

# How Numbers Actually Work

## Why This Matters

Numbers are the atoms of mathematics. Before you can understand equations, functions, or calculus, you need to *really* understand what numbers are, why they exist, and how they behave.

If you've ever wondered:

- Why can't you divide by zero?
- What's the difference between -5 and 5 besides the sign?
- Why do fractions and decimals represent the same thing?
- What does  $\sqrt{2}$  even mean?

...then you're asking the right questions. Let's answer them from scratch.

## The Big Picture: What Problem Do Numbers Solve?

Numbers exist to measure and compare things.

- "How many apples?" → Natural numbers (1, 2, 3...)
- "How much water?" → Fractions (1/2 cup, 3/4 liter)
- "What's the temperature?" → Negative numbers (-5°C)
- "How far exactly?" → Irrational numbers ( $\sqrt{2}$  meters)

Each type of number was invented to solve a specific problem humans faced.

---

## 1. Natural Numbers (Counting Numbers)

### What They Are

1, 2, 3, 4, 5, 6, ...

These are the first numbers humans invented. You can count them on your fingers. They answer "how many?"

### Mental Model

Think of natural numbers as **discrete items in an array**:

```
const apples = [🍏, 🍏, 🍏];  
console.log(apples.length); // 3
```

You can't have 2.5 apples in this array—either you have 2 or 3.

### What You Can Do

- **Add:**  $3 + 2 = 5$  (combine two groups)
- **Multiply:**  $3 \times 4 = 12$  (repeated addition:  $3 + 3 + 3 + 3$ )
- **Compare:**  $5 > 3$  (one is bigger)

### What You Can't Do

- **Subtract freely:**  $3 - 5 = ?$  (You can't have -2 apples... yet)
- **Divide freely:**  $5 \div 2 = ?$  (Not always a natural number)

This is where we hit the limits of natural numbers.

## 2. Integers (Whole Numbers Including Negatives)

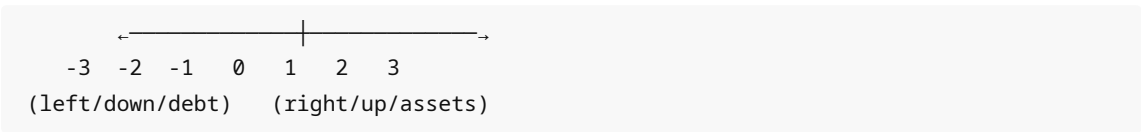
### What They Are

..., -3, -2, -1, 0, 1, 2, 3, ...

Integers extend natural numbers to include:

- **Zero:** "nothing" or "the starting point"
- **Negatives:** "opposite direction" or "debt"

### Mental Model: The Number Line



The number line is your most important mental tool. It turns numbers into **positions** or **distances**.

### Why Negatives Exist

**Problem:** You have \$10, then spend \$15. How much do you have?

Natural numbers can't answer this. But integers can: **-\$5** (you're in debt).

Think of negatives as:

- **Direction:** Moving left instead of right
- **Opposite:** The reverse of something
- **Debt vs. Assets:** Below zero

### Programming Analogy

```
let balance = 10;
balance -= 15;
console.log(balance); // -5 (totally valid)
```

In code, negative numbers are just numbers. In early math education, they're treated as scary. They're not.

### Zero: The Starting Point

Zero is special:

- It means "nothing" (0 apples)
- It's the **origin** on the number line
- It's the boundary between positive and negative
- **0 + x = x** (adding zero does nothing — the identity element)

### Integer Operations

Operation	Example	Intuition
-----------	---------	-----------

Add	$3 + (-5) = -2$	Move 3 right, then 5 left
Subtract	$3 - 5 = -2$	Same as $3 + (-5)$
Multiply	$3 \times (-2) = -6$	Flip direction (negative)
Divide	$-6 \div 3 = -2$	Reverse of multiply

### Rules for Negative Multiplication

- **Positive × Positive = Positive** (normal)
- **Positive × Negative = Negative** (flip direction)
- **Negative × Negative = Positive** (flip twice = back to original)

### Why does negative × negative = positive?

Think of it as reversing a reversal:

- Facing forward (positive)
- Turn around (negative)
- Turn around again (negative again) → You're facing forward (positive)

