How Numbers Actually Work

Why This Matters

Numbers are the atoms of mathematics. Before you can understand equations, functions, or calculus, you need to *really* understand what numbers are, why they exist, and how they behave.

If you've ever wondered:

- Why can't you divide by zero?
- What's the difference between -5 and 5 besides the sign?
- Why do fractions and decimals represent the same thing?
- What does √2 even mean?

...then you're asking the right questions. Let's answer them from scratch.

The Big Picture: What Problem Do Numbers Solve?

Numbers exist to measure and compare things.

- "How many apples?" → Natural numbers (1, 2, 3...)
- "How much water?" → Fractions (1/2 cup, 3/4 liter)
- "What's the temperature?" → Negative numbers (-5°C)
- "How far exactly?" → Irrational numbers (√2 meters)

Each type of number was invented to solve a specific problem humans faced.

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1. Natural Numbers (Counting Numbers)

What They Are

1, 2, 3, 4, 5, 6, ...

These are the first numbers humans invented. You can count them on your fingers. They answer "how many?"

Mental Model

Think of natural numbers as **discrete items in an array**:

```
const apples = [🍎, 🍎, 🍎];
console.log(apples.length); // 3
```

You can't have 2.5 apples in this array—either you have 2 or 3.

What You Can Do

- **Add**: 3 + 2 = 5 (combine two groups)
- **Multiply**: 3 × 4 = 12 (repeated addition: 3 + 3 + 3 + 3)
- **Compare**: 5 > 3 (one is bigger)

What You Can't Do

- **Subtract freely**: 3 - 5 = ? (You can't have -2 apples... yet)
- **Divide freely**: 5 ÷ 2 = ? (Not always a natural number)

This is where we hit the limits of natural numbers.

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2. Integers (Whole Numbers Including Negatives)

What They Are

..., -3, -2, -1, 0, 1, 2, 3, ...

Integers extend natural numbers to include:

- **Zero**: "nothing" or "the starting point"
- **Negatives**: "opposite direction" or "debt"

Mental Model: The Number Line

```
         ←──────────────┼──────────────→
     -3  -2  -1   0   1   2   3
  (left/down/debt)   (right/up/assets)
```

The number line is your most important mental tool. It turns numbers into **positions** or **distances**.

Why Negatives Exist

Problem: You have $10, then spend $15. How much do you have?

Natural numbers can't answer this. But integers can: **-$5** (you're in debt).

Think of negatives as:

- **Direction**: Moving left instead of right
- **Opposite**: The reverse of something
- **Debt vs. Assets**: Below zero

Programming Analogy

```
let balance = 10;
balance -= 15;
console.log(balance); // -5 (totally valid)
```

In code, negative numbers are just numbers. In early math education, they're treated as scary. They're not.

Zero: The Starting Point

Zero is special:

- It means "nothing" (0 apples)
- It's the **origin** on the number line
- It's the boundary between positive and negative
- **0 + x = x** (adding zero does nothing — the identity element)

Integer Operations

Operation	Example	Intuition

Add	3 + (-5) = -2	Move 3 right, then 5 left
Subtract	3 - 5 = -2	Same as 3 + (-5)
Multiply	3 × (-2) = -6	Flip direction (negative)
Divide	-6 ÷ 3 = -2	Reverse of multiply

Rules for Negative Multiplication

- **Positive × Positive = Positive** (normal)
- **Positive × Negative = Negative** (flip direction)
- **Negative × Negative = Positive** (flip twice = back to original)

Why does negative × negative = positive?

Think of it as reversing a reversal:

- Facing forward (positive)
- Turn around (negative)
- Turn around again (negative again) → You're facing forward (positive)

Or programmatically:

```
let direction = 1;      // forward
direction *= -1;        // -1 (backward)
direction *= -1;        // 1 (forward again)
```

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3. Rational Numbers (Fractions)

What They Are

Numbers that can be written as **one integer divided by another**: a/b (where b ≠ 0)

Examples: 1/2, 3/4, -5/6, 7/1 (which is just 7)

Mental Model: Fractions as Division

Don't think of fractions as weird symbols. Think of them as division operations that haven't been completed yet.

```
5/2 = 5 ÷ 2 = 2.5
```

The fraction bar is just a division sign:

```
  5
  ─   means   5 ÷ 2
  2
```

Why Fractions Exist

Problem: You have 1 pizza and 4 people. How much does each person get?

1 ÷ 4 = 1/4

Fractions let you represent **parts of a whole** or the result of division.

Visual: The Pizza Model

```
Original Pizza:
┌─────────┐
│  Whole  │
│    1    │
└─────────┘

Divided among 4 people:
┌───┬───┐
│1/4│1/4│
├───┼───┤
│1/4│1/4│
└───┴───┘
```

Each person gets 1/4.

Numerator and Denominator

```
  3  ← numerator (how many parts you have)
  ─
  4  ← denominator (how many parts make a whole)
```

Programming Analogy:

```
const fraction = {
  numerator: 3,
  denominator: 4,
  toDecimal() {
    return this.numerator / this.denominator; // 0.75
  }
};
```

Equivalent Fractions

1/2 = 2/4 = 3/6 = 4/8 = ...

