Numerical Optimization

Project 1 - Pahse 2

Group 27

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Implemented Tasks

1. Newton method
    - Eigenvalue Modification
    - No Modification
2. Linear Conjugate Gradient method
3. Non-Linear Conjugate Gradient method
    - Fletcher-Reevers method
    - Polak-Ribiere method
4. Quasi Newton method
    - BFGS
    - SR1
    - SR1 [within trust region framework]
5. Approximations
    - Gradient
    - Hessian

Notes:
* All runs are located in separate files by section for better readability.
* Runs were limited to 1000 maximum iterations for file size limit. [reached ¿ 500 mbs]

1 Newton Method

1.1 Exact Solutions

All the runs are located in file "1.1 exact_newton.txt"

1.1.1 Rosenbrock Function

1. **Starting point [1.2, 1.2]:**

   - Number of iterations: 8
   - Final iterate $\bar{x} \approx [1.000000000000031, 1.0000000000000497]$
   - $\nabla f(\bar{x}) \approx [7.801329535194528e-07, -2.778961505356392e-07]$
   - $||\nabla f(\bar{x})|| \approx 8.281507686704827e-07$
   - Distance $||\bar{x} - x^*|| \approx 5.865340917345583e-14$

1. **Starting point [-1.2, 1.]:**

   - Number of iterations: 22
   - Final iterate $\bar{x} \approx [1.0, 1.0]$
   - $\nabla f(\bar{x}) \approx [1.8439960666621747e-10, -1.751043754438797e-10]$
   - $||\nabla f(\bar{x})|| \approx 2.5429266060633147e-10$
   - Distance $||\bar{x} - x^*|| \approx 0.0$

1. **Starting point [0.2, 0.8]:**

   - Number of iterations: 14
   - Final iterate $\bar{x} \approx [1.0, 1.0]$
   - $\nabla f(\bar{x}) \approx [1.3306467039102766e-11, -1.4765966227514582e-11]$
   - $||\nabla f(\bar{x})|| \approx 1.9877017474833328e-11$
   - Distance $||\bar{x} - x^*|| \approx 0.0$

1.1.2 Quadratic Function

1. **Starting point [-0.2, 1.2]:**

   - Number of iterations: 29
   - Final iterate $\bar{x} \approx [-4.427683987331849e-10, 1.0000000003900889]$
   - $\nabla f(\bar{x}) \approx [-5.307632856456589e-07, 2.2351740590584064e-09]$
   - $||\nabla f(\bar{x})|| \approx 5.307679920572167e-07$
   - Distance $||\bar{x} - x^*|| \approx 5.900958955097454e-10$

1. **Starting point [3.8, 0.1]:**

   - Number of iterations: 19
   - Final iterate $\bar{x} \approx [4.000000381469727, -9.624910007318629e-27]$
   - $\nabla f(\bar{x}) \approx [1.9073486323684108e-07, 7.629394529473644e-07]$
   - $||\nabla f(\bar{x})|| \approx 7.864199876132347e-07$
   - Distance $||\bar{x} - x^*|| \approx 3.8146972691777137e-07$

1. **Starting point [1.9, 0.6]:**

   - Number of iterations: 30
   - Final iterate $\bar{x} \approx [7.371031656641207e-10, 0.9999999998855734]$
   - $\nabla f(\bar{x}) \approx [8.846634969683159e-07, 5.587945520259472e-10]$
   - $||\nabla f(\bar{x})|| \approx 8.846636734485942e-07$
   - Distance $||\bar{x} - x^*|| \approx 7.459319800676289e-10$

1.2 Approximated Solutions

All the runs are located in file "1.2 approx_newton.txt"
* Using Eigenvalue Modification.

1.2.1 Rosenbrock Function

1. **Starting point [1.2, 1.2]:**

   - Number of iterations: 8
   - Final iterate $\bar{x} \approx [0.9999999998000344, 0.9999999996000563]$
   - $\nabla f(\bar{x}) \approx [7.802436394697449e-07, -2.7793448652924537e-07]$
   - $||\nabla f(\bar{x})|| \approx 8.282679009446572e-07$
   - Distance $||\bar{x} - x^*|| \approx 4.471478950333974e-10$

1. **Starting point [-1.2, 1.]:**

   - Number of iterations: 22
   - Final iterate $\bar{x} \approx [0.9999999998000044, 0.9999999996000088]$
   - $\nabla f(\bar{x}) \approx [8.815237416425712e-11, -8.786305016742868e-11]$
   - $||\nabla f(\bar{x})|| \approx 1.2446186827908086e-10$
   - Distance $||\bar{x} - x^*|| \approx 4.472037023659638e-10$

1. **Starting point [0.2, 0.8]:**

   - Number of iterations: 9

- Final iterate $\bar{x} \approx [0.9999999998000817, 0.9999999996001488]$
- $\nabla f(\bar{x}) \approx [8.057588703555523e-07, -2.8192445044395916e-07]$
- $||\nabla f(\bar{x})|| \approx 8.53656109282177e-07$
- Distance $||\bar{x} - x^*|| \approx 4.470439265151773e-10$

