions to be identical. We assume agents' costs are private and independent, but that the prior distributions are common knowledge.

We adopt the standard solution concept of Bayes–Nash equilibrium (BNE). Informally, a BNE is a strategy mapping $b_i(\cdot)$ from costs to bids for every agent such that each agent's bid given their cost maximizes their expected utility given the strategies of other bidders. Formally, given a profile $b(\cdot)$ of bidding strategies for each agent, define the interim allocation probability of agent i bidding b to be

$$\tilde{x}_i(b) = \mathbb{E}_{c_{-i}}\left[\tilde{x}_i\big(b,\, b_{-i}(c_{-i})\big)\right].$$

Similarly, define the interim expected payments of agent i to be

$$\tilde{p}_i(b) = \mathbb{E}_{c_{-i}}\left[\tilde{p}_i\big(b,\, b_{-i}(c_{-i})\big)\right].$$

Define the interim expected utility $\tilde{u}_i(b)$ similarly. A profile of bidding strategies $b(\cdot)$ is a BNE if for every agent i with cost c_i, the following best response inequality holds for every alternate bid b:

$$\tilde{u}_i\big(b_i(c_i)\big) \;\geq\; \tilde{u}_i(b).$$

In what follows, we argue assuming agents are bidding according to an arbitrary BNE profile of bidding strategies. Since the strategies map costs to bids and bids are mapped to allocation and payments, we will often consider allocations, payments, and utilities as a function of cost, taking the bid functions as implicit. Formally, we will let

$$\begin{aligned}
x_i(c_i) &= \mathbb{E}_{c_{-i}}\left[\tilde{x}_i\big(b_i(c_i),\, b_{-i}(c_{-i})\big)\right],\\
p_i(c_i) &= \mathbb{E}_{c_{-i}}\left[\tilde{p}_i\big(b_i(c_i),\, b_{-i}(c_{-i})\big)\right],\\
u_i(c_i) &= \mathbb{E}_{c_{-i}}\left[\tilde{u}_i\big(b_i(c_i),\, b_{-i}(c_{-i})\big)\right].
\end{aligned}$$

Note that we use tildes when the argument to the function is a bid, and omit the tildes when an argument to the function is the cost instead.

We study the objective of utilitarian social welfare. In such setting the natural equivalent of the social welfare would be the social cost which could be computed as the expected cost of the winner. In other words,

$$\mathrm{SC}\big(b(\cdot)\big) = \sum_i c_i\, x_i(c_i).$$

As our benchmark, we use the efficient (first-best) cost—the expected minimum cost across bidders: $\mathrm{OPT}_{\mathrm{cost}} := \mathbb{E}_{\mathbf{c}}[\min_i c_i]$. This is the welfare of the mechanism which always allocates the agent with the lowest cost. We state performance relative to the cost benchmark by aiming at upper-bounding the efficiency of the mechanism allocation through the Price of Anarchy.

$$\mathrm{PoA}\big(b(\cdot)\big) \coloneqq \frac{\mathrm{SC}\big(b(\cdot)\big)}{\mathrm{OPT}_{\mathrm{cost}}}. \tag{1}$$

Finally, we note that it will be useful to consider allocation probabilities, expected payments, and expected utility in smaller probability spaces, conditioning, for example, on agent i having the lowest cost c_i. Given such an event $\mathcal{E} := \min_k c_k = c_i$, we will use the shorthand

$$\tilde{x}_i(b \mid \mathcal{E}) = \mathbb{E}_{c_{-i}}\left[\tilde{x}_i\big(b,\, b_{-i}(c_{-i})\big) \;\middle|\; \mathcal{E}\right]$$

to denote the allocation probability of agent i given a bid of b conditioned on this event, and so on for payments and utilities.

3 Finding a lower bound on the Efficiency

For the scope of finding a lower bound on the PoA, we need first to prove some Lemmas that would be later useful.

