

Numerical Optimization

Mohamed Abdelaziz [k12137202]

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Auxiliary Score: 30/30

1 Part I

Steepest Descent

1.1 $f(x) = x^5 - 2.5x^3 + 2x + 1$

- $f'(x) = 5x^4 - 7.5x^2 + 2$
 - $f''(x) = 20x^3 - 15x$
 - Critical points $\approx \{\pm 0.589, \pm 1.074\}$
 - Local minimizers $\approx \{-0.589, 1.074\}$
 - $\bar{x} \approx -0.58897123$
 - $\|\bar{x} - x^*\| \approx 2.87695648 \cdot e^{-5}$
 - $\nabla f(\bar{x}) = 9.75395409 \cdot e^{-09}$
 - Number of iterations = 403
-

1.2 $f(x) = x^4 - 3x^3 - 10x^2 + 24x + 7$

- $f'(x) = 4x^3 - 9x^2 - 20x + 24$
 - $f''(x) = 12x^2 - 18x - 20$
 - Critical points = $\{-1.935, 0.962, 3.224\}$
 - Local minimizers = $\{\pm -1.935, 3.224\}$
 - $\bar{x} \approx -1.9353626$
 - $\|\bar{x} - x^*\| \approx 0.0003626$
 - $\nabla f(\bar{x}) = 3.55271368 \cdot e^{-09}$
 - Number of iterations = 33
-

1.3 $f(x) = -2xe^{-x^4} + 5\cos(10x)$

- $f'(x) = -2e^{-x^4} + 40\sin(10x)$
 - $f''(x) = 8x^3 * np.exp(-x * 4) + 400 * np.cos(10 * x)$
 - Critical points = $\{\pm \frac{k\pi}{10} : k \in \mathbb{N}\}$
 - Local minimizers = $\{\pm \frac{k\pi}{10} : k \text{ is odd and } \in \mathbb{N}\}$
 - $\bar{x} \approx 0.31795764$
 - $||\bar{x} - x^*|| \approx 4.23625918 \quad e-05$
 - $\nabla f(\bar{x}) = -4.88498131 \quad e-09$
 - Number of iterations = 34
-

1.4 $f(x) = x^3 - 2xe^{-x} + e^{-2x}$

- $f'(x) = 3x^2 + 2xe^{-x} - 2e^{-x} - 2e^{-2x}$
 - $f''(x) = 6x - 2e^{-x} + 4e^{-2x}$
 - Critical points = $\{0.594\}$
 - Local minimizers = $\{\pm 0.594\}$
 - $\bar{x} \approx 0.59391612$
 - $||\bar{x} - x^*|| \approx 8.38820114 \quad e-05$
 - $\nabla f(\bar{x}) = -9.95592497 \quad e-07$
 - Number of iterations = 2383
-

1.5 $f(x) = (x - 3)(x - 5)(x - 7) + 85 - (x - 4)(x - 6)(x - 8)$

- $f'(x) = 6x - 33$
- $f''(x) = 6$
- Critical points = $\{5.5\}$
- Local minimizers = $\{\pm 5.5\}$
- $\bar{x} \approx 5.49999997$
- $||\bar{x} - x^*|| \approx 3.27825527 \quad e-08$

- $\nabla f(\bar{x}) = 7.531753 \cdot e^{-07}$
- Number of iterations = 14

Newton Method

1.6 $f(x) = x^5 - 2.5x^3 + 2x + 1$

- $f'(x) = 5x^4 - 7.5x^2 + 2$
- $f''(x) = 20x^3 - 15x$
- Critical points $\approx \{\pm 0.589, \pm 1.074\}$
- Local minimizers $\approx \{-0.589, 1.074\}$
- $\bar{x} \approx -0.58897123$
- $\|\bar{x} - x^*\| \approx 2.87676191 \cdot e^{-5}$
- $\nabla f(\bar{x}) = -5.55111512 \cdot e^{-11}$
- Number of iterations = 2

1.7 $f(x) = x^4 - 3x^3 - 10x^2 + 24x + 7$

- $f'(x) = 4x^3 - 9x^2 - 20x + 24$
- $f''(x) = 12x^2 - 18x - 20$
- Critical points = $\{-1.935, 0.962, 3.224\}$
- Local minimizers = $\{\pm -1.935, 3.224\}$
- $\bar{x} \approx -1.9353626$
- $\|\bar{x} - x^*\| \approx 0.0003626$
- $\nabla f(\bar{x}) = 3.55271368 \cdot e^{-09}$
- Number of iterations = 2

