Linear Functions

Why This Matters

Linear functions are the **simplest and most fundamental functions**. They model:

- **Constant change**: Speed, pricing, growth
- **Relationships**: Cause and effect
- **Predictions**: Trends and forecasting

In programming and data science:

- **Linear regression**: Fitting lines to data
- **Time complexity**: O(n) algorithms
- **Interpolation**: Estimating between points
- **Animation**: Linear motion

Linear functions are your mental model for "steady change."

The Big Picture: Constant Rate of Change

Linear function: Output changes by the same amount for each unit of input.

```
Every time x increases by 1, y increases by the same amount

x:  0   1   2   3   4
y:  1   3   5   7   9

Change: +2  +2  +2  +2  (constant)
```

Graphically: A straight line.

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1. The Slope-Intercept Form

The Standard Equation

```
y = mx + b

m = slope (rate of change)
b = y-intercept (starting value)
x = input (independent variable)
y = output (dependent variable)
```

This is the most important equation in algebra.

Visual Breakdown

```
y = 2x + 3
    ↑   ↑
    │   └─ b (y-intercept) = 3
    └───── m (slope) = 2
```

```
    y
    │
 7  ├───────●    (2, 7)
 6  ├───────┆
 5  ├─────●      (1, 5)
 4  ├─────┆
 3  ├───●──────    (0, 3) ← y-intercept (b=3)
 2  ├───┆
 1  ├─●          (-1, 1)
 0  ├─┆
────┼─┴────────▶ x
   -1 0  1  2

Slope m = 2: "rise 2, run 1"
```

Programming Analogy

```javascript
// Linear function as code
function linearFunction(x, m, b) {
  return m * x + b;
}

// Or as object
const line = {
  slope: 2,
  intercept: 3,
  evaluate(x) {
    return this.slope * x + this.intercept;
  }
};

line.evaluate(5);  // 2(5) + 3 = 13
```

---

2. Understanding Slope (m)

What Is Slope?

Slope = Rate of change = Rise over run

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Interpretation

m = 2: For every 1 unit right, go up 2 units

m = -3: For every 1 unit right, go down 3 units

m = 0.5: For every 1 unit right, go up 0.5 units

m = 0: Horizontal line (no change)

m = undefined: Vertical line (infinite change)

Types of Slopes

```
Positive (m > 0):    Negative (m < 0):
      ⁄                    ⁄
     ⁄ (rising)             \ (falling)
    ⁄                        \

Zero (m = 0):        Undefined:
  ─────                    │
  (horizontal)             │ (vertical)
                           │
```

Calculating Slope from Two Points

Given: (1, 3) and (4, 9)

```
m = (y₂ - y₁) / (x₂ - x₁)
  = (9 - 3) / (4 - 1)
  = 6 / 3
  = 2
```

Real-World Slope Examples

Speed:

```
Distance = rate × time
d = rt

If r = 60 mph:
d = 60t

Slope m = 60 (60 miles per hour)
```

Pricing:

```
Cost = price_per_unit × quantity + fixed_cost
C = 5q + 20

Slope m = 5 (cost increases $5 per unit)
```

Temperature conversion:

```
F = (9/5)C + 32

Slope m = 9/5 = 1.8
(Fahrenheit changes 1.8° for each 1° Celsius)
```

---

3. Understanding Y-Intercept (b)

What Is Y-Intercept?

The y-value when x = 0 (where the line crosses the y-axis)

```
y = mx + b

When x = 0:
y = m(0) + b = b
```

Visual

```
    y
    │
  5 ├──
  4 ├──
  3 ├●──────── This is b (y-intercept)
  2 ├─/
  1 ├/
  0 ├─────────▶ x
```

Interpretation

b = 3: Starting value is 3 (when x = 0)

b = -2: Starting value is -2 (below origin)

b = 0: Line passes through origin

Real-World Y-Intercept Examples

Fixed cost:

```
Total = variable_cost × units + fixed_cost
C = 5q + 100

b = 100 (fixed cost even if q = 0)
```

Initial position:

```
position = velocity × time + starting_position
s = 3t + 10

b = 10 (started at position 10)
```

Base salary:

```
Income = commission × sales + base_salary
I = 0.1s + 30000

b = 30,000 (salary even with zero sales)
```

---

4. Writing Linear Equations

From Slope and Y-Intercept

Given: m = 3, b = -2

```
y = 3x - 2
```

Done! Just plug into y = mx + b.

