

Linear Functions

Why This Matters

Linear functions are the **simplest and most fundamental functions**. They model:

- **Constant change:** Speed, pricing, growth
- **Relationships:** Cause and effect
- **Predictions:** Trends and forecasting

In programming and data science:

- **Linear regression:** Fitting lines to data
- **Time complexity:** O(n) algorithms
- **Interpolation:** Estimating between points
- **Animation:** Linear motion

Linear functions are your mental model for "steady change."

The Big Picture: Constant Rate of Change

Linear function: Output changes by the same amount for each unit of input.

Every time x increases by 1, y increases by the same amount

$x: 0 \quad 1 \quad 2 \quad 3 \quad 4$
 $y: 1 \quad 3 \quad 5 \quad 7 \quad 9$

Change: +2 +2 +2 +2 (constant)

Graphically: A straight line.

1. The Slope-Intercept Form

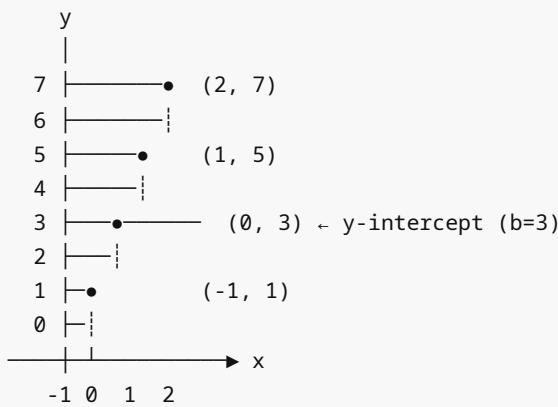
The Standard Equation

$y = mx + b$
 $m = \text{slope}$ (rate of change)
 $b = y\text{-intercept}$ (starting value)
 $x = \text{input}$ (independent variable)
 $y = \text{output}$ (dependent variable)

This is the most important equation in algebra.

Visual Breakdown

$y = 2x + 3$
↑ ↑
| ↘ b (y-intercept) = 3
——— m (slope) = 2



Slope $m = 2$: "rise 2, run 1"

Programming Analogy

```
// Linear function as code
function linearFunction(x, m, b) {
    return m * x + b;
}

// Or as object
const line = {
    slope: 2,
    intercept: 3,
    evaluate(x) {
        return this.slope * x + this.intercept;
    }
};

line.evaluate(5); // 2(5) + 3 = 13
```

2. Understanding Slope (m)

What Is Slope?

Slope = Rate of change = Rise over run

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Interpretation

$m = 2$: For every 1 unit right, go up 2 units

$m = -3$: For every 1 unit right, go down 3 units

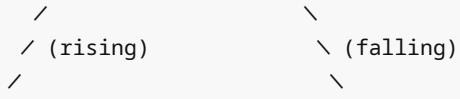
$m = 0.5$: For every 1 unit right, go up 0.5 units

m = 0: Horizontal line (no change)

m = undefined: Vertical line (infinite change)

Types of Slopes

Positive ($m > 0$): Negative ($m < 0$):



Zero ($m = 0$): Undefined:



Calculating Slope from Two Points

Given: (1, 3) and (4, 9)

$$\begin{aligned} m &= (y_2 - y_1) / (x_2 - x_1) \\ &= (9 - 3) / (4 - 1) \\ &= 6 / 3 \\ &= 2 \end{aligned}$$

Real-World Slope Examples

Speed:

$$\begin{aligned} \text{Distance} &= \text{rate} \times \text{time} \\ d &= rt \\ \\ \text{If } r &= 60 \text{ mph:} \\ d &= 60t \\ \\ \text{Slope } m &= 60 \text{ (60 miles per hour)} \end{aligned}$$

Pricing:

$$\begin{aligned} \text{Cost} &= \text{price_per_unit} \times \text{quantity} + \text{fixed_cost} \\ C &= 5q + 20 \\ \\ \text{Slope } m &= 5 \text{ (cost increases \$5 per unit)} \end{aligned}$$

Temperature conversion:

$$\begin{aligned} F &= (9/5)C + 32 \\ \\ \text{Slope } m &= 9/5 = 1.8 \\ (\text{Fahrenheit changes } 1.8^\circ \text{ for each } 1^\circ \text{ Celsius}) \end{aligned}$$

3. Understanding Y-Intercept (b)

What Is Y-Intercept?