1.8 $f(x) = -2xe^{-x^4} + 5\cos(10x)$

- $f'(x) = -2e^{-x^4} + 40\sin(10x)$
 - $f''(x) = 8x^3 * np.exp(-x * 4) + 400 * np.cos(10 * x)$
 - Critical points = $\{\pm \frac{k\pi}{10} : k \in \mathbb{N}\}$
 - Local minimizers = $\{\pm \frac{k\pi}{10} : k \text{ is odd and } \in \mathbb{N}\}$
 - $\bar{x} \approx 0.31795764$
 - $||\bar{x} - x^*|| \approx 4.23625918 \quad e-05$
 - $\nabla f(\bar{x}) \approx 0$
 - Number of iterations = 2
-

1.9 $f(x) = x^3 - 2xe^{-x} + e^{-2x}$

- $f'(x) = 3x^2 + 2xe^{-x} - 2e^{-x} - 2e^{-2x}$
 - $f''(x) = 6x - 2e^{-x} + 4e^{-2x}$
 - Critical points = $\{0.594\}$
 - Local minimizers = $\{\pm 0.594\}$
 - $\bar{x} \approx 0.59391627$
 - $||\bar{x} - x^*|| \approx 8.3725868 \quad e-05$
 - $\nabla f(\bar{x}) = -2.77555756 \quad e^{-11}$
 - Number of iterations = 2
-

1.10 $f(x) = (x - 3)(x - 5)(x - 7) + 85 - (x - 4)(x - 6)(x - 8)$

- $f'(x) = 6x - 33$
- $f''(x) = 6$
- Critical points = $\{5.5\}$
- Local minimizers = $\{\pm 5.5\}$
- $\bar{x} \approx 5.5$
- $||\bar{x} - x^*|| = 0$
- $\nabla f(\bar{x}) = 7.1054273 \quad e^{-09}$
- Number of iterations = 2

2 Part II

$$\begin{aligned}f(x) &= \frac{1}{2} \sum_{j=1}^m r_j^2(x) \\r_j(x) &= c_j^T x - b_j \\c_j^T(x) &= (1, a_j, a_j^2, a_j^3, \dots, a_j^n) \in \mathbb{R}^{n+1} \\b_j &= g(a_j) = \sin(a_j)\end{aligned}$$

Steepest Descent

2.1 Problem (11)

- $m = 100$
 - $q = 1$
 - $n = 1$
 - $\bar{x} \approx [0.00309302, 0.90142102]$
 - Number of iterations = 27
-

2.2 Problem (12)

- $m = 150$
 - $q = 2$
 - $n = 2$
 - $\bar{x} \approx [0.00849632, 0.64558447, -0.00568154]$
 - Number of iterations = 79
-

2.3 Problem (13)

- $m = 100$
 - $q = 1$
 - $n = 3$
 - $\bar{x} \approx [4.16871381e^{-05}, 9.97938123e^{-01}, 7.39845194e^{-05}, -1.57404640e^{-01}]$
 - Number of iterations = 505
-

2.4 Problem (14)

- $m = 150$
 - $q = 2$
 - $n = 4$
 - $\bar{x} \approx [9.86767585e^{-04}, 9.71720349e^{-01}, -2.06426593e^{-04}, -1.32634310e^{-01}, -1.46827021e^{-04}]$
 - Number of iterations = 3174
-

2.5 Problem (15)

- $m = 100$
 - $q = 1$
 - $n = 5$
 - $\bar{x} \approx [2.87849602e^{-07}, 9.99983413e^{-01}, -3.05409039e^{-06}, -1.66518000e^{-01}, 3.87024745e^{-06}, 8.01197670e^{-03}]$
 - Number of iterations = 3054
-

Newton Method

2.6 Problem (16)

- $m = 100$
 - $q = 1$
 - $n = 1$
 - $\bar{x} \approx [0.00309302, 0.90142101]$
 - Number of iterations = 2
-

2.7 Problem (17)

- $m = 150$
 - $q = 2$
 - $n = 2$
 - $\bar{x} \approx [0.00849633, 0.64558447, -0.00568154]$
 - Number of iterations = 3
-

