

Trigonometry

Why This Matters

Trigonometry is the mathematics of **angles, rotation, and waves**. It's essential for:

- **Game development:** Rotation, aiming, circular motion
- **Computer graphics:** 3D transformations, lighting, cameras
- **Physics simulations:** Projectiles, pendulums, waves
- **Signal processing:** Audio, video, compression (Fourier transforms)
- **Animation:** Smooth motion, easing, orbits
- **Navigation:** GPS, triangulation

Trigonometry connects **angles** with **coordinates**, making circular and periodic motion mathematically tractable.

The Big Picture: Beyond Triangles

Traditional approach: "Trig is about triangles." **Modern approach:** "Trig is about **rotation** and **periodic motion**."

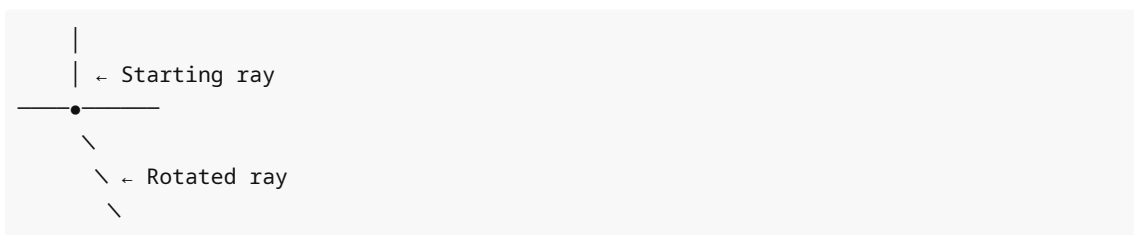
While trig started with triangles, its real power is in describing:

- **Circular motion:** Wheels, orbits, gears
 - **Waves:** Sound, light, tides, oscillations
 - **Periodic patterns:** Seasons, clocks, rhythms
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1. Angles: Rotation, Not Shapes

What Is an Angle?

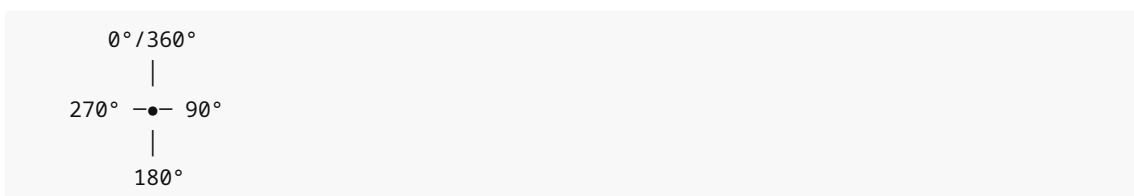
An **angle** measures **rotation** from a starting direction.



Angle = amount of rotation

Degrees

360° = one full rotation (full circle)



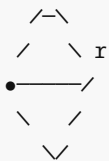
Common angles:

90° = quarter turn (right angle)
 180° = half turn (straight line)
 270° = three-quarters turn
 360° = full turn (back to start)

Radians (The Natural Unit)

Radian: The angle when arc length equals radius

Arc length = radius



Key fact: One full circle = 2π radians

360° = 2π radians
 180° = π radians
 90° = $\pi/2$ radians

Why radians? They make calculus formulas cleaner. Most programming uses radians.

Converting Between Degrees and Radians

Degrees to Radians: multiply by $\pi/180$
Radians to Degrees: multiply by $180/\pi$

Examples:

45° = $45 \times \pi/180 = \pi/4$ radians
 $\pi/6$ radians = $\pi/6 \times 180/\pi = 30^\circ$

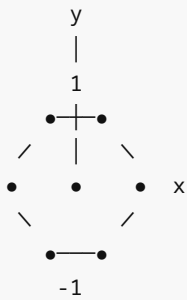
Programming:

```
const degToRad = deg => deg * Math.PI / 180;  
const radToDeg = rad => rad * 180 / Math.PI;  
  
degToRad(90);    //  $\pi/2 \approx 1.571$   
radToDeg(Math.PI); // 180
```

2. The Unit Circle (Your Mental Model)

What It Is

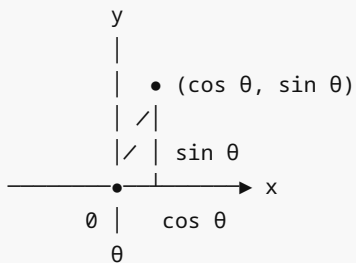
A circle with radius 1 centered at the origin.