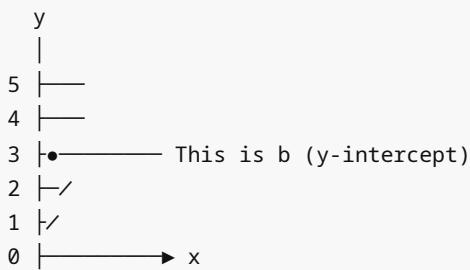
The **y**-value when **x = 0** (where the line crosses the y-axis)

$$y = mx + b$$

When $x = 0$:

$$y = m(0) + b = b$$

Visual



Interpretation

b = 3: Starting value is 3 (when $x = 0$)

b = -2: Starting value is -2 (below origin)

b = 0: Line passes through origin

Real-World Y-Intercept Examples

Fixed cost:

```
Total = variable_cost * units + fixed_cost  
C = 5q + 100  
  
b = 100 (fixed cost even if q = 0)
```

Initial position:

```
position = velocity * time + starting_position  
s = 3t + 10  
  
b = 10 (started at position 10)
```

Base salary:

```
Income = commission * sales + base_salary  
I = 0.1s + 30000  
  
b = 30,000 (salary even with zero sales)
```

4. Writing Linear Equations

From Slope and Y-Intercept

Given: $m = 3$, $b = -2$

$$y = 3x - 2$$

Done! Just plug into $y = mx + b$.

From Slope and a Point

Given: $m = 2$, point $(3, 7)$

Method: Use point-slope form

$$y - y_1 = m(x - x_1)$$

$$y - 7 = 2(x - 3)$$

$$y - 7 = 2x - 6$$

$$y = 2x + 1$$

From Two Points

Given: $(1, 3)$ and $(4, 9)$

Step 1: Find slope

$$m = (9 - 3) / (4 - 1) = 6/3 = 2$$

Step 2: Use point-slope form with either point

$$y - 3 = 2(x - 1)$$

$$y - 3 = 2x - 2$$

$$y = 2x + 1$$

From a Graph

Read directly:

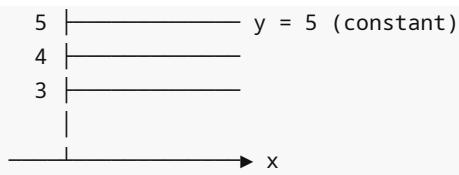
1. Find y-intercept (where line crosses y-axis): b
 2. Count rise/run from any two points: m
 3. Write $y = mx + b$
-

5. Special Cases

Horizontal Lines

$$y = 5 \quad (\text{or } y = b \text{ where } m = 0)$$



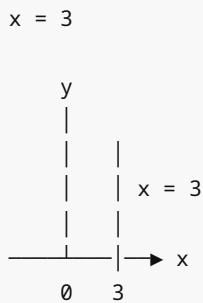


Characteristics:

- Slope $m = 0$
- y never changes
- Form: $y = b$

Example: Temperature stays at 70°F all day.

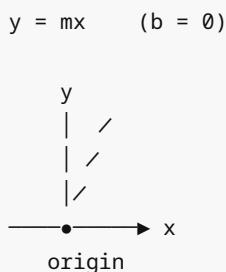
Vertical Lines



Characteristics:

- Slope undefined (division by zero)
- x never changes
- **NOT a function** (fails vertical line test)
- Form: $x = a$

Lines Through Origin



Examples:

- Direct proportionality
- $y = 2x$ (doubling)
- Distance = speed \times time (starting from origin)

6. Parallel and Perpendicular Lines

Parallel Lines

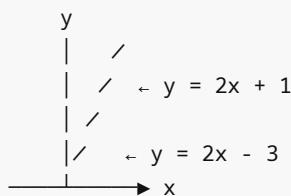
Same slope, different y-intercepts

$$y = 2x + 1$$

$$y = 2x - 3$$

Both have $m = 2$ (parallel)

Visual:



Never intersect (same direction).

