SF2743 Advanced Real Analysis I - Course summary

Leo Trolin

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Part I. Integration theory

Measures

Definition: (**Limits of sets**)

For a sequence of sets (E_n), we define:

(i) $\limsup_{n\to\infty} E_n = \overline{\lim}_{n\to\infty} := \{\text{points which belong to infinitely many } E_n\}$

$$= \lim_{n\to\infty} \bigcup_{m\geq n} E_m = \bigcap_{n=1}^{\infty} \bigcup_{m\geq n} E_m.$$

(ii) $\liminf_{n\to\infty} E_n = \underline{\lim}_{n\to\infty} := \{\text{points which belong to all but finitely many } E_n\}$

$$= \lim_{n\to\infty} \bigcap_{m\geq n} E_m = \bigcup_{n=1}^{\infty} \bigcap_{m\geq n} E_m.$$

(iii) $\lim_{n\to\infty} E_n$ as $\overline{\lim}\, E_n$ and $\underline{\lim}\, E_n$, if they coincide.

Definition/Proposition: (**Monotone increasing/decreasing sequence of sets**)

Let (E_n) be a sequence of sets.

(i) (E_n) is monotone increasing if $E_1 \subseteq E_2 \subseteq \dots$. In this case, $\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n$.

(ii) (E_n) is monotone decreasing if $E_1 \supseteq E_2 \supseteq \dots$. In this case, $\lim_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} E_n$.

Definition/Proposition: (**Algebra**)

A collection $\mathcal{A}$ of subsets of a space X is an algebra if

(i) $\varnothing \in \mathcal{A}$

(ii) $A, B \in \mathcal{A} \implies A \setminus B \in \mathcal{A}$

(iii) $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$ (or equivalently, take only disjoint A, B)

(iv) $X \in \mathcal{A}$

In this case, $A, B \in \mathcal{A} \implies A \cap B \in \mathcal{A}$.

Definition/Proposition: (**σ-algebra**)

A collection $\mathcal{A}$ of subsets of a space X is a σ-algebra if

(i) $\varnothing \in \mathcal{A}$

(ii) $A, B \in \mathcal{A} \implies A \setminus B \in \mathcal{A}$

(iii) $A_n \in \mathcal{A}$ for $n = 1, 2, \cdots \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ (or equivalently, take only disjoint A_n)

(iv) $X \in \mathcal{A}$

In this case, $A_n \in \mathcal{A} \implies \bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$.

Definition: (**Measure**)

Let $\mathcal{A}$ be a σ-algebra over a space X. A function $\mu : \mathcal{A} \to [0, \infty]$ is a measure if

(i) $\mu(\varnothing) = 0$

(ii) μ is completely additive, i.e. for disjoint sets $E_n \in \mathcal{A}$ it holds that

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$$

The triple $(X, \mathcal{A}, \mu)$ is a measure space. If $\mu(X) < \infty$, μ is a finite measure. If $\mu(X) = 1$, μ is a probability measure. If X is the countable union of sets of finite measure under μ, then μ is a σ-finite measure.

Proposition: (**Basic properties of measures**)

> Let μ be a measure with domain $\mathcal{A}$.
> (a) μ is monotone, i.e. if $E, F \in \mathcal{A}$ with $E \subseteq F$, then $\mu(E) \leq \mu(F)$.
> (b) If $E, F \in \mathcal{A}$, with $E \subseteq F$, $\mu(F) < \infty$, then $\mu(F \setminus E) = \mu(F) - \mu(E)$.
> (c) If $E, F \in \mathcal{A}$, then $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.
> (d) If (E_n) is a monotone increasing sequence in $\mathcal{A}$, then $\lim_{n\to\infty} \mu(E_n) = \mu\left(\lim_{n\to\infty} E_n\right)$.
> (e) If (E_n) is a monotone decreasing sequence in $\mathcal{A}$, and $\mu(E_N) < \infty$ for some N, then $\lim_{n\to\infty} \mu(E_n) = \mu\left(\lim_{n\to\infty} E_n\right)$.
> (f) If $E_n \in \mathcal{A}$ for $n = 1, 2, \ldots$, then $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n)$.
> (g) If $E_n \in \mathcal{A}$ for $n = 1, 2, \ldots$, then $\mu\left(\underline{\lim}_{n\to\infty} E_n\right) \leq \underline{\lim}_{n\to\infty} \mu(E_n)$.
> (h) If $E_n \in \mathcal{A}$ for $n = 1, 2, \ldots$, and $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) < \infty$, then $\mu\left(\overline{\lim}_{n\to\infty} E_n\right) \geq \overline{\lim}_{n\to\infty} \mu(E_n)$.
> (i) If $E_n \in \mathcal{A}$ for $n = 1, 2, \ldots$, and $\lim_{n\to\infty} E_n$ exists and $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) < \infty$, then $\mu\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$.

Construction of measures

Definition: (**Outer measure**)

> Let X be a space. A function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is an outer measure if
> (i) $\mu^*(\varnothing) = 0$
> (ii) μ^* is monotone, i.e. $E \subseteq F \implies \mu^*(E) \leq \mu^*(F)$.
> (iii) μ^* is countably subadditive, i.e. $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n)$.

