

# Numerical Optimization

Project 2 - Phase 2

Group 20

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## Implemented Tasks

### 1. Simplex Method

- 5 Problems with 2 Variables [Feasible + Infeasible starting points]
- 5 Problems with 10 Variables [Feasible + Infeasible starting points]

### 2. Active Set Method

- 5 Problems with 3 Starting points with varying sizes as in the project description.

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Notes:

\* All runs are located in separate files by section for better readability.

# 1 Simplex Method

Stopping Criteria:  $q = -1$  which indicates that no variable can enter the basis with a negative reduced cost, and the algorithm has converged.

- All the runs are located in file "simplex\_runs.txt"

## 1.1 Problem 1

A:  $[[1.0, 2.0], [3.0, 4.0]]$

b:  $[5.0, 6.0]$

c:  $[-7.0, 8.0]$

### 1. Feasible Starting Point $[-4, 4.5]$

- Phase 1 Iterations: 2
- Phase 1 Basis:  $[[2, 0], [2, 0]]$
- Phase 2 Iterations: 1
- Phase 2 Basis:  $[[2, 0]]$
- Optimum x:  $[2.158, 0.0]$

### 2. In-Feasible Starting Point $[0.0, 0.0]$

- Phase 1 Iterations: 1
  - Phase 1 Basis:  $[[2, 3]]$
  - Phase 2 Iterations: 2
  - Phase 2 Basis:  $[[2, 0], [2, 0]]$
  - Optimum x:  $[2.158, 0.0]$
- 

## 1.2 Problem 2

A:  $[[10.0, 11.0], [12.0, 13.0]]$

b:  $[14.0, 15.0]$

c:  $[16.0, 17.0]$

### 1. Feasible Starting Point $[1.4, 0. ]$

- Phase 1 Iterations: 2
- Phase 1 Basis:  $[[2, 1], [2, 1]]$
- Phase 2 Iterations: 2
- Phase 2 Basis:  $[[2, 3], [2, 3]]$
- Optimum x:  $[0.0, 0.0]$

## 2. In-Feasible Starting Point [0., 0. ]

- Phase 1 Iterations: 1
  - Phase 1 Basis:  $[[2, 3]]$
  - Phase 2 Iterations: 1
  - Phase 2 Basis:  $[[2, 3]]$
  - Optimum x:  $[0.0, 0.0]$
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## 1.3 Problem 3

A:  $[[15.0, 0.0], [0.0, 20.0]]$

b:  $[-25.0, 13.0]$

c:  $[3.0, -1.0]$

### 1. Feasible Starting Point [1.5, 0. ]

- Phase 1 Iterations: 2
- Phase 1 Basis:  $[[2, 1], [2, 1]]$
- Phase 2 Iterations: 1
- Phase 2 Basis:  $[[2, 1]]$
- Optimum x:  $[0.0, 0.65]$

### 2. In-Feasible Starting Point [0., 0. ]

- Phase 1 Iterations: 1
  - Phase 1 Basis:  $[[2, 3]]$
  - Phase 2 Iterations: 2
  - Phase 2 Basis:  $[[2, 1], [2, 1]]$
  - Optimum x:  $[0.0, 0.65]$
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## 1.4 Problem 4

A:  $[[15.0, 16.0], [17.0, 18.0]]$

b:  $[-8.0, -9.0]$

c:  $[3.0, -1.0]$

### 1. Feasible Starting Point [2.0, 4.0]

- Phase 1 Iterations: 1
- Phase 1 Basis:  $[[2, 3]]$

- Phase 2 Iterations: 2
- Phase 2 Basis:  $[[1, 3], [1, 3]]$
- Optimum  $x$ :  $[0.0, -0.5]$

**2. In-Feasible Starting Point  $[0., 0. ]$**

- Phase 1 Iterations: 1
  - Phase 1 Basis:  $[[2, 3]]$
  - Phase 2 Iterations: 2
  - Phase 2 Basis:  $[[1, 3], [1, 3]]$
  - Optimum  $x$ :  $[0.0, -0.5]$
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## 1.5 Problem 5

A:  $[[3, 1], [1, 2]]$   
 b:  $[9, 8]$   
 c:  $[-1, -1]$

**1. Feasible Starting Point  $[1.0, 1.0]$**

- Phase 1 Iterations: 3
- Phase 1 Basis:  $[[0, 3], [0, 1], [0, 1]]$
- Phase 2 Iterations: 1
- Phase 2 Basis:  $[[0, 1]]$
- Optimum  $x$ :  $[2.0, 3.0]$

