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1.)

a.)

let: $x_{k+1} = Ax_k + Bu_k \mid A = \begin{bmatrix} 1 & 0 \\ -1/2 & 1/2 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

algebraic multiplicity, geometric multiplicity → equal ⇒ λ_1 is not defective

$\because \lambda_1 = 1,\ \lambda_2 = 1/2,\ \because |\lambda_i| \leq 1$, and $\lambda_1 = 1,\ m_1 = 1 = g_1$ $\therefore$ <u>SYSTEM is LYAPUNOV STABLE BUT NOT ASYMPTOTIC STABLE</u>

$\because |\lambda_i|$ is not $< 1\ \forall \lambda_i$

b.) let: $\dot{x} = Ax + Bu \mid A = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$

let: $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -7-\lambda & -2 & 6 \\ 2 & -3-\lambda & -2 \\ -2 & -2 & 1-\lambda \end{vmatrix} = (-7-\lambda)\begin{vmatrix} -3-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} + 2\begin{vmatrix} 2 & -2 \\ -2 & 1-\lambda \end{vmatrix} + 6\begin{vmatrix} 2 & -3-\lambda \\ -2 & -2 \end{vmatrix} = 0$

$\Rightarrow -\lambda^3 - 9\lambda^2 - 23\lambda - 15 = 0$

$\Rightarrow \lambda_1 = -1,\ \lambda_2 = -5,\ \lambda_3 = -3$

$\because \mathrm{Re}(\lambda_i) < 0$, <u>SYSTEM is ASYMPTOTIC STABLE</u> ⟶ <u>SYSTEM is THEREFORE ALSO LYAPUNOV STABLE</u>

2.) Let: $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \Bigg|\ A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix},\ C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

Let: $\begin{cases} \hat{A} = M^{-1}AM,\ \hat{B} = M^{-1}B, \\ \hat{C} = CM,\ \hat{D} = D \end{cases}$

Let: $M = M_c$ is
- first $n_c = \text{rank}(P_c)$ columns are lin. indep columns of P_c
- last $(n - n_c)$ columns are arbitrary $\mid \text{rank}(M) = n$

$\Big|\ P_c = [A_c^0 B_c \,|\, A_c B_c \,|\, \cdots A_c^{n-1} B_c]$

Let: $P_c = [B \,|\, AB \,|\, A^2B]$

$\because A^2 = AA = \begin{bmatrix} -1 & 3 & -2 \\ -2 & 4 & -2 \\ 1 & -3 & 2 \end{bmatrix},\ P_c = \left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 & 1 \end{array}\right],\ \because \text{rank}(P_c) = 2,\quad M = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 1 & 2 & 2 \end{bmatrix}\ \therefore M^{-1} = \frac{1}{|M|}\text{Adj}(M) = \frac{1}{\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} \begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \end{bmatrix}$

$\Rightarrow M^{-1} = \frac{-1}{3} \begin{bmatrix} -4 & 2 & -1 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

$\therefore \hat{A} = \frac{-1}{3} \begin{bmatrix} -1 & 2 & -1 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix},\ \hat{B} = M^{-1}B = \frac{-1}{3} \begin{bmatrix} -4 & 2 & -1 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\ \hat{C} = CM = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 6 \end{bmatrix}$

Let: $\hat{A} = \left[\begin{array}{c|c} A_c & \overset{\text{also a matrix}}{A_{12}} \\ \hline \emptyset & A_{\bar{c}} \end{array}\right],\ \hat{B} = \begin{bmatrix} B_c \\ \emptyset_{n-n_c} \end{bmatrix},\ \hat{C} = [C_c \,|\, C_{\bar{c}}]\ \Big|\ A_c \in \mathbb{R}^{n_c \times n_c},\ A_{\bar{c}} \in \mathbb{R}^{(n-n_c)\times(n-n_c)}$, etc.

