Integrals

Why This Matters

Integrals measure accumulation. They answer:

> *"How much total change happened?"*

Integrals are everywhere:

- **Physics**: Distance from velocity, work from force
- **Statistics**: Probability, expected values, distributions
- **Economics**: Total cost, consumer surplus
- **Data science**: Area under ROC curve, cumulative distributions
- **Engineering**: Signal processing, control systems

Understanding integrals means understanding **total quantities from rates**.

The Big Picture: From Rates to Totals

The Fundamental Question

Given: Rate of change (derivative) **Find**: Total amount (original function)

```
Speed (mph) → Total distance traveled
Flow rate (gal/min) → Total water
Marginal cost → Total cost
```

Derivative vs Integral

```
        differentiation
f(x) ───────────────────────→ f'(x)
     ←───────────────────────
        integration
```

Integrals "undo" derivatives (with a twist).

---

1. The Accumulation Concept

Visual: Area Under Curve

Definite integral from a to b:

```
  y
  |     /\
  |    /  \
  |   /    \
  |  /▓▓▓▓▓▓\
──┼─────────────▶ x
  a          b
```

```
∫[a to b] f(x)dx = shaded area
```

Riemann Sums: Building Intuition

Approximate area with rectangles:

```
   |
   |   ||||||||
   |   ||||||||
   |   ||||||||
───┴─────────────
   a          b

Area ≈ f(x₁)Δx + f(x₂)Δx + ... + f(xₙ)Δx
```

As rectangles get thinner ($\Delta x \to 0$), approximation becomes exact:

```
           n
∫[a to b] f(x)dx = lim Σ f(xᵢ)Δx
                  n→∞ i=1
```

This limit of sums is the **definite integral**.

---

2. Definite vs Indefinite Integrals

Definite Integral

```
∫[a to b] f(x)dx

- Has limits: a and b
- Gives a NUMBER (the accumulated total)
- Represents area, total change, etc.
```

Example:

```
∫[0 to 3] x² dx = 9  (units depend on context)
```

Indefinite Integral

```
∫ f(x)dx = F(x) + C

- No limits
- Gives a FUNCTION (antiderivative)
- C is an arbitrary constant
```

Example:

```
∫ x² dx = x³/3 + C
```

The "+ C" is crucial because derivatives of constants are zero:

```
d/dx(x³/3) = x²
d/dx(x³/3 + 5) = x²
d/dx(x³/3 + C) = x²  for any C
```

---

3. The Fundamental Theorem of Calculus

Part 1: Connecting Derivative and Integral

If F'(x) = f(x), then:

```
∫[a to b] f(x)dx = F(b) - F(a)
```

In words:

- To find definite integral, find antiderivative F
- Evaluate at endpoints: F(b) - F(a)

Example: ∫[1 to 3] x^2 dx

```
Antiderivative: F(x) = x³/3

∫[1 to 3] x² dx = F(3) - F(1)
                = 27/3 - 1/3
                = 9 - 1/3
                = 26/3
```

Part 2: Derivative of an Integral

```
d/dx[∫[a to x] f(t)dt] = f(x)
```

Integration and differentiation are inverse operations.

---

4. Basic Integration Rules

Power Rule (Reverse of Derivative)

```
∫ xⁿ dx = xⁿ⁺¹/(n+1) + C  (if n ≠ -1)
```

Examples:

```
∫ x³ dx = x⁴/4 + C
∫ x dx = x²/2 + C
∫ 1 dx = x + C
```

Special case (n = -1):

```
∫ 1/x dx = ln|x| + C
```

Constant Multiple

$$\int cf(x)dx = c\int f(x)dx$$

Example:

$$\int 5x^2\, dx = 5\int x^2\, dx = 5\cdot x^3/3 + C = 5x^3/3 + C$$

Sum/Difference

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

Example:

