Limits

Why This Matters

Limits are the gateway to calculus. They answer the question:

> *"What value does a function approach as the input gets closer and closer to some number?"*

Limits let us:

- **Handle infinity**: Understand what happens "at the edge"
- **Deal with discontinuities**: Analyze behavior at problem points
- **Define derivatives**: Instantaneous rate of change
- **Define integrals**: Total accumulation

Without limits, there is no calculus. With limits, you can understand change itself.

The Big Picture: Approaching vs Reaching

The Core Idea

A limit is about getting arbitrarily close, not necessarily arriving.

```
"What does f(x) approach as x approaches 2?"

x values:  1.9   1.99   1.999   1.9999  ...
f(x):      3.8   3.98   3.998   3.9998  ...

The function approaches 4 (even if f(2) ≠ 4 or doesn't exist)
```

Notation:

```
lim f(x) = L
x→a

Read as: "The limit of f(x) as x approaches a equals L"
```

Why Not Just Substitute?

Sometimes substitution works:

```
f(x) = 2x + 1

lim f(x) = f(3) = 7
x→3
```

But sometimes it doesn't:

```
        x² - 4
f(x) = ──────
         x - 2
```

```
At x = 2: 0/0 (undefined!)

But the limit exists: lim f(x) = 4
                      x→2
```

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1. Intuitive Understanding

Visual: Zooming In

```
Graph of f(x) = x²:

    y
    │
  4 ├───────●  (2, 4)
    │      ╱
  3 ├─────╱
    │    ╱
  2 ├───╱
    │  ╱
  1 ├─●
────┼──────────▶ x
    0  1  2  3

As x → 2, f(x) → 4
```

Even without computing f(2), we can see where it's heading.

Table of Values

```
f(x) = (x² - 4) / (x - 2)

Approaching from the left:
x:    1.9      1.99     1.999    1.9999
f(x): 3.9      3.99     3.999    3.9999

Approaching from the right:
x:    2.1      2.01     2.001    2.0001
f(x): 4.1      4.01     4.001    4.0001

Both approach 4
```

One-Sided Limits

Left-hand limit: Approach from values less than a

```
lim f(x)  (x → a from the left)
x→a⁻
```

Right-hand limit: Approach from values greater than a

```
lim f(x)  (x → a from the right)
x→a⁺
```

Two-sided limit exists if:

```
lim f(x) = lim f(x) = L
x→a⁻     x→a⁺

Then: lim f(x) = L
      x→a
```

---

2. Evaluating Limits

Direct Substitution

If the function is continuous at a, just plug in:

```
lim (3x² + 2x - 1) = 3(2)² + 2(2) - 1
x→2                = 12 + 4 - 1
                   = 15
```

Indeterminate Form 0/0

When substitution gives 0/0, factor and simplify:

```
       x² - 9
lim ──────────
x→3    x - 3

Direct: (9-9)/(3-3) = 0/0  ✗

Factor numerator:
       (x+3)(x-3)
lim ────────────
x→3     x - 3

Cancel (valid since x ≠ 3 in the limit):
lim (x + 3) = 6
x→3
```

Rationalizing

Multiply by conjugate to eliminate square roots:

```
      √(x+1) - 1
lim ────────────
x→0      x

Direct: 0/0  ✗

Multiply by conjugate:
```

```
     √(x+1) - 1   √(x+1) + 1
   ──────────── × ────────────
         x        √(x+1) + 1

      (x+1) - 1
  = ─────────────────
    x(√(x+1) + 1)

      x
  = ─────────────────
    x(√(x+1) + 1)

  = ─────────────   (cancel x)
    √(x+1) + 1

  = 1/(1 + 1) = 1/2
```

3. Limits at Infinity

What It Means

What happens as x gets arbitrarily large (or negative)?

```
lim f(x)  (x → ∞)
x→∞

lim f(x)  (x → -∞)
x→-∞
```

Polynomial Limits at Infinity

Dominated by highest-degree term:

```
       3x² + 5x - 1
lim ──────────────────
x→∞   2x² - x + 7

For large x, only highest powers matter:
     3x²
≈ ──────── = 3/2
     2x²

lim = 3/2
x→∞
```

General rule:

```
       aₘxᵐ + ...
lim ───────────────
x→∞    bₙxⁿ + ...
```

```
If m < n: limit = 0
If m = n: limit = aₘ/bₙ
If m > n: limit = ±∞
```

Exponential vs Polynomial

Exponential grows faster:

```
lim x/eˣ = 0     (denominator grows faster)
x→∞

lim eˣ/x = ∞     (numerator grows faster)
x→∞
```

Logarithmic Growth

Logarithms grow slower than polynomials:

```
lim log(x)/x = 0
x→∞

lim x/log(x) = ∞
x→∞
```

---

4. Continuous Functions

Definition

A function is **continuous at x = a** if:

```
lim f(x) = f(a)
x→a
```

Three conditions:

1. f(a) is defined
2. lim f(x) exists x→a
3. They're equal

Visual

```
Continuous:           Discontinuous:
    |                     |
────●────             ────○──●─
  (smooth)              (break)
```

Types of Discontinuity

Jump discontinuity:

```
      |
   ───┤  •
```

```
    │
  • ├──
    │

Left and right limits exist but differ
```

Removable discontinuity:

```
    │
  ──○── • (hole at one point)
    │

Can be "fixed" by redefining f(a)
```

Infinite discontinuity:

```
    │    │
  ──┘    └──
    │ ∞  │

Vertical asymptote (limit is ∞ or -∞)
```

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5. Important Limit Patterns

Limit of sin(x)/x

```
lim sin(x)/x = 1
x→0
```

One of the most important limits in calculus.

