3. Written Questions

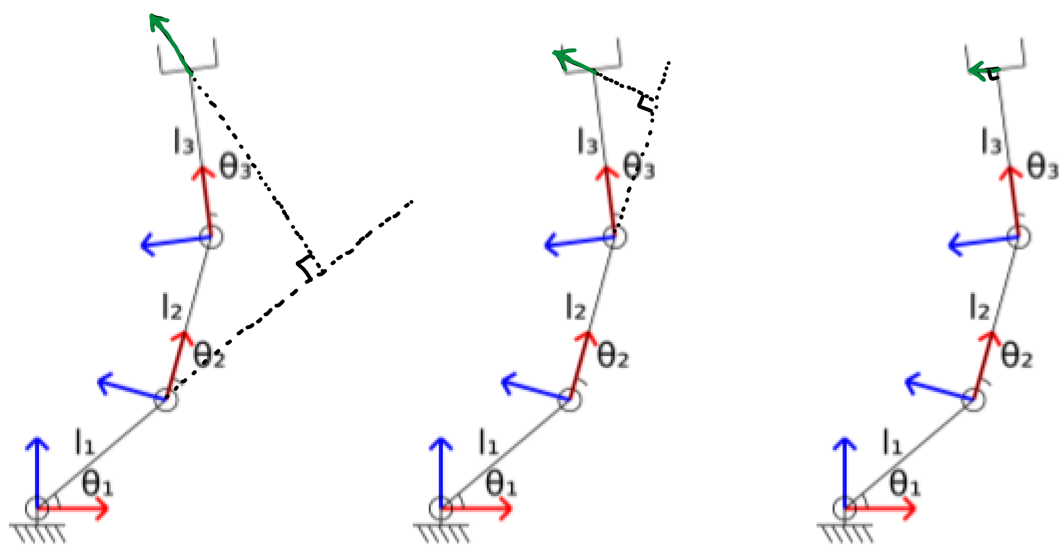
3.1 RRR Robot

$$let: \begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix} = \begin{bmatrix}\frac{\pi}{2}\\ \frac{\pi}{3}\\ \frac{\pi}{3}\end{bmatrix}$$

3.1.1.) $\dim(J) = m_{workspace} \mathrm{x} n_{joints} = \boldsymbol{2x3}$

3.1.2.) **Underconstrained** because the number of rows (constraint equations) is less than the number of columns (unknowns) $\because n_{rows} < n_{columns} \rightarrow 2 < 3$.

3.1.3.)



3.1.4.)

In order of joint index, the directions vectors are:

$$\hat{v}_1 = \hat{\omega}_1 \times \hat{r}_{12} = \hat{k} \times [\cos(\theta_1), \sin(\theta_1), 0] = [0,0,1] \times [0,1,0] = [-1,0,0]$$

$$\hat{v}_2 = \hat{\omega}_2 \times \hat{r}_{23} = \hat{k} \times [\cos(\theta_1+\theta_2), \sin(\theta_1+\theta_2), 0] = [0,0,1] \times \left[-\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right] = \left[-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0\right]$$

$$\hat{v}_3 = \hat{\omega}_3 \times \hat{r}_{3e} = \hat{k} \times [\cos(\theta_1+\theta_2+\theta_3), \sin(\theta_1+\theta_2+\theta_3), 0] = [0,0,1] \times \left[-\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\right] = \left[\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0\right]$$

Thus, **all three vectors are not linearly independent** $\because |\hat{v}_1 \hat{v}_2 \hat{v}_3| = \begin{vmatrix} -1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 \end{vmatrix} = 0$

But $\boldsymbol{\vec{v}_1}$ **and** $\boldsymbol{\vec{v}_2}$ **are linearly independent** $\because \begin{vmatrix} -1 & -\frac{1}{2} \\ 0 & -\frac{\sqrt{3}}{2} \end{vmatrix} = -\frac{\sqrt{3}}{2} \neq 0,$

$\vec{\boldsymbol{v}}_2$ **and** $\vec{\boldsymbol{v}}_3$ **are linearly independent** $\because \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{vmatrix} = -\frac{\sqrt{3}}{2} \neq 0,$

and $\vec{\boldsymbol{v}}_1$ **and** $\vec{\boldsymbol{v}}_3$ **are linearly independent** $\because \begin{vmatrix} -1 & \frac{1}{2} \\ 0 & -\frac{\sqrt{3}}{2} \end{vmatrix} = -\frac{\sqrt{3}}{2} \neq 0.$

