Powers, Roots, and Exponents

Why This Matters

Powers and exponents represent **repeated multiplication**, just like multiplication represents repeated addition. They're everywhere:

- Compound interest (money grows exponentially)
- Big-O notation (algorithm complexity: $O(n^2)$, $O(2^n)$)
- Data storage (1 KB = 2^{10} bytes)
- Squares and cubes (area, volume)
- Scientific notation (3×10^8 m/s)

Understanding exponents unlocks understanding of growth, scaling, and geometric relationships.

The Big Picture: Repeated Operations

```
Addition (repeated counting):
3 + 3 + 3 + 3 = 4 × 3 = 12

Multiplication (repeated addition):
3 × 3 × 3 × 3 = 3⁴ = 81

Exponentiation (repeated multiplication):
Power tower: 3^(3^3) = 3^27 = ...huge
```

Each operation is "one level up" from the previous.

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1. Exponents: Repeated Multiplication

What They Are

Exponent notation: b^n

```
b^n = b × b × b × ... × b  (n times)
 ↑   ↑
base exponent
```

Examples:

```
2³ = 2 × 2 × 2 = 8
5² = 5 × 5 = 25
10⁴ = 10 × 10 × 10 × 10 = 10,000
```

Reading Exponents

- **2^3**: "two to the third power" or "two cubed"
- **5^2**: "five to the second power" or "five squared"
- **10^4**: "ten to the fourth power"

Why They Exist

Problem: Writing 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 is tedious.

Solution: 2^8 (much more compact)

Exponents are **shorthand for repetition**.

Visual: Geometric Meaning

Squaring (x^2)

Area of a square:

```
Side length: 3

┌───┬───┬───┐
│ 1 │ 2 │ 3 │
├───┼───┼───┤
│ 4 │ 5 │ 6 │
├───┼───┼───┤
│ 7 │ 8 │ 9 │
└───┴───┴───┘

Area = 3² = 9 square units
```

That's why we call it "squared"—it's literally a square.

Cubing (x^3)

Volume of a cube:

```
Side length: 2

    ┌─────┬─────┐
   /     /     /│
  ┌─────┬─────┐ │
 /     /     /│ │
┌─────┬─────┐ │ │
│     │     │ │/│
│     │     │/│ │
├─────┼─────┤ │/
│     │     │/
└─────┴─────┘

Volume = 2³ = 8 cubic units
```

That's why we call it "cubed"—it's literally a cube.

Programming Analogy

```
// Exponent as loop
function power(base, exponent) {
  let result = 1;
```

```
  for (let i = 0; i < exponent; i++) {
    result *= base;
  }
  return result;
}

power(2, 3);  // 8

// Or use built-in
Math.pow(2, 3);  // 8
2 ** 3;          // 8 (ES7 exponentiation operator)
```

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2. Special Cases and Rules

Zero Exponent

Any number to the power of zero is 1:

```
5⁰ = 1
100⁰ = 1
(-3)⁰ = 1
```

Why? Pattern recognition:

```
2³ = 8
2² = 4   (divided by 2)
2¹ = 2   (divided by 2)
2⁰ = 1   (divided by 2)
```

Each time you decrease the exponent by 1, you divide by the base.

One Exponent

Any number to the power of one is itself:

```
5¹ = 5
100¹ = 100
```

This makes sense: "multiply 5 by itself once" = 5.

Negative Exponents

Negative exponent = reciprocal:

```
2⁻³ = 1 / 2³ = 1/8

x⁻ⁿ = 1 / xⁿ
```

Why? Continue the pattern:

```
2³ = 8
2² = 4    (÷ 2)
```

```
2¹ = 2    (÷ 2)
2⁰ = 1    (÷ 2)
2⁻¹ = 1/2 (÷ 2)
2⁻² = 1/4 (÷ 2)
```

Mental model: Negative exponent "flips" the number:

```
5² = 25
5⁻² = 1/25
```

Fractional Exponents (Preview)

Fractional exponents = roots:

```
x^(1/2) = √x      (square root)
x^(1/3) = ∛x      (cube root)
x^(2/3) = (∛x)²   (cube root, then squared)
```

We'll explore this more in the roots section.

---

3. Laws of Exponents

These rules make working with exponents much easier. They're not arbitrary—they come from the definition.