**2. In-Feasible Starting Point  $[0., 0. ]$**

- Phase 1 Iterations: 1
  - Phase 1 Basis:  $[[2, 3]]$
  - Phase 2 Iterations: 3
  - Phase 2 Basis:  $[[2, 1], [0, 1], [0, 1]]$
  - Optimum  $x$ :  $[2.0, 3.0]$
-

## 1.6 Problem 6

A:  $[[1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 0.0], [1.0, -1.0, 1.0, 0.0, 0.0, -1.0, 0.0, 0.0, 0.0, -1.0, 0.0, 1.0]]$

b:  $[2.0, 2.0]$

c:  $[1.0, 1.0, 1.0, -1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 0.0, 0.0]$

### 1. Feasible Starting Point $[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$

- Phase 1 Iterations: 3
- Phase 1 Basis:  $[[0, 13], [0, 11], [0, 11]]$
- Phase 2 Iterations: 2
- Phase 2 Basis:  $[[3, 11], [3, 11]]$
- Optimum x:  $[0.0, 0.0, 0.0, 2.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0]$

### 2. In-Feasible Starting Point $[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$

- Phase 1 Iterations: 1
  - Phase 1 Basis:  $[[12, 13]]$
  - Phase 2 Iterations: 3
  - Phase 2 Basis:  $[[3, 13], [3, 13]]$
  - Optimum x:  $[0.0, 0.0, 0.0, 2.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0]$
- 

## 1.7 Problem 7

A:  $[[1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 0.0], [1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 0.0, 0.0, -1.0, 0.0, 1.0]]$

b:  $[3.0, 4.0]$

c:  $[1.0, 1.0, -1.0, -1.0, 1.0, 1.0, -1.0, -1.0, 1.0, 1.0, 0.0, 0.0]$

### 1. Feasible Starting Point $[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$

- Phase 1 Iterations: 3
- Phase 1 Basis:  $[[0, 13], [0, 4], [0, 4]]$
- Phase 2 Iterations: 5
- Phase 2 Basis:  $[[7, 4], [7, 6], [7, 2], [7, 3], [7, 3]]$
- Optimum x:  $[0.0, 0.0, 0.0, 1.0, 0.0, 0.0, 0.0, 0.0, 2.0, 0.0, 0.0]$

### 2. In-Feasible Starting Point $[1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0]$

- Phase 1 Iterations: 2
  - Phase 1 Basis:  $[[0, 13], [0, 13]]$
  - Phase 2 Iterations: 2
  - Phase 2 Basis:  $[[7, 13], [7, 13]]$
  - Optimum x:  $[0.0, 0.0, 0.0, 2.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0]$
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## 1.8 Problem 8

A:  $[[10.0, 20.0, 30.0, 40.0, 50.0, 60.0, 70.0, 80.0, 90.0, 100.0, 1.0, 0.0, 0.0, 0.0],$   
 $[100.0, 90.0, 80.0, 70.0, 60.0, 50.0, 40.0, 30.0, 20.0, 10.0, 0.0, 1.0, 0.0, 0.0], [30.0,$   
 $40.0, 50.0, 60.0, 70.0, 80.0, 90.0, 100.0, 110.0, 120.0, 0.0, 0.0, 1.0, 0.0], [50.0,$   
 $30.0, 20.0, 10.0, 100.0, 90.0, 80.0, 70.0, 60.0, 50.0, 0.0, 0.0, 0.0, 1.0]]$

b:  $[100.0, 200.0, 300.0, 400.0]$

c:  $[100.0, 200.0, 300.0, 400.0, 500.0, 600.0, 700.0, 800.0, 900.0, 1000.0, 0.0,$   
 $0.0, 0.0, 0.0, 0.0, 0.0]$

### 1. Feasible Starting Point $[1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$

- Phase 1 Iterations: 5
- Phase 1 Basis:  $[[4, 15, 16, 17], [4, 15, 16, 13], [4, 15, 12, 13], [4, 2, 12,$   
 $13], [4, 2, 12, 13]]$
- Phase 2 Iterations: 3
- Phase 2 Basis:  $[[10, 2, 12, 13], [10, 11, 12, 13], [10, 11, 12, 13]]$
- Optimum x:  $[0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0]$

### 2. In-Feasible Starting Point $[1, 0, 1, 0, 1, 0, 1, 0, 1, 0]$

- Phase 1 Iterations: 3
- Phase 1 Basis:  $[[14, 15, 12, 17], [3, 15, 12, 17], [3, 15, 12, 17]]$
- Phase 2 Iterations: 2
- Phase 2 Basis:  $[[10, 15, 12, 17], [10, 15, 12, 17]]$
- Optimum x:  $[0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0]$