$$\int (x^2 + 3x - 5)dx = x^3/3 + 3x^2/2 - 5x + C$$

---

5. Common Antiderivatives

Polynomials

$$\int x^n\, dx = x^{n+1}/(n+1) + C$$

Exponential

$$\int e^x\, dx = e^x + C$$

$$\int a^x\, dx = a^x/\ln(a) + C$$

Logarithmic

$$\int 1/x\, dx = \ln|x| + C$$

Trigonometric

$$\int \sin(x)dx = -\cos(x) + C$$
$$\int \cos(x)dx = \sin(x) + C$$
$$\int \sec^2(x)dx = \tan(x) + C$$
$$\int 1/\sqrt{(1-x^2)}\, dx = \sin^{-1}(x) + C$$
$$\int 1/(1+x^2)\, dx = \tan^{-1}(x) + C$$

---

6. Integration Techniques (Brief Overview)

Substitution (Chain Rule in Reverse)

For integrals like $\int f(g(x))\cdot g'(x)dx$:

Let $u = g(x)$, then $du = g'(x)dx$

$$\int f(g(x))\cdot g'(x)dx = \int f(u)du$$

Example: $\int 2x \cdot \sin(x^2)dx$

```
Let u = x², du = 2x dx

∫ 2x·sin(x²)dx = ∫ sin(u)du
                = -cos(u) + C
                = -cos(x²) + C
```

Integration by Parts (Product Rule in Reverse)

```
∫ u dv = uv - ∫ v du
```

Example: $\int x \cdot e^x \, dx$

```
Let u = x, dv = eˣdx
Then du = dx, v = eˣ

∫ x·eˣ dx = x·eˣ - ∫ eˣ dx
          = x·eˣ - eˣ + C
          = eˣ(x - 1) + C
```

Partial Fractions (For Rational Functions)

Break complex fractions into simpler ones.

```
     1            A         B
────────── = ────── + ──────
(x-1)(x+2)    x-1       x+2
```

Then integrate each separately.

---

7. Definite Integrals: Properties

Basic Properties

```
∫[a to b] cf(x)dx = c∫[a to b] f(x)dx

∫[a to b] [f(x) + g(x)]dx = ∫[a to b] f(x)dx + ∫[a to b] g(x)dx

∫[a to a] f(x)dx = 0

∫[a to b] f(x)dx = -∫[b to a] f(x)dx
```

Additivity

```
∫[a to b] f(x)dx + ∫[b to c] f(x)dx = ∫[a to c] f(x)dx
```

Comparison

```
If f(x) ≤ g(x) on [a,b], then:
∫[a to b] f(x)dx ≤ ∫[a to b] g(x)dx
```

---

8. Applications of Integrals

Area Between Curves

Area between f(x) and g(x) from a to b:

```
A = ∫[a to b] |f(x) - g(x)|dx
```

If $f(x) \geq g(x)$:

```
A = ∫[a to b] [f(x) - g(x)]dx
```

Example: Area between $y = x^2$ and $y = x$ from 0 to 1

```
A = ∫[0 to 1] (x - x²)dx
  = [x²/2 - x³/3][0 to 1]
  = 1/2 - 1/3
  = 1/6
```

Distance from Velocity

If v(t) is velocity, total distance is:

```
distance = ∫[t₁ to t₂] v(t)dt
```

Example: $v(t) = 3t^2$ from $t = 0$ to $t = 2$

```
distance = ∫[0 to 2] 3t² dt
         = [t³][0 to 2]
         = 8 - 0
         = 8 units
```

Total Cost from Marginal Cost

If MC(x) is marginal cost:

```
Total cost = Fixed cost + ∫[0 to x] MC(q)dq
```

Example: $MC(x) = 2x + 5$, fixed cost = $100

```
Total cost = 100 + ∫[0 to x] (2q + 5)dq
           = 100 + [q² + 5q][0 to x]
           = 100 + x² + 5x
```

Average Value

Average value of f on [a,b]:

```
        1      b
f_avg = ─── ∫[a to ] f(x)dx
        b-a
```

Example: Average of f(x) = x^2 on [0, 3]

```
        1   ∫[0 to 3] x² dx
f_avg = ───
        3

       = 1/3 · [x³/3][0 to 3]
       = 1/3 · 9
       = 3
```