Definition: (**Measurable set**)

> Let μ^* be an outer measure. A set $E \subseteq X$ is (μ^*-)measurable if for all $A \subseteq X$ we have
> $$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E).$$

Theorem: (**Construction of a measure**)

> Let μ^* be an outer measure. Let $\mathcal{A}$ denote the μ^*-measurable sets. Then, $\mathcal{A}$ is a σ-algebra and $\mu^*|_{\mathcal{A}}$ is a measure.

↳ Comment:
This always gives a *complete* measure.

Definition: (**Sequential covering class**)

> Let X be a space. A sequential covering class $\mathcal{K}$ is a collection of subsets of X such that
> (i) $\varnothing \in \mathcal{K}$
> (ii) For all $A \subseteq X$, there is a sequence (E_n) in $\mathcal{K}$ such that $\bigcup_{n=1}^{\infty} E_n \supseteq A$. (or equivalently, $\mathcal{K}$ is a cover of X)

Theorem: (**Construction of an outer measure**)

> Let $\mathcal{K}$ be a sequential covering class of X. Let $\lambda : \mathcal{K} \to [0, \infty]$ be a function satisfying $\lambda(\varnothing) = 0$. For any $A \subseteq X$, define
> $$\mu^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \lambda(E_n) \middle| E_n \in \mathcal{K}, \bigcup_{n=1}^{\infty} E_n \supseteq A \right\}.$$
> Then, μ^* is an outer measure.

Definition: (**Complete measure**)

> A measure μ is complete if for $N \subseteq E \in \mathcal{A}$ and $\mu(E) = 0$ we have $N \in \mathcal{A}$.

Definition/Theorem: (**Completion**)

> Let $(X, \mathcal{A}, \mu)$ be a measure space. Let
> $$\overline{\mathcal{A}} := \left\{ E \cup N : E \in \mathcal{A} \text{ and } N \subseteq F \text{ for some } F \in \mathcal{A} \text{ with } \mu(F) = 0 \right\}.$$
> Then $\overline{\mathcal{A}}$ is a σ-algebra and $\overline{\mu} : \overline{\mathcal{A}} \to [0, \infty]$ defined by $\overline{\mu}(E \cup N) := \mu(E)$ is a complete measure, called the completion of μ.

Definition: (**Lebesgue measure on $\mathbb{R}^n$**)

> Let $\mathcal{K}$ be the collection of open intervals, which is a sequential covering class on $\mathbb{R}^n$. Let $\lambda : \mathcal{K} \to [0, \infty]$ be given by the volume. Then, the measure given by two previous theorems is the Lebesgue measure on $\mathbb{R}^n$.

↳ Comment:
All intervals are Lebesgue measurable.

Measurable functions

Definition/Proposition: (**Measurable function**)

> Let $f : X_0 \to [-\infty, \infty]$ where X_0 is a measurable set in the measure space X. This is a measurable function (on X_0) if one of the following equivalent statements hold:
> (i) $f^{-1}(U) \in \mathcal{A}$ for all open $U \subseteq \mathbb{R}$, and $f^{-1}(+\infty), f^{-1}(-\infty) \in \mathcal{A}$
> (ii) $f^{-1}((-\infty, c)) \in \mathcal{A}$ for all $c \in \mathbb{R}$, and $f^{-1}(+\infty), f^{-1}(-\infty) \in \mathcal{A}$
> (iii) $f^{-1}((-\infty, c]) \in \mathcal{A}$ for all $c \in \mathbb{R}$, and $f^{-1}(+\infty), f^{-1}(-\infty) \in \mathcal{A}$
> (iv) $f^{-1}((c, +\infty)) \in \mathcal{A}$ for all $c \in \mathbb{R}$, and $f^{-1}(+\infty), f^{-1}(-\infty) \in \mathcal{A}$
> (v) $f^{-1}((c, +\infty]) \in \mathcal{A}$ for all $c \in \mathbb{R}$, and $f^{-1}(+\infty), f^{-1}(-\infty) \in \mathcal{A}$

Proposition:

> A continuous function $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable.

Lemma:

> If f, g are measurable functions, then $\{x : f(x) < g(x)\}$ is a measurable set.

Theorem:

> If f, g are measurable functions, then $f \pm g$ and fg are measurable functions.

Theorem:

> For a sequence (f_n) of measurable functions, the following are also measurable:
> (a) $\sup f_n$
> (b) $\inf f_n$
> (c) $\overline{\lim} f_n$
> (d) $\underline{\lim} f_n$

Definition: (**Almost everywhere**)

> A property concerning points x in a measure space is said to hold almost everywhere if the set where it does not hold has measure zero.

Lemma:

> If f is a measurable function, and $f = g$ a.e. in a complete measure space, then g is measurable.

Definition: (**Almost everywhere convergence**)

> A sequence (f_n) of functions converges (pointwise) almost everywhere to g if $\lim_n f_n(x) = g(x)$ a.e. We write $\lim_n f_n = g$ a.e.