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## 1.9 Problem 9

A:  $[[1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 0.0], [1.0, 0.0, -1.0, 0.0, 1.0,$   
 $0.0, -1.0, 0.0, 1.0, 0.0, 0.0, 1.0]]$

b:  $[100.0, 30.0]$

c:  $[1.0, -1.0, -1.0, -1.0, 1.0, 1.0, 1.0, -1.0, 1.0, -1.0, 0.0, 0.0]$

### 1. Feasible Starting Point $[1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$

- Phase 1 Iterations: 3
- Phase 1 Basis:  $[[1, 13], [1, 0], [1, 0]]$
- Phase 2 Iterations: 2
- Phase 2 Basis:  $[[1, 11], [1, 11]]$
- Optimum x:  $[0.0, 100.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0]$

### 2. In-Feasible Starting Point $[0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$

- Phase 1 Iterations: 1
  - Phase 1 Basis:  $[[12, 13]]$
  - Phase 2 Iterations: 2
  - Phase 2 Basis:  $[[1, 13], [1, 13]]$
  - Optimum x:  $[0.0, 100.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0]$
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### 1.10 Problem 10

A:  $[[1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 0.0], [1.0, 0.0, -1.0, 0.0, 1.0, 0.0, -1.0, 0.0, 1.0, 0.0, 0.0, 1.0]]$

b:  $[2.0, 5.0]$

c:  $[1.0, 1.0, -1.0, -1.0, 1.0, 1.0, 1.0, -1.0, 1.0, 1.0, 0.0, 0.0]$

#### 1. Feasible Starting Point $[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$

- Phase 1 Iterations: 3
- Phase 1 Basis:  $[[12, 11], [0, 11], [0, 11]]$
- Phase 2 Iterations: 3
- Phase 2 Basis:  $[[2, 11], [3, 11], [3, 11]]$
- Optimum x:  $[0.0, 0.0, 0.0, 2.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0]$

#### 2. In-Feasible Starting Point $[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$

- Phase 1 Iterations: 1
  - Phase 1 Basis:  $[[12, 13]]$
  - Phase 2 Iterations: 4
  - Phase 2 Basis:  $[[2, 13], [2, 11], [3, 11], [3, 11]]$
  - Optimum x:  $[0.0, 0.0, 0.0, 2.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0]$
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## 2 Active set Method

Problem Reformulation:

Given:

Objective function:  $\frac{1}{2}||Mx - y||^2$

$= \frac{1}{2}(Mx - y)^T(Mx - y)$

$= \frac{1}{2}(x^T M^T Mx - 2y^T Mx + y^T y)$

Comparing it with Standard QP form in (16.1a) we get

$G = M^T M$

$c = -2M^T y$

Constraint:  $\|x\|_1 = \sum_{i=1}^n |x_i| \leq 1$

We introduce  $z \in 0, 1^n$  to overcome the difficulty in expressing the constraint in linear form, due to the absolute value.

This means ( $z_i = 1$  if  $|x_i|$  is active and  $z_i = 0$  otherwise).

We introduce 2 more constraints:

- $\sum_{i=1}^n z_i \leq 1$  (16.1d) (At most one absolute value can be active)
- $x_i \leq z_i$  (16.1e) (If  $z_i = 0$ , then  $x_i$  must be zero)

We then reformulate the constraints as:

$$\|x\|_1 = \sum_{i=1}^n |x_i| = \sum_{i=1}^n (x_i^+ + x_i^-)$$

Such that  $x_i^+$  and  $x_i^-$  are the positive and negative components of  $x_i$

$$\implies x_i^+ = x_i * z_i, \quad x_i^- = -x_i * z_i$$

$$\|x\|_1 = \sum_{i=1}^n |x_i| = \sum_{i=1}^n (x_i * z_i - x_i * z_i) = \sum_{i=1}^n (x_i * z_i + (-x_i) * z_i)$$

$$= \sum_{i=1}^n (x_i * z_i + (-x_i) * z_i)$$

$$= \sum_{i=1}^n (x_i * z_i - x_i * z_i) = \sum_{i=1}^n (x_i * z_i)$$

Comparing the constraint with the standard QP form, we can identify:

$$a_i = [z_i, -z_i] \text{ (2}^n\text{-dimensional vector)}$$

$$b_i = 0$$

## 2.1 Problem 1

- m = 1, n = 2
- M: [[0.976, 4.304]]
- y: [2.055]
- A: [[-1.0, -1.0], [-1.0, 1.0], [1.0, -1.0], [1.0, 1.0]]
- b: [-1.0, -1.0, -1.0, -1.0]
- c: [-4.013, -17.691]
- G: [[0.953, 4.202], [4.202, 18.523]]

