

Functions

Why This Matters

Functions are the beating heart of mathematics and programming.

Everything you've learned so far—numbers, algebra, coordinates—comes together in functions. A function is:

- **A relationship** between inputs and outputs
- **A rule** that transforms data
- **A mapping** from one set to another

As a developer, you already use functions constantly:

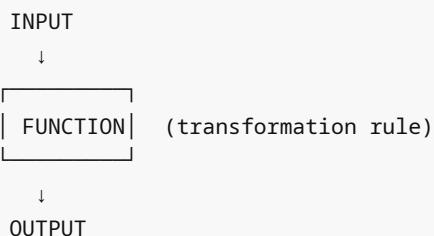
```
function double(x) {  
    return x * 2;  
}
```

Mathematical functions work **exactly the same way**. Understanding them deeply unlocks calculus, data science, ML, and advanced programming.

The Big Picture: Input → Transform → Output

The Machine Metaphor

Think of a function as a **machine**:



Example: "Doubling machine"

```
Input: 3  
↓  
[×2]  
↓  
Output: 6
```

Programming Analogy

```
// Function (math)  
f(x) = 2x + 1  
  
// Function (code)  
function f(x) {
```

```
    return 2*x + 1;  
}  
  
// Both do the same thing:  
f(3) → 2(3) + 1 → 7
```

Math functions and code functions are the same concept.

1. Function Notation

Standard Form

```
f(x) = ...  
  
f      = function name  
x      = input variable (argument)  
f(x)  = output (return value)
```

Read as: "f of x" or "f at x"

Examples

```
f(x) = 2x + 1  
  
f(3) = 2(3) + 1 = 7  
f(0) = 2(0) + 1 = 1  
f(-2) = 2(-2) + 1 = -3
```

Multiple Function Names

```
f(x) = x2  
g(x) = 3x - 2  
h(x) = √x
```

Different names for different functions (like variable names in code).

Other Notation

Sometimes you'll see:

```
y = f(x)  
y = 2x + 1
```

Here, `y` is the output when input is `x`.

2. What Makes Something a Function?

The Key Rule: One Input → One Output

A function must give exactly ONE output for each input.

Valid function:

```
f(x) = x2  
  
f(2) = 4      (always 4, never anything else)  
f(2) = 4      (consistent)
```

NOT a function:

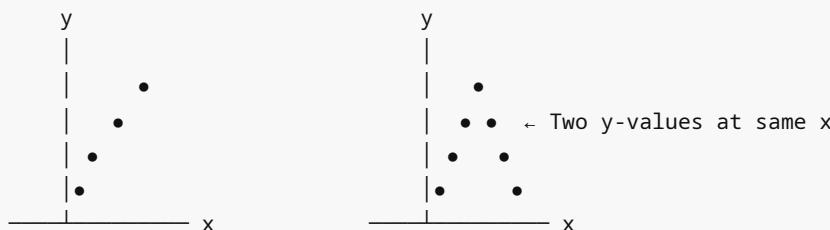
```
"What is ±√x?"  
  
√4 = 2 or -2  (two outputs for one input)
```

This is a **relation**, not a function.

Visual Test: Vertical Line Test

If a vertical line intersects a graph more than once, it's NOT a function.

Function (passes test): Not a function (fails test):



Why This Matters

Functions are predictable: Same input always gives same output.

```
// Good function (predictable)
function square(x) {
  return x * x;
}

square(5); // Always 25

// Bad "function" (unpredictable)
function random(x) {
  return Math.random(); // Different each time!
}
```

Math functions are **pure functions** in programming terms.

3. Domain and Range

Domain: All Valid Inputs

The **domain** is the set of all input values that work.

Examples:

```
f(x) = x + 5
Domain: All real numbers ( $\mathbb{R}$ )
(You can add 5 to any number)

g(x) =  $\sqrt{x}$ 
Domain:  $x \geq 0$ 
(Can't take square root of negative in real numbers)

h(x) = 1/x
Domain:  $x \neq 0$ 
(Can't divide by zero)
```

Range: All Possible Outputs

The **range** is the set of all output values the function can produce.

