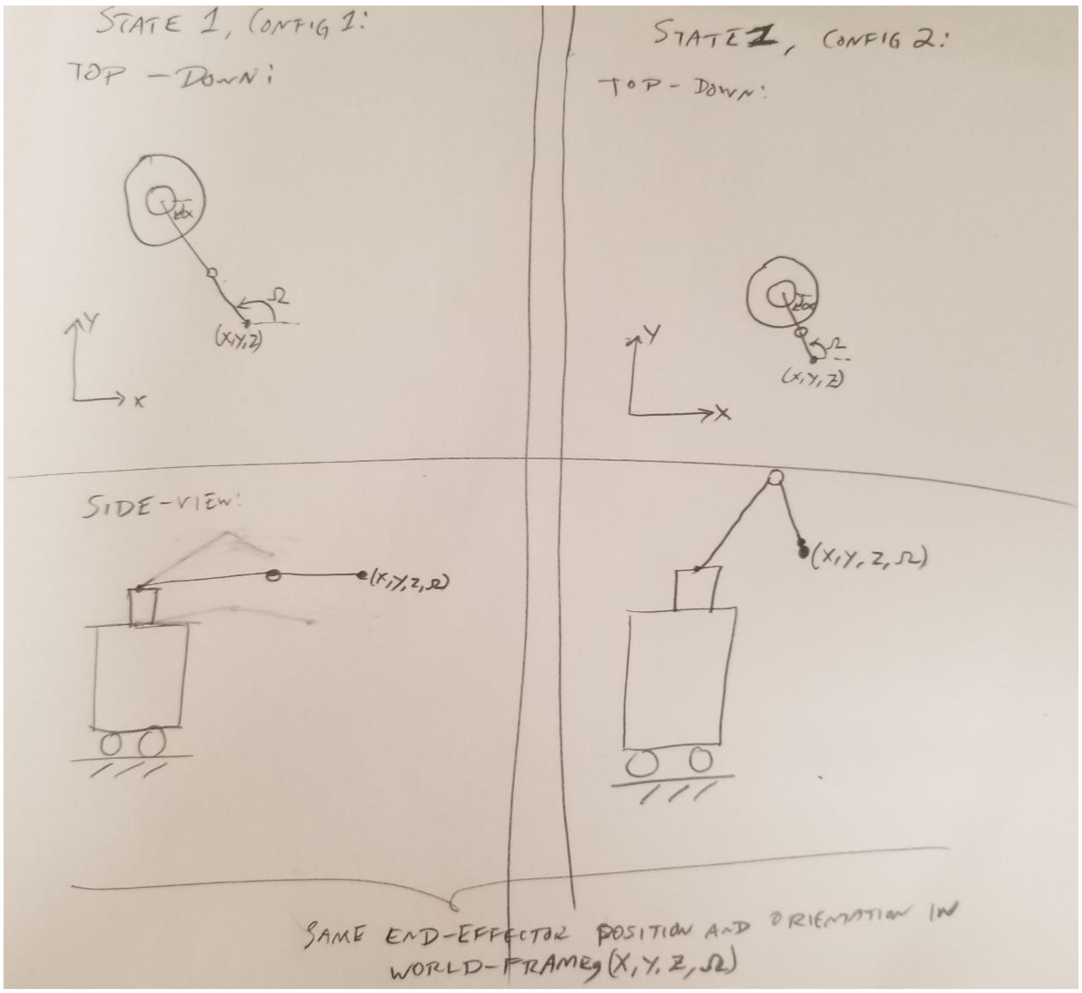
2.)

For any X, Y, Z, Ω position of the RR-Arm's end effector in the world-frame, there are infinitely many robot configurations (so long as all actuators values are considered to be able to take up any value within a bounded subset of the real numbers) since the system is overactuated and, for any position, the arm could scrunch back slightly while maintaining the same Z-height and the mobile robot platform could then translate in the x-y plane while maintaining the same orientation, Ω, to make up the difference.