Every point on the circle has coordinates (x, y) where $x^2 + y^2 = 1$

Angle on the Unit Circle

An angle θ (theta) defines a point on the circle:



Key insight:

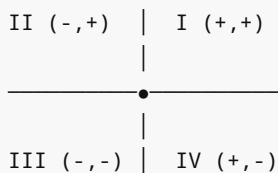
- **x-coordinate = $\cos(\theta)$**
- **y-coordinate = $\sin(\theta)$**

This is the definition of sine and cosine!

Special Angles

$\theta = 0^\circ$:	$(1, 0)$	$\cos(0) = 1, \quad \sin(0) = 0$
$\theta = 90^\circ$:	$(0, 1)$	$\cos(90^\circ) = 0, \quad \sin(90^\circ) = 1$
$\theta = 180^\circ$:	$(-1, 0)$	$\cos(180^\circ) = -1, \quad \sin(180^\circ) = 0$
$\theta = 270^\circ$:	$(0, -1)$	$\cos(270^\circ) = 0, \quad \sin(270^\circ) = -1$

Quadrants



- **Quadrant I:** Both positive
- **Quadrant II:** cos negative, sin positive
- **Quadrant III:** Both negative
- **Quadrant IV:** cos positive, sin negative

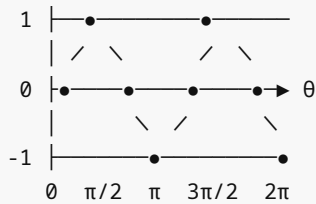
3. Sine and Cosine: The Core Functions

Definitions

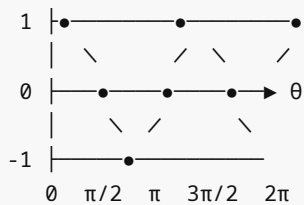
On the unit circle:

$\cos(\theta)$ = x-coordinate of point at angle θ
 $\sin(\theta)$ = y-coordinate of point at angle θ

Graph of $\sin(\theta)$:



Graph of $\cos(\theta)$:



Key Properties

Range: Both oscillate between -1 and 1

$$-1 \leq \sin(\theta) \leq 1$$
$$-1 \leq \cos(\theta) \leq 1$$

Period: Repeat every 2π (360°)

$$\sin(\theta + 2\pi) = \sin(\theta)$$
$$\cos(\theta + 2\pi) = \cos(\theta)$$

Phase shift: Cosine is sine shifted left by $\pi/2$

$$\cos(\theta) = \sin(\theta + \pi/2)$$

Pythagorean identity:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

Because: $x^2 + y^2 = 1$ on unit circle

Programming

```
Math.sin(Math.PI / 2); // 1 (sin(90°))
Math.cos(0);           // 1 (cos(0°))
Math.sin(0);           // 0 (sin(0°))
Math.cos(Math.PI);     // -1 (cos(180°))
```

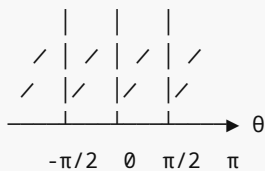
4. Tangent and Other Functions

Tangent

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{y}{x}$$

Interpretation: Slope of the line from origin to $(\cos \theta, \sin \theta)$

Graph:



Vertical asymptotes at $\pm\pi/2, \pm3\pi/2, \dots$
(where $\cos(\theta) = 0$)

Range: All real numbers $(-\infty \text{ to } +\infty)$

Period: π (repeats every 180°)

Reciprocal Functions

$$\csc(\theta) = \frac{1}{\sin(\theta)} \quad (\text{cosecant})$$

$$\sec(\theta) = \frac{1}{\cos(\theta)} \quad (\text{secant})$$

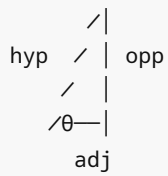
$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)} \quad (\text{cotangent})$$

Less common, but useful in some contexts.

5. Right Triangle Interpretation

SOH-CAH-TOA

For a **right triangle** with angle θ :

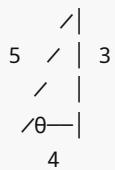


$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

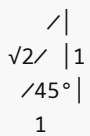
Example: Triangle with sides 3, 4, 5



$$\begin{aligned}\sin(\theta) &= 3/5 = 0.6 \\ \cos(\theta) &= 4/5 = 0.8 \\ \tan(\theta) &= 3/4 = 0.75\end{aligned}$$

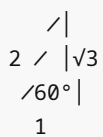
Special Right Triangles

45-45-90 triangle:



$$\begin{aligned}\sin(45^\circ) &= 1/\sqrt{2} = \sqrt{2}/2 \approx 0.707 \\ \cos(45^\circ) &= 1/\sqrt{2} = \sqrt{2}/2 \approx 0.707 \\ \tan(45^\circ) &= 1\end{aligned}$$

30-60-90 triangle:



```
sin(30°) = 1/2 = 0.5
cos(30°) = √3/2 ≈ 0.866
sin(60°) = √3/2 ≈ 0.866
cos(60°) = 1/2 = 0.5
```

6. Inverse Trig Functions

What They Do

Inverse functions answer: "What angle gives this value?"

```
sin(θ) = 0.5 → θ = ?
arcsin(0.5) = 30° (or π/6)
```

Notation

```
arcsin(x) or sin-1(x) (inverse sine)
arccos(x) or cos-1(x) (inverse cosine)
arctan(x) or tan-1(x) (inverse tangent)
```

Note: $\sin^{-1}(x) \neq 1/\sin(x)$. It means "inverse", not "reciprocal".

