# 16720 — PS4

Connor W. Colombo (cwcolomb)

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## 1 Theory

### Q1.1

Given this setup where the image coordinates are normalized so that the principle point coincides with the coordinate origin in each image, the point correspondence is $\mathbf{x}_1 = \begin{bmatrix} x_1 & y_1 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ in Camera 1 and $\mathbf{x}'_1 = \begin{bmatrix} x'_1 & y'_1 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ in Camera 2. Thus, setting up the system to solve for the fundamental matrix $\mathbf{F}$ using the result of the Longuet-Higgins Equation in the form:

$$\mathbf{x}_1^T \mathbf{F} \mathbf{x}_1$$
$$\rightarrow \mathbf{A} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = 0$$

the first row of $\mathbf{A}$ would be given by this point correspondence as:

$$\mathbf{A} = \begin{bmatrix} x_1 x'_1 & x_1 y'_1 & x_1 & y_1 x'_1 & y_1 y'_1 & y_1 & x'_1 & y'_1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

since all $x$ and $y$ coordinates are 0 in each image. Thus, thinking of this result

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = 0$$

as a system of equations, it's clear that this point correspondence contributes setting $f_{33} = 0$.

## Q1.2

First, it is worth noting that such a translation constitutes a frontoparallel relation between the cameras by definition, thus the epipolar lines would be horizontal (lie along the axis of translation, $x$) with the epipoles at infinity. This can be seen by evaluating the epipolar line as follows.

The essential matrix for a pure translation along the $x$ axis by amount $x$ would be:

$$\mathbf{E} = \mathbf{SR}$$

where $\mathbf{R}$ is the identity matrix and $S$ is the skew symmetric matrix corresponding to the translation vector $\mathbf{t} = \begin{bmatrix} x & y & z \end{bmatrix}^T = \begin{bmatrix} x & 0 & 0 \end{bmatrix}^T$:

$$\mathbf{S} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -x \\ 0 & x & 0 \end{bmatrix}$$

$$\therefore \mathbf{E} = \mathbf{SR} = \mathbf{S} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -x \\ 0 & x & 0 \end{bmatrix}$$

Since the epipolar line in the right image $\tilde{\mathbf{l}}_r$ corresponding to a point $\mathbf{p}_l = \begin{bmatrix} u_l & v_l & 1 \end{bmatrix}^T$ in the left image is given by:

$$\tilde{\mathbf{l}}_r = \mathbf{E}\mathbf{p}_l = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -x \\ 0 & x & 0 \end{bmatrix} \begin{bmatrix} u_l \\ v_l \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -x \\ xv_l \end{bmatrix}$$

the epipolar line in the right (post-translation) image corresponding to a point in the left (original) image is thus given by the equation:

$$0u + -xv + xv_l = 0 \ \rightarrow \ v = v_l$$

This result is quite evidently a horizontal line at height $v_l$.

Equivalently, since the epipolar line in the left image $\tilde{\mathbf{l}}_l$ corresponding to a point $\mathbf{p}_r = \begin{bmatrix} u_r & v_r & 1 \end{bmatrix}^T$ in the right image is given by:

$$\tilde{\mathbf{l}}_l = \mathbf{E}^T\mathbf{p}_r = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & -x & 0 \end{bmatrix} \begin{bmatrix} u_r \\ v_r \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ x \\ -xv_r \end{bmatrix}$$

the epipolar line in the left (original) image corresponding to a point in the right (post-translation) image is thus given by the equation:

$$0u + xv + -xv_r = 0 \ \rightarrow \ v = v_r$$

This result is quite evidently a horizontal line at height $v_r$.

## Q1.3

The essential matrix is given by

$$\mathbf{E} = \mathbf{S}_{\text{rel}}\mathbf{R}_{\text{rel}}$$

where $\mathbf{R}_{\text{rel}}$ is the rotation matrix and $\mathbf{S}_{\text{rel}}$ is the skew symmetric matrix corresponding to the translation vector $\mathbf{t}_{\text{rel}} = \begin{bmatrix} x & y & z \end{bmatrix}^T$ defined as follows:

$$\mathbf{S}_{\text{rel}} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

Thus, the essential matrix $\mathbf{E}$ is given by:

$$\mathbf{E} = \begin{bmatrix} 0 & -\mathbf{t}_{\text{rel}3} & \mathbf{t}_{\text{rel}2} \\ \mathbf{t}_{\text{rel}3} & 0 & -\mathbf{t}_{\text{rel}1} \\ -\mathbf{t}_{\text{rel}2} & \mathbf{t}_{\text{rel}1} & 0 \end{bmatrix} \mathbf{R}_{\text{rel}}$$

