3. Written Questions

3.1.) Robot Analysis: RRR

3.1.1.) $\dim(\boldsymbol{J}) = 3xn \rightarrow \mathbf{dim}(\boldsymbol{J}) = \boldsymbol{3x3}$ (dimension of column-space is **3**)

3.1.2.) Each row of the Jacobian describes the instantaneous motion along a particular axis (given by the row number) that result from instantaneous motions of the robot's joints as a function of the robot's joint state. As a result, this describes the instantaneous motions of each of the robot's joints required to move along a given axis as a function of the robot's joint state.

3.1.3.) Each column of the Jacobian describes the instantaneous direction of motion that would result from an instantaneous change in the state of a given joint (determined by the column number) as a function of the robot's joint state.

3.1.4.)

$$\boldsymbol{f} = \begin{bmatrix} f_x \\ f_y \\ f_\theta \end{bmatrix} = \begin{bmatrix} l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3) \\ l_1\sin(\theta_1) + l_2\sin(\theta_1+\theta_2) + l_3\sin(\theta_1+\theta_2+\theta_3) \\ \theta_1+\theta_2+\theta_3 \end{bmatrix}$$

3.1.5.)

$$\frac{\partial \boldsymbol{f}}{\partial \theta_1} = \begin{bmatrix} \frac{\partial f_x}{\partial \theta_1} \\ \frac{\partial f_y}{\partial \theta_1} \\ \frac{\partial f_\theta}{\partial \theta_1} \end{bmatrix} = \begin{bmatrix} -l_1\sin(\theta_1) - l_2\sin(\theta_1+\theta_2) - l_3\sin(\theta_1+\theta_2+\theta_3) \\ l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3) \\ 1 \end{bmatrix}$$

$$\frac{\partial \boldsymbol{f}}{\partial \theta_2} = \begin{bmatrix} \frac{\partial f_x}{\partial \theta_2} \\ \frac{\partial f_y}{\partial \theta_2} \\ \frac{\partial f_\theta}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} -l_2\sin(\theta_1+\theta_2) - l_3\sin(\theta_1+\theta_2+\theta_3) \\ l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3) \\ 1 \end{bmatrix}$$

$$\frac{\partial \boldsymbol{f}}{\partial \theta_3} = \begin{bmatrix} \frac{\partial f_x}{\partial \theta_3} \\ \frac{\partial f_y}{\partial \theta_3} \\ \frac{\partial f_\theta}{\partial \theta_3} \end{bmatrix} = \begin{bmatrix} -l_3\sin(\theta_1+\theta_2+\theta_3) \\ l_3\cos(\theta_1+\theta_2+\theta_3) \\ 1 \end{bmatrix}$$

3.1.6.)

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial \boldsymbol{f}}{\partial \theta_1} & \frac{\partial \boldsymbol{f}}{\partial \theta_2} & \frac{\partial \boldsymbol{f}}{\partial \theta_3} \end{bmatrix}$$

$$\rightarrow \boldsymbol{J} = \begin{bmatrix} -l_1 \sin(\theta_1) - l_2 \sin(\theta_1 + \theta_2) - l_3 \sin(\theta_1 + \theta_2 + \theta_3) & -l_2 \sin(\theta_1 + \theta_2) - l_3 \sin(\theta_1 + \theta_2 + \theta_3) & -l_3 \sin(\theta_1 + \theta_2 + \theta_3) \\ l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3) & l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3) & l_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ 1 & 1 & 1 \end{bmatrix}$$

3.1.7.)

$$let: \boldsymbol{\Theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \dot{\boldsymbol{\Theta}} = \begin{bmatrix} v \\ v \\ v \end{bmatrix}$$

$$\because \dot{\boldsymbol{X}} = \boldsymbol{J}\dot{\boldsymbol{\Theta}}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ l_1 + l_2 + l_3 & l_2 + l_3 & l_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v \\ v \\ v \end{bmatrix}$$

Therefore:

θ_1 contributes $\boldsymbol{0}$ to its linear velocity in x, $(\boldsymbol{l_1} + \boldsymbol{l_2} + \boldsymbol{l_3})\boldsymbol{v}$ to the end-effector's linear velocity in *y*, and $\boldsymbol{v}$ to its angular velocity.

θ_2 contributes $\boldsymbol{0}$ to its linear velocity in x, $(\boldsymbol{l_1} + \boldsymbol{l_2})\boldsymbol{v}$ to the end-effector's linear velocity in *y*, and $\boldsymbol{v}$ to its angular velocity.