Examples:

```
f(x) =  $x^2$ 
Domain: All real numbers
Range:  $y \geq 0$  (squares are never negative)

g(x) = sin(x)
Domain: All real numbers
Range:  $-1 \leq y \leq 1$  (sine oscillates between -1 and 1)
```

Visual Understanding

Domain (inputs): x-axis	Function: graph	Range (outputs): y-axis
<hr/> $f(x) = x^2$		<hr/>
<- domain: \mathbb{R} ->		↑ range: $y \geq 0$ ↑

Programming Analogy

```
// Domain: integers (implicitly)
// Range: booleans
function isEven(n) {
    return n % 2 === 0;
}

// Domain: numbers (type system)
// Range: numbers
function double(x) {
    return x * 2;
}
```

Type signatures in TypeScript are like domain/range:

```
function f(x: number): number {
    return x * 2;
}
// Domain: number, Range: number
```

4. Evaluating Functions

Substitution

Replace the variable with the input value:

$$f(x) = 3x^2 - 2x + 1$$

Find $f(4)$:

$$\begin{aligned}f(4) &= 3(4)^2 - 2(4) + 1 \\&= 3(16) - 8 + 1 \\&= 48 - 8 + 1 \\&= 41\end{aligned}$$

Multiple Inputs

$$f(x) = x^2 - 5$$

$$f(0) = 0^2 - 5 = -5$$

$$f(1) = 1^2 - 5 = -4$$

$$f(2) = 4 - 5 = -1$$

$$f(3) = 9 - 5 = 4$$

Variable Expressions

You can input expressions, not just numbers:

$$f(x) = x^2 + 1$$

$$f(a) = a^2 + 1$$

$$f(2x) = (2x)^2 + 1 = 4x^2 + 1$$

$$f(x+h) = (x+h)^2 + 1 = x^2 + 2xh + h^2 + 1$$

This is crucial for calculus.

5. Graphing Functions

Every Function Has a Graph

The graph is all points $(x, f(x))$:

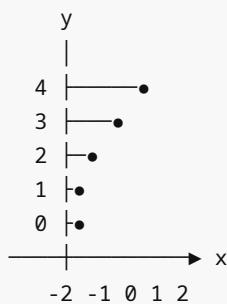
$$f(x) = x + 2$$

Points:

x: -2 -1 0 1 2

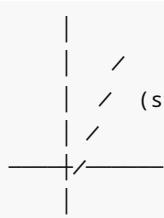
y: 0 1 2 3 4

Graph:

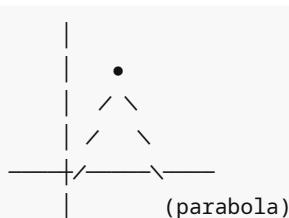


Common Function Shapes

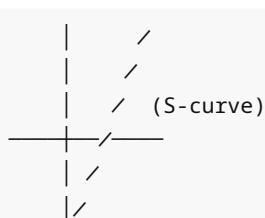
Linear: $f(x) = mx + b$



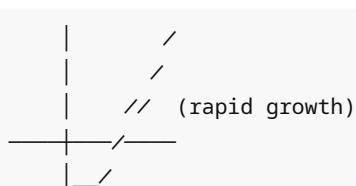
Quadratic: $f(x) = x^2$



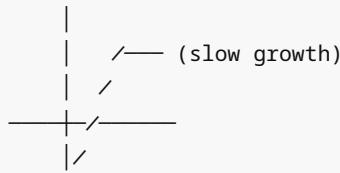
Cubic: $f(x) = x^3$



Exponential: $f(x) = 2^x$



Logarithmic: $f(x) = \log(x)$



Programming: Plotting Functions

```
// Generate points
function plotFunction(f, xMin, xMax, step) {
  const points = [];
  for (let x = xMin; x <= xMax; x += step) {
    points.push({ x, y: f(x) });
  }
  return points;
}

// Example
const f = x => x**2;
const points = plotFunction(f, -5, 5, 0.5);
// Returns [{x:-5, y:25}, {x:-4.5, y:20.25}, ...]
```

6. Composition of Functions

What It Means

Apply one function, then apply another to the result.