Probability (Area = 1)

For probability density function f(x):

```
P(a ≤ X ≤ b) = ∫[a to b] f(x)dx

∫[-∞ to ∞] f(x)dx = 1  (total probability)
```

Work and Energy

Work = force × distance

If force varies:

```
W = ∫[a to b] F(x)dx
```

Example: Spring with F(x) = kx from 0 to d

```
W = ∫[0 to d] kx dx
  = [kx²/2][0 to d]
  = kd²/2
```

---

9. Numerical Integration (Programming)

Trapezoidal Rule

Approximate area using trapezoids:

```
∫[a to b] f(x)dx ≈ (b-a)/2 · [f(a) + f(b)]
```

Better with multiple intervals:

```
function trapezoidalRule(f, a, b, n) {
  const h = (b - a) / n;
  let sum = (f(a) + f(b)) / 2;

  for (let i = 1; i < n; i++) {
```

```
    sum += f(a + i*h);
  }

  return h * sum;
}

// Example: ∫[0 to 1] x² dx (exact: 1/3)
const f = x => x**2;
trapezoidalRule(f, 0, 1, 100);  // ≈ 0.33335
```

Simpson's Rule

More accurate (uses parabolas):

```
function simpsonsRule(f, a, b, n) {
  // n must be even
  const h = (b - a) / n;
  let sum = f(a) + f(b);

  for (let i = 1; i < n; i++) {
    const coeff = (i % 2 === 0) ? 2 : 4;
    sum += coeff * f(a + i*h);
  }

  return (h / 3) * sum;
}

simpsonsRule(f, 0, 1, 100);  // ≈ 0.333333333
```

Monte Carlo Integration

Use random sampling:

```
function monteCarloIntegrate(f, a, b, numSamples) {
  let sum = 0;

  for (let i = 0; i < numSamples; i++) {
    const x = a + Math.random() * (b - a);
    sum += f(x);
  }

  return (b - a) * sum / numSamples;
}

monteCarloIntegrate(f, 0, 1, 10000);  // ≈ 0.333
```

Useful for high-dimensional integrals (where grid methods fail).

---

10. Integrals as Array Reduction

The reduce() Analogy

JavaScript reduce is like discrete integration:

```
// Sum array elements (discrete integral)
const values = [1, 2, 3, 4, 5];
const total = values.reduce((acc, val) => acc + val, 0);
// total = 15

// This is like: ∫ values dx ≈ Σ values[i]
```

Cumulative Sum (Running Integral)

```
function cumulativeSum(arr) {
  let cumsum = [0];
  for (let i = 0; i < arr.length; i++) {
    cumsum.push(cumsum[i] + arr[i]);
  }
  return cumsum;
}

cumulativeSum([1, 2, 3, 4]);  // [0, 1, 3, 6, 10]

// Like: F(x) = ∫[0 to x] f(t)dt
```

From Rates to Totals

```
// Velocities at each second
const velocities = [10, 15, 20, 25, 30];  // m/s

// Total distance (trapezoidal approximation)
let distance = 0;
for (let i = 0; i < velocities.length - 1; i++) {
  distance += (velocities[i] + velocities[i+1]) / 2 * 1;  // 1 sec intervals
}
// distance ≈ ∫ v(t) dt
```

---

11. Improper Integrals (Infinite Limits)

Infinite Upper Limit

```
∫[a to ∞] f(x)dx = lim[b→∞] ∫[a to b] f(x)dx
```

Example: ∫[1 to ∞] $1/x^2$ dx

```
= lim[b→∞] [-1/x][1 to b]
= lim[b→∞] (-1/b + 1)
```

```
= 0 + 1
= 1  (converges)
```

But: ∫[1 to ∞] 1/x dx diverges (goes to ∞)

When They Converge

```
∫[1 to ∞] 1/xᵖ dx converges if p > 1
                  diverges if p ≤ 1
```