θ_3 contributes $\boldsymbol{0}$ to its linear velocity in x, $\boldsymbol{l_3 v}$ to the end-effector's linear velocity in *y*, and $\boldsymbol{v}$ to its angular velocity.

3.1.8.)

$$\because \boldsymbol{J} = \begin{bmatrix} -l_1 - l_2 - l_3 & -l_2 - l_3 & -l_3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} where\ \boldsymbol{\Theta} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\because \dot{\boldsymbol{X}} = \boldsymbol{J}\dot{\boldsymbol{\Theta}}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -l_1 - l_2 - l_3 & -l_2 - l_3 & -l_3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

a.)

$$\rightarrow \begin{bmatrix} -1 \\ 0 \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ (l_1 + l_2 + l_3)\dot{\theta}_1 + (l_2 + l_3)\dot{\theta}_2 + l_3\dot{\theta}_3 \\ \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}$$

Therefore, **it is not possible** for the end-effector to instantaneously move in the planar direction $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

b.)

$$\rightarrow \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ (l_1 + l_2 + l_3)\dot{\theta}_1 + (l_2 + l_3)\dot{\theta}_2 + l_3\dot{\theta}_3 \\ \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}$$

$$\rightarrow \begin{cases} \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 = 1 \\ (l_1 + l_2 + l_3)\dot{\theta}_1 + (l_2 + l_3)\dot{\theta}_2 + l_3\dot{\theta}_3 = 1 \end{cases}$$

So, the full space of solutions is:

$$\therefore \begin{cases} \dot{\theta}_1 = \dot{\theta}_1 \\ \dot{\theta}_2 = \dfrac{-\left((l_1 + l_2)\dot{\theta}_1 + l_3 - 1\right)}{l_2} \\ \dot{\theta}_3 = \dfrac{l_1 x + l_2 + l_3 - 1}{l_2} \end{cases}$$

Since there are only 2 equations and 3 unknowns, a new constraint must be imposed; say, $\dot{\theta}_1 = 1$.

Therefore, one particular solution from this space is:

$$\dot{\boldsymbol{\Theta}} = \begin{bmatrix} \dot{\boldsymbol{\theta}}_1 \\ \dot{\boldsymbol{\theta}}_2 \\ \dot{\boldsymbol{\theta}}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \dfrac{\mathbf{1}}{\boldsymbol{l_2}} - \dfrac{\boldsymbol{l_3}}{\boldsymbol{l_2}} - \dfrac{\boldsymbol{l_1} + \boldsymbol{l_2}}{\boldsymbol{l_2}} \\ \dfrac{\boldsymbol{l_1}}{\boldsymbol{l_2}} + \mathbf{1} + \dfrac{\boldsymbol{l_3}}{\boldsymbol{l_2}} - \dfrac{\mathbf{1}}{\boldsymbol{l_2}} \end{bmatrix}$$

3.1.9.)

$$let: \begin{bmatrix} f_x \\ f_y \end{bmatrix} (5N) * \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} N$$

$$\because \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \boldsymbol{J}^T \begin{bmatrix} f_x \\ f_y \\ \tau \end{bmatrix} = \begin{bmatrix} -l_1 \sin(\theta_1) - l_2 \sin(\theta_1 + \theta_2) - l_3 \sin(\theta_1 + \theta_2 + \theta_3) & l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3) & 1 \\ -l_2 \sin(\theta_1 + \theta_2) - l_3 \sin(\theta_1 + \theta_2 + \theta_3) & l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3) & 1 \\ -l_3 \sin(\theta_1 + \theta_2 + \theta_3) & l_3 \cos(\theta_1 + \theta_2 + \theta_3) & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ \tau \end{bmatrix}$$

Full solution space:

$$\rightarrow \begin{bmatrix} \boldsymbol{\tau_1} \\ \boldsymbol{\tau_2} \\ \boldsymbol{\tau_3} \end{bmatrix} = \begin{bmatrix} \mathbf{-5\boldsymbol{l_1} \sin(\boldsymbol{\theta_1}) - 5\boldsymbol{l_2} \sin(\boldsymbol{\theta_1} + \boldsymbol{\theta_2}) - 5\boldsymbol{l_3} \sin(\boldsymbol{\theta_1} + \boldsymbol{\theta_2} + \boldsymbol{\theta_3}) + 5\boldsymbol{l_1} \cos(\boldsymbol{\theta_1}) + 5\boldsymbol{l_2} \cos(\boldsymbol{\theta_1} + \boldsymbol{\theta_2}) + 5\boldsymbol{l_3} \cos(\boldsymbol{\theta_1} + \boldsymbol{\theta_2} + \boldsymbol{\theta_3}) + \boldsymbol{\tau}} \\ \mathbf{-5\boldsymbol{l_2} \sin(\boldsymbol{\theta_1} + \boldsymbol{\theta_2}) - 5\boldsymbol{l_3} \sin(\boldsymbol{\theta_1} + \boldsymbol{\theta_2} + \boldsymbol{\theta_3}) + 5\boldsymbol{l_2} \cos(\boldsymbol{\theta_1} + \boldsymbol{\theta_2}) + 5\boldsymbol{l_3} \cos(\boldsymbol{\theta_1} + \boldsymbol{\theta_2} + \boldsymbol{\theta_3}) + \boldsymbol{\tau}} \\ \mathbf{-5\boldsymbol{l_3} \sin(\boldsymbol{\theta_1} + \boldsymbol{\theta_2} + \boldsymbol{\theta_3}) + 5\boldsymbol{l_3} \cos(\boldsymbol{\theta_1} + \boldsymbol{\theta_2} + \boldsymbol{\theta_3}) + \boldsymbol{\tau}} \end{bmatrix}$$

Assuming $\tau = 0$ (since unspecified, this is allowable):

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} -5l_1 \sin(\theta_1) - 5l_2 \sin(\theta_1 + \theta_2) - 5l_3 \sin(\theta_1 + \theta_2 + \theta_3) + 5l_1 \cos(\theta_1) + 5l_2 \cos(\theta_1 + \theta_2) + 5l_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ -5l_2 \sin(\theta_1 + \theta_2) - 5l_3 \sin(\theta_1 + \theta_2 + \theta_3) + 5l_2 \cos(\theta_1 + \theta_2) + 5l_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ -5l_3 \sin(\theta_1 + \theta_2 + \theta_3) + 5l_3 \cos(\theta_1 + \theta_2 + \theta_3) \end{bmatrix}$$

Assuming $\boldsymbol{\Theta} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $and\ \tau = 0$:

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} 5l_1 + 5l_2 + 5l_3 \\ 5l_2 + 5l_3 \\ 5l_3 \end{bmatrix} = (5N) \begin{bmatrix} l_1 + l_2 + l_3 \\ l_2 + l_3 \\ l_3 \end{bmatrix}$$

3.1.10.)

a.)

$$\because \boldsymbol{f_{COM1}} = \begin{bmatrix} \frac{l_1}{2}\cos(\theta_1) \\ \frac{l_1}{2}\sin(\theta_1) \\ \theta_1 \end{bmatrix}, \quad \boldsymbol{J_{COM1}} = \begin{bmatrix} -\frac{l_1}{2}\sin(\theta_1) & 0 & 0 \\ \frac{l_1}{2}\cos(\theta_1) & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{J_{COM1}^T} = \begin{bmatrix} -\frac{l_1}{2}\sin(\theta_1) & \frac{l_1}{2}\cos(\theta_1) & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\because \begin{bmatrix} f_{x_{COM1}} \\ f_{y_{COM1}} \\ \tau_{COM1} \end{bmatrix} = -\begin{bmatrix} 0 \\ -0.25g \\ 0 \end{bmatrix} N = \begin{bmatrix} 0 \\ 0.25g \\ 0 \end{bmatrix} N, \quad where: g = 9.8\frac{m}{s^2}$$

$$\because \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}_{COM1} = \boldsymbol{J_{COM1}^T} \begin{bmatrix} f_{x_{COM1}} \\ f_{y_{COM1}} \\ \tau_{COM1} \end{bmatrix}$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}_{COM1} = \begin{bmatrix} -\frac{l_1}{2}\sin(\theta_1) & \frac{l_1}{2}\cos(\theta_1) & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0.25g \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \boldsymbol{\tau_1} \\ \boldsymbol{\tau_2} \\ \boldsymbol{\tau_3} \end{bmatrix}_{\boldsymbol{COM1}} = \begin{bmatrix} \frac{\boldsymbol{(0.25kg)gl_1}}{\boldsymbol{2}}\boldsymbol{\cos(\theta_1)} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}$$

b.)