$\therefore A_c = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix},\ B_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ C_c = \begin{bmatrix} 3 & 3 \end{bmatrix}$

CONTROLLABLE FORM: $\begin{cases} \dot{\hat{x}}_c = A_c \hat{x}_c + B_c u, \\ \hat{y} = C_c \hat{x}_c + Du \end{cases} \Bigg|\ A_c = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix},\ B_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ C_c = \begin{bmatrix} 3 & 3 \end{bmatrix}$

└ check observability: $Q_c = \begin{bmatrix} C_c \\ C_c A_c \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}$

$\because n_{c\bar{o}} = \text{rank}(Q_c) = 1 < n_c$, controllable form is NOT observable

└ check stabilizability: The controllable reduced form by definition has no uncontrollable modes, therefore all (zero) of its uncontrollable modes (which there are none) are Lyapunov stable, therefore the controllable canonical form is stabilizable

└ check detectability: First, find the unobservable modes (given by $A_{c\bar{o}}$)

Let: M_{co}^{-1} be first n_{co} rows of Q_c + arbitrary rows so that $\text{rank}(M_{co}^{-1}) = n_c$,

$\Rightarrow M_{co}^{-1} = \begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix},\ \therefore M_{co} = (M_{co}^{-1})^{-1} = \frac{1}{|M_{co}^{-1}|}\text{Adj}(M_{co}^{-1}) = \frac{1}{3}\begin{bmatrix} 2 & -3 \\ -1 & 3 \end{bmatrix}$

Let: $\hat{A}_c = M_{co}^{-1} A_c M_{co} = \frac{1}{3}\begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 & 0 \\ 1 & -3 \end{bmatrix}$

Let: $\hat{A}_c = \left[\begin{array}{c|c} A_{co} & \emptyset \\ \hline A_{21} & A_{c\bar{o}} \end{array}\right]$ $\quad \therefore A_{c\bar{o}} = \begin{bmatrix} -1 \end{bmatrix} \leftarrow$ represents uncontrollable modes of A_c

$\because \lambda_{c\bar{o}_1} = -1,\ \text{Re}(\lambda_{c\bar{o}_1}) = -1 < 0 \rightarrow \lambda_{c\bar{o}_1}$ is asymptotically stable ($\therefore$ also Lyapunov stable),

$\therefore$ All unobservable modes of A_c are Lyapunov stable,

$\therefore$ the controllable canonical form is detectable

Q3.)

let: $z=\begin{bmatrix}\theta\\ \dot{x}\\ \dot{y}\\ \dot{\theta}\end{bmatrix},\ U=\begin{bmatrix}u_1\\ u_2\end{bmatrix},\quad \begin{cases} m\ddot{x}=-u_1\sin(\theta)+\varepsilon u_2\cos(\theta),\\ m\ddot{y}=u_1\cos(\theta)+\varepsilon u_2\sin(\theta)-mg,\\ J\ddot{\theta}=u_2,\end{cases}$

$$\therefore \dot{z}=\begin{bmatrix}\dot{\theta}\\ \ddot{x}\\ \ddot{y}\\ \ddot{\theta}\end{bmatrix}=f(z,u)=\begin{bmatrix}\dot{\theta}\\ -u_1/m\,\sin(\theta)+\frac{\varepsilon u_2}{m}\cos(\theta)\\ u_1/m\,\cos(\theta)+\frac{\varepsilon u_2}{m}\sin(\theta)-g\\ u_2/J\end{bmatrix}$$

LINEARIZE ABOUT: $\tilde{x}(t)=\tilde{y}(t)=\tilde{\theta}(t)=0,\ \tilde{u}_1=mg,\ \tilde{u}_2=0,\ \therefore \dot{\tilde{\theta}}(t)=0$

let: $\tilde{z}=\begin{bmatrix}\tilde{\theta}\\ \tilde{x}\\ \tilde{y}\\ \dot{\tilde{\theta}}\end{bmatrix},\ \tilde{U}=\begin{bmatrix}\tilde{u}_1\\ \tilde{u}_2\end{bmatrix},\ \delta_z=\begin{bmatrix}\delta_\theta\\ \delta_x\\ \delta_y\\ \delta_{\dot{\theta}}\end{bmatrix}=\begin{bmatrix}\theta-\tilde{\theta}\\ x-\tilde{x}\\ y-\tilde{y}\\ \dot{\theta}-\dot{\tilde{\theta}}\end{bmatrix}=\begin{bmatrix}\theta\\ x\\ y\\ \dot{\theta}\end{bmatrix},\ \delta_U=\begin{bmatrix}\delta_{u_1}\\ \delta_{u_2}\end{bmatrix}=\begin{bmatrix}u_1-\tilde{u}_1\\ u_2-\tilde{u}_2\end{bmatrix}=\begin{bmatrix}u_1-mg\\ u_2\end{bmatrix}$

