

Trigonometry

Why This Matters

Trigonometry is the mathematics of **angles, rotation, and waves**. It's essential for:

- **Game development:** Rotation, aiming, circular motion
- **Computer graphics:** 3D transformations, lighting, cameras
- **Physics simulations:** Projectiles, pendulums, waves
- **Signal processing:** Audio, video, compression (Fourier transforms)
- **Animation:** Smooth motion, easing, orbits
- **Navigation:** GPS, triangulation

Trigonometry connects **angles** with **coordinates**, making circular and periodic motion mathematically tractable.

The Big Picture: Beyond Triangles

Traditional approach: "Trig is about triangles." **Modern approach:** "Trig is about **rotation** and **periodic motion**."

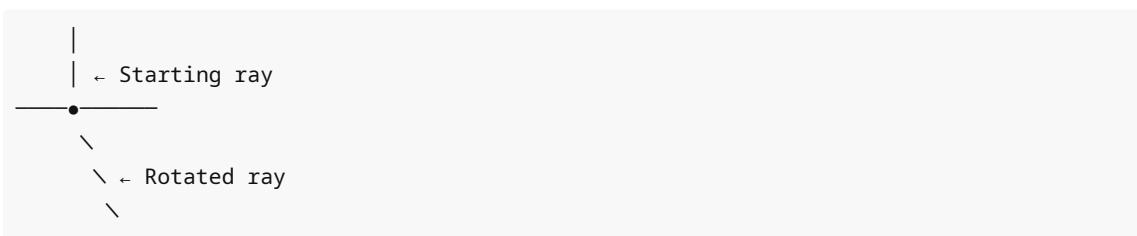
While trig started with triangles, its real power is in describing:

- **Circular motion:** Wheels, orbits, gears
 - **Waves:** Sound, light, tides, oscillations
 - **Periodic patterns:** Seasons, clocks, rhythms
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1. Angles: Rotation, Not Shapes

What Is an Angle?

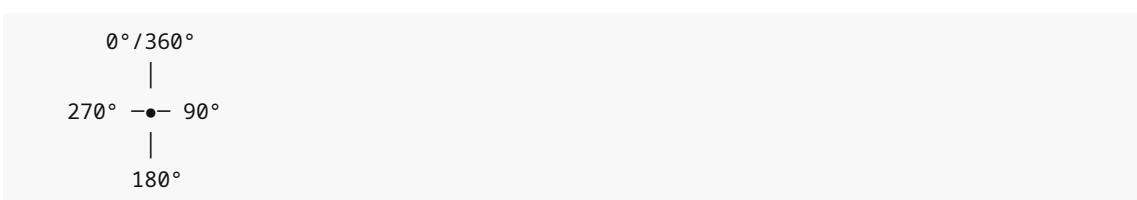
An **angle** measures **rotation** from a starting direction.



Angle = amount of rotation

Degrees

360° = one full rotation (full circle)



Common angles:

```
90° = quarter turn (right angle)
180° = half turn (straight line)
270° = three-quarters turn
360° = full turn (back to start)
```

Radians (The Natural Unit)

Radian: The angle when arc length equals radius

```
Arc length = radius
  /-\ \
  /   \ r
  •-----/
  \   /
    \v

Angle = 1 radian
```

Key fact: One full circle = 2π radians

```
360° = 2π radians
180° = π radians
90° = π/2 radians
```

Why radians? They make calculus formulas cleaner. Most programming uses radians.

Converting Between Degrees and Radians

```
Degrees to Radians: multiply by π/180
Radians to Degrees: multiply by 180/π
```

Examples:

```
45° = 45 × π/180 = π/4 radians
π/6 radians = π/6 × 180/π = 30°
```

Programming:

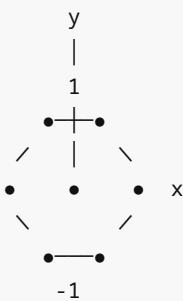
```
const degToRad = deg => deg * Math.PI / 180;
const radToDeg = rad => rad * 180 / Math.PI;

degToRad(90); // π/2 ≈ 1.571
radToDeg(Math.PI); // 180
```

2. The Unit Circle (Your Mental Model)

What It Is

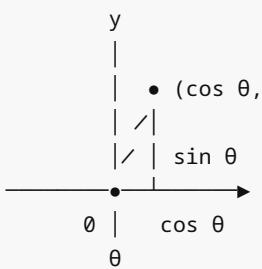
A **circle with radius 1** centered at the origin.