Theorem:

> Let (f_n) be a sequence of measurable functions.
> (a) If $\lim_n f_n = g$ then g is measurable.
> (b) If $\lim_n f_n = g$ a.e. and X is complete, then g is measurable.

Integrals

Definition: (**Simple function**)

> f is a simple function if there are mutually disjoint sets $E_1, \dots, E_m \in \mathcal{A}$ and real numbers $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ such that
> $$f(x) = \sum_{i=1}^{m} \alpha_i \chi_{E_i}(x).$$

Theorem:

> Let f be a nonnegative simple function. Then, there exists a monotone increasing sequence (f_n) (meaning $f_n(x) \leq f_{n+1}(x)$) of simple nonnegative functions such that (f_n) converges to f everywhere.

Definition: (**Almost uniform convergence**)

> A sequence (f_n) of a.e. real-valued, measurable functions converges almost uniformly to a measurable function f if for every $\varepsilon > 0$ there exists a measurable set E such that $\mu(E) < \varepsilon$ and (f_n) converges uniformly to f on $X \setminus E$.

↳ Comment:
f is necessarily a.e. real-valued.

Definition: (**Convergence in measure**)

> A sequence (f_n) of a.e. real-valued measurable functions converges in measure to a measurable function f if for every $\varepsilon > 0$ we have
> $$\lim_{n\to\infty} \mu\left(\left\{x \in X : \left|f_n(x) - f(x)\right| > \varepsilon\right\}\right) = 0.$$

↳ Comment:
If so, f is uniquely determined a.e. Also, f is necessarily a.e. real-valued.

Definition: (**Integral of simple functions**)

> A simple function $f = \sum_i^m \alpha_i \chi_{E_i}$ is integrable if $\mu(E_i) < \infty$ for all i where $\alpha_i \neq 0$. Its integral is then $\int f(x)\ \mathrm{d}\mu(x) := \sum_i^m \alpha_i \mu(E_i)$. (By convention, $0 \cdot \infty = 0$ in this sum)
>
> If $E \in \mathcal{A}$, we define the integral over E as $\int_E f\ \mathrm{d}\mu := \int \chi_E f\ \mathrm{d}\mu$.

↳ Comment:
This definition is indeed independent of the representation of f as a simple function.

Definition: (**Cauchy in the mean**)

> A sequence (f_n) of integrable (simple) functions is Cauchy in the mean if
> $$\int |f_n - f_m|\ \mathrm{d}\mu \to 0 \quad \text{as } n, m \to \infty.$$

Definition/Theorem: (**Integral**)

> Let f be an extended real-valued, measurable function. It is integrable if there is a sequence (f_n) of integrable simple functions such that:
> (i) (f_n) is Cauchy in the mean
> (ii) $\lim_n f_n = f$ a.e.
> (ii') (f_n) converges in measure to f. (We may equivalently replace (ii) with (ii') here)
>
> Its integral is then $\displaystyle\int f(x)\ \mathrm{d}\mu(x) := \lim_{n\to\infty} \int f_n(x)\ \mathrm{d}\mu(x)$.
>
> If $E \in \mathcal{A}$, we define the integral over E as $\int_E f\ \mathrm{d}\mu := \int \chi_E f\ \mathrm{d}\mu$.
>
> If f is nonnegative and not integrable, we write $\int f\ \mathrm{d}\mu = \infty$.

↳ Comment:
f is necessarily a.e. real-valued. The value of the integral is independent of our choice of (f_n).

Theorem: (**Properties of integrals**)

For integrable functions f, g and real numbers α, β:
(a) $\int \alpha f + \beta g \ \mathrm{d}\mu = \alpha \int f \ \mathrm{d}\mu + \beta \int g \ \mathrm{d}\mu$
(b) If $f \geq g$ a.e. then $\int f \ \mathrm{d}\mu \geq \int g \ \mathrm{d}\mu$
(c) $|f|$ is integrable, and $\left|\int f \ \mathrm{d}\mu\right| \leq \int |f| \ \mathrm{d}\mu$
(d) If $m \leq f \leq M$ a.e. on a measurable set E with finite measure, then $m\mu(E) \leq \int_E f \ \mathrm{d}\mu \leq M\mu(E)$
(e) If $f \geq 0$ a.e. and $E \subseteq F$ are measurable sets, then $\int_E f \ \mathrm{d}\mu \leq \int_F f \ \mathrm{d}\mu$
(f) If $f \geq m > 0$ on a measurable set E, then $\mu(E) < \infty$
(g) If f is a.e. nonnegative, then $\int f \ \mathrm{d}\mu = 0$ if and only if $f = 0$ a.e.
(h) If E has measure zero, then any measurable function h is integrable on E and $\int_E h \ \mathrm{d}\mu = 0$
(i) If f is positive on a measurable set E and $\int_E f \ \mathrm{d}\mu = 0$, then $\mu(E) = 0$

Definition: (**Convergence in the mean**)

A sequence (f_n) of integrable functions converges in the mean to an integrable function f if

$$\int |f_n - f| \ \mathrm{d}\mu \to 0 \quad \text{as } n \to \infty.$$

Lemma:

If f is integrable and (f_n) is the sequence of integrable simple functions approximating f given by definition, then (f_n) converges in the mean to f.