1.2.2 Quadratic Function

1. **Starting point [-0.2, 1.2]:**

   - Number of iterations: 6
   - Final iterate $\bar{x} \approx [-6.516034754523722e-19, 1.0]$
   - $\nabla f(\bar{x}) \approx [-2.993882863120485e-08, 1.0060964953284623e-08]$
   - $||\nabla f(\bar{x})|| \approx 3.158411745798626e-08$
   - Distance $||\bar{x} - x^*|| \approx 6.516034754523722e-19$

1. **Starting point [3.8, 0.1]:**

   - Number of iterations: 8
   - Final iterate $\bar{x} \approx [4.0, 9.244834619306816e-20]$
   - $\nabla f(\bar{x}) \approx [8.437694991091607e-15, 5.3985828940939125e-12]$
   - $||\nabla f(\bar{x})|| \approx 5.398589487921837e-12$
   - Distance $||\bar{x} - x^*|| \approx 9.244834619306816e-20$

1. **Starting point [1.9, 0.6]:**

   - Number of iterations: 8
   - Final iterate $\bar{x} \approx [4.0, 9.259077933451737e-20]$
   - $\nabla f(\bar{x}) \approx [4.4408920872884515e-16, -7.475231195641711e-15]$
   - $||\nabla f(\bar{x})|| \approx 7.488410822972015e-15$
   - Distance $||\bar{x} - x^*|| \approx 9.259077933451737e-20$

2 Conjugate Gradient Method [Linear]

2.1 Exact Solutions

All the runs are located in file "2.1 exact_cg_linear.txt"

2.1.1 Rosenbrock Function

1. **Starting point [1.2, 1.2]:**
    - Number of iterations: 1000
    - Final iterate $\bar{x} \approx [-1.3439864598640971, 1.6394023156614814]$
    - $\nabla f(\bar{x}) \approx [-94.41105136594786, -33.37945772730939]$
    - $||\nabla f(\bar{x})|| \approx 100.13807876224148$
    - Distance $||\bar{x} - x^*|| \approx 2.4296312159048927$
    - alpha $\approx 9.473244919935914e-08$

1. **Starting point [-1.2, 1.]:**
    - Number of iterations: 1000
    - Final iterate $\bar{x} \approx [25.93328313588868, 664.4834137702009]$
    - $\nabla f(\bar{x}) \approx [83573.29981758706, -1610.3520871934734]$
    - $||\nabla f(\bar{x})|| \approx 83588.81310465549$
    - Distance $||\bar{x} - x^*|| \approx 663.9517369177477$
    - alpha $\approx 2.8876742852435617e-10$

1. **Starting point [0.2, 0.8]:**
    - Number of iterations: 1000
    - Final iterate $\bar{x} \approx [9.818929297187877, 87.05171572669302]$
    - $\nabla f(\bar{x}) \approx [36778.361269383706, -1871.931363296278]$
    - $||\nabla f(\bar{x})|| \approx 36825.96889004003$
    - Distance $||\bar{x} - x^*|| \approx 86.50243518801297$
    - alpha $\approx 2.9293646396947243e-10$

2.1.2 Quadratic Function

1. **Starting point [-0.2, 1.2]:**

   - Number of iterations: 1000
   - Final iterate $\bar{x} \approx [-0.00013099340897818518, 1.1461152100959844]$
   - $\nabla f(\bar{x}) \approx [0.04284026256436663, 0.5843357469261601]$
   - $||\nabla f(\bar{x})|| \approx 0.5859040478033389$
   - Distance $||\bar{x} - x^*|| \approx 0.14611526881427161$
   - alpha $\approx 3.405561869024827e-08$

1. **Starting point [3.8, 0.1]:**

   - Number of iterations: 1000
   - Final iterate $\bar{x} \approx [3.999746852921016, 5.604964650524923e-08]$
   - $\nabla f(\bar{x}) \approx [-6.323071256023153e-05, 1.6081370774433492e-05]$
   - $||\nabla f(\bar{x})|| \approx 6.524364717625329e-05$
   - Distance $||\bar{x} - x^*|| \approx 0.0002531470851889546$
   - alpha $\approx 1.0892257588598292e-05$

1. **Starting point [1.9, 0.6]:**

   - Number of iterations: 1000
   - Final iterate $\bar{x} \approx [3.9974718646554854, 5.938187563088533e-07]$
   - $\nabla f(\bar{x}) \approx [-0.0006314391716175313, 0.00032096808935862427]$
   - $||\nabla f(\bar{x})|| \approx 0.0007083332138475224$
   - Distance $||\bar{x} - x^*|| \approx 0.0025281354142538573$
   - alpha $\approx 1.4205276716883149e-05$

2.2 Approximated Solutions

All the runs are located in file "2.2 approx_cg_linear.txt"

2.2.1 Rosenbrock Function

1. **Starting point [1.2, 1.2]:**

   - Number of iterations: 1000
   - Final iterate $\bar{x} \approx [-1.3435915482324658, 1.6383313144228253]$
   - $\nabla f(\bar{x}) \approx [-94.3890814673054, -33.381386809416824]$
   - $||\nabla f(\bar{x})|| \approx 100.11800879742627$

- Distance $||\bar{x} - x^*|| \approx 2.428968548976997$
- alpha $\approx 9.470949492146962e-08$

1. **Starting point [-1.2, 1.]:**

   - Number of iterations: 1000
   - Final iterate $\bar{x} \approx [25.894010079262813, 661.7831723657958]$
   - $\nabla f(\bar{x}) \approx [90332.73052045843, -1743.3171196898911]$
   - $||\nabla f(\bar{x})|| \approx 90349.55095550594$
   - Distance $||\bar{x} - x^*|| \approx 661.2519282539987$
   - alpha $\approx 2.1016912122169093e-10$