Lemma 3.1. *Given any Bayes–Nash equilibrium of the first-price procurement auction, let $i^\star$ be the random variable $i^\star = \arg\min_i b_i(c_i)$ (breaking ties arbitrarily). The expected welfare in any Bayes–Nash equilibrium of the first-price auction can be written as*

$$\mathrm{WELF} = \mathbb{E}[c_i\, \mathbf{1}\{i \text{ wins}\}\mathbf{1}\{i \text{ has the minimum cost}\}] + \mathbb{E}\left[\sum_{j\neq i} c_j\, \mathbf{1}\{j \text{ wins}\}\right].$$

Expressing this in integral form yields the following:

$$\sum_{i=1}^{n} \int f_i(c_i)\, \Pr\big[\mathcal{E}_i(c_i)\big]\Bigg(c_i\, x_i(c_i \mid \mathcal{E}_i(c_i)) + \mathbb{E}[\, c_{i^\star} \mid \mathcal{E}_i(c_i),\, i^\star \neq i]\Pr[i^\star \neq i \mid \mathcal{E}_i(c_i)]\Bigg)\, dc_i.$$

Given a cost profile $\mathbf{c}$, let $\tau_i(c_{-i})$ denote agent i's *threshold bid*. That is, $\tau_i(c_{-i})$ is the bid of the lowest bidder other than i; agent i wins iff $b_i(c_i) \leq \tau_i(c_{-i})$ (modulo tiebreaking). When i loses the auction, the winner pays their bid, which is $\tau_i(c_{-i})$ by definition. Upper bounding agent i's threshold bid will then translate to revenue and hence welfare. We produce such an upper bound with the following lemma:

Lemma 3.2. *Let $\tau_i(\varrho, c_i)$ be the threshold bid with quantile ϱ in the distribution of $\tau_i(c_{-i})$ conditioned on $\mathcal{E}_i(c_i)$. That is, $\tau_i(\varrho, c_i)$ is the cost such that*

$$\Pr\bigl[\tau_i(c_{-i}) \geq \tau_i(\varrho, c_i) \,\big|\, \mathcal{E}_i(c_i)\bigr] = \varrho.$$

Then, as long as $\tau_i(\varrho, c_i) \leq b_i(c_i)$,

$$\tau_i(\varrho, c_i) \;\leq\; c_i + \frac{u_i(c_i \mid \mathcal{E}_i(c_i))}{\varrho}\,. \tag{2}$$

Proof. Let $\tilde{x}_i(b)$ and $\tilde{u}_i(b)$ denote bidder i's (unconditional) winning probability and expected utility when bidding b, and let $\tilde{x}_i(b \,|\, \mathcal{E}_i(c_i))$ and $\tilde{u}_i(b \,|\, \mathcal{E}_i(c_i))$ be the corresponding quantities conditional on $\mathcal{E}_i(c_i)$. In a first-price procurement auction, $\tilde{u}_i(b) = (b - c_i)\tilde{x}_i(b)$ and independence across opponents implies $\tilde{x}_i(b) = \prod_{j\neq i}(1 - B_j(b))$, hence $\tilde{u}_i(b) = (b - c_i)\prod_{j\neq i}(1 - B_j(b))$.

Fix $j \neq i$ and write $b_j^{-1}(b) := \sup\{t : b_j(t) \leq b\}$. Since bids are weakly increasing in costs, $\{b_j(c_j) \geq b\} \iff \{c_j \geq b_j^{-1}(b)\}$. Conditioning on $\mathcal{E}_i(c_i) = \bigcap_{k\neq i}\{c_k \geq c_i\}$ and using independence, $\Pr[b_j(c_j) \geq b \,|\, \mathcal{E}_i(c_i)] = \Pr[c_j \geq \max\{b_j^{-1}(b), c_i\}]/\Pr[c_j \geq c_i]$. With F_j the CDF of c_j this equals $\min\Bigl(\frac{1-F_j(b_j^{-1}(b))}{1-F_j(c_i)}, 1\Bigr)$, and since $B_j(b) = F_j(b_j^{-1}(b))$ we obtain $\Pr[b_j(c_j) \geq b \,|\, \mathcal{E}_i(c_i)] = \min\Bigl(\frac{1-B_j(b)}{1-F_j(c_i)}, 1\Bigr)$. Conditional independence across j then yields $\tilde{x}_i(b \,|\, \mathcal{E}_i(c_i)) = \prod_{j\neq i}\min\Bigl(\frac{1-B_j(b)}{1-F_j(c_i)}, 1\Bigr)$ and $\tilde{u}_i(b \,|\, \mathcal{E}_i(c_i)) = (b - c_i)\tilde{x}_i(b \,|\, \mathcal{E}_i(c_i))$.