$$let: \boldsymbol{J_{COM2}} = \begin{bmatrix} J_{COM2}^{11} & J_{COM2}^{12} & J_{COM2}^{13} \\ J_{COM2}^{21} & J_{COM2}^{22} & J_{COM2}^{23} \\ J_{COM2}^{31} & J_{COM2}^{32} & J_{COM2}^{33} \end{bmatrix}, \quad \boldsymbol{J_{COM2}^T} = \begin{bmatrix} J_{COM2}^{11} & J_{COM2}^{21} & J_{COM2}^{31} \\ J_{COM2}^{12} & J_{COM2}^{22} & J_{COM2}^{32} \\ J_{COM2}^{13} & J_{COM2}^{23} & J_{COM2}^{33} \end{bmatrix}$$

$$\because \begin{bmatrix} f_{x_{COM2}} \\ f_{y_{COM2}} \\ \tau_{COM2} \end{bmatrix} = -\begin{bmatrix} 0 \\ -0.25g \\ 0 \end{bmatrix} N = \begin{bmatrix} 0 \\ 0.25g \\ 0 \end{bmatrix} N, \quad where: g = 9.8\frac{m}{s^2}$$

$$\because \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}_{COM2} = \boldsymbol{J_{COM2}^T} \begin{bmatrix} f_{x_{COM2}} \\ f_{y_{COM2}} \\ \tau_{COM2} \end{bmatrix}$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}_{COM2} = \begin{bmatrix} J_{COM2}^{11} & J_{COM2}^{21} & J_{COM2}^{31} \\ J_{COM2}^{12} & J_{COM2}^{22} & J_{COM2}^{32} \\ J_{COM2}^{13} & J_{COM2}^{23} & J_{COM2}^{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0.25g \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}_{COM2} = \begin{bmatrix} 0.25gJ_{COM2}^{21} \\ 0.25gJ_{COM2}^{22} \\ 0.25gJ_{COM2}^{23} \end{bmatrix}$$

$$\because f_{COM2} = \begin{bmatrix} l_1\cos(\theta_1) + \frac{l_2}{2}\cos(\theta_1+\theta_2) \\ l_1\sin(\theta_1) + \frac{l_2}{2}\sin(\theta_1+\theta_2) \\ \theta_1+\theta_2 \end{bmatrix}, \quad \begin{bmatrix} J_{COM2}^{21} \\ J_{COM2}^{22} \\ J_{COM2}^{23} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_{COM2_y}}{\partial\theta_1} \\ \frac{\partial f_{COM2_y}}{\partial\theta_2} \\ \frac{\partial f_{COM2_y}}{\partial\theta_3} \end{bmatrix} = \begin{bmatrix} l_1\cos(\theta_1) + \frac{l_2}{2}\cos(\theta_1+\theta_2) \\ \frac{l_2}{2}\cos(\theta_1+\theta_2) \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix}\tau_1\\\tau_2\\\tau_3\end{bmatrix}_{COM2} = (0.25kg)g\begin{bmatrix} l_1\cos(\theta_1)+\frac{l_2}{2}\cos(\theta_1+\theta_2)\\ \frac{l_2}{2}\cos(\theta_1+\theta_2)\\ 0\end{bmatrix}$$

c.)

$$let: \boldsymbol{J_{COM3}} = \begin{bmatrix} J^{11}_{COM3} & J^{12}_{COM3} & J^{13}_{COM3}\\ J^{21}_{COM3} & J^{22}_{COM3} & J^{23}_{COM3}\\ J^{31}_{COM3} & J^{32}_{COM3} & J^{33}_{COM3}\end{bmatrix},\ \boldsymbol{J^T_{COM3}} = \begin{bmatrix} J^{11}_{COM3} & J^{21}_{COM3} & J^{31}_{COM3}\\ J^{12}_{COM3} & J^{22}_{COM3} & J^{32}_{COM3}\\ J^{13}_{COM3} & J^{23}_{COM3} & J^{33}_{COM3}\end{bmatrix}$$

$$\because \begin{bmatrix} f_{x_{COM3}}\\ f_{y_{COM3}}\\ \tau_{COM3}\end{bmatrix} = -\begin{bmatrix}0\\-0.25g\\0\end{bmatrix}N = \begin{bmatrix}0\\0.25g\\0\end{bmatrix}N,\quad where: g = 9.8\frac{m}{s^2}$$

$$\because \begin{bmatrix}\tau_1\\\tau_2\\\tau_3\end{bmatrix}_{COM3} = \boldsymbol{J^T_{COM3}}\begin{bmatrix} f_{x_{COM3}}\\ f_{y_{COM3}}\\ \tau_{COM3}\end{bmatrix}$$

$$\begin{bmatrix}\tau_1\\\tau_2\\\tau_3\end{bmatrix}_{COM3} = \begin{bmatrix} J^{11}_{COM3} & J^{21}_{COM3} & J^{31}_{COM3}\\ J^{12}_{COM3} & J^{22}_{COM3} & J^{32}_{COM3}\\ J^{13}_{COM3} & J^{23}_{COM3} & J^{33}_{COM3}\end{bmatrix}\begin{bmatrix}0\\0.25g\\0\end{bmatrix}$$