Notation: $(f \circ g)(x) = f(g(x))$

Read as: "f composed with g" or "f of g of x"

Visual

```
x
↓
[g] ← First function
↓
g(x)
↓
[f] ← Second function
↓
f(g(x))
```

Example

$$\begin{aligned}f(x) &= 2x + 1 \\g(x) &= x^2\end{aligned}$$

Find $(f \circ g)(3)$:

Step 1: Apply g first

$$g(3) = 3^2 = 9$$

Step 2: Apply f to result

$$f(9) = 2(9) + 1 = 19$$

So: $(f \circ g)(3) = 19$

General Form

$$(f \circ g)(x) = f(g(x))$$

Replace x in f with the entire $g(x)$:

$$f(x) = 2x + 1$$

$$g(x) = x^2$$

$$f(g(x)) = 2(x^2) + 1 = 2x^2 + 1$$

Order Matters!

$$f(g(x)) \neq g(f(x))$$

$$f(x) = 2x$$

$$g(x) = x + 1$$

$$f(g(x)) = 2(x+1) = 2x + 2$$

$$g(f(x)) = 2x + 1$$

Different results!

Programming Analogy

```
const f = x => 2*x + 1;
const g = x => x**2;

// Composition: f(g(x))
const compose = (f, g) => x => f(g(x));

const fog = compose(f, g);
fog(3); // f(g(3)) = f(9) = 19

// Also chainable:
const result = f(g(3));
```

7. Inverse Functions

What They Are

An **inverse function** undoes what the original function does.

Notation: $f^{-1}(x)$ (read as "f inverse of x")

$$\begin{aligned} f(x) &= 2x && \text{(doubles)} \\ f^{-1}(x) &= x/2 && \text{(halves, undoing the double)} \\ \\ f(5) &= 10 \\ f^{-1}(10) &= 5 && \text{(undoes it)} \end{aligned}$$

The Key Property

$$\begin{aligned} f(f^{-1}(x)) &= x \\ f^{-1}(f(x)) &= x \end{aligned}$$

They cancel each other out

Visual

$$\begin{array}{ccc} x & \xrightarrow{[f]} & f(x) \\ & \xleftarrow{[f^{-1}]} & \end{array}$$

Going forward then backward returns to x

Examples

$$\begin{array}{ll} f(x) = x + 5 & f^{-1}(x) = x - 5 \quad (\text{subtract undoes add}) \\ f(x) = 3x & f^{-1}(x) = x/3 \quad (\text{divide undoes multiply}) \\ f(x) = x^2 & f^{-1}(x) = \sqrt{x} \quad (\text{root undoes square}) \\ f(x) = e^x & f^{-1}(x) = \ln(x) \quad (\text{log undoes exponential}) \end{array}$$

Finding Inverse Functions

Method:

1. Replace $f(x)$ with y
2. Swap x and y
3. Solve for y
4. Replace y with $f^{-1}(x)$

Example: Find inverse of $f(x) = 2x + 3$

$$\begin{aligned} \text{Step 1: } y &= 2x + 3 \\ \text{Step 2: } x &= 2y + 3 \quad (\text{swap } x \text{ and } y) \\ \text{Step 3: Solve for } y \\ x &= 2y + 3 \\ x - 3 &= 2y \end{aligned}$$

```
y = (x - 3) / 2
```

```
Step 4: f-1(x) = (x - 3) / 2
```

Verify:

```
f(5) = 2(5) + 3 = 13  
f-1(13) = (13 - 3) / 2 = 5 ✓
```

Not All Functions Have Inverses

A function must be **one-to-one** (each output comes from exactly one input).