Or programmatically:

```
let direction = 1;    // forward
direction *= -1;     // -1 (backward)
direction *= -1;     // 1 (forward again)
```

## 3. Rational Numbers (Fractions)

### What They Are

Numbers that can be written as **one integer divided by another**:  $a/b$  (where  $b \neq 0$ )

Examples:  $1/2$ ,  $3/4$ ,  $-5/6$ ,  $7/1$  (which is just 7)

### Mental Model: Fractions as Division

**Don't think of fractions as weird symbols. Think of them as division operations that haven't been completed yet.**

$$5/2 = 5 \div 2 = 2.5$$

The fraction bar is just a division sign:

$$\frac{5}{2} \text{ means } 5 \div 2$$

### Why Fractions Exist

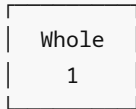
**Problem:** You have 1 pizza and 4 people. How much does each person get?

$$1 \div 4 = 1/4$$

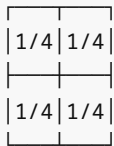
Fractions let you represent **parts of a whole** or the result of division.

## Visual: The Pizza Model

Original Pizza:



Divided among 4 people:



Each person gets 1/4.

## Numerator and Denominator

```
3 ← numerator (how many parts you have)
—
4 ← denominator (how many parts make a whole)
```

## Programming Analogy:

```
const fraction = {
  numerator: 3,
  denominator: 4,
  toDecimal() {
    return this.numerator / this.denominator; // 0.75
  }
};
```

## Equivalent Fractions

**1/2 = 2/4 = 3/6 = 4/8 = ...**

They all represent the same **value**, just written differently.

Why? Because you can multiply both top and bottom by the same number:

$$\frac{1}{2} = \frac{1 \times 2}{2 \times 2} = \frac{2}{4}$$

Think of it like this:

```
const a = 1/2;    // 0.5
const b = 2/4;    // 0.5
```

```
console.log(a === b); // true
```

## Simplifying Fractions

Find the **greatest common divisor (GCD)** and divide both parts:

$$\frac{6}{8} = \frac{6 \div 2}{8 \div 2} = \frac{3}{4}$$

**Why simplify?** Smaller numbers are easier to work with.

## Operations on Fractions

### Addition (Same Denominator)

$$\frac{1}{4} + \frac{2}{4} = \frac{1+2}{4} = \frac{3}{4}$$

Easy: just add the numerators.

### Addition (Different Denominators)

$$\frac{1}{2} + \frac{1}{3} = ?$$

Find a **common denominator** (lowest common multiple):

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

## Multiplication

$$\frac{1}{3} \times \frac{2}{4} = \frac{1 \times 2}{3 \times 4} = \frac{2}{12} = \frac{1}{6}$$

Multiply tops, multiply bottoms.

**Intuition:** "2/4 of 1/3" means "take 1/3, then take half of *that*"

## Division

$$\frac{1}{3} \div \frac{2}{4} = \frac{1}{3} \times \frac{4}{2} = \frac{1 \times 4}{3 \times 2} = \frac{4}{6}$$

Flip the second fraction and multiply. This is called **multiplying by the reciprocal**.

**Why?** Division is the inverse of multiplication. If you multiply by 2/4, you divide by 4/2.

---

## 4. Decimals (Another Way to Write Fractions)

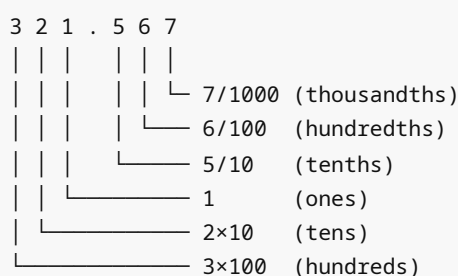
### What They Are

Numbers with a decimal point: 0.5, 3.14, -2.75

**Key insight: Decimals are just fractions in disguise.**

$$\begin{aligned}0.5 &= 5/10 = 1/2 \\0.75 &= 75/100 = 3/4 \\0.333 &= 333/1000 \approx 1/3\end{aligned}$$

### Place Value



$$321.567 = 300 + 20 + 1 + 0.5 + 0.06 + 0.007$$

### Why Decimals?

Decimals are easier to:

- Type (0.5 vs 1/2)
- Compare (0.7 vs 0.65 — just compare digit by digit)
- Use in calculators