$$\rightarrow \begin{bmatrix}\tau_1\\\tau_2\\\tau_3\end{bmatrix}_{COM3} = \begin{bmatrix}0.25gJ^{21}_{COM3}\\0.25gJ^{22}_{COM3}\\0.25gJ^{23}_{COM3}\end{bmatrix}$$

$$\because f_{COM3_y} = l_1\sin(\theta_1)+l_2\sin(\theta_1+\theta_2)+\frac{l_3}{2}\sin(\theta_1+\theta_2+\theta_3),$$

$$\begin{bmatrix}J^{21}_{COM3}\\J^{22}_{COM3}\\J^{23}_{COM3}\end{bmatrix} = \begin{bmatrix}\frac{\partial f_{COM3_y}}{\partial\theta_1}\\ \frac{\partial f_{COM3_y}}{\partial\theta_2}\\ \frac{\partial f_{COM3_y}}{\partial\theta_3}\end{bmatrix} = \begin{bmatrix} l_1\cos(\theta_1)+l_2\cos(\theta_1+\theta_2)+\frac{l_3}{2}\cos(\theta_1+\theta_2+\theta_3)\\ l_2\cos(\theta_1+\theta_2)+\frac{l_3}{2}\cos(\theta_1+\theta_2+\theta_3)\\ \frac{l_3}{2}\cos(\theta_1+\theta_2+\theta_3)\end{bmatrix}$$

$$\rightarrow \begin{bmatrix}\tau_1\\\tau_2\\\tau_3\end{bmatrix}_{COM3} = (0.25kg)g\begin{bmatrix} l_1\cos(\theta_1)+l_2\cos(\theta_1+\theta_2)+\frac{l_3}{2}\cos(\theta_1+\theta_2+\theta_3)\\ l_2\cos(\theta_1+\theta_2)+\frac{l_3}{2}\cos(\theta_1+\theta_2+\theta_3)\\ \frac{l_3}{2}\cos(\theta_1+\theta_2+\theta_3)\end{bmatrix}$$

d.)

$$\because \begin{bmatrix}\tau_1\\\tau_2\\\tau_3\end{bmatrix}_{allCOM} = \begin{bmatrix}\tau_1\\\tau_2\\\tau_3\end{bmatrix}_{COM1} + \begin{bmatrix}\tau_1\\\tau_2\\\tau_3\end{bmatrix}_{COM2} + \begin{bmatrix}\tau_1\\\tau_2\\\tau_3\end{bmatrix}_{COM3}$$

$$\begin{bmatrix}\tau_1\\\tau_2\\\tau_3\end{bmatrix}_{allCOM} = (0.25kg)g\begin{bmatrix} 2.5l_1\cos(\theta_1)+1.5l_2\cos(\theta_1+\theta_2)+0.5l_3\cos(\theta_1+\theta_2+\theta_3)\\ 1.5l_2\cos(\theta_1+\theta_2)+0.5l_3\cos(\theta_1+\theta_2+\theta_3)\\ 0.5l_3\cos(\theta_1+\theta_2+\theta_3)\end{bmatrix}$$

3.1.11.)

Based on the pattern of previous results, the mass of Joint 2 will contribute the following to the required joint torques:

$$\begin{bmatrix}\tau_1\\ \tau_2\\ \tau_3\end{bmatrix}_{J2} = m_J g \begin{bmatrix}J_{J2}^{21}\\ J_{J2}^{22}\\ J_{J2}^{23}\end{bmatrix},\ where\ m_J = (0.35kg),\qquad \begin{bmatrix}J_{J2}^{21}\\ J_{J2}^{22}\\ J_{J2}^{23}\end{bmatrix} = \begin{bmatrix}\frac{\partial f_{J3_y}}{\partial\theta_1}\\ \frac{\partial f_{J3_y}}{\partial\theta_2}\\ \frac{\partial f_{J3_y}}{\partial\theta_3}\end{bmatrix} = \begin{bmatrix}l_1\cos(\theta_1)\\ 0\\ 0\end{bmatrix}$$

$$\rightarrow \begin{bmatrix}\tau_1\\ \tau_2\\ \tau_3\end{bmatrix}_{J2} = (0.35kg)g\begin{bmatrix}l_1\cos(\theta_1)\\ 0\\ 0\end{bmatrix}$$

