24677 — P4

Connor W. Colombo (cwcolomb)

November 24th 2020

4 Project 4

4.1 Exercise 1

4.1.1 Background

For an environment containing n fixed landmarks with unknown locations given by $\mathbf{m}^j = \begin{bmatrix} m_x^j & m_y^j \end{bmatrix}^T$, $j \in [1, n]$, let the vector of desired unknown states at discrete timestep t containing the global vehicle location and the global landmark locations be:

$$\mathbf{x}_t = \begin{bmatrix} X_t \\ Y_t \\ \psi_t \\ m_x^1 \\ m_y^1 \\ \vdots \\ m_x^n \\ m_y^n \end{bmatrix}$$

and the measurement vector at t be given by:

$$\mathbf{y}_t = \begin{bmatrix} \left\|\mathbf{m}^1 - \mathbf{p}_t\right\| \\ \vdots \\ \left\|\mathbf{m}^n - \mathbf{p}_t\right\| \\ \Delta\psi^1 \\ \vdots \\ \Delta\psi^n \end{bmatrix} + \mathbf{v}_t$$

where $\left\|\mathbf{m}^j - \mathbf{p}_t\right\|$ is the distance to landmark $\mathbf{m}^j$ and $\Delta\psi^j = \text{atan2}\left(m_y^j - Y_t,\ m_x^j - X_t\right) - \psi_t$ is the bearing to landmark $\mathbf{m}^j$, $\mathbf{v}_t$ is the observation noise vector, and $\mathbf{p}_t = \begin{bmatrix} X_t & Y_t \end{bmatrix}^T$ is the position of the vehicle.

4.1.2 Linearized Kalman State Update $\mathbf{F}_t$

As a result, the update for unknown states is

$$\mathbf{x}_{t+1} = \begin{bmatrix} X_{t+1} \\ Y_{t+1} \\ \psi_{t+1} \\ m_x^1 \\ m_y^1 \\ \vdots \\ m_x^n \\ m_y^n \end{bmatrix} = \begin{bmatrix} X_t \\ Y_t \\ \psi_t \\ m_x^1 \\ m_y^1 \\ \vdots \\ m_x^n \\ m_y^n \end{bmatrix} + \dot{\mathbf{x}}_t \delta t + \omega_t$$

where ω_t is the process noise vector, and $\dot{\mathbf{x}}_t = \begin{bmatrix} \dot{X}_t & \dot{Y}_t & \dot{\psi}_t & \mathbf{0}_{2n}^T \end{bmatrix}^T$ is the temporal derivative of the states with $\mathbf{0}_{2n}$ being a vector of $2n$ zeros accounting for the fact that the m_x^j and m_y^j locations of the n fixed landmarks don't change over time. Thus it follows that:

$$\mathbf{x}_{t+1} = \begin{bmatrix} X_t \\ Y_t \\ \psi_t \\ m_x^1 \\ m_y^1 \\ \vdots \\ m_x^n \\ m_y^n \end{bmatrix} + \begin{bmatrix} \dot{x}_{t+1}\cos(\psi_t) - \dot{y}_{t+1}\sin(\psi_t) \\ \dot{x}_{t+1}\sin(\psi_t) + \dot{y}_{t+1}\cos(\psi_t) \\ \dot{\psi}_t \\ \mathbf{0}_{2n} \end{bmatrix} \delta t + \omega_t$$

Writing this as $\mathbf{x}_{t+1} = f(\mathbf{x}_t, \mathbf{u}_t) + \omega_t$, where $\mathbf{u}_{t+1} = \begin{bmatrix} \dot{x}_{t+1} & \dot{y}_{t+1} & \dot{\psi}_{t+1} \end{bmatrix}^T$ is the Kalman input vector of measurable states, allows the linearization matrix $\mathbf{F}_t$ of $f(\mathbf{x}_t, \mathbf{u}_t)$ used in the Extended Kalman Filter update to be found to be:

$$f(\mathbf{x}_t, \mathbf{u}_t) = \begin{bmatrix} X_t + (\dot{x}_{t+1}\cos(\psi_t) - \dot{y}_{t+1}\sin(\psi_t))\delta t \\ Y_t + (\dot{x}_{t+1}\sin(\psi_t) + \dot{y}_{t+1}\cos(\psi_t))\delta t \\ \psi_t + \dot{\psi}_t \delta t \\ m_x^1 \\ m_y^1 \\ \vdots \\ m_x^n \\ m_y^n \end{bmatrix}$$