$$\text{let: } A=\left[\frac{\partial f}{\partial z}\right]_{(\tilde{z},\tilde{U})}=\begin{bmatrix}\frac{\partial f_1}{\partial\theta}&\frac{\partial f_1}{\partial x}&\frac{\partial f_1}{\partial y}&\frac{\partial f_1}{\partial\dot{\theta}}\\ \frac{\partial f_2}{\partial\theta}&\frac{\partial f_2}{\partial x}&\frac{\partial f_2}{\partial y}&\frac{\partial f_2}{\partial\dot{\theta}}\\ \frac{\partial f_3}{\partial\theta}&\frac{\partial f_3}{\partial x}&\frac{\partial f_3}{\partial y}&\frac{\partial f_3}{\partial\dot{\theta}}\\ \frac{\partial f_4}{\partial\theta}&\frac{\partial f_4}{\partial x}&\frac{\partial f_4}{\partial y}&\frac{\partial f_4}{\partial\dot{\theta}}\end{bmatrix}_{(\tilde{z},\tilde{U})}=\begin{bmatrix}0&0&0&1\\ -u_1/m\,\cos(\theta)-\frac{\varepsilon u_2}{m}\sin(\theta)&0&0&0\\ -u_1/m\,\sin(\theta)+\frac{\varepsilon u_2}{m}\cos(\theta)&0&0&0\\ 0&0&0&0\end{bmatrix}_{(\tilde{z},\tilde{U})}=\begin{bmatrix}0&0&0&1\\ -g&0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{bmatrix},$$

$$\text{let: } B=\left[\frac{\partial f}{\partial u}\right]_{(\tilde{z},\tilde{U})}=\begin{bmatrix}\frac{\partial f_1}{\partial u_1}&\frac{\partial f_1}{\partial u_2}\\ \frac{\partial f_2}{\partial u_1}&\frac{\partial f_2}{\partial u_2}\\ \frac{\partial f_3}{\partial u_1}&\frac{\partial f_3}{\partial u_2}\\ \frac{\partial f_4}{\partial u_1}&\frac{\partial f_4}{\partial u_2}\end{bmatrix}_{(\tilde{z},\tilde{U})}=\begin{bmatrix}0&0\\ \frac{-1}{m}\sin(\theta)&\frac{\varepsilon}{m}\cos(\theta)\\ \frac{1}{m}\cos(\theta)&\frac{\varepsilon}{m}\sin(\theta)\\ 0&\frac{1}{J}\end{bmatrix}_{(\tilde{z},\tilde{U})}=\begin{bmatrix}0&0\\ 0&\varepsilon/m\\ 1/m&0\\ 0&1/J\end{bmatrix}$$

$$\therefore \text{LINEARIZED (LTI) MODEL: } \dot{\delta}_z=A\delta_z+B\delta_U \Rightarrow \begin{bmatrix}\dot{\delta}_\theta\\ \dot{\delta}_x\\ \dot{\delta}_y\\ \dot{\delta}_{\dot{\theta}}\end{bmatrix}=\begin{bmatrix}0&0&0&1\\ -g&0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{bmatrix}\begin{bmatrix}\theta\\ x\\ y\\ \dot{\theta}\end{bmatrix}+\begin{bmatrix}0&0\\ 0&\varepsilon/m\\ 1/m&0\\ 0&1/J\end{bmatrix}\begin{bmatrix}u_1-mg\\ u_2\end{bmatrix}$$

DETERMINE LTI MODEL STABILITY ABOUT $\{\tilde{z},\tilde{U}\}$:

$$\text{let: } |A-\lambda I|=0 \Rightarrow -\lambda\begin{vmatrix}-\lambda&0&0\\ 0&-\lambda&0\\ 0&0&-\lambda\end{vmatrix}-0+0-1\begin{vmatrix}-g&-\lambda&0\\ 0&0&-\lambda\\ 0&0&0\end{vmatrix}=-\lambda(-\lambda^3)-(0-0+0)=\lambda^4$$