Every point on the circle has coordinates (x, y) where $x^2 + y^2 = 1$

Angle on the Unit Circle

An angle θ (theta) defines a point on the circle:



Key insight:

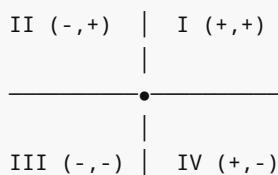
- **x-coordinate = $\cos(\theta)$**
- **y-coordinate = $\sin(\theta)$**

This is the definition of sine and cosine!

Special Angles

$\theta = 0^\circ$:	$(1, 0)$	$\cos(0^\circ) = 1, \sin(0^\circ) = 0$
$\theta = 90^\circ$:	$(0, 1)$	$\cos(90^\circ) = 0, \sin(90^\circ) = 1$
$\theta = 180^\circ$:	$(-1, 0)$	$\cos(180^\circ) = -1, \sin(180^\circ) = 0$
$\theta = 270^\circ$:	$(0, -1)$	$\cos(270^\circ) = 0, \sin(270^\circ) = -1$

Quadrants



- **Quadrant I:** Both positive
- **Quadrant II:** cos negative, sin positive
- **Quadrant III:** Both negative
- **Quadrant IV:** cos positive, sin negative

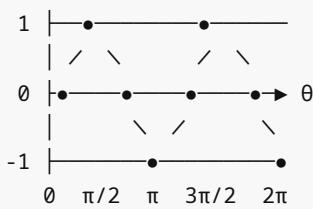
3. Sine and Cosine: The Core Functions

Definitions

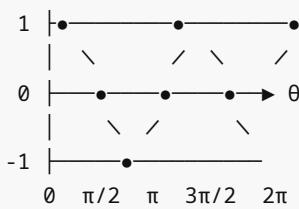
On the unit circle:

$$\cos(\theta) = \text{x-coordinate of point at angle } \theta$$
$$\sin(\theta) = \text{y-coordinate of point at angle } \theta$$

Graph of $\sin(\theta)$:



Graph of $\cos(\theta)$:



Key Properties

Range: Both oscillate between -1 and 1

$$-1 \leq \sin(\theta) \leq 1$$
$$-1 \leq \cos(\theta) \leq 1$$

Period: Repeat every 2π (360°)

$$\sin(\theta + 2\pi) = \sin(\theta)$$
$$\cos(\theta + 2\pi) = \cos(\theta)$$

Phase shift: Cosine is sine shifted left by $\pi/2$

$$\cos(\theta) = \sin(\theta + \pi/2)$$

Pythagorean identity:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

Because: $x^2 + y^2 = 1$ on unit circle

Programming

```

Math.sin(Math.PI / 2); // 1 (sin(90°))
Math.cos(0);           // 1 (cos(0°))
Math.sin(0);           // 0 (sin(0°))
Math.cos(Math.PI);    // -1 (cos(180°))

```

4. Tangent and Other Functions

Tangent

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{y}{x}$$

Interpretation: Slope of the line from origin to $(\cos \theta, \sin \theta)$

Graph:



Vertical asymptotes at $\pm\pi/2, \pm3\pi/2, \dots$
(where $\cos(\theta) = 0$)

Range: All real numbers $(-\infty \text{ to } +\infty)$

Period: π (repeats every 180°)

Reciprocal Functions

$$\csc(\theta) = \frac{1}{\sin(\theta)} \quad (\text{cosecant})$$

$$\sec(\theta) = \frac{1}{\cos(\theta)} \quad (\text{secant})$$

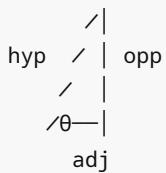
$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)} \quad (\text{cotangent})$$

Less common, but useful in some contexts.

5. Right Triangle Interpretation

SOH-CAH-TOA

For a **right triangle** with angle θ :

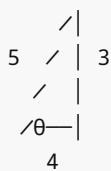


$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

Example: Triangle with sides 3, 4, 5



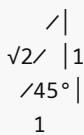
$$\sin(\theta) = 3/5 = 0.6$$

$$\cos(\theta) = 4/5 = 0.8$$

$$\tan(\theta) = 3/4 = 0.75$$

Special Right Triangles

45-45-90 triangle:

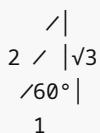


$$\sin(45^\circ) = 1/\sqrt{2} = \sqrt{2}/2 \approx 0.707$$

$$\cos(45^\circ) = 1/\sqrt{2} = \sqrt{2}/2 \approx 0.707$$

$$\tan(45^\circ) = 1$$

30-60-90 triangle:



```
sin(30°) = 1/2 = 0.5  
cos(30°) = √3/2 ≈ 0.866  
sin(60°) = √3/2 ≈ 0.866  
cos(60°) = 1/2 = 0.5
```