Joint 3 will contribute:

$$\begin{bmatrix}\tau_1\\ \tau_2\\ \tau_3\end{bmatrix}_{J3} = m_J g \begin{bmatrix}J_{J3}^{21}\\ J_{J3}^{22}\\ J_{J3}^{23}\end{bmatrix},\ where\ m_J = (0.35kg),\qquad \begin{bmatrix}J_{J3}^{21}\\ J_{J3}^{22}\\ J_{J3}^{23}\end{bmatrix} = \begin{bmatrix}\frac{\partial f_{J3_y}}{\partial\theta_1}\\ \frac{\partial f_{J3_y}}{\partial\theta_2}\\ \frac{\partial f_{J3_y}}{\partial\theta_3}\end{bmatrix} = \begin{bmatrix}l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2)\\ l_2\cos(\theta_1+\theta_2)\\ 0\end{bmatrix}$$

$$\rightarrow \begin{bmatrix}\tau_1\\ \tau_2\\ \tau_3\end{bmatrix}_{J3} = (0.35kg)g\begin{bmatrix}l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2)\\ l_2\cos(\theta_1+\theta_2)\\ 0\end{bmatrix}$$

Therefore, the total required joint torques to resist gravity are:

$$\begin{bmatrix}\tau_1\\ \tau_2\\ \tau_3\end{bmatrix}_{all} = (0.25kg)g\begin{bmatrix}2.5l_1\cos(\theta_1) + 1.5l_2\cos(\theta_1+\theta_2) + 0.5l_3\cos(\theta_1+\theta_2+\theta_3)\\ 1.5l_2\cos(\theta_1+\theta_2) + 0.5l_3\cos(\theta_1+\theta_2+\theta_3)\\ 0.5l_3\cos(\theta_1+\theta_2+\theta_3)\end{bmatrix} + (0.35kg)g\begin{bmatrix}2l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2)\\ l_2\cos(\theta_1+\theta_2)\\ 0\end{bmatrix}$$

$$\rightarrow \begin{bmatrix}\boldsymbol{\tau_1}\\ \boldsymbol{\tau_2}\\ \boldsymbol{\tau_3}\end{bmatrix}_{\boldsymbol{all}} = \left(\begin{bmatrix}\mathbf{1.325}\boldsymbol{l_1}\mathbf{cos}(\boldsymbol{\theta_1}) + \mathbf{0.725}\boldsymbol{l_2}\mathbf{cos}(\boldsymbol{\theta_1}+\boldsymbol{\theta_2}) + \mathbf{0.125}\boldsymbol{l_3}\mathbf{cos}(\boldsymbol{\theta_1}+\boldsymbol{\theta_2}+\boldsymbol{\theta_3})\\ \mathbf{0.725}\boldsymbol{l_2}\mathbf{cos}(\boldsymbol{\theta_1}+\boldsymbol{\theta_2}) + \mathbf{0.125}\boldsymbol{l_3}\mathbf{cos}(\boldsymbol{\theta_1}+\boldsymbol{\theta_2}+\boldsymbol{\theta_3})\\ \mathbf{0.125}\boldsymbol{l_3}\mathbf{cos}(\boldsymbol{\theta_1}+\boldsymbol{\theta_2}+\boldsymbol{\theta_3})\end{bmatrix}\boldsymbol{kg}\right)\boldsymbol{g}$$

3.2.) Robot Analysis: RPR

3.2.1.)

$$\boldsymbol{f} = \begin{bmatrix}f_x\\ f_y\\ \theta\end{bmatrix} = \begin{bmatrix}d_2\cos(\theta_1) + l_3\cos(\theta_1+\theta_3)\\ d_2\sin(\theta_1) + l_3\sin(\theta_1+\theta_3)\\ \theta_1+\theta_3\end{bmatrix}$$

3.2.2.)

$$\boldsymbol{J} = \begin{bmatrix}\frac{\partial\boldsymbol{f}}{\partial\theta_1} & \frac{\partial\boldsymbol{f}}{\partial d_2} & \frac{\partial\boldsymbol{f}}{\partial\theta_3}\end{bmatrix}$$

$$\frac{\partial \boldsymbol{f}}{\partial \theta_1} = \begin{bmatrix} \frac{\partial f_x}{\partial \theta_1} \\ \frac{\partial f_y}{\partial \theta_1} \\ \frac{\partial f_\theta}{\partial \theta_1} \end{bmatrix} = \begin{bmatrix} -d_2 \sin(\theta_1) - l_3 \sin(\theta_1 + \theta_3) \\ d_2 \cos(\theta_1) + l_3 \cos(\theta_1 + \theta_3) \\ 1 \end{bmatrix}$$