2.8 Problem (18)

- $m = 100$
 - $q = 1$
 - $n = 3$
 - $\bar{x} \approx [4.16622157e^{-05}, 9.97938343e^{-01}, 7.40263268e^{-05}, -1.57404981e^{-01}]$
 - Number of iterations = 2
-

2.9 Problem (19)

- $m = 150$
 - $q = 2$
 - $n = 4$
 - $\bar{x} \approx [9.86741666e^{-04}, 9.71720353e^{-01}, -2.06379856e^{-04}, -1.32634311e^{-01}, -1.46838850e^{-04}]$
 - Number of iterations = 3
-

2.10 Problem (20)

- $m = 100$
- $q = 1$
- $n = 5$
- $\bar{x} \approx [-4.87982611e^{-07}, 9.99986511e^{-01}, 2.83116348e^{-06}, -1.66537886e^{-01}, -3.73357240e^{-06}, 8.03353596e^{-03}]$
- Number of iterations = 2

3 Part III

Most of the trials in this part has not fully converged, because the hillbert matrix functions are ill-conditioned matrix, meaning that its eigenvalues are spread out over several orders of magnitude which causes very slow convergence using steepest descent, further more the condition number of the Hilbert matrix grows very quickly with n making steepest descent even slower as n increases.

3.1 n=5

- Hessian = $\begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ 1/3 & 1/4 & 1/5 & 1/6 & 1/7 \\ 1/4 & 1/5 & 1/6 & 1/7 & 1/8 \\ 1/5 & 1/6 & 1/7 & 1/8 & 1/9 \end{pmatrix},$
 - *Eigenvalues* $\lambda \in \{87055.1122, /1792.2506, /81.8198, /2.5479, /-1.1477, /-18.3242, /-245.0822, /-3967.8396\}$
 - Local minimizers = $[5, -120, 630, -1120, 630]$
 - $\bar{x} \approx [4.98888889, -119.79006035, 629.08935989, -1118.61974829, 629.32309598]$
 - $\|\bar{x} - x^*\| \approx (0.01111111, 0.20993965, 0.91064011, 0.4696116, 0.67690402)$
 - $\nabla f(\bar{x}) \approx (-4e^{-12}, 0)$
 - Number of iterations = 1,549,951
-

3.2 n=8

- Hessian = $\begin{pmatrix} 1 & 1/2 & 1/3 & \dots & 1/8 \\ 1/2 & 1/3 & 1/4 & \dots & 1/9 \\ 1/3 & 1/4 & 1/5 & \dots & 1/10 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/8 & 1/9 & 1/10 & \dots & 1/15 \end{pmatrix},$
- *Eigenvalues* $\lambda \in \{1548.8101, -122.5893, -37.0330, 9.7909, 0.0086\}$
- Local minimizers = $[-8, 504, -7560, 46200, -138600, 216216, -168168, 51480]$
- $\bar{x} \approx [-4.74286231, 123.72958515, -642.40119086, 703.32457717, 1033.52817099, -1040.46540292, -1956.32642031, 1826.88843977]$
- $\|\bar{x} - x^*\| \approx (-3.2571377, 380.27, 6917.6, 45497, 137567, 215176, 167128, 49653.1)$

- $\nabla f(\bar{x}) \approx [1.00165383e^{-04}, 6.61100170e^{-05}, -9.89659915e^{-06}, 2.15791325e^{-04}, -1.80901709e^{-04}, -2.05800000e^{-05}, 2.72295326e^{-04}, -1.04381341e^{-04}]$
 - Number of iterations = 2,000,000
-