6. Inverse Trig Functions

What They Do

Inverse functions answer: "What angle gives this value?"

```
sin(θ) = 0.5 → θ = ?  
arcsin(0.5) = 30° (or π/6)
```

Notation

```
arcsin(x) or sin⁻¹(x) (inverse sine)  
arccos(x) or cos⁻¹(x) (inverse cosine)  
arctan(x) or tan⁻¹(x) (inverse tangent)
```

Note: $\sin^{-1}(x) \neq 1/\sin(x)$. It means "inverse", not "reciprocal".

Examples

```
sin⁻¹(1) = 90° = π/2  
cos⁻¹(0) = 90° = π/2  
tan⁻¹(1) = 45° = π/4
```

Domains and Ranges

```
arcsin: Domain [-1, 1], Range [-π/2, π/2]  
arccos: Domain [-1, 1], Range [0, π]  
arctan: Domain all reals, Range (-π/2, π/2)
```

Programming

```
Math.asin(0.5); // π/6 ≈ 0.524 (30°)  
Math.acos(0); // π/2 ≈ 1.571 (90°)  
Math.atan(1); // π/4 ≈ 0.785 (45°)  
  
// atan2: handles all quadrants correctly  
Math.atan2(y, x); // angle to point (x, y)
```

7. Real-World Applications

Rotation and Direction

Point a character toward target:

```

function angleTo(from, to) {
  const dx = to.x - from.x;
  const dy = to.y - from.y;
  return Math.atan2(dy, dx); // angle in radians
}

const player = {x: 0, y: 0};
const enemy = {x: 3, y: 4};
const angle = angleTo(player, enemy); // ~0.927 rad (53°)

```

Circular Motion

Move in a circle:

```

function circularMotion(centerX, centerY, radius, angle) {
  return {
    x: centerX + radius * Math.cos(angle),
    y: centerY + radius * Math.sin(angle)
  };
}

// Orbit around (100, 100) with radius 50
for (let angle = 0; angle < 2 * Math.PI; angle += 0.1) {
  const pos = circularMotion(100, 100, 50, angle);
  // Plot or draw at pos
}

```

Projectile Motion

Initial velocity at angle θ :

```

v_x = v_0 cos(\theta) (horizontal component)
v_y = v_0 sin(\theta) (vertical component)

x(t) = v_x * t
y(t) = v_y * t - \frac{1}{2}gt^2 (with gravity)

```

```

function shoot(speed, angleDegrees) {
  const angleRad = angleDegrees * Math.PI / 180;
  return {
    vx: speed * Math.cos(angleRad),
    vy: speed * Math.sin(angleRad)
  };
}

const velocity = shoot(100, 45); // 45° angle
// vx ≈ 70.7, vy ≈ 70.7

```

Waves and Oscillation

Sine wave for smooth oscillation:

```
// Bounce up and down
function bounce(time, amplitude, frequency) {
    return amplitude * Math.sin(frequency * time);
}

// Animate
let time = 0;
function animate() {
    const y = bounce(time, 50, 2); // 50px amplitude, 2 Hz
    sprite.y = centerY + y;
    time += 0.1;
    requestAnimationFrame(animate);
}
```

Camera/3D Rotation

Rotate point around origin:

```
function rotate(point, angle) {
    const cos = Math.cos(angle);
    const sin = Math.sin(angle);
    return {
        x: point.x * cos - point.y * sin,
        y: point.x * sin + point.y * cos
    };
}

const rotated = rotate({x: 1, y: 0}, Math.PI / 2);
// Result: {x: 0, y: 1} (rotated 90°)
```

Distance and Triangulation

Find distance using angles:

If you know angle and one side, you can find others:



$$\begin{aligned} a &= c \times \sin(\theta) \\ b &= c \times \cos(\theta) \\ c &= a / \sin(\theta) = b / \cos(\theta) \end{aligned}$$

8. Transformations of Trig Functions

General Form

```
y = A sin(B(x - C)) + D  
  
A = amplitude (height)  
B = frequency (speed of oscillation)  
C = phase shift (horizontal shift)  
D = vertical shift
```

Amplitude (A)

Stretches vertically:

```
y = 2 sin(x)      (amplitude 2, oscillates -2 to 2)  
y = 0.5 sin(x)    (amplitude 0.5, oscillates -0.5 to 0.5)
```

Frequency (B)