Therefore, **the dimension of the space spanned by the three vectors is 2.**

3.1.5.) $rank(\boldsymbol{J}) = 2$

Because $\boldsymbol{J} = \begin{bmatrix} -l_1\sin(\theta_1) - l_2\sin(\theta_1+\theta_2) - l_3\sin(\theta_1+\theta_2+\theta_3) & -l_2\sin(\theta_1+\theta_2) - l_3\sin(\theta_1+\theta_2+\theta_3) & -l_3\sin(\theta_1+\theta_2+\theta_3) \\ l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3) & l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3) & l_3\cos(\theta_1+\theta_2+\theta_3) \end{bmatrix}$

3.1.6.) Right pseudo inverse, $\boldsymbol{J}^+$.

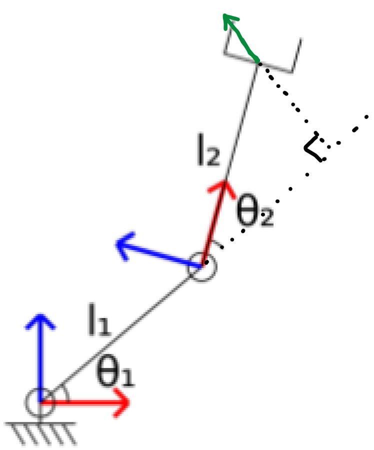
3.2 RR Robot

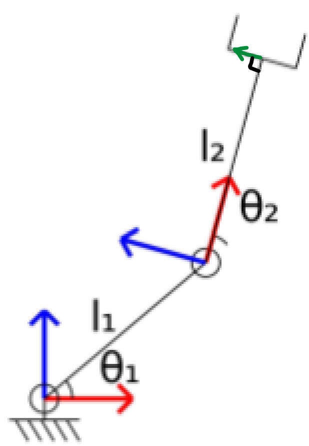
$$let\colon \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \frac{\pi}{4} \\ \frac{\pi}{4} \end{bmatrix}$$

3.2.1.) $\dim(J) = m_{workspace} \mathrm{x} n_{joints} = \boldsymbol{2x2}$

3.2.2.) **Neither** (perfectly constrained) because the number of rows (constraint equations) equals the number of columns (unknowns) $\because n_{rows} = n_{columns} \rightarrow 2 = 2.$

3.2.3.)





3.2.4.)

In order of joint index, the directions vectors are:

$$\hat{v}_1 = \widehat{\omega}_1 \times \hat{r}_{12} = \hat{k} \times [\cos(\theta_1), \sin(\theta_1), 0] = [0,0,1] \times \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right] = \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right]$$

$$\hat{v}_2 = \widehat{\omega}_2 \times \hat{r}_{2e} = \hat{k} \times [\cos(\theta_1+\theta_2), \sin(\theta_1+\theta_2), 0] = [0,0,1] \times [0,1,0] = [-1,0,0]$$

These two vectors, $\vec{v}_1$ **and** $\vec{v}_2$**, are linearly independent** $\because \begin{vmatrix} -\frac{\sqrt{2}}{2} & -1 \\ \frac{\sqrt{2}}{2} & 0 \end{vmatrix} = \frac{\sqrt{2}}{2} \neq 0,$

Therefore, **the dimension of the space spanned by the two vectors is 2.**

3.2.5.) $rank(\boldsymbol{J}) = 2$

Because $\boldsymbol{J} = \begin{bmatrix} -l_1 \sin(\theta_1) - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$

3.2.6.) Inverse, $\boldsymbol{J}^{-1}$.

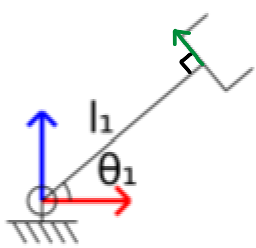
3.3 R Robot

$$let: \theta_1 = \frac{\pi}{4}$$

3.3.1.) $\dim(J) = m_{workspace} \text{x} n_{joints} = \boldsymbol{2x1}$

3.3.2.) **Overconstrained** (perfectly constrained) because the number of rows (constraint equations) is greater than the number of columns (unknowns) $\because n_{rows} > n_{columns} \rightarrow 2 > 1$.