From Slope and a Point

Given: m = 2, point (3, 7)

Method: Use point-slope form

```
y - y₁ = m(x - x₁)

y - 7 = 2(x - 3)
y - 7 = 2x - 6
y = 2x + 1
```

From Two Points

Given: (1, 3) and (4, 9)

Step 1: Find slope

```
m = (9 - 3) / (4 - 1) = 6/3 = 2
```

Step 2: Use point-slope form with either point

```
y - 3 = 2(x - 1)
y - 3 = 2x - 2
y = 2x + 1
```

From a Graph

Read directly:

1. Find y-intercept (where line crosses y-axis): b
2. Count rise/run from any two points: m
3. Write y = mx + b

---

5. Special Cases

Horizontal Lines

```
y = 5    (or y = b where m = 0)

    y
    |
```

```
  5 ├────────────── y = 5 (constant)
  4 ├──────────────
  3 ├──────────────
    │
────┴──────────────▶ x
```

Characteristics:

- Slope m = 0
- y never changes
- Form: y = b

Example: Temperature stays at 70°F all day.

Vertical Lines

```
x = 3

    y
    │
    │   │
    │   │ x = 3
    │   │
────┴───│──▶ x
    0   3
```

Characteristics:

- Slope undefined (division by zero)
- x never changes
- **NOT a function** (fails vertical line test)
- Form: x = a

Lines Through Origin

```
y = mx    (b = 0)

    y
    │  ⁄
    │ ⁄
    │⁄
────●─────▶ x
   origin
```

Examples:

- Direct proportionality
- y = 2x (doubling)
- Distance = speed × time (starting from origin)

---

6. Parallel and Perpendicular Lines

Parallel Lines

Same slope, different y-intercepts

```
y = 2x + 1
y = 2x - 3

Both have m = 2 (parallel)
```

Visual:

```
    y
    |   ⁄
    |  ⁄  ← y = 2x + 1
    | ⁄
    |⁄   ← y = 2x - 3
────┴────────▶ x
```

Never intersect (same direction).

Perpendicular Lines

Slopes are negative reciprocals:

```
m₁ × m₂ = -1

If m₁ = 2, then m₂ = -1/2
If m₁ = 3/4, then m₂ = -4/3
```

Example:

```
y = 2x + 1     (m = 2)
y = -½x + 3    (m = -1/2)

These are perpendicular (meet at 90°)
```

Visual:

```
    y
    |    ⁄
    |   ⁄ ← m = 2
────┼──⁄────
    |⟍
    | ⟍ ← m = -1/2
```

Special case: Horizontal and vertical lines are perpendicular

```
y = 5  (horizontal, m = 0)
x = 3  (vertical, m = undefined)
```

---

7. Applications and Examples

Motion at Constant Speed

```
Distance = speed × time
d = 60t

m = 60 mph (speed)
b = 0 (starts at origin)

After 3 hours: d = 60(3) = 180 miles
```

Linear Pricing

```
Cost = price_per_item × quantity + fixed_cost
C = 15q + 200

m = 15 (variable cost per unit)
b = 200 (fixed costs)

For 50 items: C = 15(50) + 200 = $950
```

Temperature Conversion

Celsius to Fahrenheit:

```
F = (9/5)C + 32

m = 9/5 = 1.8
b = 32

At 0°C: F = 32°F
At 100°C: F = (9/5)(100) + 32 = 212°F
```

Fahrenheit to Celsius (inverse):

```
C = (5/9)(F - 32)
C = (5/9)F - 160/9

m = 5/9 ≈ 0.556
b = -160/9 ≈ -17.78
```

Depreciation

```
Value = initial_value - depreciation_rate × years
V = 20000 - 2000t

m = -2000 (loses $2000/year)
b = 20000 (initial value)

After 5 years: V = 20000 - 2000(5) = $10,000
```

Salary with Commission

```
Income = commission_rate × sales + base
I = 0.08s + 35000

m = 0.08 (8% commission)
b = 35000 (base salary)

With $100k sales: I = 0.08(100000) + 35000 = $43,000
```

---

8. Finding Intersections

Where Two Lines Meet

Solve the system of equations:

```
Line 1: y = 2x + 1
Line 2: y = -x + 4

Set equal:
2x + 1 = -x + 4
3x = 3
x = 1

Substitute back:
y = 2(1) + 1 = 3

Intersection: (1, 3)
```

Visual

```
    y
    |
  4 ├───●      Line 2
  3 ├──●●      Intersection (1, 3)
  2 ├─●─●
  1 ├●──●      Line 1
────┼────────► x
    0  1  2
```

Break-Even Analysis

When do costs equal revenue?

```
Cost: C = 5q + 1000    (production cost)
Revenue: R = 10q       (sales income)

Break-even: C = R
5q + 1000 = 10q
1000 = 5q
```

```
q = 200 units

At 200 units, cost = revenue = $2000
```

---

9. Linear Regression (Data Fitting)

The Problem

Given data points, find the "best fit" line.

Data: (1, 2), (2, 4), (3, 5), (4, 7)

Goal: Find y = mx + b that best approximates the points.

Least Squares Method (Intuition)

Minimize the total squared error between predicted and actual values.

Result: Formulas for m and b:

```
m = (n·Σxy - Σx·Σy) / (n·Σx² - (Σx)²)
b = (Σy - m·Σx) / n

where n = number of points
```

(The math is complex, but computers do it instantly.)