3.)

3.1)

$$\vec{q} = \begin{bmatrix} x_i \\ y_i \\ \theta_i \\ \phi_i \end{bmatrix}$$

3.2)

* Lectures used w to represent constraints and q to represent states, where: $\vec{w}_j(\vec{q}) * \vec{q} = \vec{0}$

<u>From knife-edge constraint on rear wheels:</u>

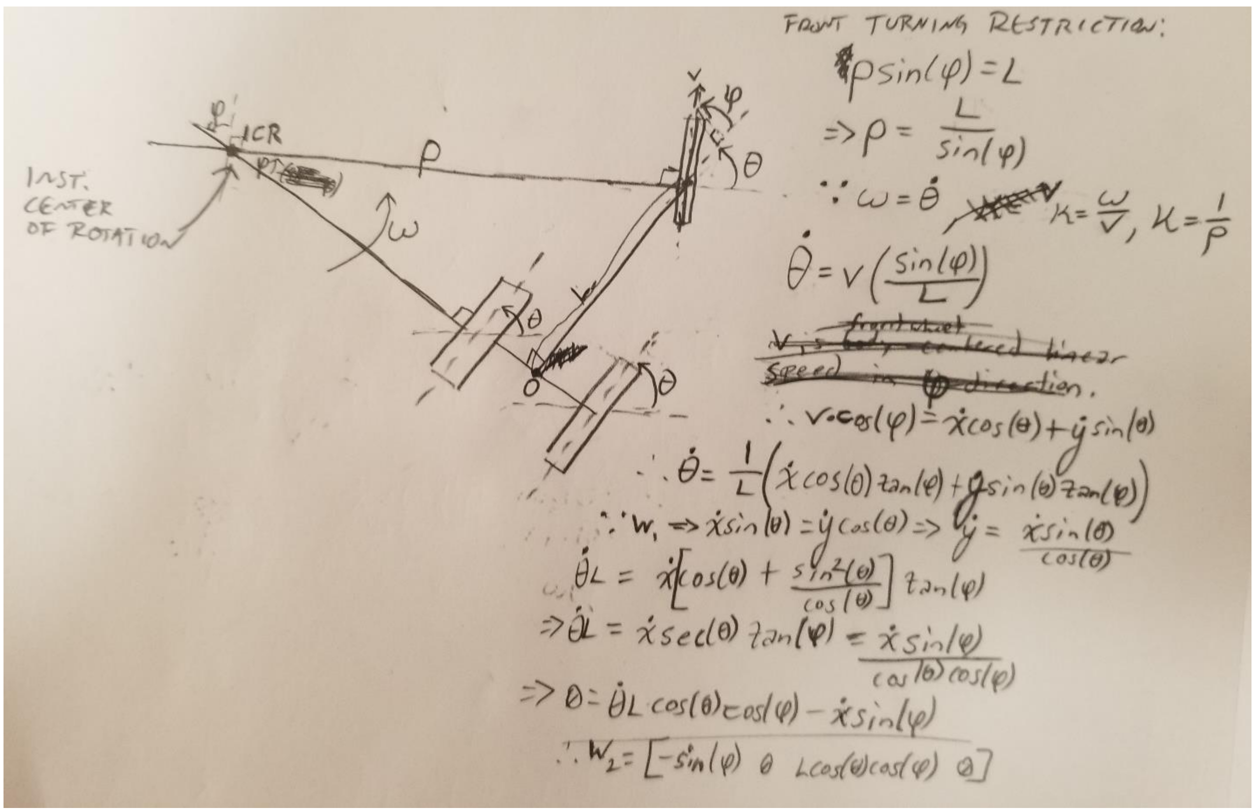
$$\frac{\delta y_i}{\delta x_i} = \tan(\theta_i)$$

$$\rightarrow \dot{y}_i \cos(\theta_i) = \dot{x}_i \sin(\theta_i)$$

$$\rightarrow -\dot{x}_i \sin(\theta_i) + \dot{y}_i \cos(\theta_i) = 0$$

$$\therefore \vec{\boldsymbol{w}}_{\mathbf{1}} = [-\sin(\boldsymbol{\theta}_i) \quad \cos(\boldsymbol{\theta}_i) \quad \mathbf{0} \quad \mathbf{0}]$$