3.3.3.)



3.3.4.) **The dimension of the space spanned by the vector is 1.**

3.3.5.) $rank(\boldsymbol{J}) = 1$

Because $\boldsymbol{J} = \begin{bmatrix} -l_1 \sin(\theta_1) \\ l_1 \cos(\theta_1) \end{bmatrix}$

3.3.6.) Left pseudoinverse , $\boldsymbol{J}^{\#}$.

3.3.7.) A left pseudoinverse is able to find the best (or, rather, least bad in terms of sum squared error) solution for a set of equations where there are no perfect solutions; whereas, a right pseudoinverse is able to only able to find the best (least norm) solution for a set of equations where multiple solutions already exist.

3.4 RRR Robot, revisited with rotation

3.4.1.) $\dim(J) = m_{workspace} \text{x} n_{joints} = \boldsymbol{3x3}$

3.4.2.) **Neither** (perfectly constrained) because the number of rows (constraint equations) equals the number of columns (unknowns) $\because n_{rows} = n_{columns} \rightarrow 3 = 3$.

3.4.3.)

$$\boldsymbol{f} = \begin{bmatrix} f_x \\ f_y \\ f_\theta \end{bmatrix} = \begin{bmatrix} l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3) \\ l_1\sin(\theta_1) + l_2\sin(\theta_1+\theta_2) + l_3\sin(\theta_1+\theta_2+\theta_3) \\ \theta_1+\theta_2+\theta_3 \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial \boldsymbol{f}}{\partial \theta_1} & \frac{\partial \boldsymbol{f}}{\partial \theta_2} & \frac{\partial \boldsymbol{f}}{\partial \theta_3} \end{bmatrix}$$

$$\rightarrow \boldsymbol{J} = \begin{bmatrix} -l_1\sin(\theta_1) - l_2\sin(\theta_1+\theta_2) - l_3\sin(\theta_1+\theta_2+\theta_3) & -l_2\sin(\theta_1+\theta_2) - l_3\sin(\theta_1+\theta_2+\theta_3) & -l_3\sin(\theta_1+\theta_2+\theta_3) \\ l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3) & l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3) & l_3\cos(\theta_1+\theta_2+\theta_3) \\ 1 & 1 & 1 \end{bmatrix}$$

3.4.4.)

$$let\colon \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.6 \\ 0.4 \end{bmatrix}$$

$$\rightarrow J = \begin{bmatrix} -0.5646l_1 - 0.9320l_2 - 0.9996l_3 & -0.9320l_2 - 0.9996l_3 & -0.9996l_3 \\ 0.8253l_1 + 0.3624l_2 - 0.0292l_3 & 0.3624l_2 - 0.0292l_3 & -0.0292l_3 \\ 1 & 1 & 1 \end{bmatrix}$$

a.)

$$\dot{X}_1 = \boldsymbol{J}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \dot{X}_1 = \begin{bmatrix} -0.5646l_1 - 0.9320l_2 - 0.9996l_3 \\ 0.8253l_1 + 0.3624l_2 - 0.0292l_3 \\ 1 \end{bmatrix}$$

b.)

$$\dot{X}_2 = \boldsymbol{J}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \dot{X}_2 = \begin{bmatrix} -0.9320l_2 - 0.9996l_3 \\ 0.3624l_2 - 0.0292l_3 \\ 1 \end{bmatrix}$$

c.)

$$\dot{X}_3 = \boldsymbol{J}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \dot{X}_3 = \begin{bmatrix} -0.9996l_3 \\ -0.0292l_3 \\ 1 \end{bmatrix}$$

3.4.5.)

