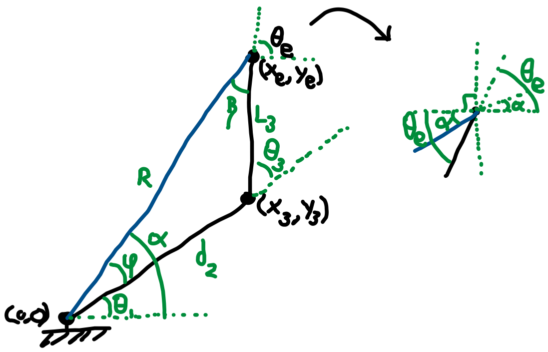
3. Written Questions

3.1 Analytic IK: RPR Robot

3.1.1.)



$$R = \sqrt{x_e^2 + y_e^2}, \quad \varphi + \theta_1 = \alpha = atan2(y_e, x_e), \quad \beta = \theta_e - \alpha$$

Law of cosines: $\because\ d_2^2 = R^2 + l_3^2 - 2Rl_3\cos(\beta)$

$$\boldsymbol{d_2 = \sqrt{x_e^2 + y_e^2 + l_3^2 - 2l_3\sqrt{x_e^2 + y_e^2}\cos\left(\theta_e - atan2(y_e, x_e)\right)}}$$

Law of cosines: $\because\ R^2 = d_2^2 + l_3^2 - 2d_2l_3\cos(\pi - \theta_3)$

$$\rightarrow \cos(\pi - \theta_3) = -\cos(\theta_3) = \frac{x_e^2 + y_e^2 - d_2^2 - l_3^2}{-2d_2l_3}$$

$$\rightarrow \boldsymbol{\theta_3 = \mathrm{acos}\left(\frac{x_e^2 + y_e^2 - d_2^2 - l_3^2}{-2d_2l_3}\right)}$$

$$\because\ \theta_e = \theta_1 + \theta_3$$

$$\boldsymbol{\theta_1 = \theta_e - \theta_3}$$

$$\therefore \begin{cases} \boldsymbol{d_2 = \sqrt{x_e^2 + y_e^2 + l_3^2 - 2l_3\sqrt{x_e^2 + y_e^2}\cos\left(\theta_e - atan2(y_e, x_e)\right)}} \\ \boldsymbol{\theta_3 = \mathrm{acos}\left(\frac{x_e^2 + y_e^2 - d_2^2 - l_3^2}{-2d_2l_3}\right)} \\ \boldsymbol{\theta_1 = \theta_e - \theta_3} \end{cases}$$

3.1.2.)

Assuming, as stated, that d_2 has no upper bound, then every position (x_e, y_e) is inherently accessible; since, the effector could be placed at any distance from the origin by just configuring d_2 and θ_3 and θ_1 could be set as if the robot where using polar coordinates. Additionally, the equations derived in 3.1.1 impose no limits on accessible values of θ_e for a given position (x_e, y_e).

3.2 Numerical IK: Cost Function

3.2.1.)

$$\theta_d = \begin{bmatrix}\theta_1\\\theta_2\end{bmatrix}, \qquad p_d = \begin{bmatrix}x_d\\y_d\end{bmatrix}, \qquad p_e = \begin{bmatrix}f_x(\theta)\\f_y(\theta)\end{bmatrix} = f(\theta), \qquad h(\theta) = \|p_d - p_e\|^2$$

$$h(\theta) = \|p_d - p_e\|^2 = \left\|\begin{matrix}x_d - f_x(\theta)\\y_d - f_y(\theta)\end{matrix}\right\|^2$$

$$\rightarrow \boldsymbol{h(\theta) = \left(x_d - f_x(\theta)\right)^2 + \left(y_d - f_y(\theta)\right)^2}$$

3.2.2.)

$$\nabla h(\theta) = \begin{bmatrix}\dfrac{\partial h}{\partial\theta_1}\\\dfrac{\partial h}{\partial\theta_2}\end{bmatrix} = \begin{bmatrix}2(f_x(\theta) - x_d)\dfrac{\partial f_x}{\partial\theta_1} + 2\left(f_y(\theta) - y_d\right)\dfrac{\partial f_y}{\partial\theta_1}\\2(f_x(\theta) - x_d)\dfrac{\partial f_x}{\partial\theta_2} + 2\left(f_y(\theta) - y_d\right)\dfrac{\partial f_y}{\partial\theta_2}\end{bmatrix}$$

$$\rightarrow \boldsymbol{\nabla h(\theta) = \begin{bmatrix}2(f_x(\theta) - x_d)\dfrac{\partial f_x}{\partial\theta_1} + 2\left(f_y(\theta) - y_d\right)\dfrac{\partial f_y}{\partial\theta_1}\\2(f_x(\theta) - x_d)\dfrac{\partial f_x}{\partial\theta_2} + 2\left(f_y(\theta) - y_d\right)\dfrac{\partial f_y}{\partial\theta_2}\end{bmatrix} = 2\left[(f_x(\theta) - x_d), \left(f_y(\theta) - y_d\right)\right]\begin{bmatrix}\dfrac{\partial f_x}{\partial\theta_1} & \dfrac{\partial f_x}{\partial\theta_2}\\\dfrac{\partial f_y}{\partial\theta_1} & \dfrac{\partial f_y}{\partial\theta_2}\end{bmatrix}}$$

3.2.3.)

$$\nabla h(\theta) = 2\left[(f_x(\theta) - x_d), \left(f_y(\theta) - y_d\right)\right]\begin{bmatrix}\dfrac{\partial f_x}{\partial\theta_1} & \dfrac{\partial f_x}{\partial\theta_2}\\\dfrac{\partial f_y}{\partial\theta_1} & \dfrac{\partial f_y}{\partial\theta_2}\end{bmatrix} = 2(f - p_d)^T\begin{bmatrix}\dfrac{\partial f_x}{\partial\theta_1} & \dfrac{\partial f_x}{\partial\theta_2}\\\dfrac{\partial f_y}{\partial\theta_1} & \dfrac{\partial f_y}{\partial\theta_2}\end{bmatrix}$$

$$\because J(\theta) = \begin{bmatrix}\dfrac{\partial f_x}{\partial\theta_1} & \dfrac{\partial f_x}{\partial\theta_2}\\\dfrac{\partial f_y}{\partial\theta_1} & \dfrac{\partial f_y}{\partial\theta_2}\end{bmatrix} for\ SO(2)$$

$$\boldsymbol{\nabla h(\theta) = 2(f - p_d)^T J(\theta) = 2J(\theta)^T(f - p_d)}$$

3.2.4.)

It looks the same. The gradient found in 3.2.3. looks the same regardless of the number of links in a serial chain of revolute joints since it is in vectorized form in terms of the Jacobian.