<u>From turning constraint imposed by front wheel:</u>



Turning radius: $R_i \sin(\phi_i) = \ell_i \rightarrow R_i = \frac{\ell_i}{\sin(\phi_i)}$

Curvature of Path of Front Wheel: $\because \kappa = \frac{\omega_f}{V_f}$ *where* $\kappa = R_i^{-1}$

$$\therefore \omega_f = \frac{V_f}{R_i} = \frac{V_f \sin(\phi_i)}{\ell_i}$$

V_f is linear speed of front wheel, ω_f is rotational speed of front wheel about ICR.

$$\therefore \omega_f = \dot{\theta}_i$$

Relation between V_f and translational speed of origin:

$$\left(V_f \circ \hat{X}_i\right) = V_f cos(\phi_i) = \dot{x}_i \cos(\theta_i) + \dot{y} \sin(\theta_i)$$
$$\rightarrow V_f = \frac{\dot{x}_i \cos(\theta_i)}{\cos(\phi_i)} + \frac{\dot{y}_i \sin(\theta_i)}{\cos(\phi_i)}$$

$$\therefore \dot{\theta}_i = \frac{1}{\ell_i}(\dot{x}_i \cos(\theta_i) \tan(\phi_i) + \dot{y}_i \sin(\theta_i) \tan(\phi_i))$$

Constraint 1: $\because\ \dot{x}_i \sin(\theta_i) = \dot{y}_i \cos(\theta_i) \ \rightarrow\ \dot{y}_i = \dot{x}_i \frac{\sin(\theta_i)}{\cos(\theta_i)}$

$$\rightarrow \dot{\theta}_i \ell_i = \dot{x}_i \left(\cos(\theta_i) + \frac{\sin^2(\theta_i)}{\cos(\theta_i)}\right) \tan(\phi_i)$$

$$\rightarrow \dot{\theta}_i \ell_i = \dot{x}_i \sec(\theta_i) \tan(\phi_i)$$

$$\rightarrow \dot{\theta}_i \ell_i = \frac{\dot{x}_i \sin(\phi_i)}{\cos(\theta_i) \cos(\phi_i)}$$

$$\rightarrow \dot{\theta} \ell_i \cos(\theta_i) \cos(\phi_i) - \dot{x}_i \sin(\phi_i) = 0$$

$$\therefore \vec{\boldsymbol{w}}_{\boldsymbol{2}} = [-\mathbf{sin}(\boldsymbol{\phi}_i) \quad \mathbf{0} \quad \boldsymbol{\ell}_i \mathbf{cos}(\boldsymbol{\theta}_i) \mathbf{cos}(\boldsymbol{\phi}_i) \quad \mathbf{0}]$$

<u>Complete Set of Constraints:</u>

$$\therefore \vec{\boldsymbol{w}}_{\boldsymbol{1}} = [-\mathbf{sin}(\boldsymbol{\theta}_i) \quad \mathbf{cos}(\boldsymbol{\theta}_i) \quad \mathbf{0} \quad \mathbf{0}]$$
$$\therefore \vec{\boldsymbol{w}}_{\boldsymbol{2}} = [-\mathbf{sin}(\boldsymbol{\phi}_i) \quad \mathbf{0} \quad \boldsymbol{\ell}_i \mathbf{cos}(\boldsymbol{\theta}_i) \mathbf{cos}(\boldsymbol{\phi}_i) \quad \mathbf{0}]$$

3.3)

where: $\dot{\vec{q}} = \vec{g}_1 u_1 + \vec{g}_2 u_2 + \cdots$

Two controllable parameters of tricycle are pedaling velocity and turning of front wheel.