$$\frac{\partial \boldsymbol{f}}{\partial d_2} = \begin{bmatrix} \frac{\partial f_x}{\partial \theta_2} \\ \frac{\partial f_y}{\partial \theta_2} \\ \frac{\partial f_\theta}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} \cos(\theta_1) \\ \sin(\theta_1) \\ 0 \end{bmatrix}$$

$$\frac{\partial \boldsymbol{f}}{\partial \theta_3} = \begin{bmatrix} \frac{\partial f_x}{\partial \theta_3} \\ \frac{\partial f_y}{\partial \theta_3} \\ \frac{\partial f_\theta}{\partial \theta_3} \end{bmatrix} = \begin{bmatrix} -l_3 \sin(\theta_1 + \theta_3) \\ l_3 \cos(\theta_1 + \theta_3) \\ 1 \end{bmatrix}$$

$$\therefore \boldsymbol{J} = \begin{bmatrix} \boldsymbol{-d_2 \sin(\theta_1) - l_3 \sin(\theta_1 + \theta_3)} & \boldsymbol{\cos(\theta_1)} & \boldsymbol{-l_3 \sin(\theta_1 + \theta_3)} \\ \boldsymbol{d_2 \cos(\theta_1) + l_3 \cos(\theta_1 + \theta_3)} & \boldsymbol{\sin(\theta_1)} & \boldsymbol{l_3 \cos(\theta_1 + \theta_3)} \\ \boldsymbol{1} & \boldsymbol{0} & \boldsymbol{1} \end{bmatrix}$$

3.2.3.)

Any COM will require a corresponding joint torque vector determined by:

$$let: \boldsymbol{J_{COMi}} = \begin{bmatrix} J^{11}_{COMi} & J^{12}_{COMi} & J^{13}_{COMi} \\ J^{21}_{COMi} & J^{22}_{COMi} & J^{23}_{COMi} \\ J^{31}_{COMi} & J^{32}_{COMi} & J^{33}_{COMi} \end{bmatrix}, \ \boldsymbol{J^T_{COMi}} = \begin{bmatrix} J^{11}_{COMi} & J^{21}_{COMi} & J^{31}_{COMi} \\ J^{12}_{COMi} & J^{22}_{COM3} & J^{32}_{COMi} \\ J^{13}_{COMi} & J^{23}_{COMi} & J^{33}_{COMi} \end{bmatrix}$$

$$\because \begin{bmatrix} f_{x_{COMi}} \\ f_{y_{COMi}} \\ \tau_{COMi} \end{bmatrix} = -\begin{bmatrix} 0 \\ -m_{Ji}g \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ m_{Ji}g \\ 0 \end{bmatrix}, \quad where: g = 9.8\frac{m}{s^2}$$

$$\because \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}_{COMi} = \boldsymbol{J^T_{COMi}} \begin{bmatrix} f_{x_{COMi}} \\ f_{y_{COMi}} \\ \tau_{COMi} \end{bmatrix}$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}_{COMi} = \begin{bmatrix} J^{11}_{COMi} & J^{21}_{COMi} & J^{31}_{COMi} \\ J^{12}_{COMi} & J^{22}_{COMi} & J^{32}_{COMi} \\ J^{13}_{COMi} & J^{23}_{COMi} & J^{33}_{COMi} \end{bmatrix} \begin{bmatrix} 0 \\ m_{Ji}g \\ 0 \end{bmatrix}$$

a.) The COM of Revolute Joint 1 will require no torque to support it (since the forward kinematic map to its center is all 0s).

$$\begin{bmatrix}\tau_1\\\tau_2\\\tau_3\end{bmatrix}_{J1} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

b.) The COM of the Prismatic Joint 2 will require:

$$\begin{bmatrix}\tau_1\\\tau_2\\\tau_3\end{bmatrix}_{J2} = m_{PJ}\begin{bmatrix}J_{J2}^{21}\\J_{J2}^{22}\\J_{J2}^{23}\end{bmatrix},\ where\ m_{PJ} = (1.0kg)g, \qquad \begin{bmatrix}J_{J2}^{21}\\J_{J2}^{22}\\J_{J2}^{23}\end{bmatrix} = \begin{bmatrix}\frac{\partial f_{J2_y}}{\partial\theta_1}\\\frac{\partial f_{J2_y}}{\partial\theta_2}\\\frac{\partial f_{J2_y}}{\partial\theta_3}\end{bmatrix} = \begin{bmatrix}\frac{d_2}{2}\cos(\theta_1)\\0\\0\end{bmatrix}$$

$$\rightarrow \begin{bmatrix}\tau_1\\\tau_2\\\tau_3\end{bmatrix}_{J2} = (1.0kg)g\begin{bmatrix}\frac{d_2}{2}\cos(\theta_1)\\0\\0\end{bmatrix}$$