$\therefore \lambda_1=0,\ m_1=4 \leftarrow$ algebraic multiplicity

FIND REAL EIGENVECTORS:

$$\text{let: } 0=(A-\lambda_1 I)v_1=\begin{bmatrix}-0&0&0&1\\ -g&-0&0&0\\ 0&0&-0&0\\ 0&0&0&-0\end{bmatrix}\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}\Rightarrow\begin{cases}d=0,\\ ga=0,\\ 0b=0,\\ 0c=0\end{cases}\Rightarrow \text{any } v_1=\begin{bmatrix}0\\ b\\ c\\ 0\end{bmatrix} \text{ for any } \{b,c\} \text{ is a valid eigenvector}$$

$\therefore$ linearly independent eigenvectors of λ_1 will be, for example: $v_1=\begin{bmatrix}0\\ 1\\ 0\\ 0\end{bmatrix},\ v_2=\begin{bmatrix}0\\ 0\\ 1\\ 0\end{bmatrix}$

$\therefore q_1=2$

$\because Re(\lambda_1)\geq 0$, the model is NOT asymptotically stable

$\because q_1<m_1$, λ_1 is degenerate; so, the model is NOT Lyapunov stable

$\therefore$ the LTI model is NOT stable around the given equilibrium solution

(NOTE: if we were trying to use this LTI model to assess the stability of the original nonlinear model about the given equilibrium, we wouldn't have even considered the degeneracy of λ_1 and, instead, immediately considered it unstable b/c $Re(\lambda_1)=0$ as per Lyapunov's Indirect Method.)

Q4.) let: $\dot{x} = \begin{bmatrix} a & 0 \\ 1 & -1 \end{bmatrix} x$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $V(x) = x_1^2 + x_2^2$

- Clearly, $V(x) > 0 \ \forall x \neq \emptyset$ $\therefore$ $V(x)$ is positive definite

SOLVE CONDITION FOR $\dot{V}(x)$ NEGATIVE DEFINITENESS (CRITERION FOR ASYMPTOTIC STABILITY):

let: $\emptyset > \dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$

$\Rightarrow \emptyset > 2x_1(ax_1) + 2x_2(x_1 - x_2) = 2ax_1^2 + 2x_1x_2 - 2x_2^2$

$\Rightarrow \emptyset > 2a\gamma^2 + 2\gamma - 2 \quad | \, \gamma = \frac{x_1}{x_2}$

$\Rightarrow 1 - \gamma > a\gamma^2$

$\Rightarrow a < \frac{1}{\gamma^2} - \frac{1}{\gamma} = \left(\frac{x_2}{x_1}\right)^2 - \frac{x_2}{x_1} = \frac{x_2^2 - x_1x_2}{x_1^2}$

$\therefore$ $\dot{V}(x)$ is negative definite (and therefore the model is asymptotically stable about the origin) when $a < \frac{x_2^2 - x_1x_2}{x_1^2} \ \forall \{x_1, x_2\} \neq \emptyset$.

Q5.) let: $\begin{cases} \dot{x}_1 = x_2 - x_1 x_2^2, \\ \dot{x}_2 = -x_1^3 \end{cases} \Rightarrow \dot{x} = f(x) = \begin{bmatrix} x_2 - x_1 x_2^2 \\ -x_1^3 \end{bmatrix} \Big| \; x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

a.) FIND EQUILIBRIUM:

let: $\dot{\bar{x}} = f(\bar{x}) = 0 \Rightarrow \begin{cases} \bar{x}_2 - \bar{x}_1 \bar{x}_2^2 = 0, \\ -\bar{x}_1^3 = 0 \end{cases} \Rightarrow \begin{cases} \bar{x}_1 = 0, \\ \bar{x}_2 = 0 \end{cases} \Rightarrow \bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

let: $A = \left[\frac{\partial f}{\partial x}\right]_{\bar{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{\bar{x}} = \begin{bmatrix} -x_2^2 & 1-2x_1 \\ -3x_1^2 & 0 \end{bmatrix}_{\bar{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\therefore$ LINEARIZED SYSTEM: $\dot{\delta}_x = A \delta_x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \delta_x \; \Big| \; \delta_x = x - \bar{x} = x$