3.3 n=12

- Hessian =
$$\begin{pmatrix} & 1/2 & 1/3 & \dots & 1/12 \\ 1/2 & 1/3 & 1/4 & \dots & 1/13 \\ 1/3 & 1/4 & 1/5 & \dots & 1/14 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/12 & 1/13 & 1/14 & \dots & 1/23 \end{pmatrix},$$
 - *Eigenvalues* $\lambda \in \{3.53e+08, 10799.9829, 374.9138, 20.1487, 1.4916, 0.1688, -0.8901, -5.5429, -58.0767, -239.5943, -1470.2308, -8197.4079\}$
 - Local minimizers = $[-1.28758021e+01, 1.82700490e+03, -6.35497518e+04, 9.48423728e+05 - 7.55539458e+06, 3.58352883e+07 - 1.07176657e+08, 2.07239527e+08 - 2.58446725e+09, 8.81021812e+07 - 1.67167856e+08, 2.00603085e+08, 8.81021812e+07, 1.67167856e+08]$
 - $\bar{x} \approx [2.88956174, -102.85260253, 731.95872666, -1432.73684337, -182.00373731, 1317.20529959, 1336.56]$
 - $\|\bar{x} - x^*\| \approx [1.57653638e+01, 1.92985750e+03, 6.42817105e+04, 9.49856465e+05, 7.55521258e+06, 3.58339711e+07, 1.07177994e+08, 2.07239445e+08, 2.58445309e+08, 2.00603085e+08, 8.81021812e+07, 1.67167856e+08]$
 - $\nabla f(\bar{x}) \approx (1.43356586e-04, 1.15933381e-04, -7.15451371e-05, 2.16071195e-04, 2.00548845e-04, -8.17737107e-05, -2.27452069e-04, -1.06640900e-04, 1.61651583e-04, 3.54704771e-04, 2.46762356e-04, -3.44236096e-04)$
 - Number of iterations = 2,000,000
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3.4 n=20

- Hessian =
$$\begin{pmatrix} & 1 & 1/2 & 1/3 & \dots & 1/20 \\ 1 & 1/2 & 1/3 & 1/4 & \dots & 1/21 \\ 1/2 & 1/3 & 1/4 & 1/5 & \dots & 1/22 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/20 & 1/21 & 1/22 & \dots & 1/39 \end{pmatrix},$$

- *Eigenvalues* $\lambda \in$
 $\{5.51e+14, 5.50e+07, 2.15e+05, 4112.0135, 156.3323, 12.5478, 0.9226, -1.3929, -13.6734, -124.3451,$
 $-780.9774, -3434.5738, -11724.2594, -31310.2004, -65545.8529, -111036.2281, -152381.1091,$
 $-160056.3357, -86658.6756, -12220.4411\}$
 - Local minimizers = $[-3.06198188e+01, 5.79631108e+03, -2.67656855e+05, 5.30590627e+06,$
 $-5.62934738e+07, 3.56180768e+08, -1.41454296e+093.54689896e+09, -5.26881741e+09,$
 $3.31570124e+09, 2.23492076e+09-4.62144407e+09, -4.79904240e+08, 5.96741325e+09, -4.19120773e+09,$
 $-4.68358103e+08, 1.17802999e+09, 4.13333400e+08, -7.15516924e+08, 1.98562765e+08]$
 - $\bar{x} \approx -5.03021766e-02, 2.88787282e+01, -3.42804782e+02, 9.41297643e+02,$
 $-1.45629090e+02, -9.42460729e+02, -7.64172779e+02, -2.34752591e+01, 7.29408069e+02,$
 $1.14570172e+03, 1.10996315e+03, 6.76255222e+02, -4.93988646e+00, -7.43086406e+02,$
 $-1.34428010e+03, -1.63219190e+03, -1.45815316e+03, -7.04402591e+02, 7.16638196e+02, 2.86517034e+03]$
 - $\|\bar{x}-x^*\| \approx [3.05695166e+01, 5.76743235e+03, 2.67314050e+05, 5.30496497e+06,$
 $5.62933282e+07, 3.56181710e+08, 1.41454220e+09, 3.54689898e+09, 5.26881814e+09,$
 $3.31570009e+09, 2.23491965e+09, 4.62144475e+09, 4.79904235e+08, 5.96741399e+09,$
 $4.19120639e+09, 4.68356471e+08, 1.17803145e+09, 4.13334104e+08, 7.15517641e+08, 1.98559900e+08]$
 - $\nabla f(\bar{x}) \approx [0.00289293, 0.00178108, 0.00164508, 0.00077264, 0.00083569, 0.00108468, 0.00110518,$
 $0.00089555, 0.00059735, 0.00034262, 0.00020566, 0.00020344, 0.00031179, 0.00048262, 0.00065757,$
 $0.00077734, 0.00078725, 0.00064025, 0.00029804, -0.00026889]$
 - Number of iterations = 2,000,000
-