Let $b^* = b_i(c_i)$ be bidder i's equilibrium bid. Define the ratio $R_i(b; c_i) := \frac{\tilde{u}_i(b \,|\, \mathcal{E}_i(c_i))}{\tilde{u}_i(b)} = \prod_{j\neq i}\min\Bigl(\frac{1}{1-B_j(b)}, \frac{1}{1-F_j(c_i)}\Bigr)$. Because $B_j(b)$ is nondecreasing in b, each factor and hence $R_i(b; c_i)$ is nondecreasing in b. But since $\tilde{u}_i(b^*) \geq u_i(b)$ for any deviation bid $b \leq b^*$ by the definition of BNE, this implies that $\tilde{u}_i(b^* \,|\, \mathcal{E}_i(c_i)) \geq \tilde{u}_i(b \,|\, \mathcal{E}_i(c_i)) = (b - c_i)\tilde{x}_i(b \,|\, \mathcal{E}_i(c_i))$.

Take $b = \tau := \tau_i(\varrho, c_i)$ and assume $\tau \leq b^*$. By definition of τ, $\tilde{x}_i(\tau \,|\, \mathcal{E}_i(c_i)) = \Pr[\tau_i(c_{-i}) \geq \tau \,|\, \mathcal{E}_i(c_i)] = \varrho$, hence $u_i(c_i \,|\, \mathcal{E}_i(c_i)) = \tilde{u}_i(b^* \,|\, \mathcal{E}_i(c_i)) \geq (\tau - c_i)\varrho$, which rearranges to 2. □

Lemma 3.3. *For any agent i with cost c_i, the following inequality holds:*

$$\mathbb{E}[\, c_{i^\star} \mid \mathcal{E}_i(c_i),\, i^\star \neq i\,]\Pr[\, i^\star \neq i \mid \mathcal{E}_i(c_i)\,] \;\leq\; c_i\Bigl(1 - x_i(c_i \mid \mathcal{E}_i(c_i))\Bigr) - u_i(c_i \mid \mathcal{E}_i(c_i))\ln(x_i(c_i \mid \mathcal{E}_i(c_i)))\,.$$

Proof. Fix i and condition on $\mathcal{E}_i(c_i)$. On any realization in which i does not win, we have $b_i(c_i) > \tau_i(c_{-i})$ and the winner's bid is at most $\tau_i(c_{-i})$; since agents do not underbid, $c_{i^\star} \leq \tau_i(c_{-i})$ on $\{i^\star \neq i\}$. Therefore, $\mathbb{E}[c_{i^\star} \mid \mathcal{E}_i(c_i),\, i^\star \neq i]\Pr[i^\star \neq i \mid \mathcal{E}_i(c_i)] \leq \mathbb{E}[\tau_i(c_{-i}) \mid \mathcal{E}_i(c_i),\, i^\star \neq i]\Pr[i^\star \neq i \mid \mathcal{E}_i(c_i)]$.

Using $\mathbb{E}[Z \mid A]\Pr[A] = \mathbb{E}[Z\mathbf{1}_A]$ and the tail-integral identity, and adopting the convention $i^\star \neq i \iff \tau_i(c_{-i}) \leq b_i(c_i)$,

$$\begin{aligned}
&\mathbb{E}[\tau_i(c_{-i}) \mid \mathcal{E}_i(c_i),\, i^\star \neq i]\Pr[i^\star \neq i \mid \mathcal{E}_i(c_i)] \\
&= \mathbb{E}[\tau_i(c_{-i})\mathbf{1}\{i^\star \neq i\} \mid \mathcal{E}_i(c_i)] \\
&= \int_0^\infty \Pr[\tau_i(c_{-i})\mathbf{1}\{i^\star \neq i\} > y \mid \mathcal{E}_i(c_i)]\,dy \\
&= \int_0^\infty \Pr[\tau_i(c_{-i}) > y,\; \tau_i(c_{-i}) \leq b_i(c_i) \mid \mathcal{E}_i(c_i)]\,dy.
\end{aligned}$$