Since the fundamental matrix $\mathbf{F}$ and essential matrix $\mathbf{E}$ are related by:

$$\mathbf{F} = \mathbf{K}_1^{-T}\mathbf{E}\mathbf{K}_2^{-1}$$

and since the camera is the same for each image giving that $\mathbf{K}_1 = \mathbf{K}_2 = \mathbf{K}$, the fundamental matrix can be expressed as:

$$\mathbf{F} = \mathbf{K}^{-T}\mathbf{E}\mathbf{K}^{-1}$$

$$\rightarrow \mathbf{F} = \mathbf{K}^{-T}\mathbf{S}_{\text{rel}}\mathbf{R}_{\text{rel}}\mathbf{K}^{-1}$$

$$\rightarrow \mathbf{F} = \mathbf{K}^{-T} \begin{bmatrix} 0 & -\mathbf{t}_{\text{rel}3} & \mathbf{t}_{\text{rel}2} \\ \mathbf{t}_{\text{rel}3} & 0 & -\mathbf{t}_{\text{rel}1} \\ -\mathbf{t}_{\text{rel}2} & \mathbf{t}_{\text{rel}1} & 0 \end{bmatrix} \mathbf{R}_{\text{rel}}\mathbf{K}^{-1}$$

## Q1.4

For a point on the object $\mathbf{P} = \begin{bmatrix} x & y & z \end{bmatrix}^T$ viewed through a plane mirror with normal vector $\hat{\mathbf{n}}$ to the view, the reflected point $\mathbf{P}_r = \begin{bmatrix} x_r & y_r & z_r \end{bmatrix}^T$ will be given by:

$$\mathbf{P}_r = \mathbf{R}\mathbf{P} + 2m\hat{\mathbf{n}}$$

where m is the distance to the mirror plane from the focal point of the camera along $\hat{\mathbf{n}}$ and $\mathbf{R}$ is the general **reflection** matrix for a plane mirror with normal $\hat{\mathbf{n}}$ given by:

$$\mathbf{R} = \mathbf{I} - 2\hat{\mathbf{n}}\hat{\mathbf{n}}^T = \begin{bmatrix} 1 - 2\hat{\mathbf{n}}_1^2 & -2\hat{\mathbf{n}}_1\hat{\mathbf{n}}_2 & -2\hat{\mathbf{n}}_1\hat{\mathbf{n}}_3 \\ -2\hat{\mathbf{n}}_2\hat{\mathbf{n}}_1 & 1 - 2\hat{\mathbf{n}}_y^2 & -2\hat{\mathbf{n}}_2\hat{\mathbf{n}}_3 \\ -2\hat{\mathbf{n}}_3\hat{\mathbf{n}}_1 & -2\hat{\mathbf{n}}_3\hat{\mathbf{n}}_2 & 1 - 2\hat{\mathbf{n}}_3^2 \end{bmatrix}$$

To obtain a fundamental matrix relationship between the projected images of these points, $\mathbf{p} = \begin{bmatrix} u & v & 1 \end{bmatrix}^T$ and $\mathbf{p}_r = \begin{bmatrix} u_r & v_r & 1 \end{bmatrix}^T$, one can simply substitute their defining intrinsic relationship to the 3D points into the given reflection relationship as follows:

$$\because \begin{cases} \mathbf{P}_r = \mathbf{R}\mathbf{P} + 2m\hat{\mathbf{n}}, \\ \mathbf{p}_r \equiv \mathbf{K}\mathbf{P}_r \;\rightarrow\; \mathbf{P}_r \equiv \mathbf{K}^{-1}\mathbf{p}_r \;\rightarrow\; \mathbf{P}_r = k_r\mathbf{K}^{-1}\mathbf{p}_r, \\ \mathbf{p} \equiv \mathbf{K}\mathbf{P} \;\rightarrow\; \mathbf{P} \equiv \mathbf{K}^{-1}\mathbf{p} \;\rightarrow\; \mathbf{P} = k\mathbf{K}^{-1}\mathbf{p} \end{cases}$$

$$k_r\mathbf{K}^{-1}\mathbf{p}_r = k\mathbf{R}\mathbf{K}^{-1}\mathbf{p} + 2m\hat{\mathbf{n}}$$

$$\rightarrow 0 = \mathbf{p}_r^T\mathbf{K}^{-T}(2m\hat{\mathbf{n}} \times \mathbf{R}\mathbf{K}^{-1}\mathbf{p})$$

$$\rightarrow 0 = \mathbf{p}_r^T\mathbf{K}^{-T}\mathbf{S}_t\mathbf{R}\mathbf{K}^{-1}\mathbf{p}$$

where $\mathbf{S}_t$ is the skew symmetric matrix corresponding to the translation vector $\mathbf{t} = 2m\hat{\mathbf{n}}$.