1. **Starting point [0.2, 0.8]:**

   - Number of iterations: 1000
   - Final iterate $\bar{x} \approx [9.84327459748534, 87.53707308978255]$
   - $\nabla f(\bar{x}) \approx [36843.27343853511, -1870.5963375396095]$
   - $||\nabla f(\bar{x})|| \approx 36890.729571596756$
   - Distance $||\bar{x} - x^*|| \approx 86.98774927857886$
   - alpha $\approx 3.1730384284222637e-10$

2.2.2 Quadratic Function

1. **Starting point [-0.2, 1.2]:**

   - Number of iterations: 21
   - Final iterate $\bar{x} \approx [-4.833003494058178e-10, 1.000000015038993]$
   - $\nabla f(\bar{x}) \approx [-1.1515377344982489e-07, 1.1934534212025934e-07]$
   - $||\nabla f(\bar{x})|| \approx 1.6584240177209008e-07$
   - Distance $||\bar{x} - x^*|| \approx 1.5046756722730036e-08$

1. **Starting point [3.8, 0.1]:**

   - Number of iterations: 20
   - Final iterate $\bar{x} \approx [4.000000238598754, -1.342889995353454e-10]$
   - $\nabla f(\bar{x}) \approx [1.190307990333904e-07, -1.684640791450059e-07]$
   - $||\nabla f(\bar{x})|| \approx 2.062728219632974e-07$
   - Distance $||\bar{x} - x^*|| \approx 2.3859879187810766e-07$

1. **Starting point [1.9, 0.6]:**

- Number of iterations: 20
- Final iterate $\bar{x} \approx [8.535169233797172, -0.00024370879375275887]$
- $\nabla f(\bar{x}) \approx [2.267249278986583, 3.7421978995588745]$
- $||\nabla f(\bar{x})|| \approx 4.37543876800122$
- Distance $||\bar{x} - x^*|| \approx 4.5351692403453265$
- alpha $\approx 3.5061282166303206e-09$

3 Conjugate Gradient Method [NonLinear]

3.1 Fletcher-Reevers Method

3.1.1 Exact Solutions

All the runs are located in file "3.1.1 exact_cg_nonlinear_fr.txt"

1. **Rosenbrock Function:**
   (a) **Starting point [1.2, 1.2]:**
   - Number of iterations: 42
   - Final iterate $\bar{x} \approx [1.0000002499766405, 1.0000005022142877]$
   - $\nabla f(\bar{x}) \approx [2.066543628209518e-06, -7.80724018767387e-07]$
   - $||\nabla f(\bar{x})|| \approx 2.209102206955047e-06$
   - Distance $||\bar{x} - x^*|| \approx 5.609879781037809e-07$

   (b) **Starting point [-1.2, 1.]:**
   - Number of iterations: 43
   - Final iterate $\bar{x} \approx [1.0000010805700774, 1.0000021656249312]$
   - $\nabla f(\bar{x}) \approx [-9.219940274208115e-06, 5.665549629441102e-06]$
   - $||\nabla f(\bar{x})|| \approx 1.082154107618804e-05$
   - Distance $||\bar{x} - x^*|| \approx 2.420240284490937e-06$

   (c) **Starting point [0.2, 0.8]:**
   - Number of iterations: 48
   - Final iterate $\bar{x} \approx [1.0000002161220691, 1.0000004332591392]$
   - $\nabla f(\bar{x}) \approx [6.703775802508991e-06, -3.148942440134306e-06]$
   - $||\nabla f(\bar{x})|| \approx 7.406513923674488e-06$
   - Distance $||\bar{x} - x^*|| \approx 0.0009903520314283058$

2. **Quadratic Function:**
   (a) **Starting point [-0.2, 1.2]:**
   - Number of iterations: 8
   - Final iterate $\bar{x} \approx [0.009337441078452807, 0.806294333149626]$
   - $\nabla f(\bar{x}) \approx [3.4669618112563794, -0.7441177288828611]$
   - $||\nabla f(\bar{x})|| \approx 3.5459181314784893$
   - Distance $||\bar{x} - x^*|| \approx 0.1939305885512692$

   (b) **Starting point [3.8, 0.1]:**
   - Number of iterations: 22
   - Final iterate $\bar{x} \approx [3.7939040327469202, 8.130189902116892e-05]$
   - $\nabla f(\bar{x}) \approx [-0.05142764329362186, 0.1452999740081748]$
   - $||\nabla f(\bar{x})|| \approx 0.15413268615550785$

- Distance $||\bar{x} - x^*|| \approx 0.20609598328929488$

(c) **Starting point [1.9, 0.6]:**

- Number of iterations: 9
- Final iterate $\bar{x} \approx [1.4904943812377376, 0.006396595283162027]$
- $\nabla f(\bar{x}) \approx [-0.5843883829102441, 1.7792327347216323]$
- $||\nabla f(\bar{x})|| \approx 1.8727463539907023$
- Distance $||\bar{x} - x^*|| \approx 2.509513771038243$

3.1.2 Approximated Solutions

All the runs are located in file "3.1.2 approx_cg_nonlinear_fr.txt"

