1.)

a.)

$$f(x) = x - x^2 + \frac{x^3}{3}$$
$$\therefore f'(x) = 1 - 2x + x^2 = (x-1)^2$$
$$\therefore f''(x) = 2(x-1)$$
Optimal points are:
let: $f'(x_c) = 0$
$\rightarrow (x_c - 1)^2 = 0$
$x_c = \{0, 0\}$
$\because f''(0) = 0$ and $f''(0-\varepsilon) < 0 \; and \; f''(0+\varepsilon) > 0$ where ε is small

Optimal point is an inflection point at $x_c = 0$

b.)

$f(x,y) = 2xy^2 + 2x - x^2 - 2y^2$
$\therefore f_x(x,y) = 2y^2 + 2 - 2x$
$\therefore f_y(x,y) = 4xy - 4y$
$\therefore f_{xx}(x,y) = -2$
$\therefore f_{yy}(x,y) = 4x - 4$
$\therefore f_{xy}(x,y) = 4y$
$\therefore$ Hessian: $H(x,y) = f_{xx}f_{yy} - f_{xy}^2 = -16y^2 - 8x + 8$

Optimal points are:

$(x_c, y_c) \mid f_x(x_c, y_c) = 0$ and $f_y(x_c, y_c) = 0$

$$\rightarrow \begin{cases} f_x = 2y_c^2 + 2 - 2x_c = 0, \\ f_y = 4x_c y_c - 4y_c = 0 \end{cases}$$
$$\rightarrow \begin{cases} x_c = 1, \\ y_c = 0 \end{cases} \rightarrow (x_c, y_c) = (1,0)$$

$\because H(1,0) = 0$, **The optimal point is a degenerate point at $(x_c, y_c) = (1, 0)$**

c.)

$f(x,y) = \frac{1}{3}x^3 + \frac{1}{2}y^2 - 2xy + 4x$
$\therefore f_x(x,y) = x^2 - 2y + 4$
$\therefore f_y(x,y) = y - 2x$
$\therefore f_{xx}(x,y) = 2x$
$\therefore f_{yy}(x,y) = 1$
$\therefore f_{xy}(x,y) = -2$
$\therefore$ Hessian: $H(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 2x - 4$

Optimal points are:

$(x_c, y_c) \mid f_x(x_c, y_c) = 0$ and $f_y(x_c, y_c) = 0$

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$\rightarrow \begin{cases} f_x = x_c^2 - 2y_c + 4 = 0, \\ f_y = y_c - 2x_c = 0 \end{cases}$

$\rightarrow \begin{cases} x_c = 2, \\ y_c = 4 \end{cases} \rightarrow (x_c, y_c) = (2,4)$

$\because H(2,4) = 0,$ **The optimal point is a degenerate point at** $(\boldsymbol{x_c}, \boldsymbol{y_c}) = (\mathbf{2}, \mathbf{4})$