Visual reasoning:

```
For small angles (in radians):
sin(x) ≈ x

So sin(x)/x ≈ x/x = 1
```

Limit of $(1 + 1/n)^n$

```
lim (1 + 1/n)ⁿ = e ≈ 2.71828...
n→∞
```

Definition of Euler's number e.

Generalization:

```
lim (1 + x/n)ⁿ = eˣ
n→∞
```

Squeeze Theorem

If g(x) ≤ f(x) ≤ h(x) and:

```
lim g(x) = lim h(x) = L
x→a       x→a
```

Then:

```
lim f(x) = L
x→a
```

Example: lim x^2 sin(1/x) x→0

```
-1 ≤ sin(1/x) ≤ 1  (always true)

Multiply by x²:
-x² ≤ x² sin(1/x) ≤ x²

lim (-x²) = 0,  lim (x²) = 0
x→0           x→0

Therefore: lim x² sin(1/x) = 0
          x→0
```

---

6. L'Hôpital's Rule (Preview)

For indeterminate forms 0/0 or ∞/∞:

```
       f(x)         f'(x)
lim ──────── = lim ────────
x→a    g(x)   x→a   g'(x)
```

(Take derivatives of numerator and denominator separately)

Example:

```
       x² - 1
lim ──────────  = 0/0
x→1    x - 1

Apply L'Hôpital:
       2x
lim ────── = 2/1 = 2
x→1     1
```

Note: We haven't learned derivatives yet, but this preview shows where limits lead.

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7. Programming Perspective

Numerical Limit Estimation

```javascript
function estimateLimit(f, a, epsilon = 1e-6) {
  // Approach from both sides
  const leftValue = f(a - epsilon);
  const rightValue = f(a + epsilon);

  // If close, return average
  if (Math.abs(leftValue - rightValue) < epsilon) {
    return (leftValue + rightValue) / 2;
  }

  return null;  // Limit doesn't exist or needs smaller epsilon
}

// Example: lim (x² - 4)/(x - 2) as x → 2
const f = x => (x*x - 4) / (x - 2);
estimateLimit(f, 2);  // ≈ 4
```

Asymptotic Analysis (Big-O)

Limits describe algorithm behavior as n → ∞:

```
T(n) = 3n² + 100n + 5000

lim T(n)/n² = 3
n→∞

So T(n) is O(n²) (quadratic)
```

Understanding:

```
For large n, only highest-order term matters
Just like polynomial limits at infinity
```

---

8. Real-World Applications

Instantaneous Velocity

Average velocity:

```
        distance
v_avg = ────────
          time
```

Instantaneous velocity: Average velocity over an infinitely small time interval

```
       s(t + h) - s(t)
v(t) = lim ──────────────
       h→0        h
```

This is the **derivative** (next chapter).

Marginal Cost

Average cost of next unit:

$$\text{Marginal cost} = \lim_{x \to 0} \frac{C(x + 1) - C(x)}{1}$$

Tangent Lines

Slope of tangent = limit of slopes of secant lines:

$$m = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Common Mistakes & Misconceptions

❌ "The limit equals the function value"

Not always:

$$f(x) = \frac{x^2 - 1}{x - 1}$$

$f(1)$ is undefined, but $\lim_{x \to 1} f(x) = 2$

❌ "If f(a) doesn't exist, the limit doesn't exist"

The limit can exist even if f(a) doesn't:

$f(x) = (x-2)/(x-2)$ with hole at $x=2$

$\lim_{x \to 2} f(x) = 1$, but $f(2)$ undefined

❌ "Limits always give a number"

Limits can be ∞, -∞, or not exist at all.

❌ "lim (f+g) = lim f + lim g always"

Only if both limits exist individually.

❌ "Approaching from one side is enough"

Need both sides to match for the limit to exist.

Tiny Practice

Evaluate:

1. lim (3x + 2) x→4
2. lim $(x^2 - 9)/(x - 3)$ x→3
3. lim $(x^3 + 2x)$ x→0
4. lim 5 x→100

Determine if continuous at the given point: 5. $f(x) = x^2$ at x = 2 6. f(x) = 1/x at x = 0

Limits at infinity: 7. lim (3x + 1)/(x - 2) x→∞

8. lim $(x^2 + 5)/(x + 1)$ x→∞

▶ Answers

Summary Cheat Sheet

Definition

```
lim f(x) = L
x→a

"f(x) approaches L as x approaches a"
```

One-Sided Limits

```
lim f(x)  (from left)
x→a⁻

lim f(x)  (from right)
x→a⁺

Limit exists if both equal
```

Continuity

```
f is continuous at a if:
lim f(x) = f(a)
x→a
```

Evaluation Techniques

Case	Method
Continuous	Direct substitution
0/0 form	Factor and simplify
∞/∞ form	Divide by highest power

Square roots	Rationalize

Limits at Infinity

```
       aₘxᵐ + ...
lim ───────────────
x→∞    bₙxⁿ + ...

m < n: → 0
m = n: → aₘ/bₙ
m > n: → ±∞
```

Key Limits

```
lim sin(x)/x = 1
x→0

lim (1 + 1/n)ⁿ = e
n→∞
```

Programming

```
// Numerical estimation
function limit(f, a, h = 1e-6) {
  return (f(a + h) - f(a - h)) / (2*h);
}

// Check continuity
function isContinuous(f, a, epsilon = 1e-6) {
  try {
    const limit = (f(a + epsilon) + f(a - epsilon)) / 2;
    return Math.abs(f(a) - limit) < epsilon;
  } catch {
    return false;
  }
}
```

Next Steps

Limits are the foundation of calculus. You now understand:

- Approaching vs reaching
- Evaluating limits
- Continuity
- Behavior at infinity

Next, we'll use limits to define **Derivatives**—the mathematics of instantaneous change.

Continue to: 11-derivatives.md