$\because eig(A) \rightarrow \lambda_1 = 0, \; m_1 = 2$

$\because Re(\lambda_1) = 0$, The system should not be considered stable according to Lyapunov's indirect method.

b.) let: $V(x) = x_1^4 + 2x_2^2$

- Clearly, $V(x) > 0 \; \forall x \neq 0 \; \therefore V(x)$ is positive definite

CHECK $\dot{V}(x)$ NEGATIVE DEFINITENESS:

let: $0 \leq \dot{V}(x) = 4x_1^3 \dot{x}_1 + 4x_2 \dot{x}_2 = 4x_1^3(x_2 - x_1 x_2^2) + 4x_2(-x_1^3)$

$= 4x_1^3 x_2 - 4x_1^4 x_2^2 - 4x_1^3 x_2$

$0 \leq -4x_1^4 x_2^2$

$\therefore$ it is clear that $\dot{V}(x) = -4x_1^4 x_2^2 < 0 \;\; \forall x \neq 0$

$\therefore V(x)$ is positive definite and $\dot{V}(x)$ is negative definite.

$\therefore$ the model is asymptotically stable about the origin according to Lyapunov's direct method.

c.-d.) [SEE NEXT PAGE]

q5

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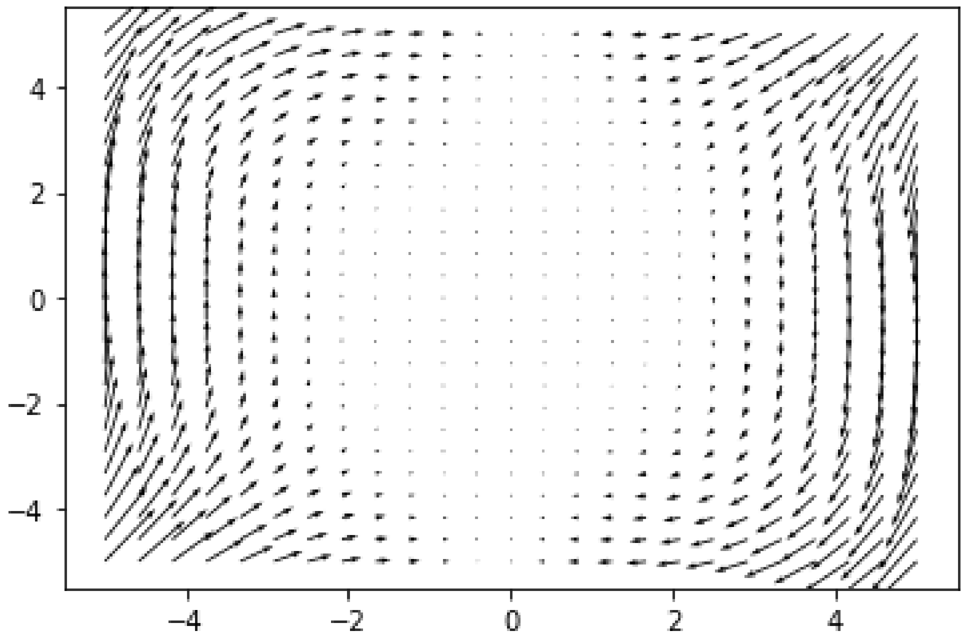
[63]:
```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D
from matplotlib import cm
```

[50]:
```
def sys(X):
    x1, x2 = X
    return [x2 - x1*np.power(x2,2), -np.power(x1,3)]
def linsys(X):
    x1, x2 = X
    return [-x2, 0]
```

[51]:
```
def phase_portrait2D(f, xmin=-5, xmax=5, ymin=-5, ymax=5, num=25):
    x1 = np.linspace(xmin,xmax, num=num)
    x2 = np.linspace(ymin,ymax, num=num)
    x1, x2 = np.meshgrid(x1,x2)
    u, v = np.zeros(x1.shape), np.zeros(x2.shape)
    Nr, Nc = x1.shape
    for r in range(Nr):
        for c in range(Nc):
            x = x1[r,c]
            y = x2[r,c]
            xd = f([x,y])
            u[r,c] = xd[0]
            v[r,c] = xd[1]
    Q = plt.quiver(x1,x2, u,v)
```