Let $\tau_i(\varrho, c_i)$ denote the ϱ-quantile of $\tau_i(c_{-i})$ conditional on $\mathcal{E}_i(c_i)$, i.e. $\Pr[\tau_i(c_{-i}) \geq \tau_i(\varrho, c_i) \mid \mathcal{E}_i(c_i)] = \varrho$. Since $x_i(c_i \mid \mathcal{E}_i(c_i)) = \Pr[\tau_i(c_{-i}) \geq b_i(c_i) \mid \mathcal{E}_i(c_i)]$, we have $b_i(c_i) = \tau_i(x_i(c_i \mid \mathcal{E}_i(c_i)), c_i)$, and the quantile-integral identity gives

$$\begin{aligned}
&\int_0^\infty \Pr[\tau_i(c_{-i}) > y,\; \tau_i(c_{-i}) \leq b_i(c_i) \mid \mathcal{E}_i(c_i)]\,dy \\
&= \int_{x_i(c_i|\mathcal{E}_i(c_i))}^1 \tau_i(\varrho, c_i)\,d\varrho.
\end{aligned}$$

By Lemma 3.2, $\tau_i(\varrho, c_i) \leq c_i + \frac{u_i(c_i|\mathcal{E}_i(c_i))}{\varrho}$ for all $\varrho \in [x_i(c_i \mid \mathcal{E}_i(c_i)), 1]$, hence

$$\begin{aligned}
\int_{x_i}^1 \tau_i(\varrho, c_i)\,d\varrho &\leq \int_{x_i}^1 \left(c_i + \frac{u_i}{\varrho}\right)d\varrho \\
&= c_i\int_{x_i}^1 d\varrho \;+\; u_i\int_{x_i}^1 \frac{1}{\varrho}\,d\varrho \\
&= c_i\bigl[\varrho\bigr]_{x_i}^1 \;+\; u_i\bigl[\ln\varrho\bigr]_{x_i}^1 \\
&= c_i(1 - x_i) \;-\; u_i\ln(x_i),
\end{aligned}$$

where $x_i := x_i(c_i \mid \mathcal{E}_i(c_i))$ and $u_i := u_i(c_i \mid \mathcal{E}_i(c_i))$. Combining the bounds yields the claim. □

Theorem 3.4 (PoA bound via monopoly prices)**.** *Assume an n-bidder first-price procurement auction with independent costs and an atomless joint distribution. Let $\mathrm{SC}_{\mathrm{OPT}} := \mathbb{E}[\min_i c_i]$ denote the optimal expected social* *cost. For each opponent j, let B_j be the CDF of bids and assume B_j is MHR, i.e. its hazard rate $h_{B_j}(b) := \frac{B_j'(b)}{1-B_j(b)}$ is nondecreasing. Define the (bid-side) monopoly price $r_j \in \arg\max_{b\geq 0} b(1-B_j(b))$ and let $r := \max_j r_j$. Then the Price of Anarchy satisfies*

$$\mathrm{PoA} \;=\; \frac{\mathbb{E}[\mathrm{SC}_{\mathrm{BNE}}]}{\mathbb{E}[\mathrm{SC}_{\mathrm{OPT}}]} \;\leq\; 1+\frac{r/e}{\mathbb{E}[\min_i c_i]}\,.$$

Proof. Fix i and condition on $\mathcal{E}_i(c_i)$. Using Lemma 3.3 (the bound on the losing-winner term) and the notation $x_i(c_i \,|\, \mathcal{E}_i(c_i))$ and $u_i(c_i \,|\, \mathcal{E}_i(c_i))$, we can upper bound the equilibrium expected social cost as