c.) The COM of Revolute Joint 3 will require:

$$\begin{bmatrix}\tau_1\\\tau_2\\\tau_3\end{bmatrix}_{J3} = m_{RJ}\begin{bmatrix}J_{J3}^{21}\\J_{J3}^{22}\\J_{J3}^{23}\end{bmatrix},\ where\ m_{RJ} = (0.2kg)g, \qquad \begin{bmatrix}J_{J3}^{21}\\J_{J3}^{22}\\J_{J3}^{23}\end{bmatrix} = \begin{bmatrix}\frac{\partial f_{J3_y}}{\partial\theta_1}\\\frac{\partial f_{J3_y}}{\partial\theta_2}\\\frac{\partial f_{J3_y}}{\partial\theta_3}\end{bmatrix} = \begin{bmatrix}d_2\cos(\theta_1)\\0\\0\end{bmatrix}$$

$$\rightarrow \begin{bmatrix}\tau_1\\\tau_2\\\tau_3\end{bmatrix}_{J3} = (0.2kg)g\begin{bmatrix}d_2\cos(\theta_1)\\0\\0\end{bmatrix}$$

d.) The COM of the distal Link 3 will require:

$$\begin{bmatrix}\tau_1\\\tau_2\\\tau_3\end{bmatrix}_{COM3} = m_{L3}\begin{bmatrix}J_{J3}^{21}\\J_{J3}^{22}\\J_{J3}^{23}\end{bmatrix},\ where\ m_{L3} = (0.5kg)g, \qquad \begin{bmatrix}J_{COM3}^{21}\\J_{COM3}^{22}\\J_{COM3}^{23}\end{bmatrix} = \begin{bmatrix}\frac{\partial f_{COM3_y}}{\partial\theta_1}\\\frac{\partial f_{COM3_y}}{\partial\theta_2}\\\frac{\partial f_{COM3_y}}{\partial\theta_3}\end{bmatrix} = \begin{bmatrix}d_2\cos(\theta_1) + \frac{l_3}{2}\cos(\theta_1+\theta_3)\\\frac{l_3}{2}\cos(\theta_1+\theta_3)\\0\end{bmatrix}$$

$$\rightarrow \begin{bmatrix}\tau_1\\\tau_2\\\tau_3\end{bmatrix}_{J3} = (0.5kg)g\begin{bmatrix}d_2\cos(\theta_1) + \frac{l_3}{2}\cos(\theta_1+\theta_3)\\\frac{l_3}{2}\cos(\theta_1+\theta_3)\\0\end{bmatrix}$$

e.) Total required joint torque vector to resist the effects of gravity:

$$\begin{bmatrix}\tau_1\\\tau_2\\\tau_3\end{bmatrix}_{all} = \begin{bmatrix}\tau_1\\\tau_2\\\tau_3\end{bmatrix}_{J1} + \begin{bmatrix}\tau_1\\\tau_2\\\tau_3\end{bmatrix}_{J2} + \begin{bmatrix}\tau_1\\\tau_2\\\tau_3\end{bmatrix}_{J3} + \begin{bmatrix}\tau_1\\\tau_2\\\tau_3\end{bmatrix}_{COM3}$$

$$\rightarrow \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}_{all} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + (1.0kg)g \begin{bmatrix} \frac{d_2}{2}\cos(\theta_1) \\ 0 \\ 0 \end{bmatrix} + (0.2kg)g \begin{bmatrix} d_2\cos(\theta_1) \\ 0 \\ 0 \end{bmatrix} + (0.5kg)g \begin{bmatrix} d_2\cos(\theta_1) + \frac{l_3}{2}\cos(\theta_1+\theta_3) \\ \frac{l_3}{2}\cos(\theta_1+\theta_3) \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \boldsymbol{\tau_1} \\ \boldsymbol{\tau_2} \\ \boldsymbol{\tau_3} \end{bmatrix}_{\boldsymbol{all}} = \left(\begin{bmatrix} \mathbf{1.2}\boldsymbol{d_2}\,\mathbf{cos}(\boldsymbol{\theta_1}) + \mathbf{0.25}\boldsymbol{l_3}\,\mathbf{cos}(\boldsymbol{\theta_1}+\boldsymbol{\theta_3}) \\ \mathbf{0.25}\boldsymbol{l_3}\,\mathbf{cos}(\boldsymbol{\theta_1}+\boldsymbol{\theta_3}) \\ \mathbf{0} \end{bmatrix} \boldsymbol{kg} \right) \boldsymbol{g}$$