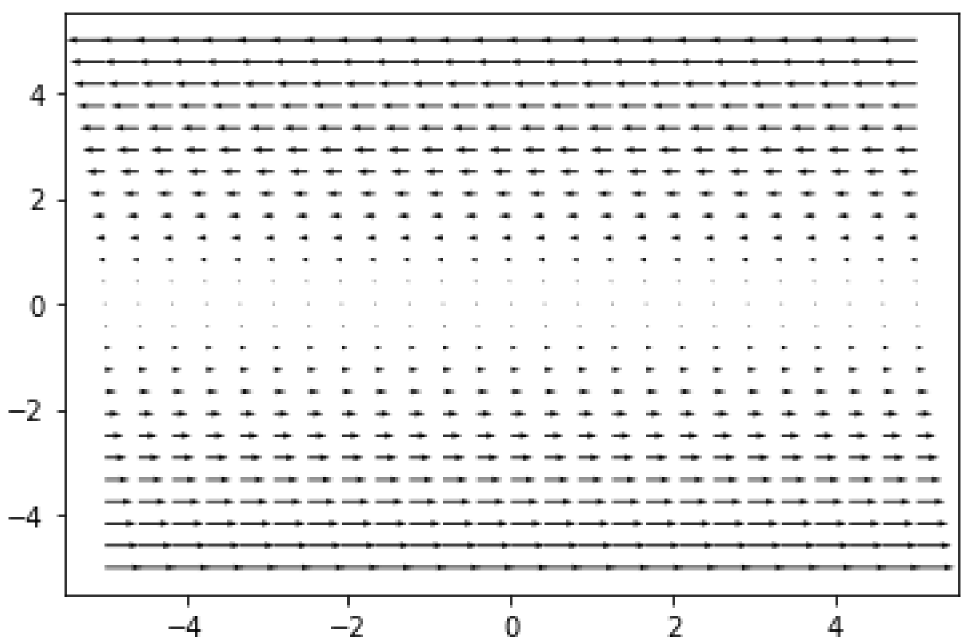
0.1 c.i. Original System Phase Portrait Plot

[52]:
```
phase_portrait2D(sys)
```



0.2 c.ii. Linearized System Phase Portrait Plot

[111]:
```
phase_portrait2D(linsys)
```



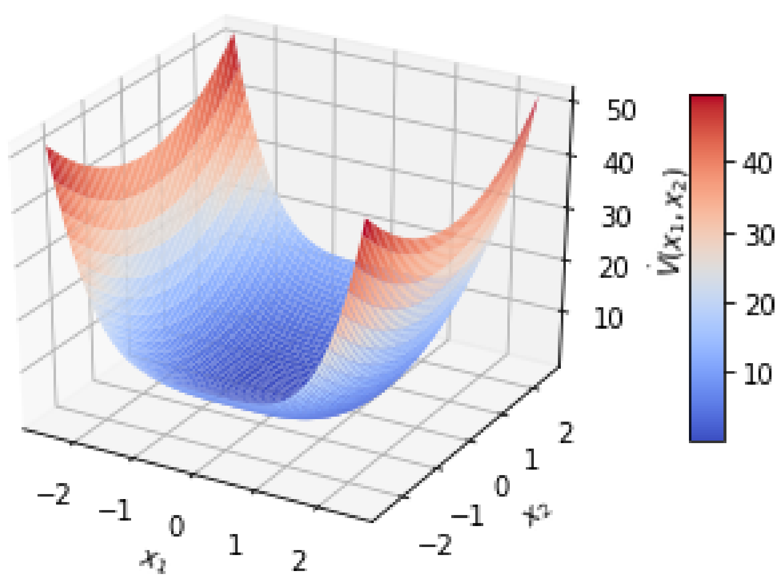
0.3 d. 3D Plot of Lyapunov Function Temporal Derivative

[123]:
```
fig = plt.figure()
ax = fig.gca(projection='3d')

x1 = np.linspace(-2.5,2.5, num=100)
x2 = np.linspace(-2.5,2.5, num=100)
x1, x2 = np.meshgrid(x1,x2)
Vd = x1**4 + 2*x2**2

surf = ax.plot_surface(x1,x2,Vd, cmap=cm.coolwarm, linewidth=2,␣
↪antialiased=True)
ax.set_xlabel(r'$x_1$')
ax.set_ylabel(r'$x_2$')
ax.set_zlabel(r'$\dot{V}(x_1,x_2)$')

# Add a color bar which maps values to colors.
fig.colorbar(surf, shrink=0.6, aspect=10)
plt.show()
```