$$\begin{aligned}
\mathbb{E}[\mathrm{SC}_{\mathrm{BNE}}] &= \sum_{i=1}^{n} \int f_i(c_i)\ \Pr[\mathcal{E}_i(c_i)] \Big(c_i\, x_i(c_i \,|\, \mathcal{E}_i(c_i)) \\
&\quad + \mathbb{E}[c_{i^\star} \mid \mathcal{E}_i(c_i),\, i^\star \neq i] \Pr[i^\star \neq i \mid \mathcal{E}_i(c_i)] \Big)\, dc_i \\
&\leq \sum_{i=1}^{n} \int f_i(c_i)\ \Pr[\mathcal{E}_i(c_i)] \Big(c_i \\
&\quad - u_i(c_i \,|\, \mathcal{E}_i(c_i)) \ln x_i(c_i \,|\, \mathcal{E}_i(c_i)) \Big)\, dc_i \\
&\leq \sum_{i=1}^{n} \int f_i(c_i)\ \Pr[\mathcal{E}_i(c_i)] \Big(c_i + \\
&\quad (b_i(c_i) - c_i) \cdot \max_{x \in (0,1]} (-x \ln x) \Big)\, dc_i \\
&= \sum_{i=1}^{n} \int f_i(c_i)\ \Pr[\mathcal{E}_i(c_i)] \Big(c_i + \tfrac{1}{e}(b_i(c_i) - c_i) \Big)\, dc_i.
\end{aligned}$$

For each (i, c_i), define $x_i(b, c_i) := x_i(b \,|\, \mathcal{E}_i(c_i))$ and $u_i(b; c_i) := (b - c_i) x_i(b, c_i)$. The BNE first-order condition gives $b_i(c_i) - c_i = \frac{1}{-\partial_b \ln x_i(b,c_i)|_{b=b_i(c_i)}}$. Moreover, writing $\bar{B}_j(b) := 1 - B_j(b)$ and $\bar{F}_j(c) := 1 - F_j(c)$ and defining the active set $\mathcal{A}_i(b, c_i) := \{j \neq i \,:\, \bar{B}_j(b) \leq \bar{F}_j(c_i)\}$, we have (away from kink points) $-\partial_b \ln x_i(b, c_i) = \sum_{j \in \mathcal{A}_i(b,c_i)} h_{B_j}(b)$. Hence $\mathbb{E}[\mathrm{SC}_{\mathrm{BNE}}]$ is upper bounded by:

$$\sum_{i=1}^{n} \int f_i(c_i)\ \Pr[\mathcal{E}_i(c_i)] \Big(c_i + \tfrac{1}{e} \cdot \frac{1}{\sum_{j \in \mathcal{A}_i(b_i(c_i), c_i)} h_{B_j}(b_i(c_i))} \Big)\, dc_i.$$

Let $r_j \in \arg\max_{b \geq 0} b\bar{B}_j(b) \implies h_{B_j}(r_j) = \frac{1}{r_j}$. Define $r := \max_j r_j$. Under MHR, $h_{B_j}(b) \geq 1/r$ for all $b \geq r$ and all j, and if $b_i(c_i) \leq r$ then trivially $b_i(c_i) - c_i \leq r$. In either case, $\frac{1}{\sum_{j \in \mathcal{A}_i(b_i(c_i),c_i)} h_{B_j}(b_i(c_i))} \leq r$, so

$$\begin{aligned}
\mathbb{E}[\mathrm{SC}_{\mathrm{BNE}}] &\leq \sum_{i=1}^{n} \int f_i(c_i)\ \Pr[\mathcal{E}_i(c_i)] \Big(c_i + \tfrac{r}{e} \Big)\, dc_i \\
&= \sum_{i=1}^{n} \int f_i(c_i)\ \Pr[\mathcal{E}_i(c_i)]\, c_i\, dc_i \;+\; \tfrac{r}{e} \sum_{i=1}^{n} \int f_i(c_i)\ \Pr[\mathcal{E}_i(c_i)]\, dc_i.
\end{aligned}$$

Because the joint distribution is atomless, the events $\{\mathcal{E}_i\}_{i=1}^n$ partition the sample space, hence $\sum_{i=1}^n \int f_i(c_i) \Pr[\mathcal{E}_i(c_i)]\, dc_i = \sum_{i=1}^n \Pr[\mathcal{E}_i] = 1$, and $\sum_{i=1}^n \int f_i(c_i) \Pr[\mathcal{E}_i(c_i)]\, c_i\, dc_i = \mathbb{E}[\min_i c_i] = \mathrm{SC}_{\mathrm{OPT}}$. Therefore $\mathbb{E}[\mathrm{SC}_{\mathrm{BNE}}] \leq \mathrm{SC}_{\mathrm{OPT}} + \frac{r}{e}$, and dividing by $\mathrm{SC}_{\mathrm{OPT}}$ yields $\mathrm{PoA} \leq 1 + \frac{r/e}{\mathbb{E}[\min_i c_i]}$. □