### 1. Starting Point [1.764, 0.4]

- Solution: [0.206, 0.908]
- Iterations: 2
- Time Elapsed: 0:00:00.000997

### 2. Starting Point [0.979, 2.241]

- Solution: [0.206, 0.908]
- Iterations: 2



- Time Elapsed: 0:00:00.000998

### 3. Starting Point [1.868, 0.977]

- Solution: [0.206, 0.908]
  - Iterations: 2
  - Time Elapsed: 0:00:00.000998
- 

## 2.2 Problem 2

- $m = 2, n = 4$
- M: [[0.976, 4.304, 2.055, 0.898], [-1.527, 2.918, -1.248, 7.835]]
- y: [9.273, -2.331]
- A: [[-1.0, -1.0, -1.0, -1.0], [-1.0, -1.0, -1.0, 1.0], [-1.0, -1.0, 1.0, -1.0], [-1.0, -1.0, 1.0, 1.0], [-1.0, 1.0, -1.0, -1.0], [-1.0, 1.0, -1.0, 1.0], [-1.0, 1.0, 1.0, -1.0], [-1.0, 1.0, 1.0, 1.0], [1.0, -1.0, -1.0, -1.0], [1.0, -1.0, -1.0, 1.0], [1.0, -1.0, 1.0, -1.0], [1.0, -1.0, 1.0, 1.0], [1.0, 1.0, -1.0, -1.0], [1.0, 1.0, -1.0, 1.0], [1.0, 1.0, 1.0, -1.0], [1.0, 1.0, 1.0, 1.0]]
- b: [-1.0, -1.0, -1.0, -1.0, -1.0, -1.0, -1.0, -1.0, -1.0, -1.0, -1.0, -1.0, -1.0, -1.0, -1.0, -1.0]
- c: [-25.225, -66.216, -43.938, 19.883]
- G: [[3.285, -0.254, 3.912, -11.088], [-0.254, 27.037, 5.203, 26.726], [3.912, 5.203, 5.782, -7.936], [-11.088, 26.726, -7.936, 62.2]]

### 1. Starting Point [1.764, 0.4, 0.979, 2.241]

- Solution: [-0.0, -0.0, 0.0, -0.0]
- Iterations: 40
- Time Elapsed: 0:00:00.021439

### 2. Starting Point [1.868, 0.977, 0.95, 0.151]

- Solution: [0.0, 0.0, 0.0, 0.0]
- Iterations: 5000
- Time Elapsed: 0:00:01.759338

### 3. Starting Point [0.103, 0.411, 0.144, 1.454]

- Solution: [0.0, 0.0, 0.0, 0.0]
  - Iterations: 5000
  - Time Elapsed: 0:00:01.767278
-

### 2.3 Problem 3

- [illegible]

- Solution: [0.0, 0.0, -0.0, 0.0, 0.0, -0.0]
- Iterations: 128
- Time Elapsed: 0:00:00.055708

- Solution: [-0.25, 0.25, 0.25, -0.25, 0.0, 0.0]
- Iterations: 5000
- Time Elapsed: 0:00:01.999589

- Solution: [-0.0, 0.0, -0.0, -0.0, 0.0, -0.0]
- Iterations: 5000
- Time Elapsed: 0:00:01.941713

[illegible]

[illegible]

[illegible]



- c: [-172.449, 128.81, 128.279, 100.588, 175.793, 105.015, 28.623, 118.534]
- G: [[237.308, -102.614, 57.186, -132.671, -84.06, 37.4, -37.434, -164.633], [-102.614, 76.042, 12.291, 76.633, 55.158, 27.741, -3.675, 105.644], [57.186, 12.291, 120.093, -17.051, 54.937, 101.119, -23.319, -39.983], [-132.671, 76.633, -17.051, 134.996, 74.14, 36.636, -53.658, 92.559], [-84.06, 55.158, 54.937, 74.14, 96.005, 63.656, -19.202, 32.902], [37.4, 27.741, 101.119, 36.636, 63.656, 119.678, -73.241, -23.216], [-37.434, -3.675, -23.319, -53.658, -19.202, -73.241, 97.929, 30.913], [-164.633, 105.644, -39.983, 92.559, 32.902, -23.216, 30.913, 191.14]]