Theorem: (**Completeness**)

If (f_n) is a sequence of integrable functions that is Cauchy in the mean, then there exists an integrable function f such that (f_n) converges in the mean to f.

Theorem: (**Properties of the integral as a set function**)

Let f be integral and define the set function $\lambda(E) := \int_E f \ \mathrm{d}\mu$. Then the following properties hold.
(a) Complete additivity: For disjoint sets $E_n \in \mathcal{A}$, we have $\lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \lambda(E_n)$.
(b) Absolute continuity: For any $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(E) < \delta$ implies $\left|\lambda(E)\right| < \varepsilon$.

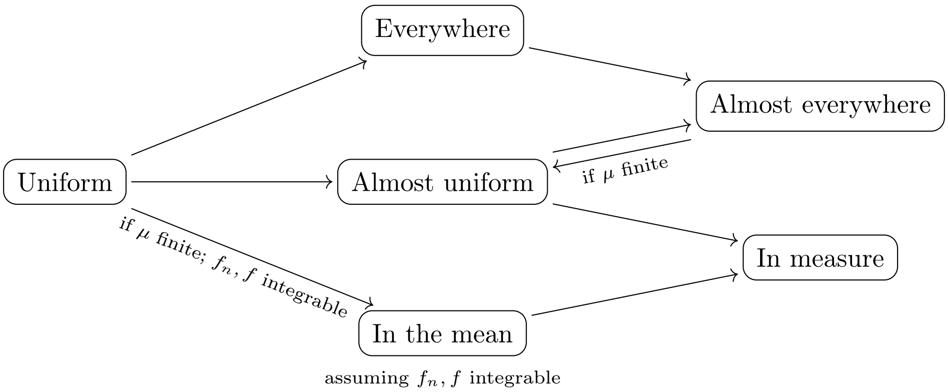
Theorem:

Let f be integrable and (E_n) be a sequence of measurable sets with limit E where $\mu(E) = 0$. Then,

$$\lim_{n\to\infty} \int_{E_n} f \ \mathrm{d}\mu = 0.$$

Diagram on types of convergence

Let (f_n) be a sequence of measurable, a.e. real-valued functions, and let f be another such function. The following diagram illustrates implications of (f_n) converging to f in various senses.



Dominated convergence theorem & more

Theorem: (**Lebesgue's dominated convergence theorem**)

> Let (f_n) be a sequence of integrable functions that converges a.e. *or* in measure to a measurable function f. If $|f_n| \leq g$ a.e. for all n, for some integrable function g, then f is integrable and (f_n) converges in mean to f. In particular,
> $$\lim_{n\to\infty} \int f_n \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu.$$

Corollary:

> If f is measurable and $|f| \leq g$ a.e. for some integrable g, then f is integrable.

Theorem: (**Lebesgue's monotone convergence theorem**)

> Let (f_n) be a monotone increasing sequence of nonnegative integrable functions and let $f := \lim_n f_n$. Then,
> $$\lim_{n\to\infty} \int f_n \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu. \quad \text{(possibly as “}\infty = \infty\text{”)}$$

Theorem: (**Fatou's lemma**)

> Let (f_n) be a sequence of nonnegative integrable functions. Then,
> $$\int \varliminf_{n\to\infty} f_n \, \mathrm{d}\mu \leq \varliminf_{n\to\infty} \int f_n \, \mathrm{d}\mu.$$
> In particular, if $\varliminf \int f_n \, \mathrm{d}\mu < \infty$, then $\varliminf f_n$ is integrable.

Connections to $\mathbb{R}$

Theorem: (**Riemann and Lebesgue integrals agree**)

> If a bounded function f on a bounded real interval $[a, b]$ is Riemann integrable, then it is also Lebesgue integrable and the two integrals agree with each other.

Theorem: (**Generalisation of Fundamental theorem of calculus**)

> Let f be Lebesgue integrable on the real interval (a, b). Then, its indefinite integral
>
> $$g(x) := \int_a^x f(t)\ \mathrm{d}t + C$$
>
> for any $C \in \mathbb{R}$ is differentiable a.e. and satisfies $g'(x) = f(x)$ a.e.

↳ Comment:
Also, g is (absolutely) continuous.

Ergodic theory

In this section, we fix $(X, \mathcal{A}, \mu)$ as a probability space.

Definition: (**Measure-preserving**)

> A map $T : X \to X$ is measure-preserving if for $A \in \mathcal{A}$, we have $T^{-1}(A) \in \mathcal{A}$ and $\mu(T^{-1}(A)) = \mu(A)$.
>
> We say that μ is T-invariant.

↳ Comment:
Then, T^n is also measure-preserving for $n = 1, 2, \ldots$

Definition: (**Essentially T-invariant function**)

> A measurable function $f : X \to \mathbb{R}$ is essentially T-invariant if $f(T(x)) = f(x)$ a.e.