But fractions are better for:

- Exact values (1/3 is exact, 0.333... is approximate)
- Showing relationships

### Converting Between Fractions and Decimals

**Fraction → Decimal:** Just divide

$$3/4 = 3 \div 4 = 0.75$$

**Decimal → Fraction:** Use place value

$$0.75 = 75/100 = 3/4 \text{ (simplified)}$$

### Terminating vs Repeating Decimals

**Terminating:** Ends after a certain point

$$1/2 = 0.5$$

$$1/4 = 0.25$$

**Repeating:** Goes on forever

$$1/3 = 0.333333\dots$$

$$1/7 = 0.142857142857\dots$$

We write repeating decimals with a bar:

$$1/3 = 0.\overline{3} \quad (\text{the } 3 \text{ repeats})$$

## 5. Irrational Numbers (Numbers That Can't Be Fractions)

### What They Are

Numbers that **cannot** be written as a fraction of two integers.

Famous examples:

- $\pi$  (pi)  $\approx 3.14159\dots$
- $\sqrt{2}$  (square root of 2)  $\approx 1.41421\dots$
- $e$  (Euler's number)  $\approx 2.71828\dots$

### Why They Exist

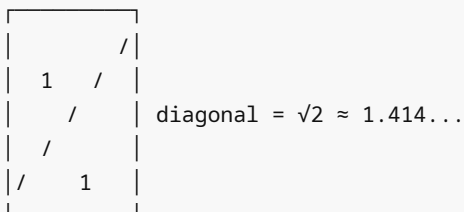
**Problem:** What's the diagonal of a  $1 \times 1$  square?

Using the Pythagorean theorem:  $\text{diagonal}^2 = 1^2 + 1^2 = 2$

So:  $\text{diagonal} = \sqrt{2}$

But  $\sqrt{2}$  cannot be written as a fraction. It's been proven mathematically.

### Visual: Why $\sqrt{2}$ is Irrational



No matter how you try to express it as a ratio of whole numbers, you can't. The decimal goes on forever *without repeating*.

### Decimal Expansion

**Rational:** Eventually repeats

$$1/3 = 0.333333\dots \quad (\text{repeats})$$

$$1/7 = 0.142857142857\dots \quad (\text{repeats})$$

**Irrational:** Never repeats, goes on forever

```
√2 = 1.41421356237309504880168872420969807856967187537694...
π  = 3.14159265358979323846264338327950288419716939937510...
```

## Programming Note

```
console.log(Math.sqrt(2)); // 1.4142135623730951
console.log(Math.PI);      // 3.141592653589793
```

Computers approximate irrationals with floating-point numbers. They can't store infinite decimals.

## Why This Matters

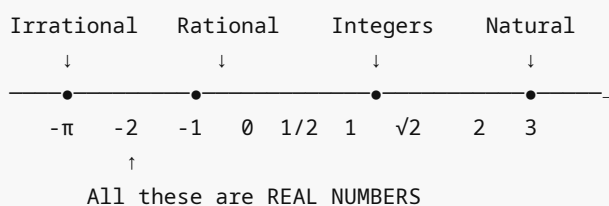
Irrational numbers show up everywhere:

- Circles ( $\pi$ )
- Right triangles ( $\sqrt{2}$ )
- Natural growth ( $e$ )

You can't avoid them, so embrace them as "numbers that can't be written as simple fractions."

---

## 6. The Complete Number Line



## Real Numbers

**Real numbers** = All the numbers on the number line

This includes:

- Natural numbers (1, 2, 3, ...)
- Zero (0)
- Negative integers (-1, -2, -3, ...)
- Fractions ( $1/2$ ,  $3/4$ , ...)
- Irrational numbers ( $\pi$ ,  $\sqrt{2}$ , ...)

If you can point to it on the number line, it's a real number.

---

## 7. Absolute Value (Distance, Not Direction)

### What It Means

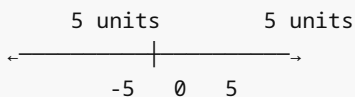
The **absolute value** of a number is its distance from zero, ignoring direction.

**Notation:**  $|x|$  (read as "absolute value of x")



```
|5| = 5 (distance from 0 is 5)
|-5| = 5 (distance from 0 is 5)
|0| = 0 (distance from 0 is 0)
```

## Visual



Both -5 and 5 are 5 units away from zero.