They all represent the same **value**, just written differently.

Why? Because you can multiply both top and bottom by the same number:

```
1     1×2     2
─  =  ───  =  ─
2     2×2     4
```

Think of it like this:

```
const a = 1/2;      // 0.5
const b = 2/4;      // 0.5
```

```
console.log(a === b); // true
```

Simplifying Fractions

Find the **greatest common divisor (GCD)** and divide both parts:

```
6       6÷2       3
──  =  ───   =   ─
8       8÷2       4
```

Why simplify? Smaller numbers are easier to work with.

Operations on Fractions

Addition (Same Denominator)

```
1       2       1+2       3
─   +   ─   =   ───   =   ─
4       4        4        4
```

Easy: just add the numerators.

Addition (Different Denominators)

```
1       1       ?
─   +   ─   =
2       3
```

Find a **common denominator** (lowest common multiple):

```
1       1       3       2       5
─   +   ─   =   ─   +   ─   =   ─
2       3       6       6       6
```

Multiplication

```
1       2       1×2       2       1
─   ×   ─   =   ───   =   ─   =   ─
3       4       3×4      12       6
```

Multiply tops, multiply bottoms.

Intuition: "2/4 of 1/3" means "take 1/3, then take half of *that*"

Division

```
1       2       1       4       1×4       4
─   ÷   ─   =   ─   ×   ─   =   ───   =   ─
3       4       3       2       3×2       6
```

Flip the second fraction and multiply. This is called **multiplying by the reciprocal**.

Why? Division is the inverse of multiplication. If you multiply by 2/4, you divide by 4/2.

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4. Decimals (Another Way to Write Fractions)

What They Are

Numbers with a decimal point: 0.5, 3.14, -2.75

Key insight: Decimals are just fractions in disguise.

```
0.5   =  5/10   =  1/2
0.75  =  75/100 =  3/4
0.333 =  333/1000 ≈ 1/3
```

Place Value

```
  3 2 1 . 5 6 7
  │ │ │   │ │ │
  │ │ │   │ │ └─ 7/1000 (thousandths)
  │ │ │   │ └─── 6/100  (hundredths)
  │ │ │   └───── 5/10   (tenths)
  │ │ └───────── 1      (ones)
  │ └─────────── 2×10   (tens)
  └───────────── 3×100  (hundreds)
```

321.567 = 300 + 20 + 1 + 0.5 + 0.06 + 0.007

Why Decimals?

Decimals are easier to:

- Type (0.5 vs 1/2)
- Compare (0.7 vs 0.65 — just compare digit by digit)
- Use in calculators

But fractions are better for:

- Exact values (1/3 is exact, 0.333... is approximate)
- Showing relationships

Converting Between Fractions and Decimals

Fraction → Decimal: Just divide

```
3/4 = 3 ÷ 4 = 0.75
```

Decimal → Fraction: Use place value

```
0.75 = 75/100 = 3/4 (simplified)
```

Terminating vs Repeating Decimals

Terminating: Ends after a certain point

```
1/2 = 0.5
1/4 = 0.25
```

Repeating: Goes on forever

```
1/3 = 0.333333...
1/7 = 0.142857142857...
```

We write repeating decimals with a bar:

```
1/3 = 0.3̄  (the 3 repeats)
```

---

5. Irrational Numbers (Numbers That Can't Be Fractions)

What They Are

Numbers that **cannot** be written as a fraction of two integers.

Famous examples:

- **π** (pi) ≈ 3.14159...
- **√2** (square root of 2) ≈ 1.41421...
- **e** (Euler's number) ≈ 2.71828...

Why They Exist

Problem: What's the diagonal of a 1×1 square?

Using the Pythagorean theorem: $\text{diagonal}^2 = 1^2 + 1^2 = 2$

So: diagonal = $\sqrt{2}$

But $\sqrt{2}$ cannot be written as a fraction. It's been proven mathematically.

Visual: Why √2 is Irrational

```
┌─────────┐
│        /│
│  1   /  │
│    /    │ diagonal = √2 ≈ 1.414...
│  /      │
│/    1   │
└─────────┘
```

No matter how you try to express it as a ratio of whole numbers, you can't. The decimal goes on forever *without repeating*.

Decimal Expansion

Rational: Eventually repeats

```
1/3  = 0.333333...  (repeats)
1/7  = 0.142857142857...  (repeats)
```

Irrational: Never repeats, goes on forever

```
√2 = 1.41421356237309504880168872420969807856967187537694...
π  = 3.14159265358979323846264338327950288419716939937510...
```

Programming Note

```
console.log(Math.sqrt(2));  // 1.4142135623730951
console.log(Math.PI);       // 3.141592653589793
```

Computers approximate irrationals with floating-point numbers. They can't store infinite decimals.

Why This Matters

Irrational numbers show up everywhere:

- Circles (π)
- Right triangles (√2)
- Natural growth (e)

You can't avoid them, so embrace them as "numbers that can't be written as simple fractions."