Changes period:

```
Period = 2π / B  
  
y = sin(2x)      (period = π, faster)  
y = sin(0.5x)    (period = 4π, slower)
```

Phase Shift (C)

Horizontal shift:

```
y = sin(x - π/2)  (shifted right by π/2)  
y = sin(x + π/4)  (shifted left by π/4)
```

Vertical Shift (D)

Moves up/down:

```
y = sin(x) + 1   (oscillates 0 to 2)  
y = sin(x) - 2   (oscillates -3 to -1)
```

Example: Ocean Wave

```
function oceanHeight(x, time) {  
  const amplitude = 2;      // 2m waves  
  const frequency = 0.5;    // slower waves  
  const speed = 0.1;        // wave moves  
  return amplitude * Math.sin(frequency * (x - speed * time));  
}
```

9. Important Identities

Pythagorean Identity

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

Variations:

$$1 + \tan^2(\theta) = \sec^2(\theta)$$

$$1 + \cot^2(\theta) = \csc^2(\theta)$$

Even/Odd Functions

$$\cos(-\theta) = \cos(\theta) \quad (\text{even function, symmetric})$$

$$\sin(-\theta) = -\sin(\theta) \quad (\text{odd function, antisymmetric})$$

$$\tan(-\theta) = -\tan(\theta) \quad (\text{odd function})$$

Sum/Difference

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

Double Angle

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

$$= 2\cos^2(\theta) - 1$$

$$= 1 - 2\sin^2(\theta)$$

You don't need to memorize these unless you use them often. They're derivable from the unit circle.

Common Mistakes & Misconceptions

✗ "Trig is only for triangles"

Trig is fundamentally about rotation and periodic motion.

✗ "Degrees and radians are the same"

They're different units. Most math/programming uses radians.

✗ " $\sin^{-1}(x)$ means $1/\sin(x)$ "

No! $\sin^{-1}(x)$ means $\arcsin(x)$ (inverse function). The reciprocal is $\csc(x) = 1/\sin(x)$.

✗ " $\tan(90^\circ) = 0$ "

No! $\tan(90^\circ)$ is **undefined** ($\cos(90^\circ) = 0$, division by zero).

✗ "Sine and cosine can be greater than 1"

Not for real angles. They're always in $[-1, 1]$.

✗ "Trig functions only work for 0° to 90° "

They work for all angles, including negative and $> 360^\circ$.

Tiny Practice

Convert:

1. 180° to radians
2. $\pi/3$ radians to degrees

Evaluate (without calculator): 3. $\sin(0^\circ)$ 4. $\cos(90^\circ)$ 5. $\tan(45^\circ)$ 6. $\sin(180^\circ)$

Find angles: 7. If $\sin(\theta) = 0.5$, what is θ (0° to 90°)? 8. If $\cos(\theta) = 0$, what is θ (0° to 180°)?

Application: 9. A point moves in a circle of radius 5. At angle 60° , what are its (x, y) coordinates?

► Answers

Summary Cheat Sheet

Angles

$$360^\circ = 2\pi \text{ radians (full circle)}$$

$$180^\circ = \pi \text{ radians}$$

$$90^\circ = \pi/2 \text{ radians}$$

$$\text{Deg to Rad: } \times \pi/180$$

$$\text{Rad to Deg: } \times 180/\pi$$

Unit Circle

$$\text{Point at angle } \theta: (\cos \theta, \sin \theta)$$

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

Core Functions

$$\sin(\theta) = y\text{-coordinate}$$

$$\cos(\theta) = x\text{-coordinate}$$

$$\tan(\theta) = \sin(\theta)/\cos(\theta) = y/x$$

Special Values

	0°	30°	45°	60°	90°
sin:	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
cos:	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
tan:	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undefined

Properties

Range: $-1 \leq \sin, \cos \leq 1$

Period: 2π (360°)

\tan period: π (180°)

Programming

```
Math.sin(theta), Math.cos(theta), Math.tan(theta) // radians
Math.asin(x), Math.acos(x), Math.atan(x) // inverse
Math.atan2(y, x) // angle to (x,y)

// Circular motion
x = centerX + radius * Math.cos(angle);
y = centerY + radius * Math.sin(angle);

// Rotation
x' = x*cos(theta) - y*sin(theta);
y' = x*sin(theta) + y*cos(theta);
```

Next Steps

Trigonometry connects angles, coordinates, and periodic motion. You now understand:

- The unit circle
- Sine, cosine, tangent
- Applications in rotation and waves

Next, we'll explore **Polynomials**—functions with multiple power terms.

Continue to: [09-polynomials.md](#)