## Mental Model: Distance

Think of absolute value as `Math.abs()` in code:

```
Math.abs(5); // 5
Math.abs(-5); // 5
Math.abs(0); // 0
```

It strips away the sign and gives you the magnitude.

## When You Use It

- **Distance:** "How far apart are -3 and 2?"

$$|-3 - 2| = |-5| = 5$$

- **Error/Difference:** "How much did I miss by?"

$$|\text{actual} - \text{expected}|$$

- **Magnitude:** "How big is this, regardless of direction?"

## Rules

```
|-x| = |x| (sign doesn't matter)
|x × y| = |x| × |y| (distribute through multiplication)
|x| ≥ 0 (always non-negative)
|x| = 0 ⇔ x = 0 (only zero has distance zero)
```

## Common Mistakes & Misconceptions

### ✗ "Negative numbers are smaller than zero"

Not in terms of magnitude. -1000 is "more negative" than -1, but both are negative.

### ✗ "Fractions are different from decimals"

They're the same thing in different notation:  $0.5 = 1/2$

## ✗ "You can't divide by zero because it's zero"

You can't divide by zero because **it's undefined**. Division by zero would break all of mathematics.

Think about it:

$$10 \div 0 = ?$$

This would mean: "What number, times 0, gives 10?" But **any number  $\times 0 = 0$** , so there's no answer.

## ✗ "Irrational numbers are rare"

Most numbers are irrational! Rationals are actually the rare ones.

## ✗ "Numbers are just symbols"

Numbers represent quantities, distances, and comparisons. They have meaning.

---

## Real-World Examples

### Money (Rationals and Negatives)

Balance: \$50.25 (decimal/rational)  
Withdrawal: -\$75 (negative)  
New balance: -\$24.75 (debt)

### Temperature (Integers and Negatives)

Freezing:  $0^{\circ}\text{C}$   
Cold:  $-10^{\circ}\text{C}$   
Hot:  $35^{\circ}\text{C}$

### Distances (Rationals and Irrationals)

Diagonal of square:  $\sqrt{2}$  meters (irrational)  
Half the distance: 0.5 km (rational decimal)

### Programming (All Types)

```
const items = 5;           // natural number
let offset = -10;          // negative integer
const ratio = 0.75;        // decimal/rational
const pi = Math.PI;        // irrational (approximated)
const distance = Math.abs(x); // absolute value
```

## Tiny Practice

Try these to test your understanding:

1. **Place on the number line:** -2, 0.5, 3, -1.5, 2
2. **Absolute value:**  $|-7|$ ,  $|3|$ ,  $|-0.5|$
3. **Convert:**  $3/4$  to decimal, 0.2 to fraction
4. **True or false:** Is -5 an integer? Is 0.5 an integer?
5. **Simplify:**  $6/8$ ,  $10/15$
6. **Compute:**  $|-3 - 5|$ ,  $|4 - 1|$

► Answers

---

## Summary Cheat Sheet

Type	Examples	What It Solves
Natural	1, 2, 3, ...	Counting
Integers	..., -2, -1, 0, 1, 2, ...	Direction, debt
Rational	$1/2$ , $3/4$ , 0.75	Parts of a whole
Irrational	$\pi$ , $\sqrt{2}$ , e	Exact geometric values
Real	All of the above	Everything on the number line

### Key Concepts

- **Number line:** Visual representation of numbers as positions
- **Negative:** Opposite direction, below zero
- **Fraction = Division:**  $3/4$  means  $3 \div 4$
- **Decimal = Fraction:** 0.5 means  $5/10 = 1/2$
- **Absolute value:** Distance from zero ( $|x|$ )
- **Zero:** The origin, boundary, identity element

### Mental Models

- Natural numbers = items in an array
  - Integers = positions on a line
  - Fractions = slicing a whole into parts
  - Decimals = place value system
  - Irrationals = infinite non-repeating decimals
  - Absolute value = `Math.abs()`
- 

## Next Steps

Now that you understand what numbers *are*, you're ready to learn what you can **do** with them.

In the next section, we'll explore **Algebra Foundations**—how to use numbers as variables and solve for unknowns.

Continue to: [01-algebra-foundations.md](#)