<u>Control Vector for Rear Wheels:</u>

Direction of travel when pedaling at u_1 with a fixed front wheel orientation.

$$\dot{x}_i = u_1 \cos(\theta_i)$$

$$\dot{y}_i = u_1 \sin(\theta_i)$$

$$\dot{\theta}_i = \frac{u_1}{R_i}$$

$$\because \tan(\phi_i) = \frac{\ell_i}{R_i}$$

$$\dot{\theta}_i = u_1 \frac{\ell_i}{\ell_i R_i} = u_1 \frac{\tan(\phi_i)}{\ell_i}$$

$$\therefore \vec{\boldsymbol{g}}_{\mathbf{1}} = \begin{bmatrix} \mathbf{cos}(\boldsymbol{\theta_i}) \\ \mathbf{sin}(\boldsymbol{\theta_i}) \\ \boldsymbol{tan}(\boldsymbol{\phi}_i)/\boldsymbol{\ell}_i \\ \mathbf{0} \end{bmatrix}$$

<u>Control Vector for Turning of Front Wheel:</u>

Turning front wheel affects only ϕ_i and angle of turn u_2 directly controls ϕ_i.

$$\therefore \vec{\boldsymbol{g}}_{\mathbf{2}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}$$

3.4)

Yes.

<u>First Control Vector:</u>

$\vec{g}_1 \circ \vec{w}_1 = -\sin(\theta_i)\cos(\theta_i) + \cos(\theta_i)\sin(\theta_i) + 0 + 0 = 0 \quad \therefore \vec{\boldsymbol{g}}_{\mathbf{1}} \perp \vec{\boldsymbol{w}}_{\mathbf{1}}$

$\vec{g}_1 \circ \vec{w}_2 = -\cos(\theta_i)\sin(\phi_i) + 0 + \frac{\ell_i}{\ell_i}\frac{\cos(\theta_i)\cos(\phi_i)\sin(\phi_i)}{\cos(\phi_i)} + 0 = 0 \quad \therefore \vec{\boldsymbol{g}}_{\mathbf{1}} \perp \vec{\boldsymbol{w}}_{\mathbf{2}}$

<u>Second Control Vector:</u>

$\vec{g}_2 \circ \vec{w}_1 = 0 + 0 + 0 + 0 = 0 \quad \therefore \vec{\boldsymbol{g}}_{\mathbf{2}} \perp \vec{\boldsymbol{w}}_{\mathbf{1}}$

$\vec{g}_2 \circ \vec{w}_2 = 0 + 0 + 0 + 0 = 0 \quad \therefore \vec{\boldsymbol{g}}_{\mathbf{2}} \perp \vec{\boldsymbol{w}}_{\mathbf{2}}$

3.5)

<u>First Degree of Control:</u>

$\vec{g}_1$ is the change in state (effectively the direction of travel) when pedaling with a fixed front wheel orientation.

<u>Second Degree of Control:</u>

$\vec{g}_2$ is the change in the front wheel orientation when turning it.

3.6)

Need four linearly independent control vectors. Must find $\vec{g}_3$ and $\vec{g}_4$ using Lie Brackets.

$$\vec{g}_3 = [\vec{g}_1, \vec{g}_2] = \left[\frac{\delta \vec{g}_2}{\delta \vec{q}}\right]\vec{g}_1 - \left[\frac{\delta \vec{g}_1}{\delta \vec{q}}\right]\vec{g}_2$$

$$\rightarrow \vec{g}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta_i) \\ \sin(\theta_i) \\ tan(\phi_i)/\ell_i \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -\sin(\theta_i) & 0 \\ 0 & 0 & \cos(\theta_i) & 0 \\ 0 & 0 & 0 & \sec^2(\phi_i)/\ell_i \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\rightarrow \boldsymbol{\vec{g}_3} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \left(\dfrac{\mathbf{-1}}{\boldsymbol{\ell_i} \mathbf{\cos^2}(\boldsymbol{\phi_i})}\right) \\ \mathbf{0} \end{bmatrix}$$