Bad example: $f(x) = x^2$

```
f(2) = 4  
f(-2) = 4 ← Same output from two inputs  
  
If you try to invert:  
f-1(4) = 2 or -2? (ambiguous!)
```

Solution: Restrict domain to $x \geq 0$, then $f^{-1}(x) = \sqrt{x}$ works.

Programming Analogy

```
// Function and its inverse  
function encode(x) {  
    return x * 2 + 3;  
}  
  
function decode(y) {  
    return (y - 3) / 2;  
}  
  
const original = 10;  
const encoded = encode(original); // 23  
const decoded = decode(encoded); // 10  
console.log(decoded === original); // true  
  
// Encryption/decryption is inverse functions
```

8. Types of Functions

Linear Functions

```
f(x) = mx + b
```

```
m = slope
```

```
b = y-intercept
```

Graph: Straight line

Quadratic Functions

$$f(x) = ax^2 + bx + c$$

Graph: Parabola (U-shaped)

Polynomial Functions

$$f(x) = a_n x^n + \dots + a_1 x + a_0$$

Sum of power terms

Rational Functions

$$f(x) = P(x) / Q(x)$$

Ratio of polynomials

$$\text{Example: } f(x) = (x+1)/(x-2)$$

Exponential Functions

$$f(x) = a^x$$

Base raised to variable power

$$\text{Example: } f(x) = 2^x$$

Logarithmic Functions

$$f(x) = \log_b(x)$$

Inverse of exponential

Trigonometric Functions

$$f(x) = \sin(x), \cos(x), \tan(x), \dots$$

Based on angles and circles

Piecewise Functions

Different rules for different inputs:

$$f(x) = \begin{cases} x + 1, & \text{if } x < 0 \\ x^2, & \text{if } 0 \leq x < 2 \\ 5, & \text{if } x \geq 2 \end{cases}$$

Programming:

```
function f(x) {  
    if (x < 0) return x + 1;  
    if (x < 2) return x**2;  
    return 5;  
}
```

9. Function Transformations

Vertical Shift

$f(x) + k \rightarrow$ Shift up by k
 $f(x) - k \rightarrow$ Shift down by k

If $f(x) = x^2$:
 $f(x) + 2 = x^2 + 2$ (parabola shifted up 2)

Horizontal Shift

$f(x - h) \rightarrow$ Shift right by h
 $f(x + h) \rightarrow$ Shift left by h

If $f(x) = x^2$:
 $f(x - 2) = (x-2)^2$ (parabola shifted right 2)

Note: Sign is opposite! +2 shifts left, -2 shifts right.

Vertical Stretch/Compression

$a \cdot f(x)$ where $a > 0$

$a > 1$: Stretch (taller)
 $0 < a < 1$: Compress (shorter)
 $a < 0$: Stretch + flip

If $f(x) = x^2$:
 $2f(x) = 2x^2$ (twice as tall)
 $0.5f(x) = 0.5x^2$ (half as tall)

Horizontal Stretch/Compression

$f(bx)$ where $b > 0$

$b > 1$: Compress (narrower)
 $0 < b < 1$: Stretch (wider)

If $f(x) = x^2$:

```
f(2x) = (2x)2 = 4x2 (narrower)
f(0.5x) = (0.5x)2 (wider)
```

Reflection

```
-f(x) → Flip over x-axis
f(-x) → Flip over y-axis

If f(x) = x2:
-f(x) = -x2 (upside-down parabola)
f(-x) = (-x)2 = x2 (same, since x2 is even)
```

10. Real-World Applications

Physics: Position Function

```
s(t) = -16t2 + v0t + s0

s(t) = position at time t
v0 = initial velocity
s0 = initial position

Models projectile motion
```

Economics: Cost Function

```
C(x) = fixed_cost + variable_cost × x

C(100) = total cost to produce 100 units
```

Computer Science: Hash Functions

```
hash(x) → fixed-size output

Used in: hash tables, cryptography, checksums
```