Examples

```
sin-1(1) = 90° = π/2
cos-1(0) = 90° = π/2
tan-1(1) = 45° = π/4
```

Domains and Ranges

```
arcsin: Domain [-1, 1], Range [-π/2, π/2]
arccos: Domain [-1, 1], Range [0, π]
arctan: Domain all reals, Range (-π/2, π/2)
```

Programming

```
Math.asin(0.5); // π/6 ≈ 0.524 (30°)
Math.acos(0);   // π/2 ≈ 1.571 (90°)
Math.atan(1);   // π/4 ≈ 0.785 (45°)

// atan2: handles all quadrants correctly
Math.atan2(y, x); // angle to point (x, y)
```

7. Real-World Applications

Rotation and Direction

Point a character toward target:

```
function angleTo(from, to) {
  const dx = to.x - from.x;
  const dy = to.y - from.y;
  return Math.atan2(dy, dx); // angle in radians
}

const player = {x: 0, y: 0};
const enemy = {x: 3, y: 4};
const angle = angleTo(player, enemy); // ~0.927 rad (53°)
```

Circular Motion

Move in a circle:

```
function circularMotion(centerX, centerY, radius, angle) {
  return {
    x: centerX + radius * Math.cos(angle),
    y: centerY + radius * Math.sin(angle)
  };
}

// Orbit around (100, 100) with radius 50
for (let angle = 0; angle < 2 * Math.PI; angle += 0.1) {
  const pos = circularMotion(100, 100, 50, angle);
  // Plot or draw at pos
}
```

Projectile Motion

Initial velocity at angle θ :

$v_x = v_0 \cos(\theta)$ (horizontal component)
 $v_y = v_0 \sin(\theta)$ (vertical component)
 $x(t) = v_x \times t$
 $y(t) = v_y \times t - \frac{1}{2}gt^2$ (with gravity)

```
function shoot(speed, angleDegrees) {
  const angleRad = angleDegrees * Math.PI / 180;
  return {
    vx: speed * Math.cos(angleRad),
    vy: speed * Math.sin(angleRad)
  };
}

const velocity = shoot(100, 45); // 45° angle
// vx ≈ 70.7, vy ≈ 70.7
```

Waves and Oscillation

Sine wave for smooth oscillation:

```
// Bounce up and down
function bounce(time, amplitude, frequency) {
  return amplitude * Math.sin(frequency * time);
}

// Animate
let time = 0;
function animate() {
  const y = bounce(time, 50, 2); // 50px amplitude, 2 Hz
  sprite.y = centerY + y;
  time += 0.1;
  requestAnimationFrame(animate);
}
```

Camera/3D Rotation

Rotate point around origin:

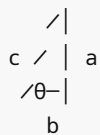
```
function rotate(point, angle) {
  const cos = Math.cos(angle);
  const sin = Math.sin(angle);
  return {
    x: point.x * cos - point.y * sin,
    y: point.x * sin + point.y * cos
  };
}

const rotated = rotate({x: 1, y: 0}, Math.PI / 2);
// Result: {x: 0, y: 1} (rotated 90°)
```

Distance and Triangulation

Find distance using angles:

If you know angle and one side, you can find others:


$$a = c \times \sin(\theta)$$
$$b = c \times \cos(\theta)$$
$$c = a / \sin(\theta) = b / \cos(\theta)$$

8. Transformations of Trig Functions

General Form

$$y = A \sin(B(x - C)) + D$$

A = amplitude (height)

B = frequency (speed of oscillation)

C = phase shift (horizontal shift)

D = vertical shift

Amplitude (A)

Stretches vertically:

$$\begin{aligned} y &= 2 \sin(x) && (\text{amplitude } 2, \text{ oscillates } -2 \text{ to } 2) \\ y &= 0.5 \sin(x) && (\text{amplitude } 0.5, \text{ oscillates } -0.5 \text{ to } 0.5) \end{aligned}$$

Frequency (B)

Changes period:

$$\text{Period} = 2\pi / B$$

$$\begin{aligned} y &= \sin(2x) && (\text{period} = \pi, \text{ faster}) \\ y &= \sin(0.5x) && (\text{period} = 4\pi, \text{ slower}) \end{aligned}$$