3.5 n=30

- Hessian =
$$\begin{pmatrix} 1 & 1/2 & 1/3 & \dots & 1/30 \\ 1/2 & 1/3 & 1/4 & \dots & 1/31 \\ 1/3 & 1/4 & 1/5 & \dots & 1/32 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/30 & 1/31 & 1/32 & \dots & 1/59 \end{pmatrix},$$

- *Eigenvalues* $\lambda \in$
 $\{51.30e+24, 2.71e+10, 5.70e+07, 2.24e+05, 3617.6886, 116.8753, 9.9089, 1.3278, -4.7526,$
 $-27.3374, -141.4316, -685.4242, -3051.3098, -12481.6667, -46147.4284, -152498.9364,$
 $-449950.8891, -1182552.2831, -2750902.1395, -5458738.7743, -9102221.2213, -11110549,$
 $-9153253.5203, -3693137.9142, 6345777.9014, 24957847.6934, 70364783.4911, 160440005.6745,$
 $315801782.0742, 534437200.9579\}$

- Local minimizers = $[-7.09110073e+01, 1.21386137e+04, -5.09455681e+05, 9.11337230e+06, -8.56771813e+07, 4.65726408e+08, -1.51330544e+09, 2.87011360e+09, -2.81033085e+09, 8.97538752e+08, -2.12632140e+08, 1.35664526e+09, -9.14466496e+08, 1.59701836e+09, -4.54610631e+09, 6.56972375e+08, 5.34940917e+09, -2.00704004e+09, -8.27746637e+08, -4.32648733e+08, -2.21193601e+09, 3.86720013e+09, -3.96275416e+09, 3.55276469e+09, 2.59846246e+08, -8.78551536e+08, 5.97786826e+08, -4.68162177e+09, 5.22987290e+09, -1.62469308e+09,]$
 - $\bar{x} \approx [-3.51297069, 79.07184292, -315.71165516, 70.39404261, 583.60794576, 343.62308551, -231.09213981, -669.96597158, -778.52276799, -576.79957483, -183.11702924, 266.93337863, 661.46259988, 926.57322926, 1026.32617362, 956.42391909, 736.07699598, 400.23479828, -6.9339659, -437.09671311, -841.44591795, -1173.23901168, -1389.43057021, -1451.61863758, -1326.49166026, -985.9246594, -406.83886922, 429.08980117, 1535.81123442, 2923.32617488]$
 - $\|\bar{x} - x^*\| \approx 6.73980367e+01, 1.20595418e+04, 5.09139970e+05, 9.11330191e+06, 8.56777649e+07, 4.65726065e+08, 1.51330521e+09, 2.87011427e+09, 2.81033007e+09, 8.97539328e+08, 2.12631957e+08, 1.35664499e+09, 9.14467158e+08, 1.59701743e+09, 4.54610734e+09, 6.56971419e+08, 5.34940844e+09, 2.00704044e+09, 8.27746630e+08, 4.32648296e+08, 2.21193517e+09, 3.86720130e+09, 3.96275278e+09, 3.55276614e+09, 2.59847573e+08, 8.78550550e+08, 5.97787233e+08, 4.68162220e+09, 5.22987137e+09, 1.62469601e+09]$
 - $\nabla f(\bar{x}) \approx [1.85173576e-03, 1.21584169e-03, 7.24059939e-04, 1.17944936e-03, 6.17742555e-04, 2.45304691e-04, 2.59834450e-04, 4.56954041e-04, 6.41262999e-04, 7.19102299e-04, 6.78345150e-04, 5.49969172e-04, 3.78884180e-04, 2.07155780e-04, 6.64331708e-05, -2.40586436e-05, -5.66312285e-05, -3.30106190e-05, 3.80908128e-05, 1.43243740e-04, 2.66217564e-04, 3.89493323e-04, 4.95396755e-04, 5.66904702e-04, 5.88186136e-04, 5.44936176e-04, 4.24553973e-04, 2.16206468e-04, -8.91885266e-05, -4.99033982e-04]$
 - Number of iterations = 2,000,000
-

4 Part IV

4.1 $f(\mathbf{x}) = 3x_1^4 - 4x_1^3 + 2x_2^2$

- $\nabla f(x) = [12x_1^3 - 12x_1^2, 4x_2]$
- Hessian = $\begin{pmatrix} 36x_1^2 - 24x_1 & 0 \\ 0 & 4 \end{pmatrix}$,
- Critical points = $\{(0, 0), (1, 0)\}$