All three vectors are linearly independent $\because \left|\dot{X}_1\dot{X}_2\dot{X}_3\right| = |\boldsymbol{J}|$

$$\rightarrow \left|\dot{X}_1\dot{X}_2\dot{X}_3\right| = \begin{vmatrix} -0.5646l_1 - 0.9320l_2 - 0.9996l_3 & -0.9320l_2 - 0.9996l_3 & -0.9996l_3 \\ 0.8253l_1 + 0.3624l_2 - 0.0292l_3 & 0.3624l_2 - 0.0292l_3 & -0.0292l_3 \\ 1 & 1 & 1 \end{vmatrix} = 0.56l_1l_2 \neq 0$$

Therefore, **$\vec{v}_1$ and $\vec{v}_2$ are linearly independent**, **$\vec{v}_2$ and $\vec{v}_3$ are linearly independent**, and **$\vec{v}_1$ and $\vec{v}_3$ are linearly independent**.

As such, **the dimension of the space spanned by the three vectors is 3.**

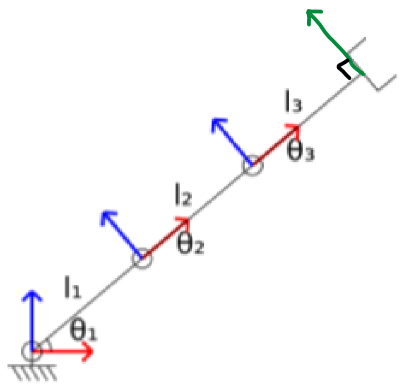
3.4.6.) $\boldsymbol{rank(J) = 3}$ based on the results of the previous problem $\because \left|\dot{X}_1\dot{X}_2\dot{X}_3\right| = |\boldsymbol{J}|$.

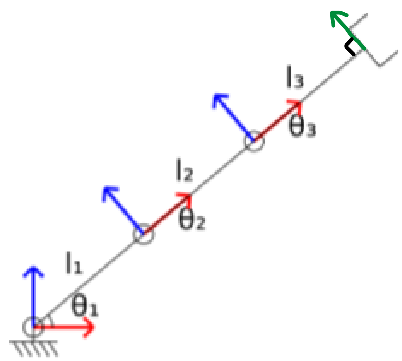
3.4.7.) Inverse, $\boldsymbol{J}^{-1}$.

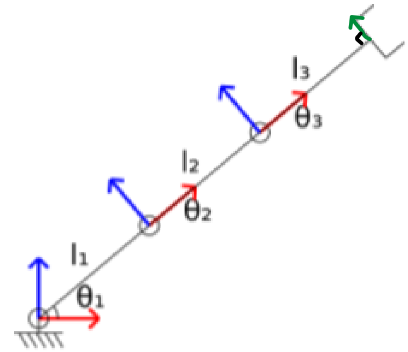
3.5 RRR Robot Singular Configuration

3.5.1.) $\mathrm{dims}(J) = m_{workspace}\mathrm{x}n_{joints} = \boldsymbol{3x3}$ but $\boldsymbol{rank(J) = 1}$; so, the two values differ due to the loss of available instantaneous motions in this singular configuration.

3.5.2.)







3.5.3.)

Very visibly, by the nature of this singularity, each of these vectors is a linear multiple of the other. Specifically:

$$\theta_1 = \theta_1, \quad \theta_2 = \theta_3 = 0$$

$$\therefore \vec{v}_1 = l_1 \begin{bmatrix} \cos\left(\theta_1 + \frac{\pi}{2}\right) \\ \sin\left(\theta_1 + \frac{\pi}{2}\right) \\ 0 \end{bmatrix}, \qquad \vec{v}_2 = (l_1 + l_2) \begin{bmatrix} \cos\left(\theta_1 + \frac{\pi}{2}\right) \\ \sin\left(\theta_1 + \frac{\pi}{2}\right) \\ 0 \end{bmatrix}, \qquad \vec{v}_3 = (l_1 + l_2 + l_3) \begin{bmatrix} \cos\left(\theta_1 + \frac{\pi}{2}\right) \\ \sin\left(\theta_1 + \frac{\pi}{2}\right) \\ 0 \end{bmatrix}$$

As a result, the **dimension of the space spanned by these vectors is 1**, **$\vec{v}_1$ and $\vec{v}_2$ are not linearly independent**, **$\vec{v}_2$ and $\vec{v}_3$ are not linearly independent**, **$\vec{v}_1$ and $\vec{v}_3$ are not linearly independent**, and **all three vectors are not linearly independent.**