Law 1: Multiplying Same Base

When multiplying, add the exponents:

```
xᵃ × xᵇ = x^(a+b)
```

Example:

```
2³ × 2² = (2×2×2) × (2×2) = 2⁵ = 32
```

Count the 2's: 3 + 2 = 5

Why it works:

```
x³ × x² = (x×x×x) × (x×x) = x×x×x×x×x = x⁵
```

Law 2: Dividing Same Base

When dividing, subtract the exponents:

```
xᵃ / xᵇ = x^(a-b)
```

Example:

```
2⁵ / 2² = 32 / 4 = 8 = 2³
```

Why it works:

$$x^5 / x^2 = (x \times x \times x \times x \times x) / (x \times x) = x \times x \times x = x^3$$

Cancel out pairs from top and bottom.

Law 3: Power of a Power

When raising a power to a power, multiply the exponents:

$$(x^a)^b = x^{(a \times b)}$$

Example:

$$(2^3)^2 = (8)^2 = 64 = 2^6$$

Why it works:

$$(x^3)^2 = x^3 \times x^3 = x^6$$

Law 4: Power of a Product

Distribute the exponent to each factor:

$$(xy)^n = x^n \times y^n$$

Example:

$$(2 \times 3)^2 = 6^2 = 36$$
$$2^2 \times 3^2 = 4 \times 9 = 36$$

Why it works:

$$(xy)^3 = (xy) \times (xy) \times (xy) = x \times x \times x \times y \times y \times y = x^3 y^3$$

Law 5: Power of a Quotient

Distribute the exponent to numerator and denominator:

$$(x/y)^n = x^n / y^n$$

Example:

$$(2/3)^2 = 4/9$$

Check: $2^2/3^2 = 4/9$ ✓

Summary Table

Rule	Formula	Example
Multiply	$x^a \times x^b = x^{(a+b)}$	$2^3 \times 2^2 = 2^5$
Divide	$x^a / x^b = x^{(a-b)}$	$2^5 / 2^2 = 2^3$

Power of Power	$(x^a)^b = x^{(a×b)}$	$(2^3)^2 = 2^6$
Power of Product	$(xy)^n = x^n y^n$	$(2×3)^2 = 2^2×3^2$
Power of Quotient	$(x/y)^n = x^n/y^n$	$(2/3)^2 = 4/9$

Programming Application

```
// These laws apply in code too
Math.pow(2, 3) * Math.pow(2, 2) === Math.pow(2, 5);  // true

// Bit shifting uses powers of 2
1 << 3  // 2³ = 8
1 << 5  // 2⁵ = 32
```

---

4. Roots: Undoing Powers

What They Are

A **root** is the inverse operation of a power.

Square root (√): What number, when squared, gives you this?

```
√25 = 5     because 5² = 25
√9 = 3      because 3² = 9
√2 ≈ 1.414 because 1.414² ≈ 2
```

Notation:

```
√x    = square root (most common)
∛x    = cube root
∜x    = fourth root
ⁿ√x   = nth root
```

Visual: Square Root as Side Length

```
Area = 25 square units
Side length = √25 = 5

┌─────┐
│  5  │ 5
│     │
└─────┘
   5
```

If you know the area, the square root gives you the side length.

Cube Root

What number, when cubed, gives you this?

```
∛8 = 2     because 2³ = 8
∛27 = 3    because 3³ = 27
∛64 = 4    because 4³ = 64
```

Fractional Exponent Notation

Roots can be written as fractional exponents:

```
√x = x^(1/2)
∛x = x^(1/3)
∜x = x^(1/4)
```

Why? It follows the power rules:

```
(x^(1/2))² = x^(1/2 × 2) = x¹ = x ✓
```

Combining Roots and Powers

```
x^(2/3) means:
1. Take the cube root: ∛x
2. Then square it: (∛x)²

Or equivalently:
1. Square it first: x²
2. Then take cube root: ∛(x²)
```