[ ]:

Q6.)

a.) let: $\begin{cases} X_{k+1} = AX_k + Bu_k, \\ Y_k = CX_k + Du_k \end{cases} \quad \Big| \; A = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}, B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, C = [5 \; 5], D = \emptyset$

let: $G(z) = C(zI - A)^{-1}B + D$

$$\Rightarrow G(z) = \begin{bmatrix} z-1 & 0 \\ \frac{1}{2} & z-\frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= [5 \; 5] \left(\frac{2}{(2z-1)(z-1)} \right) \begin{bmatrix} z-\frac{1}{2} & 0 \\ -\frac{1}{2} & z-1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= [5 \; 5] \left(\frac{2}{(2z-1)(z-1)} \right) \begin{bmatrix} z-\frac{1}{2} \\ \frac{1}{2}-z \end{bmatrix}$$

$$= \left(\frac{2}{(2z-1)(z-1)} \right) [\emptyset]$$

$\Rightarrow G(z) = \emptyset$

$\because G(z) = \frac{Y(z)}{U(z)}$, the output $(Y(z))$ will be bounded for any input $(U(z))$, including finite bounded inputs, because the output will always be $\emptyset$.

b.) let: $\begin{cases} \dot{X} = AX + BU, \\ Y = CX + DU \end{cases} \quad \Big| \; A = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}, D = \emptyset$

let: $G(s) = C(sI - A)^{-1}B + D$

$$= \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} s+7 & 2 & -6 \\ -2 & s+3 & 2 \\ 2 & 2 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$\because (sI-A)^{-1} = \frac{1}{|sI-A|} \mathrm{Adj}(sI-A)$, $|sI-A|$ will contain set of all possible poles of $G(s)$ (though, some may be canceled in $G(s)$)

$\therefore$ if the real part of all roots of $|sI-A|$ are <0, the real part of all poles will necessarily be $<\emptyset$.

$\because |sI-A| = 2\begin{vmatrix} 2 & -6 \\ s+3 & 2 \end{vmatrix} - 2\begin{vmatrix} s+7 & -6 \\ -2 & 2 \end{vmatrix} + (s-1)\begin{vmatrix} s+7 & 2 \\ -2 & s+3 \end{vmatrix}$

$$= 2(4+6s+18) - 2(2s+14-12) + (s-1)((s+7)(s+3)+4)$$

$$= 12s+44-4s-4+s^3+9s^2+15s-25$$

$$= s^3+9s^2+23s+15$$

$$= (s+1)(s+3)(s+5) \quad \Rightarrow \quad s_0 = -1, \; s_1 = -3, \; s_2 = -5$$

$\because$ Re of all roots of $|sI-A|$ are <0, Re of all poles of $G(s)$ are <0,

$\therefore$ System is BIBO stable

Q7.)

a.)

$$\dot{X}=\begin{bmatrix}\dot{T}_c\\ \dot{T}_H\end{bmatrix}=AX+BU, \quad Y=\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}T_c\\ T_H\end{bmatrix}=CX+DU$$

$$A=\begin{bmatrix}\frac{-f_c-B}{V_c} & B/V_c\\ \frac{-B}{V_H} & \frac{B-f_H}{V_H}\end{bmatrix}, B=\begin{bmatrix}f_c/V_c & \emptyset\\ \emptyset & f_H/V_H\end{bmatrix}, \quad X=\begin{bmatrix}T_c\\ T_H\end{bmatrix}, \quad U=\begin{bmatrix}u_1\\ u_2\end{bmatrix}=\begin{bmatrix}T_{Ci}\\ T_{Hi}\end{bmatrix},$$

$$C=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, D=\emptyset, \quad f_c=f_H=0.1\ m^3/min, \quad B=0.2\ m^3/min, \quad V_H=V_c=1m^3$$

b.) let $U=\emptyset$

$$\therefore Y=X=e^{A(t-t_0)}X_0+\int_{t_0}^{t}e^{A(t-\tau)}BU\,d\tau=e^{A(t-t_0)}X_0$$