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6. The Complete Number Line

```
 Irrational    Rational     Integers     Natural
     ↓            ↓            ↓            ↓
 ────●──────────●─────────────●────────────●─────→
   -π    -2    -1    0   1/2   1    √2     2    3
              ↑
        All these are REAL NUMBERS
```

Real Numbers

Real numbers = All the numbers on the number line

This includes:

- Natural numbers (1, 2, 3, ...)
- Zero (0)
- Negative integers (-1, -2, -3, ...)
- Fractions (1/2, 3/4, ...)
- Irrational numbers (π, √2, ...)

If you can point to it on the number line, it's a real number.

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7. Absolute Value (Distance, Not Direction)

What It Means

The **absolute value** of a number is its distance from zero, ignoring direction.

Notation: |x| (read as "absolute value of x")

```
|5|  = 5   (distance from 0 is 5)
|-5| = 5   (distance from 0 is 5)
|0|  = 0   (distance from 0 is 0)
```

Visual

```
         5 units         5 units
   ←─────────────┼─────────────→
            -5    0    5
```

Both -5 and 5 are 5 units away from zero.

Mental Model: Distance

Think of absolute value as `Math.abs()` in code:

```
Math.abs(5);   // 5
Math.abs(-5);  // 5
Math.abs(0);   // 0
```

It strips away the sign and gives you the magnitude.

When You Use It

- **Distance**: "How far apart are -3 and 2?"

  ```
  |-3 - 2| = |-5| = 5
  ```

- **Error/Difference**: "How much did I miss by?"

  ```
  |actual - expected|
  ```

- **Magnitude**: "How big is this, regardless of direction?"

Rules

```
|-x| = |x|            (sign doesn't matter)
|x × y| = |x| × |y|  (distribute through multiplication)
|x| ≥ 0               (always non-negative)
|x| = 0  ⟺  x = 0    (only zero has distance zero)
```

Common Mistakes & Misconceptions

❌ "Negative numbers are smaller than zero"

Not in terms of magnitude. -1000 is "more negative" than -1, but both are negative.

❌ "Fractions are different from decimals"

They're the same thing in different notation: 0.5 = 1/2

❌ "You can't divide by zero because it's zero"

You can't divide by zero because **it's undefined**. Division by zero would break all of mathematics.

Think about it:

```
10 ÷ 0 = ?
```

This would mean: "What number, times 0, gives 10?" But **any number × 0 = 0**, so there's no answer.

❌ "Irrational numbers are rare"

Most numbers are irrational! Rationals are actually the rare ones.

❌ "Numbers are just symbols"

Numbers represent quantities, distances, and comparisons. They have meaning.

Real-World Examples

Money (Rationals and Negatives)

```
Balance: $50.25  (decimal/rational)
Withdrawal: -$75 (negative)
New balance: -$24.75 (debt)
```

Temperature (Integers and Negatives)

```
Freezing: 0°C
Cold: -10°C
Hot: 35°C
```

Distances (Rationals and Irrationals)

```
Diagonal of square: √2 meters (irrational)
Half the distance: 0.5 km (rational decimal)
```

Programming (All Types)

```
const items = 5;              // natural number
let offset = -10;             // negative integer
const ratio = 0.75;           // decimal/rational
const pi = Math.PI;           // irrational (approximated)
const distance = Math.abs(x); // absolute value
```

Tiny Practice

Try these to test your understanding:

1. **Place on the number line**: -2, 0.5, 3, -1.5, 2
2. **Absolute value**: |-7|, |3|, |-0.5|
3. **Convert**: 3/4 to decimal, 0.2 to fraction
4. **True or false**: Is -5 an integer? Is 0.5 an integer?
5. **Simplify**: 6/8, 10/15
6. **Compute**: |-3 - 5|, |4 - 1|

▶ Answers

Summary Cheat Sheet

Type	Examples	What It Solves
Natural	1, 2, 3, ...	Counting
Integers	..., -2, -1, 0, 1, 2, ...	Direction, debt
Rational	1/2, 3/4, 0.75	Parts of a whole
Irrational	π, √2, e	Exact geometric values
Real	All of the above	Everything on the number line

Key Concepts

- **Number line**: Visual representation of numbers as positions
- **Negative**: Opposite direction, below zero
- **Fraction = Division**: 3/4 means 3 ÷ 4
- **Decimal = Fraction**: 0.5 means 5/10 = 1/2
- **Absolute value**: Distance from zero (|x|)
- **Zero**: The origin, boundary, identity element

Mental Models

- Natural numbers = items in an array
- Integers = positions on a line
- Fractions = slicing a whole into parts
- Decimals = place value system
- Irrationals = infinite non-repeating decimals
- Absolute value = `Math.abs()`

Next Steps

Now that you understand what numbers *are*, you're ready to learn what you can **do** with them.

In the next section, we'll explore **Algebra Foundations**—how to use numbers as variables and solve for unknowns.

Continue to: 01-algebra-foundations.md