Definition/Proposition: (**Ergodic**)

> Let T be a measure-preserving transformation. It is ergodic if any of the following equivalent statements hold:
> (i) For all $A \in \mathcal{A}$: $T^{-1}(A) = A \implies \mu(A) = 0$ or $\mu(A) = 1$.
> (ii) For all $A \in \mathcal{A}$: $\mu(T^{-1}(A)\Delta A) = 0 \implies \mu(A) = 0$ or $\mu(A) = 1$. $(A\Delta B = (A \setminus B) \cup (B \setminus A))$
> (iii) Every T-invariant function is constant a.e.

Theorem: (**Birkhoff's ergodic theorem**)

> Let $T : X \to X$ be measure preserving and f be integrable. Then,
>
> $$\bar{f}(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x))$$
>
> exists for a.e. $x \in X$. Also, $\bar{f}$ is integrable and $\int \bar{f}\ \mathrm{d}\mu = \int f\ \mathrm{d}\mu$.

↳ Comment:
$\bar{f}$ is essentially T-invariant.

Corollary:

> If T is ergodic and f is integrable, then
>
> $$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = \int f\ \mathrm{d}\mu \quad \text{a.e.}$$

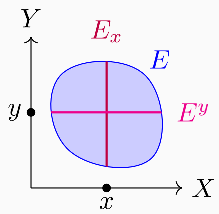
Product measures & Fubini's theorems

In this section, we fix $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ as two measure spaces.

Definition: ($\mathcal{A} \times \mathcal{B}$)

> The Cartesian product $\mathcal{A} \times \mathcal{B}$ is the σ-algebra generated by sets $A \times B \subseteq X \times Y$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Definition: (**X-section, Y-section**)

> Let $E \subseteq X \times Y$ and $x \in X$. An X-section of E is
>
> $$E_x := \{y \in Y : (x, y) \in E\} \subseteq Y.$$
>
> Analogously for $y \in Y$, a Y-section of E is
>
> $$E^y := \{x \in X : (x, y) \in E\} \subseteq X.$$
>
> 
>

Theorem:

> Suppose X and Y are σ-finite and fix a set $E \in \mathcal{A} \times \mathcal{B}$. Consider the functions
>
> $$\begin{aligned} x &\mapsto \nu(E_x), \\ y &\mapsto \mu(E^y). \end{aligned}$$
>
> These functions are measurable and
>
> $$\int \nu(E_x) \, \mathrm{d}\mu(x) = \int \mu(E^y) \, \mathrm{d}\nu(y) \quad = \text{area of } E.$$

Definition/Theorem: (**Product measure**)

> Suppose X and Y are σ-finite. Define for $E \in \mathcal{A} \times \mathcal{B}$,
>
> $$\lambda(E) := \int \nu(E_x) \, \mathrm{d}\mu(x) = \int \mu(E^y) \, \mathrm{d}\nu(y).$$
>
> Then, λ is the unique σ-finite measure on $\mathcal{A} \times \mathcal{B}$ such that for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have $\lambda(A \times B) = \mu(A)\nu(B)$. We write $\lambda = \mu \times \nu$ which is called the product measure.

Theorem: (**Fubini's theorem**)

> Let f be nonnegative and measurable on $X \times Y$. Then,
>
> $$\int f \, \mathrm{d}(\mu \times \nu) = \int \underbrace{\int f \, \mathrm{d}\mu}_{\substack{\text{measurable} \\ \text{function in } y}} \, \mathrm{d}\nu = \int \underbrace{\int f \, \mathrm{d}\nu}_{\substack{\text{measurable} \\ \text{function in } x}} \, \mathrm{d}\mu.$$

Theorem: (**Fubini's theorem**)

> Let f be integrable on $X \times Y$. Then, almost every section $x \mapsto f(x,y)$ and $y \mapsto f(x,y)$ are integrable on X and Y respectively. Also,
>
> $$\int f \, \mathrm{d}(\mu \times \nu) = \int \underbrace{\int f \, \mathrm{d}\mu}_{\substack{\text{integrable} \\ \text{function in } y}} \, \mathrm{d}\nu = \int \underbrace{\int f \, \mathrm{d}\nu}_{\substack{\text{integrable} \\ \text{function in } x}} \, \mathrm{d}\mu.$$

Theorem: (**Fubini's theorem**)

> Let f be a measurable function on $X \times Y$. If
>
> $$\iint |f| \, \mathrm{d}\mu \, \mathrm{d}\nu < \infty \quad \text{or} \quad \iint |f| \, \mathrm{d}\nu \, \mathrm{d}\mu < \infty,$$
>
> then f is integrable.

Part II. Functional Analysis

Types of spaces

Definition: (**Topological space**)

> A topologcal space X has a topology $\mathcal{T}$ of subsets of X satisfying:
> (i) $\varnothing, X \in \mathcal{T}$
> (ii) $\mathcal{T}$ is closed under arbitrary unions
> (iii) $\mathcal{T}$ is closed under finite intersections

Definition: (**Metric space**)

> A metric space X has a metric $\rho : X \times X \to [0, \infty)$ satisfying:
> (i) Positive definiteness: $\rho(x, y) \geq 0$ with equality precisely when $x = y$
> (ii) Symmetry: $\rho(x, y) = \rho(y, x)$
> (iii) Triangle inequality: $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$

Definition: (**Complete metric space**)

> A metric space X is complete if every Cauchy sequence converges.