Example:

```
8^(2/3) = (∛8)² = 2² = 4

Or: 8^(2/3) = ∛(8²) = ∛64 = 4
```

Programming

```
Math.sqrt(25);        // 5 (square root)
Math.pow(25, 0.5);    // 5 (same thing)
Math.cbrt(8);         // 2 (cube root)
Math.pow(8, 1/3);     // 2 (same thing)

// Fourth root
Math.pow(16, 1/4);    // 2 (∜16 = 2)

// General: nth root of x
Math.pow(x, 1/n);
```

---

5. Principal vs Multiple Roots

The Square Root Issue

Every positive number has TWO square roots:

```
√25 = ±5

Because:
5² = 25   ✓
(-5)² = 25 ✓
```

Convention: The radical symbol √ means the **positive root** (principal root).

```
√25 = 5    (principal root)
```

If you want both, write:

```
x² = 25
x = ±5  (plus or minus 5)
```

Odd vs Even Roots

Even roots (√, ∜, etc.):

- Only defined for non-negative numbers (in real numbers)
- $\sqrt{-4}$ is not a real number (involves imaginary numbers)
- Always give positive results (principal root)

Odd roots (∛, ∜∜∜, etc.):

- Defined for all real numbers
- $\sqrt[3]{-8} = -2$ (because $(-2)^3 = -8$)
- Preserve the sign

---

6. Simplifying Radicals

Perfect Squares

Some numbers are **perfect squares**:

```
1² = 1
2² = 4
3² = 9
4² = 16
5² = 25
6² = 36
...
10² = 100
```

These are easy to take the square root of.

Simplifying Non-Perfect Squares

Factor out perfect squares:

```
√12 = √(4 × 3) = √4 × √3 = 2√3
√18 = √(9 × 2) = √9 × √2 = 3√2
√50 = √(25 × 2) = √25 × √2 = 5√2
```

Method:

1. Find the largest perfect square factor
2. Split the radical
3. Simplify

Why it works:

```
√(a × b) = √a × √b
```

Rationalizing the Denominator

Don't leave radicals in the denominator:

```
1/√2  →  multiply top and bottom by √2

1/√2 × √2/√2 = √2/2
```

This is preferred because it's easier to approximate:

```
√2/2 ≈ 1.414/2 ≈ 0.707
```

---

7. Exponential Growth vs Polynomial Growth

Polynomial Growth (Powers)

```
Linear:     y = x     (doubles when x doubles)
Quadratic:  y = x²    (quadruples when x doubles)
Cubic:      y = x³    (8× when x doubles)
```

Graph intuition:

```
x:  1   2   3   4   5
x²: 1   4   9  16  25  (getting steeper)
x³: 1   8  27  64 125  (even steeper)
```

Exponential Growth (Base)

```
y = 2ˣ

x:  1   2   3   4   5
2ˣ: 2   4   8  16  32  (doubling each time)
```

Key difference:

- **Polynomial**: x increases, y increases by power
- **Exponential**: x increases, y multiplies by base

Exponential grows much faster:

```
At x = 10:
x² = 100
2ˣ = 1024

At x = 20:
x² = 400
2ˣ = 1,048,576  (exponential explodes)
```

Big-O Notation

```
O(n)     = linear    (fast)
O(n²)    = quadratic (slower)
O(2ⁿ)    = exponential (very slow)
O(log n) = logarithmic (very fast - next chapter!)
```

Why it matters: Algorithm efficiency

```
// O(n²) - nested loops
for (let i = 0; i < n; i++) {
  for (let j = 0; j < n; j++) {
    // n × n operations
  }
}

// O(2ⁿ) - exponential (bad!)
function fibonacci(n) {
  if (n <= 1) return n;
  return fibonacci(n-1) + fibonacci(n-2);  // doubles work each level
}
```

---

8. Real-World Applications

Data Storage (Powers of 2)

```
1 KB = 2¹⁰ bytes = 1,024 bytes
1 MB = 2²⁰ bytes = 1,048,576 bytes
1 GB = 2³⁰ bytes = 1,073,741,824 bytes
```

Why powers of 2? Binary system (computers use base-2).