4 Examples of Worst-Case Instances where $\mathrm{PoA} = \infty$ (non-MHR)

Setup. Consider one low bidder with deterministic cost c_L and n high bidders whose costs are always strictly larger than c_L. The social cost optimal allocation always serves the low bidder, hence $\mathbb{E}[\mathrm{SC}_{\mathrm{OPT}}] = c_L$. Fix $u_L > 0$ and define $b_0 := c_L + u_L$. We construct bid distributions as follows. The highest bid among the high bidders has survival function $1 - F_H(b) = r(b) = \frac{u_L}{b - c_L}$ for $b \in [b_0, \infty)$, which satisfies $r(b_0) = 1$ and $r(b) \downarrow 0$ as $b \to \infty$. For the low bidder we choose the power survival $1 - G_L(b) = f(b) := r(b)^\beta = \left(\frac{u_L}{b - c_L}\right)^\beta$ on $[b_0, \infty)$ with parameter $\beta > 0$. Such framework matches (in the reverse setting) the one used to find the worst-case instance of PoA in standard auction by Hartline, Hoy, and Taggart [6].

FOC-implied cost map. Let $\alpha := \frac{n-1}{n}$. Define the competing term $F_c(b) := r(b)^\alpha f(b)$. The equilibrium first-order condition yields the induced bid-to-cost map $c(b) = b + \frac{F_c(b)}{F_c'(b)} = b + \frac{1}{\frac{f'(b)}{f(b)} - \frac{\alpha}{b - c_L}}$ on $[b_0, \infty)$. For the power survival, $\frac{f'(b)}{f(b)} = -\frac{\beta}{b - c_L}$ and hence $c(b) = b - \frac{b - c_L}{\alpha + \beta} = c_L + (b - c_L)\frac{\alpha + \beta - 1}{\alpha + \beta}$. In particular, $c(b) > c_L$ for all $b \geq b_0$ whenever $\beta > 1 - \alpha = 1/n$.

Expected social cost. Since $1 - F_H(b) = \frac{u_L}{b - c_L}$, we have $dF_H(b) = \frac{u_L}{(b - c_L)^2}\, db$. Using $c_L(1 - f) + cf = c_L + (c - c_L)f$, a direct calculation gives $\mathbb{E}[\mathrm{SC}] = c_L + \int_{b_0}^\infty (c(b) - c_L) f(b)\, dF_H(b) = c_L + u_L \frac{\alpha + \beta - 1}{\beta(\alpha + \beta)}$. Thus, for any fixed c_L and any fixed $\beta > 1/n$, $\mathbb{E}[\mathrm{SC}]$ grows linearly in u_L and can be made arbitrarily large by increasing u_L, while $\mathbb{E}[\mathrm{SC}_{\mathrm{OPT}}] = c_L$ remains constant. Therefore $\mathrm{PoA} = \mathbb{E}[\mathrm{SC}]/\mathbb{E}[\mathrm{SC}_{\mathrm{OPT}}] \to \infty$ as $u_L \to \infty$.

Failure of MHR. The constructed high-bid distribution is not MHR (as well as the individual hazard rate for each bidder): $F_H(b) = 1 - \frac{u_L}{b - c_L}$ has density $F_H'(b) = \frac{u_L}{(b - c_L)^2}$ and hazard rate

$h_{F_H}(b) = \frac{F_H'(b)}{1-F_H(b)} = \frac{1}{b-c_L}$, which is strictly *decreasing* in b. Hence this family violates the MHR condition required by the PoA bound, and indeed yields instances with unbounded PoA.