1. **Rosenbrock Function:**

(a) **Starting point [1.2, 1.2]:**

- Number of iterations: 51
- Final iterate $\bar{x} \approx [0.9999996309579877, 0.9999992592423275]$
- $\nabla f(\bar{x}) \approx [-1.7275594154366476e-06, 4.921685725775551e-07]$
- $||\nabla f(\bar{x})|| \approx 1.7962993730714375e-06$
- Distance $||\bar{x} - x^*|| \approx 8.275952732156662e-07$

(b) **Starting point [-1.2, 1.]:**

- Number of iterations: 52
- Final iterate $\bar{x} \approx [0.9999995610283555, 0.9999991221303434]$
- $\nabla f(\bar{x}) \approx [1.0209245293784626e-05, -5.512538214643864e-06]$
- $||\nabla f(\bar{x})|| \approx 1.1602446597014463e-05$
- Distance $||\bar{x} - x^*|| \approx 9.81504579050044e-07$

(c) **Starting point [0.2, 0.8]:**

- Number of iterations: 45
- Final iterate $\bar{x} \approx [1.0000006596417301, 1.0000013234346878]$
- $\nabla f(\bar{x}) \approx [3.787732271796431e-06, -1.2310122697964804e-06]$
- $||\nabla f(\bar{x})|| \approx 3.982751181180872e-06$
- Distance $||\bar{x} - x^*|| \approx 1.4787178855686127e-06$

2. **Quadratic Function:**

(a) **Starting point [-0.2, 1.2]:**

- Number of iterations: 29
- Final iterate $\bar{x} \approx [1.8182928549841966e{-}09, 0.9999999917470224]$
- $\nabla f(\bar{x}) \approx [-3.8109549169690315e-06, -6.679081684910364e-08]$

- $||\nabla f(\bar{x})|| \approx 3.811540160143378e-06$
- Distance $||\bar{x} - x^*|| \approx 8.450906977604731e-09$

(b) **Starting point [3.8, 0.1]:**

- Number of iterations: 129
- Final iterate $\bar{x} \approx [3.9999999852107178, -1.2162441527703464e-10]$
- $\nabla f(\bar{x}) \approx [-6.666551434344903e-09, 2.458469843069588e-06]$
- $||\nabla f(\bar{x})|| \approx 2.4584788817865876e-06$
- Distance $||\bar{x} - x^*|| \approx 1.4789782343576279e-08$

(c) **Starting point [1.9, 0.6]:**

- Number of iterations: 107
- Final iterate $\bar{x} \approx [4.000000224426159, -2.365522071938385e-10]$
- $\nabla f(\bar{x}) \approx [1.1291264469076883e-07, 1.0933820377977017e-06]$
- $||\nabla f(\bar{x})|| \approx 1.0991967730619112e-06$
- Distance $||\bar{x} - x^*|| \approx 2.2442628410087405e-07$

3.2 Polak-Ribiere Method

3.2.1 Exact Solutions

All the runs are located in file "3.2.1 exact_cg_nonlinear_pr.txt"

1. **Rosenbrock Function:**

(a) **Starting point [1.2, 1.2]:**

- Number of iterations: 62
- Final iterate $\bar{x} \approx [1.000000146435967, 1.0000002947949158]$
- $\nabla f(\bar{x}) \approx [-4.7631236183420384e-07, 3.8459209150687457e-07]$
- $||\nabla f(\bar{x})|| \approx 6.121964904225682e-07$
- Distance $||\bar{x} - x^*|| \approx 3.29161867160577e-07$

(b) **Starting point [-1.2, 1.]:**

- Number of iterations: 56
- Final iterate $\bar{x} \approx [1.00000026139386, 1.0000005244568229]$
- $\nabla f(\bar{x}) \approx [-1.4482633002275743e-07, 3.338069376468411e-07]$
- $||\nabla f(\bar{x})|| \approx 3.638704954912155e-07$
- Distance $||\bar{x} - x^*|| \approx 5.859878062143473e-07$

(c) **Starting point [0.2, 0.8]:**

- Number of iterations: 77

- Final iterate $\bar{x} \approx [1.000000639402956, 1.0000012824849385]$
- $\nabla f(\bar{x}) \approx [-1.9264217092665674e-07, 7.357235709548604e-07]$
- $||\nabla f(\bar{x})|| \approx 7.605262512746728e-07$
- Distance $||\bar{x} - x^*|| \approx 1.433040040415721e-06$

2. **Quadratic Function:**

   (a) **Starting point [-0.2, 1.2]:**
   - Number of iterations: 46
   - Final iterate $\bar{x} \approx [4.773710630206499e-10, 1.000000021534833]$
   - $\nabla f(\bar{x}) \approx [3.0807682593432987e-07, 8.66167032116146e-08]$
   - $||\nabla f(\bar{x})|| \approx 3.200215367018606e-07$
   - Distance $||\bar{x} - x^*|| \approx 2.1540123413572898e-08$

   (b) **Starting point [3.8, 0.1]:**
   - Number of iterations: 1000
   - Final iterate $\bar{x} \approx [3.7947821569126825, 9.462483837924252e-05]$
   - $\nabla f(\bar{x}) \approx [-0.05118944916446478, 0.20395050736448816]$
   - $||\nabla f(\bar{x})|| \approx 0.21027641132564887$
   - Distance $||\bar{x} - x^*|| \approx 0.20521786490281713$