$$\therefore \mathbf{F}_t = \left[\frac{\partial f}{\partial \mathbf{x}}\right]\Bigg|_{\mathbf{x}_t, \mathbf{u}_{t+1}} = \begin{bmatrix} \left[\frac{\partial f_1}{\partial X}\right] & \left[\frac{\partial f_1}{\partial Y}\right] & \left[\frac{\partial f_1}{\partial \psi}\right] & \mathbf{0}_{2n}^T \\ \left[\frac{\partial f_2}{\partial X}\right] & \left[\frac{\partial f_2}{\partial Y}\right] & \left[\frac{\partial f_2}{\partial \psi}\right] & \mathbf{0}_{2n}^T \\ \left[\frac{\partial f_3}{\partial X}\right] & \left[\frac{\partial f_3}{\partial Y}\right] & \left[\frac{\partial f_3}{\partial \psi}\right] & \mathbf{0}_{2n}^T \\ & \mathbf{0}_{2n,3} & & \mathbf{I}_{2n} \end{bmatrix}_{\mathbf{x}_t, \mathbf{u}_{t+1}}$$

$$\rightarrow \mathbf{F}_t = \left[\frac{\partial f}{\partial \mathbf{x}}\right]\Bigg|_{\mathbf{x}_t, \mathbf{u}_{t+1}} = \begin{bmatrix} 1 & 0 & (-\dot{x}_{t+1}\sin(\psi_t) - \dot{y}_{t+1}\cos(\psi_t))\delta t & \mathbf{0}_{2n}^T \\ 0 & 1 & (\dot{x}_{t+1}\cos(\psi_t) - \dot{y}_{t+1}\sin(\psi_t))\delta t & \mathbf{0}_{2n}^T \\ 0 & 0 & 1 & \mathbf{0}_{2n}^T \\ & \mathbf{0}_{2n\times 3} & & \mathbf{I}_{2n} \end{bmatrix}$$

where $\mathbf{0}_{2n\times 3}$ is an $2n \times 3$ matrix of zeros and $\mathbf{I}_{2n}$ is a $2n \times 2n$ identity matrix.

4.1.3 Linearized Kalman Measurement Update $\mathbf{H}_t$

To find $\mathbf{H}_t$, the measurement vector must be written in terms of $\mathbf{x}_t$ as follows:

$$\text{let: } \mathbf{y}_t = h(\mathbf{x}_t) + \mathbf{v}_t$$

$$\text{where } \; h(\mathbf{x}_t) = \begin{bmatrix} \|\mathbf{m}^1 - \mathbf{p}_t\| \\ \vdots \\ \|\mathbf{m}^n - \mathbf{p}_t\| \\ \Delta\psi^1 \\ \vdots \\ \Delta\psi^2 \end{bmatrix} = \begin{bmatrix} \left\| \begin{bmatrix} m_x^1 \\ m_y^1 \end{bmatrix} - \begin{bmatrix} X_t \\ Y_t \end{bmatrix} \right\| \\ \vdots \\ \left\| \begin{bmatrix} m_x^n \\ m_y^n \end{bmatrix} - \begin{bmatrix} X_t \\ Y_t \end{bmatrix} \right\| \\ \text{atan2}\left(m_y^1 - Y_t,\ m_x^1 - X_t\right) - \psi_t \\ \vdots \\ \text{atan2}\left(m_y^n - Y_t,\ m_x^n - X_t\right) - \psi_t \end{bmatrix}$$