1. **Starting Point [0.103, 0.411, 0.144, 1.454, 0.761, 0.122, 0.444, 0.334]**
  - Solution: [-0.183, -0.035, -0.183, 0.113, -0.148, 0.148, 0.035, -0.113]
  - Iterations: 1155
  - Time Elapsed: 0:00:00.055708
2. **Starting Point [0.888, 1.981, 0.348, 0.156, 1.23, 1.202, 0.387, 0.302]**
  - Solution: [-0.283, -0.142, -0.142, 0.0, -0.142, 0.0, -0.142, -0.0]
  - Iterations: 196
  - Time Elapsed: 0:00:00.133230
3. **Starting Point [1.494, 0.205, 0.313, 0.854, 2.553, 0.654, 0.864, 0.742]**
  - Solution: [-0.259, -0.129, -0.065, -0.0, -0.065, -0.065, -0.129, -0.065]
  - Iterations: 693
  - Time Elapsed: 0:00:00.300796

- m = 5, n = 10
- M: [[0.976, 4.304, 2.055, 0.898, -1.527, 2.918, -1.248, 7.835, 9.273, -2.331], [5.835, 0.578, 1.361, 8.512, -8.579, -8.257, -9.596, 6.652, 5.563, 7.4], [9.572, 5.983, -0.77, 5.611, -7.635, 2.798, -7.133, 8.893, 0.437, -1.707], [-4.709, 5.485, -0.877, 1.369, -9.624, 2.353, 2.242, 2.339, 8.875, 3.636], [-2.81, -1.259, 3.953, -8.795, 3.335, 3.413, -5.792, -7.421, -3.691, -2.726]]

- [illegible]



[illegible]

[illegible]

[illegible]

[illegible]

[illegible]



[illegible]













[illegible]

[illegible]

[illegible]

- c: [-283.085, -54.905, 44.689, -171.854, 17.997, -5.948, 87.516, -228.59, 77.376, 84.205]
- G: [[156.694, 42.558, -4.405, 122.515, -88.679, -39.21, -119.765, 141.434, 14.275, 15.099], [42.558, 86.323, -4.765, 60.935, -114.194, 33.135, -34.004, 112.95, 99.065, 7.409], [-4.405, -4.765, 23.062, -26.859, 12.692, 4.03, -34.99, -13.08, 3.919, -7.368], [122.515, 60.935, -26.859, 183.971, -159.739, -78.763, -68.803, 182.032, 102.744, 80.274], [-88.679, -114.194, 12.692, -159.739, 237.971, 33.762, 97.79, -184.194, -162.95, -90.988], [-39.21, 33.135, 4.03, -78.763, 33.762, 101.712, 41.139, -27.007, -9.374, -73.432], [-119.765, -34.004, -34.99, -68.803, 97.79, 41.139, 183.091, -88.82, -26.795, -31.985], [141.434, 112.95, -13.08, 182.032, -184.194, -27.007, -88.82, 245.289, 161.706, 44.518], [14.275, 99.065, 3.919, 102.744, -162.95, -9.374, -26.795, 161.706, 209.524, 61.14], [15.099, 7.409, -7.368, 80.274, -90.988, -73.432, -31.985, 44.518, 61.14, 83.764]]

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- Iterations: 235
  - Time Elapsed: 0:00:00.238247
3. **Starting Point** [2.553, 0.654, 0.864, 0.742, 2.27, 1.454, 0.046, 0.187, 1.533, 1.469]
- Solution: [0.0, -0.28, 0.0, -0.0, -0.0, -0.0, -0.143, -0.0, 0.0, -0.137]
  - Iterations: 719
  - Time Elapsed: 0:00:06.585050
4. **Starting Point** [0.155, 0.378, 0.888, 1.981, 0.348, 0.156, 1.23, 1.202, 0.387, 0.302]
- Solution: [0.0, -0.0, -0.0, 0.0, -0.0, -0.0, -0.0, -0.0, 0.0, -0.0]
  - Iterations: 111
  - Time Elapsed: 0:00:00.074798
5. **Starting Point** [1.049, 1.42, 1.706, 1.951, 0.51, 0.438, 1.253, 0.777, 1.614, 0.213]
- Solution: [0.0, -0.28, -0.0, 0.0, -0.0, -0.0, -0.143, 0.0, 0.0, -0.137]
  - Iterations: 584
  - Time Elapsed: 0:00:04.275077