$$\vec{g}_4 = [\vec{g}_2, \vec{g}_3] = \left[\frac{\delta \vec{g}_3}{\delta \vec{q}}\right]\vec{g}_2 - \left[\frac{\delta \vec{g}_2}{\delta \vec{q}}\right]\vec{g}_3$$

$$\rightarrow \vec{g}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \left(\dfrac{-2\sin(\phi_i)}{\ell_i \cos^3(\phi_i)}\right) \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \left(\dfrac{-1}{\ell_i \cos^2(\phi_i)}\right) \\ 0 \end{bmatrix}$$

$$\rightarrow \boldsymbol{\vec{g}_4} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \left(\dfrac{\mathbf{-2\sin}(\boldsymbol{\phi_i})}{\boldsymbol{\ell_i} \mathbf{\cos^3}(\boldsymbol{\phi_i})}\right) \\ \mathbf{0} \end{bmatrix}$$

3.7)

<u>Test of Linear Independence of New Motion $\vec{g}_3$ from Previous Allowable Motions $\vec{g}_1, \vec{g}_2$:</u>

Let: $a\vec{g}_1 + b\vec{g}_2 + c\vec{g}_3 = \vec{0}$

$$\rightarrow a \begin{bmatrix} \cos(\theta_i) \\ \sin(\theta_i) \\ tan(\phi_i)/\ell_i \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ \left(\dfrac{-1}{\ell_i \cos^2(\phi_i)}\right) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \mathrm{a}\cos(\theta_i) \\ \mathrm{a}\sin(\theta_i) \\ \dfrac{atan(\phi_i)}{\ell_i} - \dfrac{c}{\ell_i \cos^2(\phi_i)} \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{Solve\ System} \{a, b, c\} = \{0,0,0\}$$

Therefore, $\vec{g}_1, \vec{g}_2, \vec{g}_3$ are linearly independent

<u>Test of Linear Independence of New Motion $\vec{g}_4$ from Previous Allowable Motions $\vec{g}_1, \vec{g}_2$:</u>

Let: $a\vec{g}_1 + b\vec{g}_2 + c\vec{g}_4 = \vec{0}$

$$\rightarrow a\begin{bmatrix} \cos(\theta_i) \\ \sin(\theta_i) \\ tan(\phi_i)/\ell_i \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ \left(\dfrac{-2\sin(\phi_i)}{\ell_i \cos^3(\phi_i)}\right) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \mathrm{acos}(\theta_i) \\ \mathrm{asin}(\theta_i) \\ \dfrac{atan(\phi_i)}{\ell_i} - c\left(\dfrac{2\sin(\phi_i)}{\ell_i \cos^3(\phi_i)}\right) \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{Solve\ System} \{a, b, c\} = \{0,0,0\}$$

Therefore, $\vec{g}_1, \vec{g}_2, \vec{g}_4$ are linearly independent

3.8)

New motion $\vec{g}_3 = \begin{bmatrix} 0 \\ 0 \\ \left(\frac{-1}{\ell_i \cos^2(\phi_i)}\right) \\ 0 \end{bmatrix}$ is a rotation of O_i about the instantaneous center of rotation, C_i, due to the orientation of the front steering wheel.

New motion $\vec{g}_4 = \begin{bmatrix} 0 \\ 0 \\ \left(\frac{-2\sin(\phi_i)}{\ell_i \cos^3(\phi_i)}\right) \\ 0 \end{bmatrix}$ is a rotation of O_i about the instantaneous center of rotation, C_i, due to the orientation of the front steering wheel.

3.9)

A tricycle and Ackerman-like cart have mathematically similar control kinematics in terms of their respective control vectors, g, and constraints w. The primary difference between them is that the tricycle has an unbounded turning radius whereas the linkage between the two front steering wheels of the cart imposes a lower bound on the cart's turning radius.