

Limits

Why This Matters

Limits are the gateway to calculus. They answer the question:

"What value does a function approach as the input gets closer and closer to some number?"

Limits let us:

- **Handle infinity:** Understand what happens "at the edge"
- **Deal with discontinuities:** Analyze behavior at problem points
- **Define derivatives:** Instantaneous rate of change
- **Define integrals:** Total accumulation

Without limits, there is no calculus. With limits, you can understand change itself.

The Big Picture: Approaching vs Reaching

The Core Idea

A limit is about getting arbitrarily close, not necessarily arriving.

"What does $f(x)$ approach as x approaches 2?"

x values: 1.9 1.99 1.999 1.9999 ...
 $f(x)$: 3.8 3.98 3.998 3.9998 ...

The function approaches 4 (even if $f(2) \neq 4$ or doesn't exist)

Notation:

$$\lim_{x \rightarrow a} f(x) = L$$

Read as: "The limit of $f(x)$ as x approaches a equals L "

Why Not Just Substitute?

Sometimes substitution works:

$$f(x) = 2x + 1$$

$$\lim_{x \rightarrow 3} f(x) = f(3) = 7$$

But sometimes it doesn't:

$$f(x) = \frac{x^2 - 4}{x - 2}$$

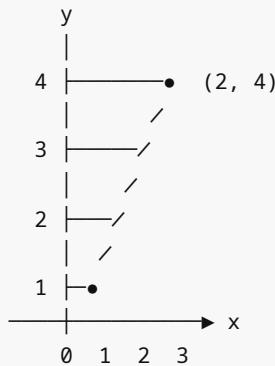
At $x = 2$: $0/0$ (undefined!)

But the limit exists: $\lim_{x \rightarrow 2} f(x) = 4$

1. Intuitive Understanding

Visual: Zooming In

Graph of $f(x) = x^2$:



As $x \rightarrow 2$, $f(x) \rightarrow 4$

Even without computing $f(2)$, we can see where it's heading.

Table of Values

$$f(x) = (x^2 - 4) / (x - 2)$$

Approaching from the left:

x:	1.9	1.99	1.999	1.9999
$f(x)$:	3.9	3.99	3.999	3.9999

Approaching from the right:

x:	2.1	2.01	2.001	2.0001
$f(x)$:	4.1	4.01	4.001	4.0001

Both approach 4

One-Sided Limits

Left-hand limit: Approach from values less than a

$$\lim_{x \rightarrow a^-} f(x) \quad (x \rightarrow a \text{ from the left})$$

Right-hand limit: Approach from values greater than a

$$\lim_{x \rightarrow a^+} f(x) \quad (x \rightarrow a \text{ from the right})$$

Two-sided limit exists if:

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = L$$

$$\text{Then: } \lim_{x \rightarrow a} f(x) = L$$

2. Evaluating Limits

Direct Substitution

If the function is continuous at a , just plug in:

$$\begin{aligned} \lim_{x \rightarrow 2} (3x^2 + 2x - 1) &= 3(2)^2 + 2(2) - 1 \\ &= 12 + 4 - 1 \\ &= 15 \end{aligned}$$

Indeterminate Form 0/0

When substitution gives 0/0, factor and simplify:

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$$

$$\text{Direct: } (9-9)/(3-3) = 0/0 \quad x$$

Factor numerator:

$$\lim_{x \rightarrow 3} \frac{(x+3)(x-3)}{x - 3}$$

Cancel (valid since $x \neq 3$ in the limit):

$$\lim_{x \rightarrow 3} (x + 3) = 6$$

Rationalizing

Multiply by conjugate to eliminate square roots:

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x}$$

$$\text{Direct: } 0/0 \quad x$$

Multiply by conjugate:

$$\begin{aligned}
 & \frac{\sqrt{x+1} - 1}{x} \times \frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1} \\
 &= \frac{(x+1) - 1}{x(\sqrt{x+1} + 1)} \\
 &= \frac{x}{x(\sqrt{x+1} + 1)} \\
 &= \frac{1}{\sqrt{x+1} + 1} \quad (\text{cancel } x) \\
 &= 1/(1 + 1) = 1/2
 \end{aligned}$$

3. Limits at Infinity

What It Means

What happens as x gets arbitrarily large (or negative)?