Thus, based on the epipolar geometry relationship defining $\mathbf{F}$ with relation to $\mathbf{p}$ and $\mathbf{p}_r$, $\mathbf{p}_r^T\mathbf{F}\mathbf{p} = 0$, $\mathbf{F}$ can be said to be:

$$\mathbf{F} = \mathbf{K}^{-T}\mathbf{S}_t\mathbf{R}\mathbf{K}^{-1} = \mathbf{K}^{-T}\mathbf{S}_t(I - 2\hat{\mathbf{n}}\hat{\mathbf{n}}^T)\mathbf{K}^{-1}$$

Implicitly, this result aligns with the intuitive result that comes from viewing this operation as equivalent to applying the reflection transform $\mathbf{R}$ and translation $\mathbf{t} = 2m\hat{\mathbf{n}}$. Thus, the essential matrix relating the projected points to each other should be:

$$\mathbf{E} = \mathbf{S}_t\mathbf{R}$$

where the reflection transform $\mathbf{R}$ substitutes the usual rotation transform (which was deemed acceptable by @532 on Piazza). Thus, the equivalent fundamental matrix will be given by:

$$\mathbf{F} = \mathbf{K}^{-T}\mathbf{E}\mathbf{K}^{-1} = \mathbf{K}^{-T}\mathbf{S}_t\mathbf{R}\mathbf{K}^{-1} = \mathbf{K}^{-T}\mathbf{S}_t(I - 2\hat{\mathbf{n}}\hat{\mathbf{n}}^T)\mathbf{K}^{-1}$$

(the same as above)

To check if this $\mathbf{F}$ is skew-symmetric, one must check if $\mathbf{F}^T = -\mathbf{F}$ as follows:

$$\mathbf{F}^T = \mathbf{K}^{-T}\mathbf{R}^T\mathbf{S}_t^T\mathbf{K}^{-1} = \mathbf{K}^{-T}(\mathbf{S}_t\mathbf{R})^T\mathbf{K}^{-1}$$

Notably, since $\mathbf{S}_t$ is itself skew symmetric, $\mathbf{S}_t^T = -\mathbf{S}_t$ and since $\mathbf{R}$ is symmetric $\mathbf{R}^T = \mathbf{R}$. Therefore:

$$\mathbf{F}^T = -\mathbf{K}^{-T}\mathbf{S}_t\mathbf{R}\mathbf{K}^{-1}$$

$$\therefore \mathbf{F}^T = -\mathbf{F}$$

Thus, viewing a point in a mirror is equivalent to getting a second view of the point separated by a skew-symmetric fundamental matrix.

# 2 Practice

## 2.1 Fundamental Matrix Estimation

### Q2.1

For images `../data/im1.png` and `./data/im2.png` using the correspondences in `../data/some_corresp.npz` with $M = 640$, the following output was produced using `test_q2_1.py` (even though this wasn't one of the requested files, it was included in the submission per post `@496` on Piazza).

$$\mathbf{F} = \begin{bmatrix} 1.14759838 \times 10^{-6} & 2.72474196 \times 10^{-5} & -2.43621967 \times 10^{-1} \\ 1.32691730 \times 10^{-5} & 4.35063985 \times 10^{-7} & -3.92697458 \times 10^{-3} \\ 2.33504509 \times 10^{-1} & -1.84813374 \times 10^{-4} & 1 \end{bmatrix}$$