3.5.4.) $rank(\boldsymbol{J}) = \mathbf{1}$

$$let: \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.6 \\ 0.4 \end{bmatrix}$$

$$\rightarrow J = \begin{bmatrix} -0.5646l_1 - 0.9320l_2 - 0.9996l_3 & -0.9320l_2 - 0.9996l_3 & -0.9996l_3 \\ 0.8253l_1 + 0.3624l_2 - 0.0292l_3 & 0.3624l_2 - 0.0292l_3 & -0.0292l_3 \\ 1 & 1 & 1 \end{bmatrix}$$

a.)

$$\dot{X}_1 = \boldsymbol{J} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \dot{X}_1 = \begin{bmatrix} -0.5646l_1 - 0.9320l_2 - 0.9996l_3 \\ 0.8253l_1 + 0.3624l_2 - 0.0292l_3 \\ 1 \end{bmatrix}$$

b.)

$$\dot{X}_2 = \boldsymbol{J} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \dot{X}_2 = \begin{bmatrix} -0.9320l_2 - 0.9996l_3 \\ 0.3624l_2 - 0.0292l_3 \\ 1 \end{bmatrix}$$

c.)

$$\dot{X}_3 = \boldsymbol{J} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \dot{X}_3 = \begin{bmatrix} -0.9996l_3 \\ -0.0292l_3 \\ 1 \end{bmatrix}$$

3.4.5.)

All three vectors are linearly independent $\because \left|\dot{X}_1 \dot{X}_2 \dot{X}_3\right| = |\boldsymbol{J}|$

$$\rightarrow \left|\dot{X}_1\dot{X}_2\dot{X}_3\right| = \begin{vmatrix} -0.5646l_1 - 0.9320l_2 - 0.9996l_3 & -0.9320l_2 - 0.9996l_3 & -0.9996l_3 \\ 0.8253l_1 + 0.3624l_2 - 0.0292l_3 & 0.3624l_2 - 0.0292l_3 & -0.0292l_3 \\ 1 & 1 & 1 \end{vmatrix} = 0.56l_1l_2 \neq 0$$

Therefore, $\vec{\boldsymbol{v}}_{\mathbf{1}}$ **and** $\vec{\boldsymbol{v}}_{\mathbf{2}}$ **are linearly independent**, $\vec{\boldsymbol{v}}_{\mathbf{2}}$ **and** $\vec{\boldsymbol{v}}_{\mathbf{3}}$ **are linearly independent**, and $\vec{\boldsymbol{v}}_{\mathbf{1}}$ **and** $\vec{\boldsymbol{v}}_{\mathbf{3}}$ **are linearly independent**.

As such, **the dimension of the space spanned by the three vectors is 3.**

3.5.6.) $\boldsymbol{rank(J) = 3}$ based on the results of the previous problem $\because \left|\dot{X}_1\dot{X}_2\dot{X}_3\right| = |\boldsymbol{J}|$.

3.6 Local Inverse Kinematics

$$\exists\, f: Q \rightarrow W, \quad J$$

3.6.1.) $\dot{\Theta}_0 = J^{-1} * (p_1 - p_0)$

3.6.2.) $\dot{\Theta}_{dt} = J^{-1} * \left(p_1 - f\left(\Theta_0 + \dot{\Theta}_0\right)\right)$

3.6.3.) To command the robot from p_0 to p_1, a modification of the above command from 3.6.2: $\dot{\Theta}_{dt,n} = J^{-1} * \left(p_1 - f\left(\Theta_0 + \sum_i^{n-1} \dot{\Theta}_{dt,i}\right)\right)$, would be run for each time instant after the initial condition until $f\left(\Theta_0 + \sum_i^{n-1} \dot{\Theta}_{dt,i}\right) = p_1$. However, since at each step the control would be commanding a velocity that would take 1 second to cover the remaining distance to p_1, each theta would need to be multiplied by the factor $\frac{\|p_1 - p_0\|}{\left\|p_1 - f\left(\Theta_0 + \sum_i^{n-1} \dot{\Theta}_{dt,i}\right)\right\|}$ to ensure that the total journey takes approximately 1 second. Likely, this would take slightly longer than 1 second because, as the robot gets close to its target, this simple algorithm would drastically increase the required speed until it would be requesting velocities beyond the joint's physical limits.