Perpendicular Lines

Slopes are negative reciprocals:

$$m_1 \times m_2 = -1$$

If $m_1 = 2$, then $m_2 = -1/2$

If $m_1 = 3/4$, then $m_2 = -4/3$

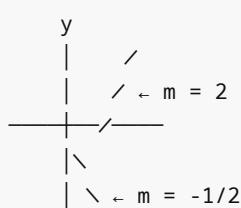
Example:

$$y = 2x + 1 \quad (m = 2)$$

$$y = -\frac{1}{2}x + 3 \quad (m = -1/2)$$

These are perpendicular (meet at 90°)

Visual:



Special case: Horizontal and vertical lines are perpendicular

$$y = 5 \quad (\text{horizontal}, m = 0)$$

$$x = 3 \quad (\text{vertical}, m = \text{undefined})$$

7. Applications and Examples

Motion at Constant Speed

Distance = speed × time
 $d = 60t$

$m = 60$ mph (speed)
 $b = 0$ (starts at origin)

After 3 hours: $d = 60(3) = 180$ miles

Linear Pricing

Cost = price_per_item × quantity + fixed_cost
 $C = 15q + 200$

$m = 15$ (variable cost per unit)
 $b = 200$ (fixed costs)

For 50 items: $C = 15(50) + 200 = \$950$

Temperature Conversion

Celsius to Fahrenheit:

$$F = (9/5)C + 32$$

$m = 9/5 = 1.8$
 $b = 32$

At 0°C : $F = 32^\circ\text{F}$
At 100°C : $F = (9/5)(100) + 32 = 212^\circ\text{F}$

Fahrenheit to Celsius (inverse):

$$C = (5/9)(F - 32)$$
$$C = (5/9)F - 160/9$$

$m = 5/9 \approx 0.556$
 $b = -160/9 \approx -17.78$

Depreciation

Value = initial_value - depreciation_rate × years
 $V = 20000 - 2000t$

$m = -2000$ (loses \$2000/year)
 $b = 20000$ (initial value)

After 5 years: $V = 20000 - 2000(5) = \$10,000$

Salary with Commission

Income = commission_rate × sales + base
 $I = 0.08s + 35000$

$m = 0.08$ (8% commission)
 $b = 35000$ (base salary)

With \$100k sales: $I = 0.08(100000) + 35000 = \$43,000$

8. Finding Intersections

Where Two Lines Meet

Solve the system of equations:

Line 1: $y = 2x + 1$
Line 2: $y = -x + 4$

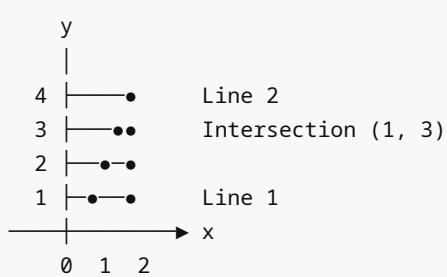
Set equal:

$$\begin{aligned} 2x + 1 &= -x + 4 \\ 3x &= 3 \\ x &= 1 \end{aligned}$$

Substitute back:
 $y = 2(1) + 1 = 3$

Intersection: (1, 3)

Visual



Break-Even Analysis

When do costs equal revenue?

Cost: $C = 5q + 1000$ (production cost)
Revenue: $R = 10q$ (sales income)

$$\begin{aligned} \text{Break-even: } C &= R \\ 5q + 1000 &= 10q \\ 1000 &= 5q \end{aligned}$$

```
q = 200 units  
  
At 200 units, cost = revenue = $2000
```

9. Linear Regression (Data Fitting)

The Problem

Given data points, find the "best fit" line.

Data: (1, 2), (2, 4), (3, 5), (4, 7)

Goal: Find $y = mx + b$ that best approximates the points.

Least Squares Method (Intuition)

Minimize the total squared error between predicted and actual values.

Result: Formulas for m and b:

$$m = \frac{(n \cdot \sum xy - \sum x \cdot \sum y)}{(n \cdot \sum x^2 - (\sum x)^2)}$$
$$b = \frac{(\sum y - m \cdot \sum x)}{n}$$

where n = number of points

(The math is complex, but computers do it instantly.)

Programming

```
function linearRegression(points) {  
    const n = points.length;  
    const sumX = points.reduce((sum, p) => sum + p.x, 0);  
    const sumY = points.reduce((sum, p) => sum + p.y, 0);  
    const sumXY = points.reduce((sum, p) => sum + p.x * p.y, 0);  
    const sumX2 = points.reduce((sum, p) => sum + p.x * p.x, 0);  
  
    const m = (n * sumXY - sumX * sumY) / (n * sumX2 - sumX * sumX);  
    const b = (sumY - m * sumX) / n;  
  
    return { slope: m, intercept: b };  
}  
  
const data = [  
    {x: 1, y: 2},  
    {x: 2, y: 4},  
    {x: 3, y: 5},  
    {x: 4, y: 7}  
];  
  
const line = linearRegression(data);
```

```
// { slope: 1.7, intercept: 0.5 }
// Best fit: y = 1.7x + 0.5
```