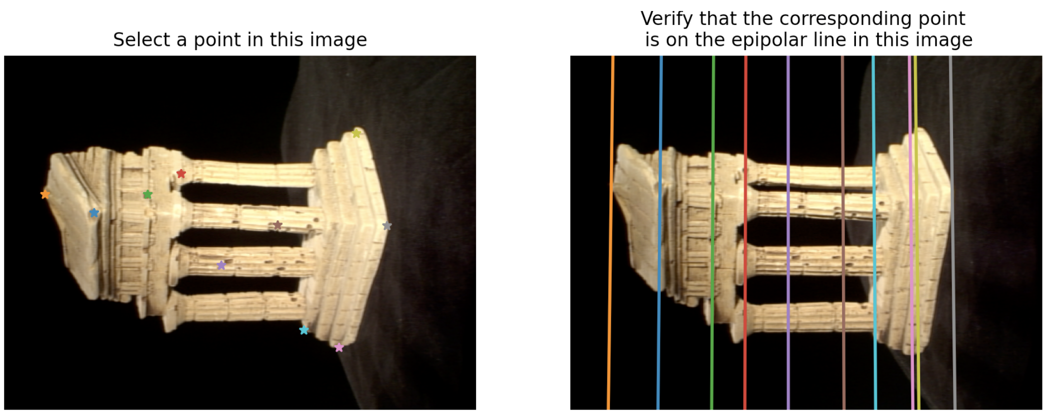


Figure 1: Sample output from `displayEpipolarF`

## 2.2 Metric Reconstruction

### Q3.1

For images `../data/im1.png` and `./data/im2.png` using the correspondences in `../data/some_corresp.npz` with $M = 640$ and the intrinsic camera matrices given in `../data/intrinsics.npz` , the following output was produced using `test_q3_1.py` (even though this wasn't one of the requested files, it was included in the submission per post `@496` on Piazza).

$$\mathbf{E} = \begin{bmatrix} 2.65280695 & 6.32134239 \times 10^{1} & -3.59648270 \times 10^{2} \\ 3.07841941 \times 10^{1} & 1.01299028 & 2.92920974 \times 10^{-1} \\ 3.60528213 \times 10^{2} & 1.24513910 \times 10^{1} & 8.15202725 \times 10^{-2} \end{bmatrix}$$

**Q3.2**

For points given by:

$$\text{let: } \mathbf{x}_{1i} = \begin{bmatrix} u_{1i} \\ v_{1i} \\ 1 \end{bmatrix}, \quad \tilde{\mathbf{w}}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix}$$

Projection from 3D to 2D gives that:

$$\begin{aligned} \mathbf{x}_{1i} = \mathbf{C}_1\tilde{\mathbf{w}}_i \;\rightarrow\; \mathbf{x}_{1i} \times \mathbf{C}_1\tilde{\mathbf{w}}_i = 0 \;\rightarrow\; \mathbf{A}_{1i}\tilde{\mathbf{w}}_i = 0 \quad | \quad \mathbf{A}_{1i} = [\mathbf{x}_{1i}]_\times \mathbf{C}_1 \\ \mathbf{x}_{2i} = \mathbf{C}_2\tilde{\mathbf{w}}_i \;\rightarrow\; \mathbf{x}_{2i} \times \mathbf{C}_2\tilde{\mathbf{w}}_i = 0 \;\rightarrow\; \mathbf{A}_{2i}\tilde{\mathbf{w}}_i = 0 \quad | \quad \mathbf{A}_{2i} = [\mathbf{x}_{2i}]_\times \mathbf{C}_2 \end{aligned}$$

where $[\mathbf{x}_{ki}]_\times$ is the skew symmetric matrix corresponding to the vector $\mathbf{x}_{ki}$. Thus, the respective components $\mathbf{A}_{1i}$ and $\mathbf{A}_{2i}$ of $\mathbf{A}_i$ can be found as follows:

$$\mathbf{A}_{1i} = \begin{bmatrix} 0 & -1 & v_{1i} \\ 1 & 0 & -u_{1i} \\ -v_{1i} & u_{1i} & 0 \end{bmatrix} \mathbf{C}_1$$

Since $\text{rank}([\mathbf{x}_{1i}]_\times) = 2$, only "equations" resulting from the multiplication of the first two rows of $[\mathbf{x}_{1i}]_\times$ with $\mathbf{C}_1$ are necessary (distinct). Likewise, the same goes for the two rows of $[\mathbf{x}_{2i}]_\times$ and $\mathbf{C}_2$. Thus, a complete $4 \times 4$ matrix $\mathbf{A}_i$ such that $\mathbf{A}_i\tilde{\mathbf{w}}_i = 0$ can be defined as:

$$\mathbf{A}_i = \begin{bmatrix} \begin{bmatrix} 0 & -1 & v_{1i} \\ 1 & 0 & -u_{1i} \end{bmatrix} \mathbf{C}_1 \\ \\ \begin{bmatrix} 0 & -1 & v_{2i} \\ 1 & 0 & -u_{2i} \end{bmatrix} \mathbf{C}_2 \end{bmatrix}$$