Programming

```javascript
function linearRegression(points) {
  const n = points.length;
  const sumX = points.reduce((sum, p) => sum + p.x, 0);
  const sumY = points.reduce((sum, p) => sum + p.y, 0);
  const sumXY = points.reduce((sum, p) => sum + p.x * p.y, 0);
  const sumX2 = points.reduce((sum, p) => sum + p.x * p.x, 0);

  const m = (n * sumXY - sumX * sumY) / (n * sumX2 - sumX * sumX);
  const b = (sumY - m * sumX) / n;

  return { slope: m, intercept: b };
}

const data = [
  {x: 1, y: 2},
  {x: 2, y: 4},
  {x: 3, y: 5},
  {x: 4, y: 7}
];

const line = linearRegression(data);
```

```
// { slope: 1.7, intercept: 0.5 }
// Best fit: y = 1.7x + 0.5
```

Use Cases

- **Trend analysis**: Sales over time
- **Predictions**: Forecasting
- **Correlation**: Relationship between variables
- **Machine learning**: Linear models

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10. Inequalities with Lines

Linear Inequalities

Instead of y = mx + b, use <, >, ≤, ≥

```
y > 2x + 1    (above the line)
y ≤ -x + 3    (on or below the line)
```

Graphing Inequalities

y > 2x + 1:

```
     y
     │ ▓▓▓▓▓▓▓
     │▓▓▓▓╱▓▓     ← Shaded region (solutions)
     │▓▓╱▓▓▓
     │▓╱▓▓▓       ← Dashed line (not included)
─────┼╱──────▶ x
```

Steps:

1. Graph the line y = 2x + 1
2. Dashed line if < or > (not included)
3. Solid line if ≤ or ≥ (included)
4. Shade above for >, ≥
5. Shade below for <, ≤

Testing Points

Check if a point satisfies the inequality:

Is (2, 6) a solution to y > 2x + 1?

```
6 > 2(2) + 1
6 > 5
True ✓
```

Is (0, 0) a solution?

```
0 > 2(0) + 1
0 > 1
False ✗
```

Common Mistakes & Misconceptions

❌ "Slope is always positive"

Slope can be negative, zero, or undefined.

❌ "Steep lines have small slopes"

Opposite! Steep lines have large absolute slopes:

```
m = 10: Very steep
m = 0.1: Very gradual
```

❌ "Parallel lines have the same equation"

Same slope, but different y-intercepts:

```
y = 2x + 1
y = 2x + 5
(parallel but different)
```

❌ "b is always positive"

Y-intercept can be negative:

```
y = 2x - 5  (b = -5)
```

❌ "All linear equations look like y = mx + b"

Other forms exist:

```
Standard form: Ax + By = C
Point-slope: y - y₁ = m(x - x₁)
```

❌ "Lines always have exactly one intersection"

- **Same line**: Infinite intersections
- **Parallel lines**: Zero intersections
- **Different non-parallel**: Exactly one

Tiny Practice

Write equations:

1. Slope 3, y-intercept -2
2. Slope -1/2, passes through (0, 4)
3. Passes through (1, 5) and (3, 11)

Find slope and y-intercept: 4. y = -2x + 7 5. 3x + 2y = 6

Determine: 6. Are y = 3x + 1 and y = 3x - 2 parallel? 7. Are y = 2x + 1 and y = -½x + 3 perpendicular?

Solve: 8. Find intersection of y = 2x + 1 and y = x + 3 9. Convert 25°C to Fahrenheit using F = (9/5)C + 32

▶ Answers

Summary Cheat Sheet

Standard Form

```
y = mx + b

m = slope (rate of change)
b = y-intercept (starting value)
```

Finding Slope

```
      rise    Δy    y₂-y₁
m = ────── = ── = ─────
      run     Δx    x₂-x₁

Positive: ⁄ rising
Negative: ＼ falling
Zero: ─ horizontal
Undefined: │ vertical
```

Writing Equations

Given	Method
m and b	$y = mx + b$
m and point	$y - y_1 = m(x - x_1)$
Two points	Find m, then use point-slope

Special Lines

```
Horizontal: y = b     (m = 0)
Vertical:   x = a     (m = undefined)
Origin:     y = mx    (b = 0)
```

Relationships

```
Parallel: Same slope (m₁ = m₂)
Perpendicular: m₁ × m₂ = -1
```

Programming

```
// Function
const f = (x, m, b) => m * x + b;

// Object
```

```javascript
const line = {
  slope: 2,
  intercept: 3,
  eval(x) { return this.slope * x + this.intercept; }
};

// Regression
const {slope, intercept} = linearRegression(data);
```

Next Steps

Linear functions are your foundation for understanding all functions. You now grasp:

- Constant rate of change
- Slope and intercept
- Graphing and applications

Next, we'll explore **Trigonometry**—functions based on angles, rotation, and waves.

Continue to: 08-trigonometry.md