NOTE: $A=\begin{bmatrix}\frac{-f_c-B}{V_c} & B/V_c\\ \frac{-B}{V} & \frac{B-f_H}{V_H}\end{bmatrix}=\begin{bmatrix}-3/10 & 2/10\\ -2/10 & 1/10\end{bmatrix}, \quad B=\begin{bmatrix}f_c/V_c & \emptyset\\ \emptyset & f_H/V\end{bmatrix}=\begin{bmatrix}1/10 & 0\\ 0 & 1/10\end{bmatrix}$

Let: $A=M^{-1}JM$, $\because$ eig$(A)\rightarrow\lambda_1=-1/10$, $m_1=2$

REAL EIGENVECTOR(S):

Let: $\emptyset=(A-\lambda_1I)v_1=\begin{bmatrix}\frac{-2}{10} & \frac{2}{10}\\ \frac{-2}{10} & \frac{2}{10}\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}\Rightarrow\begin{cases}a=b,\\ a=b\end{cases}\Rightarrow v_1=\begin{bmatrix}1\\ 1\end{bmatrix}\therefore q_1=1\Rightarrow J$ is Type II_2 Jordan Matrix, $J=\begin{bmatrix}\lambda_1 & 1\\ 0 & \lambda_1\end{bmatrix}$

$(m_1-q_1)=1$ GENERATED E-VECTORS:

Let: $v_1=(A-\lambda_2I)v_2=\begin{bmatrix}\frac{-2}{10} & \frac{2}{10}\\ \frac{-2}{10} & \frac{2}{10}\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}\Rightarrow\begin{cases}-a+b=5,\\ -a+b=5\end{cases}\Rightarrow v_2=\begin{bmatrix}-5\\ 0\end{bmatrix}$

$$\therefore M=[v_1|v_2]=\begin{bmatrix}1 & -5\\ 1 & 0\end{bmatrix}, \quad M^{-1}=\frac{1}{5}\begin{bmatrix}0 & 5\\ -1 & 1\end{bmatrix}, \quad J=\frac{1}{10}\begin{bmatrix}-1 & 10\\ 0 & -1\end{bmatrix}$$

$$\because e^{At}=Me^{Jt}M^{-1}=\frac{1}{5}\begin{bmatrix}1 & -5\\ 1 & 0\end{bmatrix}\begin{bmatrix}e^{-t/10} & te^{-t/10}\\ 0 & e^{-t/10}\end{bmatrix}\begin{bmatrix}0 & 5\\ -1 & 1\end{bmatrix}$$

$$\Rightarrow e^{At}=\frac{1}{5}\begin{bmatrix}(5-t)e^{-t/10} & te^{-t/10}\\ -te^{-t/10} & (t+5)e^{-t/10}\end{bmatrix}$$

$$\therefore Y=\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix}=\frac{1}{5}\begin{bmatrix}(5-t)e^{(t_0-t)/10} & (t-t_0)e^{(t_0-t)/10}\\ (t_0-t)e^{(t_0-t)/10} & (t-t_0+5)e^{(t_0-t)/10}\end{bmatrix}X_0$$

c.)

Let: $G(s)=C(sI-A)^{-1}B+D$

$$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}s+\frac{3}{10} & 2/10\\ -2/10 & s-\frac{1}{10}\end{bmatrix}^{-1}\begin{bmatrix}1/10 & 0\\ 0 & 1/10\end{bmatrix}$$

$\because (sI-A)^{-1}=\frac{1}{|sI-A|}Adj(sI-A)$, $|sI-A|$ will contain set of all possible poles of $G(s)$ (though, some may be canceled in $G(s)$)

$\therefore$ if the real part of all roots of $|sI-A|$ are <0, the real part of all poles will necessarily be <0.

$\because |sI-A|=(s+\frac{3}{10})(s-\frac{1}{10})-(\frac{-2}{10})(\frac{2}{10})=s^2+\frac{1}{5}s+\frac{1}{100}=\frac{1}{100}(s+\frac{1}{10})^2$, $s_1=s_2=\frac{-1}{10}$ $\leftarrow$ double root of $|sI-A|$

$\because$ Re of all roots of $|sI-A|$ are <0, Re of all poles of $G(s)$ are <0,

$\therefore$ System is BIBO stable