Definition: (**Normed space**)

> A real or complex vector space X is a normed space if it has a norm $\|\cdot\|$ satisfying:
> (i) Positive definiteness: $\|x\| \geq 0$ with equality if and only if $x = 0$
> (ii) Homogeneity: $\|sx\| = |s| \cdot \|x\|$
> (iii) Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$

↳ Comment:
A norm defines a metric via $\rho(x, y) := \|x - y\|$.

Definition: (**Inner product space**)

> A real or complex vector space X is a inner product space if it has an inner product $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}/\mathbb{C}$ satisfying:
> (i) Positive definiteness: $\langle x, x \rangle \geq 0$ with equality if and only if $x = 0$
> (ii) Hermitian symmetry: $\langle x, y \rangle = \overline{\langle y, x \rangle}$
> (iii) Sesquilinearity: $\langle sx + ty, z \rangle = s \langle x, z \rangle + t \langle y, z \rangle$

↳ Comment:
An inner product defines a norm via $\|x\| := \sqrt{\langle x, x \rangle}$.

Definition: (**Banach space**)

> A Banach space is a complete normed space (with the metric $\rho(x, y) = \|x - y\|$).

Definition: (**Hilbert space**)

> A Hilbert space is a complete inner product space (with the metric $\rho(x, y) = \sqrt{\langle x - y, x - y \rangle}$).

Below is a diagram on the various types of spaces. An arrow "$\to$" represents additional imposed structure on the space; in other words, an arrow represents a canonical inclusion "$\supset$".

$$\begin{tikzcd}
 & \text{Real/complex vector space} \arrow[dr, bend left] & & \\
\text{Topological space} \arrow[r] & \text{Metric space} \arrow[r] \arrow[dd] & \text{Normed space} \arrow[r] \arrow[d] & \text{Inner product space} \arrow[d] \\
 & & \text{Banach space} & \text{Hilbert space} \\
 & \text{Complete metric space} \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

Baire's theorem

Theorem: (**Baire's theorem for open sets**)

> Let X be a complete metric space. Let $V_1, V_2, \ldots \subseteq X$ be open dense subsets. Then, $\bigcap_{n=1}^{\infty} V_n$ is dense in X.

Theorem: (**Baire's theorem for closed sets**)

> Let X be a complete metric space. Let $F_1, F_2, \ldots \subseteq X$ be closed sets with empty interior. Then, $X \setminus \bigcup_{n=1}^{\infty} F_n$ is dense in X.

Definition: (**Nowhere dense**)

> A subset $A \subseteq X$ of a metric space is nowhere dense if $\text{int}(\overline{A}) = \varnothing$, that is, its closure has empty interior.

Definition: (**Meager**)

> A subset $A \subseteq X$ of a metric space is meager or of Baire's first category if it is the countable union of nowhere dense subsets of X.

Theorem: (**Baire's theorem in terms of Baire categories**)

> A complete metric space $X \neq \varnothing$ is not meager.

Completions

Definition: (**Isometry**)

> A function $f : X \to Y$ between metric spaces (X, ρ_X) and (Y, ρ_Y) is an isometry if for all $x_1, x_2 \in X$ we have $\rho_X(x_1, x_2) = \rho_Y(f(x_1), f(x_2))$.

Definition: (**Bounded**)

> A subset $A \subseteq X$ of a metric space is bounded if there exists $C \in \mathbb{R}$ such that $\rho(a_1, a_2) \leq C$ for all $a_1, a_2 \in A$.
>
> A function $f : Z \to X$ with a metric space codomain is bounded if its image is bounded.

Proposition:

> Let X be a complete metric space and $S \subseteq X$ a closed subset. Then, S is also a complete metric space (by restricting the metric to S).

Definition/Proposition: ($B(X, Y)$)

For any set X and a metric space Y, we define $B(X, Y)$ as the set of bounded functions $X \to Y$. If the metric for Y is ρ_Y, then the supremum metric on $B(X, Y)$ is given by

$$\rho(f, g) := \sup_{x \in X} \rho_Y(f(x), g(x)).$$

Theorem:

If Y is a complete metric space, then $B(X, Y)$ is a complete metric space.

Theorem:

Let X and Y be complete metric spaces. Then, the bounded continuous functions $X \to Y$ form a closed subset of $B(X, Y)$.

Theorem: (**Existence and uniqueness of completions**)

Let X be a metric space and fix $x_0 \in X$. For any $x \in X$, define

$$\iota(x) : y \mapsto \rho(y, x) - \rho(y, x_0).$$

Then, $\iota : X \to B(X, \mathbb{R})$ is bounded and an isometry. Moreover, $\overline{\iota(X)}$ is a complete metric space which contains the dense isometric copy $\iota(X)$ of X.