Compound Interest (Exponential Growth)

```
A = P(1 + r)ⁿ

P = principal ($1000)
r = interest rate (5% = 0.05)
n = years
```

```
A = final amount

After 10 years:
A = 1000(1.05)¹⁰ ≈ $1,629
```

Half-Life (Exponential Decay)

```
Remaining = Initial × (1/2)^(t / half-life)

If half-life is 5 years and t = 10:
Remaining = Initial × (1/2)² = Initial / 4

After 10 years, only 1/4 remains.
```

Area and Volume

```
Square area: s²
Circle area: πr²
Cube volume: s³
Sphere volume: (4/3)πr³
```

Distance Formula (Pythagorean Theorem)

```
c² = a² + b²
c = √(a² + b²)

Distance between points (x₁,y₁) and (x₂,y₂):
d = √((x₂-x₁)² + (y₂-y₁)²)
```

Scientific Notation

```
Speed of light: 3 × 10⁸ m/s
Electron mass: 9.1 × 10⁻³¹ kg
```

Much easier than writing all the zeros.

Common Mistakes & Misconceptions

❌ "$(x + y)^2 = x^2 + y^2$"

No! You must expand:

```
(x + y)² = (x + y)(x + y) = x² + 2xy + y²
```

❌ "$\sqrt{(x^2 + y^2)} = x + y$"

No! Roots don't distribute over addition:

```
√(3² + 4²) = √(9 + 16) = √25 = 5
```

```
But 3 + 4 = 7 ≠ 5
```

❌ "$x^0 = 0$"

No! $x^0 = 1$ (for $x \neq 0$)

❌ "$\sqrt{4} = \pm 2$"

No! $\sqrt{4} = 2$ (principal root only) The equation $x^2 = 4$ has solutions $x = \pm 2$, but $\sqrt{4}$ means +2.

❌ "$2^3 \times 3^3 = 6^{3}$"

No! Different bases don't combine:

```
2³ × 3³ = 8 × 27 = 216
6³ = 216 ✓ (happens to equal, but not by the rule)

But: 2³ × 3³ = (2×3)³ = 6³ (power of product rule)
```

Tiny Practice

Simplify:

1. $2^3 \times 2^4$
2. $5^6 / 5^2$
3. $(3^2)^3$
4. $(2 \times 5)^3$
5. 10^0
6. 2^{-3}
7. $\sqrt{36}$
8. $\sqrt[3]{27}$
9. $\sqrt{18}$ (simplify)
10. 8^(2/3)

Evaluate:

11. What is the area of a square with side 7?
12. What is the side length of a square with area 64?
13. If $2^x = 32$, what is x?
14. If $x^2 = 49$, what are the possible values of x?

▶ Answers

Summary Cheat Sheet

Exponent Basics

```
xⁿ = x × x × ... × x  (n times)

x⁰ = 1
x¹ = x
```

```
x⁻ⁿ = 1/xⁿ
x^(1/n) = ⁿ√x
```

Exponent Laws

Operation	Rule	Example
Multiply	$x^a \cdot x^b$ = x^(a+b)	$2^3 \cdot 2^2 = 2^5$
Divide	x^a / x^b = x^(a-b)	$2^5 / 2^2 = 2^3$
Power of Power	$(x^a)^b$ = x^(ab)	$(2^3)^2 = 2^6$
Power of Product	$(xy)^n = x^n y^n$	$(2{\cdot}3)^2 = 4{\cdot}9$
Power of Quotient	$(x/y)^n = x^n/y^n$	$(2/3)^2 = 4/9$

Roots

```
√x  = x^(1/2)  (square root)
∛x  = x^(1/3)  (cube root)
ⁿ√x = x^(1/n)  (nth root)

√(x²) = |x|    (absolute value for real numbers)
(√x)² = x      (when x ≥ 0)
```

Growth Comparison

```
Polynomial:   y = xⁿ   (faster as n increases)
Exponential:  y = aˣ   (much faster than polynomial)

O(n)    < O(n²)   < O(n³)   < O(2ⁿ)
linear    quadratic  cubic     exponential
```

Perfect Squares to Memorize

```
1²=1    2²=4    3²=9     4²=16    5²=25
6²=36   7²=49   8²=64    9²=81    10²=100
11²=121  12²=144  13²=169  14²=196  15²=225
```

Next Steps

You now understand powers, roots, and exponents—how repeated multiplication works and how to undo it. This foundation is critical for the next topic.

Next, we'll explore **Logarithms**—the inverse of exponentials, and one of the most powerful tools in mathematics and computer science.

Continue to: 04-logarithms.md