Use Cases

- **Trend analysis:** Sales over time
- **Predictions:** Forecasting
- **Correlation:** Relationship between variables
- **Machine learning:** Linear models

10. Inequalities with Lines

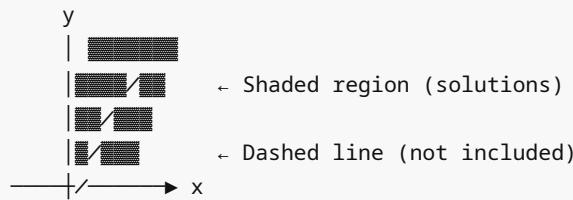
Linear Inequalities

Instead of $y = mx + b$, use $<$, $>$, \leq , \geq

$y > 2x + 1$ (above the line)
 $y \leq -x + 3$ (on or below the line)

Graphing Inequalities

$y > 2x + 1$:



Steps:

1. Graph the line $y = 2x + 1$
2. Dashed line if $<$ or $>$ (not included)
3. Solid line if \leq or \geq (included)
4. Shade above for $>$, \geq
5. Shade below for $<$, \leq

Testing Points

Check if a point satisfies the inequality:

Is $(2, 6)$ a solution to $y > 2x + 1$?

$6 > 2(2) + 1$
 $6 > 5$
True ✓

Is $(0, 0)$ a solution?

$0 > 2(0) + 1$
 $0 > 1$
False ✗

Common Mistakes & Misconceptions

✗ "Slope is always positive"

Slope can be negative, zero, or undefined.

✗ "Steep lines have small slopes"

Opposite! Steep lines have large absolute slopes:

$m = 10$: Very steep

$m = 0.1$: Very gradual

✗ "Parallel lines have the same equation"

Same slope, but different y-intercepts:

$y = 2x + 1$

$y = 2x + 5$

(parallel but different)

✗ "b is always positive"

Y-intercept can be negative:

$y = 2x - 5$ ($b = -5$)

✗ "All linear equations look like $y = mx + b$ "

Other forms exist:

Standard form: $Ax + By = C$

Point-slope: $y - y_1 = m(x - x_1)$

✗ "Lines always have exactly one intersection"

- **Same line:** Infinite intersections
- **Parallel lines:** Zero intersections
- **Different non-parallel:** Exactly one

Tiny Practice

Write equations:

1. Slope 3, y-intercept -2
2. Slope $-1/2$, passes through $(0, 4)$
3. Passes through $(1, 5)$ and $(3, 11)$

Find slope and y-intercept: 4. $y = -2x + 7$ 5. $3x + 2y = 6$

Determine: 6. Are $y = 3x + 1$ and $y = 3x - 2$ parallel? 7. Are $y = 2x + 1$ and $y = -\frac{1}{2}x + 3$ perpendicular?

Solve: 8. Find intersection of $y = 2x + 1$ and $y = x + 3$ 9. Convert 25°C to Fahrenheit using $F = (9/5)C + 32$

► Answers

Summary Cheat Sheet

Standard Form

```
y = mx + b  
  
m = slope (rate of change)  
b = y-intercept (starting value)
```

Finding Slope

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Positive: / rising

Negative: \ falling

Zero: - horizontal

Undefined: | vertical

Writing Equations

Given	Method
m and b	$y = mx + b$
m and point	$y - y_1 = m(x - x_1)$
Two points	Find m , then use point-slope

Special Lines

Horizontal: $y = b$ ($m = 0$)

Vertical: $x = a$ ($m = \text{undefined}$)

Origin: $y = mx$ ($b = 0$)

Relationships

Parallel: Same slope ($m_1 = m_2$)

Perpendicular: $m_1 \times m_2 = -1$

Programming

```
// Function  
const f = (x, m, b) => m * x + b;  
  
// Object
```

```
const line = {
  slope: 2,
  intercept: 3,
  eval(x) { return this.slope * x + this.intercept; }
};

// Regression
const {slope, intercept} = linearRegression(data);
```

Next Steps

Linear functions are your foundation for understanding all functions. You now grasp:

- Constant rate of change
- Slope and intercept
- Graphing and applications

Next, we'll explore **Trigonometry**—functions based on angles, rotation, and waves.

Continue to: [08-trigonometry.md](#)