   (c) **Starting point [1.9, 0.6]:**
   - Number of iterations: 9
   - Final iterate $\bar{x} \approx [1.371954568102016, 0.009184729550996003]$
   - $\nabla f(\bar{x}) \approx [-0.5783843876835827, 2.5951063726842376]$
   - $||\nabla f(\bar{x})|| \approx 2.658778965138406$
   - Distance $||\bar{x} - x^*|| \approx 2.628061481658446$

3.2.2 Approximated Solutions

All the runs are located in file "3.2.2 approx_cg_nonlinear_pr.txt"

1. **Rosenbrock Function:**

   (a) **Starting point [1.2, 1.2]:**
   - Number of iterations: 57
   - Final iterate $\bar{x} \approx [0.9999997437575463, 0.9999994866576619]$
   - $\nabla f(\bar{x}) \approx [-1.6908650808487344e-07, -1.7149925924324796e-07]$
   - $||\nabla f(\bar{x})|| \approx 2.4083654859119443e-07$
   - Distance $||\bar{x} - x^*|| \approx 5.737425826499231e-07$

   (b) **Starting point [-1.2, 1.]:**

- Number of iterations: 68
- Final iterate $\bar{x} \approx [0.9999995120152905, 0.9999990233419435]$
- $\nabla f(\bar{x}) \approx [-7.00019335575671e{-}07, -1.3777512464625737e{-}07]$
- $||\nabla f(\bar{x})|| \approx 7.134487053398413e-07$
- Distance $||\bar{x} - x^*|| \approx 1.0917829620113772e-06$

(c) **Starting point [0.2, 0.8]:**

- Number of iterations: 77
- Final iterate $\bar{x} \approx [0.9999999805321218, 0.9999999627067655]$
- $\nabla f(\bar{x}) \approx [-6.955444000490959e-07, 3.285043015300706e-07]$
- $||\nabla f(\bar{x})|| \approx 7.692184920836318e-07$
- Distance $||\bar{x} - x^*|| \approx 4.206879628725704e-08$

2. **Quadratic Function:**

(a) **Starting point [-0.2, 1.2]:**

- Number of iterations: 103
- Final iterate $\bar{x} \approx [4.000000167769102, -1.2373089474566375e-10]$
- $\nabla f(\bar{x}) \approx [8.363708919190659e-08, -2.593599884595895e-07]$
- $||\nabla f(\bar{x})|| \approx 2.725119562555987e-07$
- Distance $||\bar{x} - x^*|| \approx 1.6776914734831218e-07$

(b) **Starting point [3.8, 0.1]:**

- Number of iterations: 75
- Final iterate $\bar{x} \approx [4.000000770649915, -1.4195117037264182e-10]$
- $\nabla f(\bar{x}) \approx [3.8504105556198156e-07, 8.587983411144745e-07]$
- $||\nabla f(\bar{x})|| \approx 9.411648129680892e-07$
- Distance $||\bar{x} - x^*|| \approx 7.706499285115392e-07$

(c) **Starting point [1.9, 0.6]:**

- Number of iterations: 45
- Final iterate $\bar{x} \approx [-3.2292471658668086e{-}09, 1.000000009971095]$
- $\nabla f(\bar{x}) \approx [-9.504466025014588e-07, 7.331026898433423e-08]$
- $||\nabla f(\bar{x})|| \approx 9.53269709864486e-07$
- Distance $||\bar{x} - x^*|| \approx 1.0480971995405012e-08$

4 Quasi-Newton Method

4.1 BFGS Method

4.1.1 Exact Solutions

All the runs are located in file "4.1.1 exact_qn_bfgs.txt"

1. **Rosenbrock Function:**
   (a) **Starting point [1.2, 1.2]:**
   - Number of iterations: 13
   - Final iterate $\bar{x} \approx [0.9999999934373244, 0.9999999884322573]$
   - $\nabla f(\bar{x}) \approx [-6.361687862245309e-07, 3.1152171953152674e-07]$
   - $||\nabla f(\bar{x})|| \approx 7.083477297953824e-07$
   - Distance $||\bar{x} - x^*|| \approx 1.329967598999978e-08$

   (b) **Starting point [-1.2, 1.]:**
   - Number of iterations: 33
   - Final iterate $\bar{x} \approx [0.9999996791119133, 0.9999993575683815]$
   - $\nabla f(\bar{x}) \approx [-3.7955703334679935e-07, -1.311096120559796e-07]$
   - $||\nabla f(\bar{x})|| \approx 4.01563534121928e-07$
   - Distance $||\bar{x} - x^*|| \approx 7.181138827191322e-07$

   (c) **Starting point [0.2, 0.8]:**
   - Number of iterations: 23
   - Final iterate $\bar{x} \approx [0.9999999990750245, 0.9999999981212548]$
   - $\nabla f(\bar{x}) \approx [9.667680201279789e-09, -5.758815646572657e-09]$
   - $||\nabla f(\bar{x})|| \approx 1.1252910651268288e-08$
   - Distance $||\bar{x} - x^*|| \approx 2.0941019853046453e-09$