- *Eigenvalues* λ :
 $\lambda_{0,1} = 0, \lambda_{0,2} = 4,$
 $\lambda_{1,1} = 12, \lambda_{1,2} = 4$
 - Local minimizers = $(1, 0)$
 - $\bar{x} \approx (1.032660625, 0)$
 - $\|\bar{x} - x^*\| \approx (0.032660625, 0)$
 - $\nabla f(\bar{x}) \approx (-4e^{-12}, 0)$
 - Number of iterations = 1
-

4.2 $f(\mathbf{x}) = x_1^4 - 2x_1^2x_2 + 2x_2^2$

- $\nabla f(x) = [4x_1^3 - 4x_1x_2, 4x_1^3 - 4x_1x_2]$
 - Hessian = $\begin{pmatrix} 12x_1^2 - 4x_2 & -4x_1 \\ -4x_1 & 4 \end{pmatrix},$
 - Critical points = $\{(0, 0)\}$
 - *Eigenvalues* λ :
 $\lambda_{0,1} = 0, \lambda_{0,2} = 4$
 - Local minimizers = $(0, 0)$
 - $\bar{x} \approx (1.15860403e^{-03}, 6.68702018e^{-07})$
 - $\|\bar{x} - x^*\| \approx (1.15860403e^{-03}, 6.68702018e^{-07})$
 - $\nabla f(\bar{x}) \approx (3.15342613e^{-9}, -9.98844356e^{-9})$
 - Number of iterations = 2180
-

4.3 $f(\mathbf{x}) = 3x_1^4 - 4x_1^3 + 2x_2^2 + x_1^2 - x_2^2$

- $\nabla f(x) = [12x_1^3 - 12x_1^2 + 2x_1, 4x_2 - 2x_2]$
- Hessian = $\begin{pmatrix} -24x_1 + 36x_1^2 + 2 & 0 \\ 0 & 2 \end{pmatrix},$
- Critical points = $\{(0, 0), (\frac{1}{2} + \frac{1}{2\sqrt{3}}, 0)\}$
- *Eigenvalues* λ :
 $\lambda_{0,1} = 2, \lambda_{0,2} = 4$

- Local minimizers = $(0, 0)$
 - $\bar{x} \approx (4.92429539e^{-09}, 2.86102721e^{-12})$
 - $\|\bar{x} - x^*\| \approx (4.92429539e^{-09}, 2.86102721e^{-12})$
 - $\nabla f(\bar{x}) \approx (9.94403080e^{-9}, 5.77985296e^{-12})$
 - Number of iterations = 1230
-

4.4 $f(\mathbf{x}) = x_1^4 + x_2^2 - 2x_1^2 + 2$

- $\nabla f(x) = [x_2 + 2, -2x_1]$
 - Hessian = $\begin{pmatrix} 4x_1^2 + 2 & 0 \\ 0 & 2 \end{pmatrix}$
 - Critical points = $\{(\pm 1, 0)\}$
 - Local minimizers = $\{(\pm 1, 0)\}$
 - $\bar{x} \approx (1.099999997, 8.40880791e^{-23})$
 - $\|\bar{x} - x^*\| \approx (0.099999997e^{-09}, 8.40880791e^{-23})$
 - $\nabla f(\bar{x}) \approx (0, 0)$
 - Number of iterations = 3
-

4.5 $f(\mathbf{x}) = 2x_1^4 - 4x_1^3 + 2x_1^2 + x_2^2$

- $\nabla f(x) = [8x_1^3 - 12x_1^2 + 4x_1, 2x_2]$
- Hessian = $\begin{pmatrix} 24x_1^2 - 24x_1 + 4 & 0 \\ 0 & 2 \end{pmatrix}$
- Critical points = $\{(0, 0), (1, 0)\}$
- *Eigenvalues* λ :
 $\lambda_{0,1} = 2, \lambda_{0,2} = 4$
- Local minimizers = $(0, 0)$
- $\bar{x} \approx (1.00000000e^{-12}, 2.40880791e^{-23})$
- $\|\bar{x} - x^*\| \approx ((1.00000000e^{-12}, 2.40880791e^{-23}))$
- $\nabla f(\bar{x}) \approx (3.99999989e^{-12}, 0)$
- Number of iterations = 2