Further, given two complete metric spaces $\hat{X}_1$, $\hat{X}_2$ with isometries $\iota_1 : X \to \hat{X}_1$, $\iota_2 : X \to \hat{X}_2$ such that $\iota_1(X)$ and $\iota_2(X)$ are dense in $\hat{X}_1$ and $\hat{X}_2$ respectively, there exists a surjective isometry $\hat{X}_1 \to \hat{X}_2$.

L^p-spaces

Definition: (**$\mathcal{L}^p$-space**)

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $p > 0$. We denote

$$\mathcal{L}^p(X, \mu) := \left\{ f : X \to [-\infty, \infty] : f \text{ is measurable}, |f|^p \text{ is integrable} \right\},$$
$$\|f\|_p := \left(\int |f|^p \; \mathrm{d}\mu \right)^{1/p}.$$

For $p = \infty$ we similarly denote

$$\mathcal{L}^\infty(X, \mu) := \left\{ f : X \to [-\infty, \infty] : f \text{ is measurable and essentially bounded} \right\},$$
$$\|f\|_\infty := \operatorname*{ess\,sup}_{x \in X} \left| f(x) \right|.$$

Theorem: (**Hölder's inequality**)

Let p and q be extended real numbers such that $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$. For any $f \in \mathcal{L}^p(X, \mu)$ and $g \in L^q(X, \mu)$ we have $fg \in \mathcal{L}^1(X, \mu)$ and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

↳ Comment:
For $p = q = 2$, this is Cauchy-Schwarz.

Theorem: (**Minkowski's inequality**)

Let $1 \leq p \leq \infty$. For any $f, g \in \mathcal{L}^p(X, \mu)$, we have $f + g \in \mathcal{L}^p(X, \mu)$ and

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p .$$

("Triangle inequality")

Definition/Theorem: (**L^p-space**)

Let $1 \leq p \leq \infty$. Denote $L^p(X, \mu)$ as the class of all functions in $\mathcal{L}^p(X, \mu)$ under the equivalence relation given by almost everywhere equality. We give this space the norm

$$\rho([f], [g]) := \|f - g\|_p$$

for representatives $f, g \in \mathcal{L}^p(X, \mu)$. Then, $L^p(X, \mu)$ is a complete metric space.

Operator norms and Banach spaces

Theorem:

Let X be a Banach space and (x_n) be a sequence in X. If $\sum_{n=1}^{\infty} \|x_n\|$ converges in $\mathbb{R}$, then $\sum_{n=1}^{\infty} x_n$ converges to some $x \in X$. ("absolute convergence $\implies$ convergence")

Definition/Proposition: (**Operator norm**)

Let $T : X \to Y$ be a linear transformation between normed spaces. The operator norm of T is defined as any of the equivalent:

(i) $\|T\|_{\text{op}} := \sup \left\{ \frac{\|Tx\|_Y}{\|x\|_X} : 0 \neq x \in X \right\}$ or 0 if $X = \{0\}$

(ii) $\|T\|_{\text{op}} := \sup \left\{ \|Tx\|_Y : x \in X, \|x\|_X \leq 1 \right\}$

(iii) $\|T\|_{\text{op}} := \sup \left\{ \|Tx\|_Y : x \in X, \|x\|_X = 1 \right\}$ or 0 if $X = \{0\}$

Example:

(a) If $T : \mathbb{C}^n \to \mathbb{C}^n$ is given by the matrix A, then $\|T\|_{\text{op}}$ is the square root of the largest singular value of A.

(b) If T is an operator on the set of bounded, continuous functions over the measure space (Ω, μ), and is given by

$$(Tf)(x) = \int_\Omega K(x, y) f(y) \, \mathrm{d}\mu(y),$$

then

$$\|T\|_{\text{op}} \leq \sup_x \int_\Omega \left| K(x, y) \right| \, \mathrm{d}\mu(y).$$

In this case of a finite Ω with the counting measure, we can think of $K(x, y)$ as a matrix, and thus this says that $\|T\|_{\text{op}}$ is at most the largest row-sum of $|K|$.

(c) If T is an operator on $L^2(\Omega, \mu)$ given by

$$(Tf)(x) = \int_\Omega K(x, y) f(y) \, \mathrm{d}y,$$

then

$$\|T\|_{\text{op}} \leq \sqrt{\iint \left| K(x, y) \right|^2 \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y)}.$$

Proposition:

> Let $T : X \to Y$ be a linear transformation between normed spaces. Then (unless $X = \{0\}$), $\|T\|_{\text{op}}$ is the smallest possible constant such that for all $x \in X$ we have
>
> $$\|Tx\|_Y \leq \|T\|_{\text{op}} \|x\|_X .$$

Theorem:

> Let $T : X \to Y$ be a linear transformation between normed spaces. Then, the following are equivalent:
> (i) T is continuous on X
> (ii) T is continuous at a point in X
> (iii) $\|T\|_{\text{op}} < \infty$

Definition/Theorem: ($\mathcal{B}(X, Y)$)

> Let X, Y be normed spaces. Then, the operator norm is a norm on the space of all continuous linear transformations $X \to Y$, written $\mathcal{B}(X, Y)$. Furthermore, if Y is complete, then $\mathcal{B}(X, Y)$ is complete with respect to the operator norm and is therefore a Banach space.