2. **Quadratic Function:**
   (a) **Starting point [-0.2, 1.2]:**
   - Number of iterations: 16 -¿ diverges at 13
   - Final iterate $\bar{x} \approx [-0.05346227141439418, 1.2291276627394798]$
   - $\nabla f(\bar{x}) \approx [-48.2452681628738, 1.9169815189568309]$
   - $||\nabla f(\bar{x})|| \approx 48.28333789467777$
   - Distance $||\bar{x} - x^*|| \approx 0.23528217165191917$
   - rho $||\bar{x} - x^*|| \approx 9.354952124946182e+30$

   (b) **Starting point [3.8, 0.1]:**
   - Number of iterations: 13

- Final iterate $\bar{x} \approx [3.999999999858759, -3.862899813854707e-12]$
- $\nabla f(\bar{x}) \approx [-3.917310916506123e-11, -1.8698611541996652e-08]$
- $||\nabla f(\bar{x})|| \approx 1.8698652575278575e-08$
- Distance $||\bar{x} - x^*|| \approx 1.4129361162427862e-10$

(c) **Starting point [1.9, 0.6]:**

- Number of iterations: 19
- Final iterate $\bar{x} \approx [4.000000000544454, 2.6029506363324995e-12]$
- $\nabla f(\bar{x}) \approx [1.387165937834461e-10, 1.3049029432865988e-08]$
- $||\nabla f(\bar{x})|| \approx 1.3049766719493253e-08$
- Distance $||\bar{x} - x^*|| \approx 5.444604815689273e-10$

4.1.2 Approximated Solutions

All the runs are located in file "4.1.2 approx_qn_bfgs.txt"

1. **Rosenbrock Function:**

   (a) **Starting point [1.2, 1.2]:**

   - Number of iterations: 13
   - Final iterate $\bar{x} \approx [0.9999999932373403, 0.9999999880322893]$
   - $\nabla f(\bar{x}) \approx [-6.361687408350495e-07, 3.1152171954048724e-07]$
   - $||\nabla f(\bar{x})|| \approx 7.083476890349218e-07$
   - Distance $||\bar{x} - x^*|| \approx 1.3746260046734159e-08$

   (b) **Starting point [-1.2, 1.]:**

   - Number of iterations: 33
   - Final iterate $\bar{x} \approx [0.9999996844444438, 0.9999993688326836]$
   - $\nabla f(\bar{x}) \approx [-6.081896994143403e-07, -1.1260714582718505e-08]$
   - $||\nabla f(\bar{x})|| \approx 6.082939373087809e-07$
   - Distance $||\bar{x} - x^*|| \approx 7.056539452030539e-07$

   (c) **Starting point [0.2, 0.8]:**

   - Number of iterations: 23
   - Final iterate $\bar{x} \approx [0.9999999988750274, 0.9999999977212608]$
   - $\nabla f(\bar{x}) \approx [9.667610587120734e-09, -5.758826748652036e-09]$
   - $||\nabla f(\bar{x})|| \approx 1.1252856525576031e-08$
   - Distance $||\bar{x} - x^*|| \approx 2.5413019476684327e-09$

2. **Quadratic Function:**

(a) **Starting point [-0.2, 1.2]:**

- Number of iterations: 10
- Final iterate $\bar{x} \approx [-2.558477348502419e{-}11, 1.0000000001109866]$
- $\nabla f(\bar{x}) \approx [[-7.466251263393606e-09, 8.367222472184019e-10]$
- $||\nabla f(\bar{x})|| \approx 7.51298955457259e-09$
- Distance $||\bar{x} - x^*|| \approx 1.1389730305632127e-10$

(b) **Starting point [3.8, 0.1]:**

- Number of iterations: 11
- Final iterate $\bar{x} \approx [3.9999999956597296, 1.1825078675977688e-10]$
- $\nabla f(\bar{x}) \approx [-1.9336336810942954e{-}09, 5.598692413123059e{-}07]$
- $||\nabla f(\bar{x})|| \approx 5.598725804206075e-07$
- Distance $||\bar{x} - x^*|| \approx 4.341880939495625e-09$

(c) **Starting point [1.9, 0.6]:**

- Number of iterations: 10 -¿ diverges at 9
- Final iterate $\bar{x} \approx [1.5831470743104705, -0.024385554687575438]$
- $\nabla f(\bar{x}) \approx [-0.9747695437489767, -23.3644448690562]$
- $||\nabla f(\bar{x})|| \approx 23.384769823596454$
- Distance $||\bar{x} - x^*|| \approx 2.416975945203315$
- rho $= \infty$

4.2 SR1 Method

4.2.1 Exact Solutions

All the runs are located in file "4.2.1 exact_qn_sr1.txt"

1. **Rosenbrock Function:**

(a) **Starting point [1.2, 1.2]:**

- Number of iterations: 1000
- Final iterate $\bar{x} \approx [0.04206416827900303, 0.0816424551860706]$
- $\nabla f(\bar{x}) \approx [-4.7631236183420384e-07, 3.8459209150687457e-07]$
- $||\nabla f(\bar{x})|| \approx 0.09184162858864073$
- Distance $||\bar{x} - x^*|| \approx 0.26160661852730166$

(b) **Starting point [-1.2, 1.]:**

- Number of iterations: 1000
- Final iterate $\bar{x} \approx [1.5138090727188056, 2.3942457202334397]$
- $\nabla f(\bar{x}) \approx [-61.11594677243852, 20.525562317533907]$