Data Processing: Map Function

```
// Array.map is function application
const data = [1, 2, 3, 4];
const doubled = data.map(x => x * 2); // [2, 4, 6, 8]

// This is: f(x) = 2x applied to each element
```

Machine Learning: Activation Functions

```
 $\sigma(x) = 1 / (1 + e^{-x})$  (sigmoid function)
```

Maps any input to (0, 1)
Used in neural networks

Game Development: Easing Functions

```
// Linear  
 $f(t) = t$   
  
// Ease-in (quadratic)  
 $f(t) = t^2$   
  
// Ease-out  
 $f(t) = 1 - (1-t)^2$   
  
 $t \in [0, 1]$  (time)
```

Common Mistakes & Misconceptions

✗ "f(x) and f·x are the same"

No!

```
 $f(x) =$  function applied to  $x$   
 $f \cdot x =$   $f$  multiplied by  $x$  (different notation)
```

✗ "f(a + b) = f(a) + f(b)"

Usually **false**:

```
 $f(x) = x^2$   
  
 $f(2 + 3) = f(5) = 25$   
 $f(2) + f(3) = 4 + 9 = 13$   
  
 $25 \neq 13$ 
```

✗ "(f ∘ g)(x) = f(x) · g(x)"

No! Composition ≠ multiplication:

```
( $f \circ g$ )(x) =  $f(g(x))$  (composition)  
( $f \cdot g$ )(x) =  $f(x) \cdot g(x)$  (multiplication)
```

✗ "All functions have inverses"

Only **one-to-one** functions have inverses.

✗ "Domain is always all real numbers"

Many functions have restrictions:

\sqrt{x} : $x \geq 0$
 $1/x$: $x \neq 0$
 $\log(x)$: $x > 0$

Tiny Practice

Evaluate:

1. $f(x) = 3x - 2$, find $f(5)$
2. $g(x) = x^2 + 1$, find $g(-3)$
3. $h(x) = 2x^2 - x + 4$, find $h(2)$

Composition:

4. $f(x) = 2x$, $g(x) = x + 1$, find $f(g(3))$
5. $f(x) = x^2$, $g(x) = 3x$, find $(f \circ g)(2)$

Inverse:

6. Find the inverse of $f(x) = 3x - 6$
7. If $f(x) = x + 5$, what is $f^{-1}(10)$?

Domain/Range:

8. What is the domain of $f(x) = \sqrt{x-3}$?
9. What is the range of $f(x) = x^2 + 1$?

► Answers

Summary Cheat Sheet

Function Basics

$f(x) = \dots$ (function notation)

Input: x

Output: $f(x)$

Graph: all points $(x, f(x))$

Domain and Range

Domain: Valid inputs (x-values)

Range: Possible outputs (y-values)

Operations

Operation	Notation	Meaning
Evaluate	$f(3)$	Substitute 3 for x
Compose	$(f \circ g)(x)$	$f(g(x))$

Inverse	$f^{-1}(x)$	Undoes f
Add	$(f + g)(x)$	$f(x) + g(x)$
Multiply	$(f \cdot g)(x)$	$f(x) \cdot g(x)$

Transformations

$f(x) + k$	→ Shift up
$f(x - h)$	→ Shift right
$a \cdot f(x)$	→ Vertical stretch
$f(bx)$	→ Horizontal compression
$-f(x)$	→ Flip over x-axis
$f(-x)$	→ Flip over y-axis

Programming Parallels

```
// Math function
f(x) = 2x + 1

// Code function
const f = x => 2*x + 1;

// Composition
const compose = (f, g) => x => f(g(x));

// Inverse (conceptually)
encode(x) and decode(y) are inverses
```

Next Steps

Functions are the foundation of everything in advanced mathematics. You now understand:

- What functions are
- How to evaluate and graph them
- Composition and inverses
- Transformations

Next, we'll apply this to **Linear Functions**—the simplest and most fundamental type.

Continue to: [07-linear-functions.md](#)