$$\lim_{x \rightarrow \infty} f(x) \quad (x \rightarrow \infty)$$

$$\lim_{x \rightarrow -\infty} f(x) \quad (x \rightarrow -\infty)$$

Polynomial Limits at Infinity

Dominated by highest-degree term:

$$\lim_{x \rightarrow \infty} \frac{3x^2 + 5x - 1}{2x^2 - x + 7}$$

For large x , only highest powers matter:

$$\approx \frac{3x^2}{2x^2} = 3/2$$

$$\lim_{x \rightarrow \infty} = 3/2$$

General rule:

$$\lim_{x \rightarrow \infty} \frac{a_m x^m + \dots}{b_n x^n + \dots}$$

```
If m < n: limit = 0
If m = n: limit = am/bn
If m > n: limit = ±∞
```

Exponential vs Polynomial

Exponential grows faster:

$$\lim_{x \rightarrow \infty} x/e^x = 0 \quad (\text{denominator grows faster})$$

$$\lim_{x \rightarrow \infty} e^x/x = \infty \quad (\text{numerator grows faster})$$

Logarithmic Growth

Logarithms grow slower than polynomials:

$$\lim_{x \rightarrow \infty} \log(x)/x = 0$$

$$\lim_{x \rightarrow \infty} x/\log(x) = \infty$$

4. Continuous Functions

Definition

A function is **continuous at $x = a$** if:

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Three conditions:

1. $f(a)$ is defined
2. $\lim_{x \rightarrow a} f(x)$ exists
3. They're equal

Visual



Types of Discontinuity

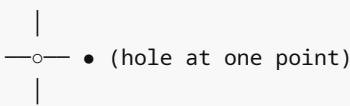
Jump discontinuity:





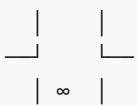
Left and right limits exist but differ

Removable discontinuity:



Can be "fixed" by redefining $f(a)$

Infinite discontinuity:



Vertical asymptote (limit is ∞ or $-\infty$)

5. Important Limit Patterns

Limit of $\sin(x)/x$

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$$

One of the most important limits in calculus.

Visual reasoning:

For small angles (in radians):
 $\sin(x) \approx x$

So $\sin(x)/x \approx x/x = 1$

Limit of $(1 + 1/n)^n$

$$\lim_{n \rightarrow \infty} (1 + 1/n)^n = e \approx 2.71828\dots$$

Definition of Euler's number e .

Generalization:

$$\lim_{n \rightarrow \infty} (1 + x/n)^n = e^x$$

Squeeze Theorem

If $g(x) \leq f(x) \leq h(x)$ and:

$$\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L$$

Then:

$$\lim_{x \rightarrow a} f(x) = L$$

Example: $\lim_{x \rightarrow 0} x^2 \sin(1/x)$

$$-1 \leq \sin(1/x) \leq 1 \quad (\text{always true})$$

Multiply by x^2 :

$$-x^2 \leq x^2 \sin(1/x) \leq x^2$$

$$\lim_{x \rightarrow 0} (-x^2) = 0, \quad \lim_{x \rightarrow 0} (x^2) = 0$$

$$\text{Therefore: } \lim_{x \rightarrow 0} x^2 \sin(1/x) = 0$$

6. L'Hôpital's Rule (Preview)

For indeterminate forms $0/0$ or ∞/∞ :

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

(Take derivatives of numerator and denominator separately)

Example:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 0/0$$

Apply L'Hôpital:

$$\lim_{x \rightarrow 1} \frac{2x}{1} = 2/1 = 2$$

Note: We haven't learned derivatives yet, but this preview shows where limits lead.

7. Programming Perspective

Numerical Limit Estimation

```

function estimateLimit(f, a, epsilon = 1e-6) {
    // Approach from both sides
    const leftValue = f(a - epsilon);
    const rightValue = f(a + epsilon);

    // If close, return average
    if (Math.abs(leftValue - rightValue) < epsilon) {
        return (leftValue + rightValue) / 2;
    }

    return null; // Limit doesn't exist or needs smaller epsilon
}

// Example: lim (x^2 - 4)/(x - 2) as x → 2
const f = x => (x*x - 4) / (x - 2);
estimateLimit(f, 2); // ≈ 4

```

Asymptotic Analysis (Big-O)