$$\therefore \mathbf{H}_t = \left.\left[\frac{\partial h}{\partial \mathbf{x}}\right]\right|_{\mathbf{x}_t} = \begin{bmatrix} \left[\frac{\partial h_1}{\partial X}\right] & \left[\frac{\partial h_1}{\partial Y}\right] & \left[\frac{\partial h_1}{\partial \psi}\right] & \left[\frac{\partial h_1}{\partial m_x^1}\right] & \left[\frac{\partial h_1}{\partial m_y^1}\right] & \cdots & \left[\frac{\partial h_1}{\partial m_x^n}\right] & \left[\frac{\partial h_1}{\partial m_y^n}\right] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \left[\frac{\partial h_n}{\partial X}\right] & \left[\frac{\partial h_n}{\partial Y}\right] & \left[\frac{\partial h_n}{\partial \psi}\right] & \left[\frac{\partial h_n}{\partial m_x^1}\right] & \left[\frac{\partial h_n}{\partial m_y^1}\right] & \cdots & \left[\frac{\partial h_n}{\partial m_x^n}\right] & \left[\frac{\partial h_n}{\partial m_y^n}\right] \\ \left[\frac{\partial h_{n+1}}{\partial X}\right] & \left[\frac{\partial h_{n+1}}{\partial Y}\right] & \left[\frac{\partial h_{n+1}}{\partial \psi}\right] & \left[\frac{\partial h_{n+1}}{\partial m_x^1}\right] & \left[\frac{\partial h_{n+1}}{\partial m_y^1}\right] & \cdots & \left[\frac{\partial h_{n+1}}{\partial m_x^n}\right] & \left[\frac{\partial h_{n+1}}{\partial m_y^n}\right] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \left[\frac{\partial h_{2n}}{\partial X}\right] & \left[\frac{\partial h_{2n}}{\partial Y}\right] & \left[\frac{\partial h_{2n}}{\partial \psi}\right] & \left[\frac{\partial h_{2n}}{\partial m_x^1}\right] & \left[\frac{\partial h_{2n}}{\partial m_y^1}\right] & \cdots & \left[\frac{\partial h_{2n}}{\partial m_x^n}\right] & \left[\frac{\partial h_{2n}}{\partial m_y^n}\right] \end{bmatrix}_{\mathbf{x}_t}$$

$$\rightarrow \mathbf{H}_t = \begin{bmatrix} \mathbf{H}_{A_t} & \mathbf{H}_{B_t} \\ \mathbf{H}_{C_t} & \mathbf{H}_{D_t} \end{bmatrix}$$

$$\text{where } \; \mathbf{H}_{A_t} = \begin{bmatrix} \frac{X_t - m_x^1}{\|\mathbf{m}^1 - \mathbf{p}_t\|} & \frac{Y_t - m_y^1}{\|\mathbf{m}^1 - \mathbf{p}_t\|} & 0 \\ \vdots & \vdots & \vdots \\ \frac{X_t - m_x^n}{\|\mathbf{m}^n - \mathbf{p}_t\|} & \frac{Y_t - m_y^n}{\|\mathbf{m}^n - \mathbf{p}_t\|} & 0 \end{bmatrix},$$

$$\mathbf{H}_{B_t} = \begin{bmatrix} \frac{m_x^1 - X_t}{\|\mathbf{m}^1 - \mathbf{p}_t\|} & \frac{m_y^1 - Y_t}{\|\mathbf{m}^1 - \mathbf{p}_t\|} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{m_x^n - X_t}{\|\mathbf{m}^n - \mathbf{p}_t\|} & \frac{m_y^n - Y_t}{\|\mathbf{m}^n - \mathbf{p}_t\|} \end{bmatrix},$$

$$\mathbf{H}_{C_t} = \begin{bmatrix} \frac{m_y^1 - Y_t}{(m_x^1 - X_t)^2 + \left(m_y^1 - Y_t\right)^2} & \frac{X_t - m_x^1}{(m_x^1 - X_t)^2 + \left(m_y^1 - Y_t\right)^2} & -1 \\ \vdots & \vdots & \vdots \\ \frac{m_y^n - Y_t}{(m_x^n - X_t)^2 + \left(m_y^n - Y_t\right)^2} & \frac{X_t - m_x^n}{(m_x^n - X_t)^2 + \left(m_y^n - Y_t\right)^2} & -1 \end{bmatrix},$$

$$\mathbf{H}_{D_t} = \begin{bmatrix} \frac{Y_t - m_y^1}{(m_x^1 - X_t)^2 + \left(m_y^1 - Y_t\right)^2} & \frac{m_x^1 - X_t}{(m_x^1 - X_t)^2 + \left(m_y^1 - Y_t\right)^2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{Y_t - m_y^n}{(m_x^n - X_t)^2 + \left(m_y^n - Y_t\right)^2} & \frac{m_x^n - X_t}{(m_x^n - X_t)^2 + \left(m_y^n - Y_t\right)^2} \end{bmatrix}$$