Theorem 4.1 (Bounded-support PoA via value-auction reduction)**.** *Consider a first-price procurement auction with n bidders and independent private costs $c_i \in [\underline{c}, \bar{c}]$. Let c^{eq} be the equilibrium cost of the winning bidder and $c^\star := \min_i c_i$ the welfare-optimal cost. Assume the corresponding first-price* value *auction obtained by $v_i := \bar{c} - c_i$ is $(1 - \frac{1}{e^2})$-efficient at every Bayes–Nash equilibrium (as in Jin and Lu [2]). Then $\mathbb{E}[c^{\mathrm{eq}}] \le (1 - \frac{1}{e^2})\mathbb{E}[c^\star] + \frac{1}{e^2}\bar{c}$, and hence*

$$\mathrm{PoA} \;=\; \frac{\mathbb{E}[c^{\mathrm{eq}}]}{\mathbb{E}[c^\star]} \;\le\; 1 + \frac{1}{e^2}\Big(\frac{\bar{c}}{\mathbb{E}[c^\star]} - 1\Big) \;\le\; 1 + \frac{1}{e^2}\Big(\frac{\bar{c}}{\underline{c}} - 1\Big).$$

Proof. Define the affine transformation $v_i := \bar{c} - c_i \in [0, \bar{c} - \underline{c}]$ and $\tilde{b}_i := \bar{c} - b_i$. Then the procurement winner (minimum bid) coincides with the value-auction winner (maximum bid), and utilities are preserved pointwise: $u_i^{\mathrm{proc}}(b, c) = (b_i - c_i)\mathbf{1}\{i \text{ wins}\} = (v_i - \tilde{b}_i)\mathbf{1}\{i \text{ wins}\} = u_i^{\mathrm{val}}(\tilde{b}, v)$. Thus any BNE of the procurement game maps to a BNE of the value game with the same allocation and interim utilities. In particular, letting v^{eq} denote the equilibrium value of the winning bidder in the value auction, we have $v^{\mathrm{eq}} = \bar{c} - c^{\mathrm{eq}}$ and $\max_i v_i = \bar{c} - c^\star$. By $(1 - \frac{1}{e^2})$-efficiency of the value auction, $\mathbb{E}[v^{\mathrm{eq}}] \ge (1 - \frac{1}{e^2})\mathbb{E}[\max_i v_i]$, i.e. $\mathbb{E}[\bar{c} - c^{\mathrm{eq}}] \ge (1 - \frac{1}{e^2})\mathbb{E}[\bar{c} - c^\star]$. Rearranging yields $\mathbb{E}[c^{\mathrm{eq}}] \le (1 - \frac{1}{e^2})\mathbb{E}[c^\star] + \frac{1}{e^2}\bar{c}$. Dividing by $\mathbb{E}[c^\star]$ gives $\mathrm{PoA} \le 1 + \frac{1}{e^2}(\frac{\bar{c}}{\mathbb{E}[c^\star]} - 1)$, and since $\mathbb{E}[c^\star] \ge \underline{c}$ the final bound follows. □

References

1. Koutsoupias E and Papadimitriou C. Worst-case equilibria. Computer Science Review 2009;3:65–9.
2. Jin Y and Lu P. First Price Auction is 1-1/e2 Efficient. J. ACM 2023;70.
3. Syrgkanis V and Tardos E. Composable and efficient mechanisms. In: *Proceedings of the Forty-Fifth Annual ACM Symposium on Theory of Computing*. STOC '13. Palo Alto, California, USA: Association for Computing Machinery, 2013:211–20. DOI: 10.1145/2488608.2488635. URL: https://doi.org/10.1145/2488608.2488635.
4. Roughgarden T. Intrinsic Robustness of the Price of Anarchy. J. ACM 2015;62.
5. Hoy D, Taggart S, and Wang Z. A tighter welfare guarantee for first-price auctions. In: *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*. STOC 2018. Los Angeles, CA, USA: Association for Computing Machinery, 2018:132–7. DOI: 10.1145/3188745.3188944. URL: https://doi.org/10.1145/3188745.3188944.
6. Hartline J, Hoy D, and Taggart S. Price of anarchy for auction revenue. In: *Proceedings of the Fifteenth ACM Conference on Economics and Computation*. EC '14. Palo Alto, California, USA: Association for Computing Machinery, 2014:693–710. DOI: 10.1145/2600057.2602878. URL: https://doi.org/10.1145/2600057.2602878.