---

12. Connection to Other Concepts

Integration and Probability

Cumulative Distribution Function (CDF):

```
F(x) = ∫[-∞ to x] f(t)dt

where f(t) is probability density function (PDF)
```

Integration and Machine Learning

Loss over dataset:

```
Total loss = ∫ L(f(x), y)·p(x,y) dx dy

In practice: Average over samples
```

Area Under ROC Curve (AUC):

```
AUC = ∫[0 to 1] TPR(FPR) d(FPR)
```

Differential Equations

Many solutions involve integrals:

```
dy/dx = f(x)  →  y = ∫ f(x)dx
```

Common Mistakes & Misconceptions

❌ "Forgetting the + C"

Indefinite integrals always have an arbitrary constant.

```
∫ x dx = x²/2 + C  (not just x²/2)
```

❌ "∫ f·g = (∫f)·(∫g)"

No! Integration doesn't distribute over multiplication.

❌ "∫ f/g = (∫f)/(∫g)"

No! Use substitution or other techniques.

❌ "Area is always positive"

```
∫[-1 to 1] x dx = 0  (areas cancel)

For geometric area, use: ∫ |f(x)|dx
```

❌ "Definite integral needs + C"

No! Definite integrals give numbers, not functions.

Tiny Practice

Find antiderivatives:

1. ∫ $(3x^2 - 2x + 1)dx$
2. ∫ $(x^3 + 1/x)dx$
3. ∫ e^x dx
4. ∫ sin(x)dx
5. ∫ $(2x + 1)^3 \cdot 2$ dx (hint: substitution)

Evaluate definite integrals: 6. ∫[0 to 2] x^2 dx 7. ∫[1 to e] (1/x) dx 8. ∫[-π to π] sin(x)dx

Applications: 9. Find area under $y = x^2$ from x = 0 to x = 3 10. If v(t) = 2t + 1, find distance from t = 0 to t = 3

▶ Answers

Summary Cheat Sheet

Definitions

```
Definite:   ∫[a to b] f(x)dx = F(b) - F(a)  (number)

Indefinite: ∫ f(x)dx = F(x) + C  (function)
```

Fundamental Theorem

```
∫[a to b] f(x)dx = F(b) - F(a)

where F'(x) = f(x)
```

Key Rules

Integral	Result
∫ x^n dx	$x^{n+1}/(n+1) + C$
∫ 1/x dx	ln|x| + C
∫ e^x dx	$e^x + C$

∫ sin(x)dx	-cos(x) + C
∫ cos(x)dx	sin(x) + C
∫ cf(x)dx	c∫f(x)dx
∫ [f+g]dx	∫f dx + ∫g dx

Applications

```
Area: ∫[a to b] f(x)dx

Distance: ∫[t₁ to t₂] v(t)dt

Average: (1/(b-a))∫[a to b] f(x)dx

Work: ∫[a to b] F(x)dx
```

Programming

```
// Trapezoidal
const integrate = (f, a, b, n) => {
  const h = (b-a)/n;
  let sum = (f(a) + f(b))/2;
  for (let i = 1; i < n; i++) sum += f(a + i*h);
  return h * sum;
};

// Reduce analogy
const total = arr.reduce((sum, x) => sum + x, 0);
```

Congratulations! 🎉

You've completed the entire mathematics curriculum from **numbers to calculus**!

You now understand:

- ✓ Number systems and arithmetic
- ✓ Algebraic manipulation
- ✓ Functions and their properties
- ✓ Coordinate geometry
- ✓ Trigonometry
- ✓ Limits and continuity
- ✓ Derivatives (rates of change)
- ✓ Integrals (accumulation)

What's Next?

Keep practicing:

- Work through problems in each chapter

- Apply concepts to programming projects
- Explore Khan Academy, 3Blue1Brown, or Brilliant

Advanced topics (when ready):

- Multivariable calculus (functions of x, y, z)
- Differential equations (modeling change)
- Linear algebra (vectors, matrices, transformations)
- Real analysis (rigorous foundations)
- Probability and statistics

Return to: README.md to review any topics

You've built a solid mathematical foundation. Now go apply it! 🚀