Limits describe algorithm behavior as $n \rightarrow \infty$:

$$T(n) = 3n^2 + 100n + 5000$$

$$\lim_{n \rightarrow \infty} T(n)/n^2 = 3$$

So $T(n)$ is $O(n^2)$ (quadratic)

Understanding:

For large n , only highest-order term matters
Just like polynomial limits at infinity

8. Real-World Applications

Instantaneous Velocity

Average velocity:

$$v_{\text{avg}} = \frac{\text{distance}}{\text{time}}$$

Instantaneous velocity: Average velocity over an infinitely small time interval

$$v(t) = \lim_{h \rightarrow 0} \frac{s(t + h) - s(t)}{h}$$

This is the **derivative** (next chapter).

Marginal Cost

Average cost of next unit:

$$\text{Marginal cost} = \lim_{x \rightarrow 0} \frac{C(x + 1) - C(x)}{1}$$

Tangent Lines

Slope of tangent = limit of slopes of secant lines:

$$m = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

Common Mistakes & Misconceptions

✗ "The limit equals the function value"

Not always:

$$f(x) = \frac{x^2 - 1}{x - 1}$$

$f(1)$ is undefined, but $\lim_{x \rightarrow 1} f(x) = 2$

✗ "If $f(a)$ doesn't exist, the limit doesn't exist"

The limit can exist even if $f(a)$ doesn't:

$$f(x) = (x-2)/(x-2) \text{ with hole at } x=2$$

$\lim_{x \rightarrow 2} f(x) = 1$, but $f(2)$ undefined

✗ "Limits always give a number"

Limits can be ∞ , $-\infty$, or not exist at all.

✗ " $\lim (f+g) = \lim f + \lim g$ always"

Only if both limits exist individually.

✗ "Approaching from one side is enough"

Need both sides to match for the limit to exist.

Tiny Practice

Evaluate:

1. $\lim_{x \rightarrow 4} (3x + 2)$
2. $\lim_{x \rightarrow 3} (x^2 - 9)/(x - 3)$
3. $\lim_{x \rightarrow 0} (x^3 + 2x)$
4. $\lim_{x \rightarrow 100} 5$

Determine if continuous at the given point: 5. $f(x) = x^2$ at $x = 2$ 6. $f(x) = 1/x$ at $x = 0$

Limits at infinity: 7. $\lim_{x \rightarrow \infty} (3x + 1)/(x - 2)$

8. $\lim_{x \rightarrow \infty} (x^2 + 5)/(x + 1)$

► Answers

Summary Cheat Sheet

Definition

$$\lim_{x \rightarrow a} f(x) = L$$

" $f(x)$ approaches L as x approaches a "

One-Sided Limits

$$\lim_{x \rightarrow a^-} f(x) \text{ (from left)}$$
$$\lim_{x \rightarrow a^+} f(x) \text{ (from right)}$$

Limit exists if both equal

Continuity

f is continuous at a if:
 $\lim_{x \rightarrow a} f(x) = f(a)$

Evaluation Techniques

Case	Method
Continuous	Direct substitution
0/0 form	Factor and simplify
∞/∞ form	Divide by highest power

[Square roots](#)[Rationalize](#)

Limits at Infinity

$$\lim_{x \rightarrow \infty} \frac{a_m x^m + \dots}{b_n x^n + \dots}$$

$m < n: \rightarrow 0$
 $m = n: \rightarrow a_m/b_n$
 $m > n: \rightarrow \pm\infty$

Key Limits

$$\lim_{x \rightarrow 0} \sin(x)/x = 1$$

$$\lim_{n \rightarrow \infty} (1 + 1/n)^n = e$$

Programming

```
// Numerical estimation
function limit(f, a, h = 1e-6) {
    return (f(a + h) - f(a - h)) / (2*h);
}

// Check continuity
function isContinuous(f, a, epsilon = 1e-6) {
    try {
        const limit = (f(a + epsilon) + f(a - epsilon)) / 2;
        return Math.abs(f(a) - limit) < epsilon;
    } catch {
        return false;
    }
}
```

Next Steps

Limits are the foundation of calculus. You now understand:

- Approaching vs reaching
- Evaluating limits
- Continuity
- Behavior at infinity

Next, we'll use limits to define **Derivatives**—the mathematics of instantaneous change.

Continue to: [11-derivatives.md](#)