- $||\nabla f(\bar{x})|| \approx 64.47059530004749$
- Distance $||\bar{x} - x^*|| \approx 1.4859074303594493$

(c) **Starting point [0.2, 0.8]:**

- Number of iterations: 1000
- Final iterate $\bar{x} \approx [0.4526567196822766, 0.2024504247699525]$
- $\nabla f(\bar{x}) \approx [-0.6515028409679398, -0.4895362207133269]$
- $||\nabla f(\bar{x})|| \approx 0.8149243297261309$
- Distance $||\bar{x} - x^*|| \approx 0.967300362585787$

2. **Quadratic Function:**

(a) **Starting point [-0.2, 1.2]:**

- Number of iterations: 1000
- Final iterate $\bar{x} \approx [-0.0011159542151841786, 1.1823939693347825]$
- $\nabla f(\bar{x}) \approx [-0.7539845753333217, 0.7289016727744723]$
- $||\nabla f(\bar{x})|| \approx 1.0487089150064441$
- Distance $||\bar{x} - x^*|| \approx 2.1540123413572898e-08$

(b) **Starting point [3.8, 0.1]:**

- Number of iterations: 1000
- Final iterate $\bar{x} \approx [3.794720678079162, 8.989096280748178e-05]$
- $\nabla f(\bar{x}) \approx [-0.05121154183453141, 0.18340664003078994]$
- $||\nabla f(\bar{x})|| \approx 0.19042220885299518$
- Distance $||\bar{x} - x^*|| \approx 0.20527934160227673$

(c) **Starting point [1.9, 0.6]:**

- Number of iterations: 9
- Final iterate $\bar{x} \approx [1.5284816372318362, 0.00834402035290613]$
- $\nabla f(\bar{x}) \approx [-0.5456853815798153, 3.409988291480307]$
- $||\nabla f(\bar{x})|| \approx 3.4533741013250636$
- Distance $||\bar{x} - x^*|| \approx 2.4715324477287113$

4.2.2 Approximated Solutions

All the runs are located in file "4.2.2 approx_qn_sr1.txt"

1. **Rosenbrock Function:**

(a) **Starting point [1.2, 1.2]:**

- Number of iterations: 1000
- Final iterate $\bar{x} \approx [1.111802358357418, 1.2365126963259327]$
- $\nabla f(\bar{x}) \approx [0.042064168322142326, 0.08164245535700887]$

- $||\nabla f(\bar{x})|| \approx 0.09184162876035414$
- Distance $||\bar{x} - x^*|| \approx 0.26160661852797873$

(b) **Starting point [-1.2, 1.]:**

- Number of iterations: 1000
- Final iterate $\bar{x} \approx [1.5138090670758741, 2.394245717383411]$
- $\nabla f(\bar{x}) \approx [-61.11595516744561, 20.525565167295312]$
- $||\nabla f(\bar{x})|| \approx 64.4706041655118$
- Distance $||\bar{x} - x^*|| \approx 1.485907425733973$

(c) **Starting point [0.2, 0.8]:**

- Number of iterations: 1000
- Final iterate $\bar{x} \approx [0.4526567196806503, 0.20245042476877273]$
- $\nabla f(\bar{x}) \approx [-0.6515028407871881, -0.4895362206436271]$
- $||\nabla f(\bar{x})|| \approx 0.8149243295397568$
- Distance $||\bar{x} - x^*|| \approx 0.96730036258768$

2. **Quadratic Function:**

(a) **Starting point [-0.2, 1.2]:**

- Number of iterations: 1000
- Final iterate $\bar{x} \approx [-0.0006399145594388124, 1.0572846945395347]$
- $\nabla f(\bar{x}) \approx [-0.100349292951607, 0.4571276116163009]$
- $||\nabla f(\bar{x})|| \approx 0.4680124292130617$
- Distance $||\bar{x} - x^*|| \approx 0.05728826859952371$

(b) **Starting point [3.8, 0.1]:**

- Number of iterations: 1000
- Final iterate $\bar{x} \approx [3.7973602446971038, 9.337090061698302e-05]$
- $\nabla f(\bar{x}) \approx [-0.10112320410638614, -0.0006115727811326077]$
- $||\nabla f(\bar{x})|| \approx 0.10112505342400792$
- Distance $||\bar{x} - x^*|| \approx 0.2026397768142838$

(c) **Starting point [1.9, 0.6]:**

- Number of iterations: 1000
- Final iterate $\bar{x} \approx [1.7029963707700075, 0.005203128015321194]$
- $\nabla f(\bar{x}) \approx [-1.124264226715077, -0.0253539197414554]$
- $||\nabla f(\bar{x})|| \approx 1.1245500756824955$
- Distance $||\bar{x} - x^*|| \approx 2.2970095222347036$

4.3 SR1 Method [Trust Region]

4.3.1 Exact Solutions

All the runs are located in file "4.3.1 exact_qn_sr1_trust.txt"

1. **Rosenbrock Function:**
   (a) **Starting point [1.2, 1.2]:**
   - Number of iterations: 1000
   - Final iterate $\bar{x} \approx [0.9999830425178828, 0.9999660345566617]$
   - $\nabla f(\bar{x}) \approx [-1.3608644500628941e-05, -1.0153332041795693e-05]$
   - $||\nabla f(\bar{x})|| \approx 1.6978968069215908e-05$
   - Distance $||\bar{x} - x^*|| \approx 3.796323933654216e-05$