Phase Shift (C)

Horizontal shift:

$$\begin{aligned} y &= \sin(x - \pi/2) && (\text{shifted right by } \pi/2) \\ y &= \sin(x + \pi/4) && (\text{shifted left by } \pi/4) \end{aligned}$$

Vertical Shift (D)

Moves up/down:

$$\begin{aligned} y &= \sin(x) + 1 && (\text{oscillates } 0 \text{ to } 2) \\ y &= \sin(x) - 2 && (\text{oscillates } -3 \text{ to } -1) \end{aligned}$$

Example: Ocean Wave

```
function oceanHeight(x, time) {  
  const amplitude = 2;      // 2m waves  
  const frequency = 0.5;    // slower waves  
  const speed = 0.1;        // wave moves  
  return amplitude * Math.sin(frequency * (x - speed * time));  
}
```

9. Important Identities

Pythagorean Identity

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

Variations:

$$1 + \tan^2(\theta) = \sec^2(\theta)$$

$$1 + \cot^2(\theta) = \csc^2(\theta)$$

Even/Odd Functions

$$\cos(-\theta) = \cos(\theta) \quad (\text{even function, symmetric})$$

$$\sin(-\theta) = -\sin(\theta) \quad (\text{odd function, antisymmetric})$$

$$\tan(-\theta) = -\tan(\theta) \quad (\text{odd function})$$

Sum/Difference

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

Double Angle

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

$$= 2\cos^2(\theta) - 1$$

$$= 1 - 2\sin^2(\theta)$$

You don't need to memorize these unless you use them often. They're derivable from the unit circle.

Common Mistakes & Misconceptions

✗ "Trig is only for triangles"

Trig is fundamentally about rotation and periodic motion.

✗ "Degrees and radians are the same"

They're different units. Most math/programming uses radians.

✗ " $\sin^{-1}(x)$ means $1/\sin(x)$ "

No! $\sin^{-1}(x)$ means $\arcsin(x)$ (inverse function). The reciprocal is $\csc(x) = 1/\sin(x)$.

✗ " $\tan(90^\circ) = 0$ "

No! $\tan(90^\circ)$ is **undefined** ($\cos(90^\circ) = 0$, division by zero).

✗ "Sine and cosine can be greater than 1"

Not for real angles. They're always in $[-1, 1]$.

✗ "Trig functions only work for 0° to 90° "

They work for all angles, including negative and $> 360^\circ$.

Tiny Practice

Convert:

- 1. 180° to radians
- 2. $\pi/3$ radians to degrees

Evaluate (without calculator): 3. $\sin(0^\circ)$ 4. $\cos(90^\circ)$ 5. $\tan(45^\circ)$ 6. $\sin(180^\circ)$

Find angles: 7. If $\sin(\theta) = 0.5$, what is θ (0° to 90°)? 8. If $\cos(\theta) = 0$, what is θ (0° to 180°)?

Application: 9. A point moves in a circle of radius 5. At angle 60° , what are its (x, y) coordinates?

► Answers

Summary Cheat Sheet

Angles

$360^\circ = 2\pi$ radians (full circle)
 $180^\circ = \pi$ radians
 $90^\circ = \pi/2$ radians

Deg to Rad: $\times \pi/180$
Rad to Deg: $\times 180/\pi$

Unit Circle

Point at angle θ : $(\cos \theta, \sin \theta)$

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

Core Functions

$\sin(\theta)$ = y-coordinate
 $\cos(\theta)$ = x-coordinate
 $\tan(\theta) = \sin(\theta)/\cos(\theta) = y/x$

Special Values

	0°	30°	45°	60°	90°
sin:	0	1/2	$\sqrt{2}/2$	$\sqrt{3}/2$	1
cos:	1	$\sqrt{3}/2$	$\sqrt{2}/2$	1/2	0
tan:	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undefined

Properties

Range: $-1 \leq \sin, \cos \leq 1$
Period: 2π (360°)
tan period: π (180°)

Programming

```
Math.sin(θ), Math.cos(θ), Math.tan(θ) // radians
Math.asin(x), Math.acos(x), Math.atan(x) // inverse
Math.atan2(y, x) // angle to (x,y)

// Circular motion
x = centerX + radius * Math.cos(angle);
y = centerY + radius * Math.sin(angle);

// Rotation
x' = x*cos(θ) - y*sin(θ);
y' = x*sin(θ) + y*cos(θ);
```

Next Steps

Trigonometry connects angles, coordinates, and periodic motion. You now understand:

- The unit circle
- Sine, cosine, tangent
- Applications in rotation and waves

Next, we'll explore **Polynomials**—functions with multiple power terms.

Continue to: [09-polynomials.md](#)