   (b) **Starting point [-1.2, 1.]:**
   - Number of iterations: 1000
   - Final iterate $\bar{x} \approx [0.9979938481937891, 0.9959874909111021]$
   - $\nabla f(\bar{x}) \approx [-0.0023236495005287848, -0.0008460243091423791]$
   - $||\nabla f(\bar{x})|| \approx 0.0024728736589173963$
   - Distance $||\bar{x} - x^*|| \approx 0.004486075596560071$

   (c) **Starting point [0.2, 0.8]:**
   - Number of iterations: 1000
   - Final iterate $\bar{x} \approx [1.000033418187719, 1.0000671545311057]$
   - $\nabla f(\bar{x}) \approx [1.000033418187719, 1.0000671545311057]$
   - $||\nabla f(\bar{x})|| \approx 8.728434551424043e-05$
   - Distance $||\bar{x} - x^*|| \approx 7.501004145084355e-05$

2. **Quadratic Function:**
   (a) **Starting point [-0.2, 1.2]:**
   - Number of iterations: 1000
   - Final iterate $\bar{x} \approx [-0.007791940092785499, 1.130474610655278]$
   - $\nabla f(\bar{x}) \approx [-5.84620608500771, 0.5346973052376088]$
   - $||\nabla f(\bar{x})|| \approx 5.870607021135849$
   - Distance $||\bar{x} - x^*|| \approx 0.13070707079594418$

   (b) **Starting point [3.8, 0.1]:**
   - Number of iterations: 1000
   - Final iterate $\bar{x} \approx [3.9945416187071876, 1.3741812311951485e-06]$
   - $\nabla f(\bar{x}) \approx [-0.0013632166160585932, 0.0011251956922249352]$
   - $||\nabla f(\bar{x})|| \approx 0.0017676042792717473$

- Distance $||\bar{x} - x^*|| \approx 0.005458381465791705$

(c) **Starting point [1.9, 0.6]:**

- Number of iterations: 9
- Final iterate $\bar{x} \approx [1.5365643821046733, 0.004705379783804337]$
- $\nabla f(\bar{x}) \approx [-0.5907412512800368, 0.8882488921858505]$
- $||\nabla f(\bar{x})|| \approx 1.0667526988169724$
- Distance $||\bar{x} - x^*|| \approx 2.4634401117368854$

4.3.2 Approximated Solutions

All the runs are located in file "4.3.2 approx_qn_sr1_trust.txt"

1. **Rosenbrock Function:**

   (a) **Starting point [1.2, 1.2]:**

   - Number of iterations: 1000
   - Final iterate $\bar{x} \approx [0.9999830425178828, 0.9999660345566617]$
   - $\nabla f(\bar{x}) \approx [-1.3608644500628941e-05, -1.0153332041795693e-05]$
   - $||\nabla f(\bar{x})|| \approx 1.6978968069215908e-05$
   - Distance $||\bar{x} - x^*|| \approx 3.796323933654216e-05$

   (b) **Starting point [-1.2, 1.]:**

   - Number of iterations: 1000
   - Final iterate $\bar{x} \approx [0.9979938481937891, 0.9959874909111021]$
   - $\nabla f(\bar{x}) \approx [-0.0023236495005287848, -0.0008460243091423791]$
   - $||\nabla f(\bar{x})|| \approx 0.0024728736589173963$
   - Distance $||\bar{x} - x^*|| \approx 0.004486075596560071$

   (c) **Starting point [0.2, 0.8]:**

   - Number of iterations: 1000
   - Final iterate $\bar{x} \approx [1.000033418187719, 1.0000671545311057]$
   - $\nabla f(\bar{x}) \approx [-5.998341940803055e-05, 6.340777845004197e-05]$
   - $||\nabla f(\bar{x})|| \approx 8.728434551424043e-05$
   - Distance $||\bar{x} - x^*|| \approx 7.501004145084355e-05$

2. **Quadratic Function:**

   (a) **Starting point [-0.2, 1.2]:**

   - Number of iterations: 1000
   - Final iterate $\bar{x} \approx [-0.007791940092785499, 1.130474610655278]$
   - $\nabla f(\bar{x}) \approx [-5.84620608500771, 0.5346973052376088]$

- $||\nabla f(\bar{x})|| \approx 5.870607021135849$
- Distance $||\bar{x} - x^*|| \approx 0.13070707079594418$

(b) **Starting point [3.8, 0.1]:**

- Number of iterations: 1000
- Final iterate $\bar{x} \approx [3.9945416187071876, 1.3741812311951485e-06]$
- $\nabla f(\bar{x}) \approx [-0.0013632166160585932, 0.0011251956922249352]$
- $||\nabla f(\bar{x})|| \approx 0.0017676042792717473$
- Distance $||\bar{x} - x^*|| \approx 0.005458381465791705$

(c) **Starting point [1.9, 0.6]:**

- Number of iterations: 1000
- Final iterate $\bar{x} \approx [1.5365643821046733, 0.004705379783804337]$
- $\nabla f(\bar{x}) \approx [-0.5907412512800368, 0.8882488921858505]$
- $||\nabla f(\bar{x})|| \approx 1.0667526988169724$
- Distance $||\bar{x} - x^*|| \approx 2.4634401117368854$