↳ Comment:
Operators in $\mathcal{B}(X, Y)$ are called bounded operators, even though they may not be bounded as functions.

Theorem: (**Banach-Steinhaus/The uniform boundedness principle**)

> Let X and Y be normed spaces and $\mathcal{C}$ be a collection of continuous linear transformations $X \to Y$. Suppose there is a non-meager subset $A \subseteq X$ such that for all $x \in A$ we have
>
> $$\sup_{T \in \mathcal{C}} \|Tx\|_Y < \infty.$$
>
> Then,
>
> $$\sup_{T \in \mathcal{C}} \|T\|_{\text{op}} < \infty.$$
>
> (Pointwise boundedness on a non-meager set implies uniform boundedness)

Corollary:

> Let X be a Banach space, Y be a normed space and $\mathcal{C}$ be a collection of continuous linear transformations $X \to Y$. Suppose that for all $x \in X$ we have
>
> $$\sup_{T \in \mathcal{C}} \|Tx\|_Y < \infty.$$
>
> Then,
>
> $$\sup_{T \in \mathcal{C}} \|T\|_{\text{op}} < \infty.$$
>
> (If X is complete, then pointwise boundedness everywhere implies uniform boundedness)

↳ Comment:
This is Banach-Steinhaus in the case where $A = X$, which indeed is non-meager by Baire's theorem.

Mixed results on linearity and Banach spaces

Theorem: (**Riesz's lemma**)

> Let X be a normed space and $Y \subsetneq X$ be a closed linear subspace. Then, for any $\varepsilon > 0$ there exists $z \in Z$ satisfying $\|z\| = 1$ and
> $$\|z - y\| > 1 - \varepsilon, \quad \forall y \in Y.$$

↳ Comment:
If X has an inner product, this also holds for $\varepsilon = 0$.

Theorem:

> Let X be a normed space and $Y \subseteq X$ be a finite-dimensional subspace. Then, Y is closed.

Theorem:

> Let X be a normed space. If every bounded subset of X has compact closure, then X is finite-dimensional.

Theorem:

> Let X be a Banach space and Y be a normed space. Let $T_n : X \to Y$ be a sequence of continuous linear transformations for $n = 1, 2, \ldots$ If $\lim_{n\to\infty} T_n x =: Tx$ exists for each $x \in X$, then T is continuous.
>
> (Pointwise convergence of linear transforms from a Banach space preserves continuity)

Theorem: (**Open mapping theorem**)

> Let X, Y be Banach spaces and $T : X \to Y$ be a continuous linear transformation which is surjective. Then, there exists $\delta > 0$ such that for all $y \in Y$ with $\|y\|_Y < \delta$ there is $x \in X$ with $\|x\| < 1$ and $Tx = y$.

Corollary:

> Let X, Y be Banach spaces and $T : X \to Y$ be a continuous linear transformation which is surjective. Then, T is an open map, i.e. for all open $V \subseteq X$, $T(X)$ is open in Y.

Corollary:

> Let X, Y be Banach spaces and $T : X \to Y$ be a continuous linear transformation which is bijective. Then T^{-1} is continuous.

Theorem: (**Closed graph theorem**)

> Let X, Y be Banach spaces and $T : X \to Y$ be a linear transformation. If $\mathrm{graph}(T)$ is closed in $X \times Y$, then T is continuous.

↳ Comment:
The converse always holds. In fact, it requires only that X is a topological space and Y is a Hausdorff space.

Convexity & duality

Definition: (**Convex**)

> Let V be a vector space. A subset $C \subseteq V$ is convex if for all $x, y \in C$ and $0 \leq t \leq 1$ we have $tx + (1-t)y \in C$.

Theorem: (**Hahn-Banach**)

> Let X be a real vector space and $S \subseteq X$ be a linear subspace. Suppose $f : S \to \mathbb{R}$ is linear and satisfies $f(v) \leq p(v)$ for $v \in S$, where $p : X \to \mathbb{R}$ is sublinear, i.e.
> (i) $p(x+y) \leq p(x) + p(y),\ x, y \in X$
> (ii) $p(tx) = tp(x),\ x \in X, t > 0$
> Then, f can be extended to a linear function $F : X \to \mathbb{R}$ such that $F \leq p$ on X and $F|_S = f$.

↳ Comment:
An example of such a p is a norm.

Theorem: (**Hahn-Banach over $\mathbb{C}$**)

> Let X be a complex vector space and $S \subseteq X$ be a linear subspace. Suppose $f : S \to \mathbb{C}$ is linear and satisfies $|f(v)| \leq p(v)$ where $p : X \to [0, \infty)$ satisfies
> (i) $p(x+y) \leq p(x) + p(y),\ x, y \in X$
> (ii) $p(tx) = |t|\, p(x),\ x \in X, t \in \mathbb{C}$
> Then f can be extended to a linear function $F : X \to \mathbb{C}$ such that $|F| \leq p$ on X and $F|_S = f$.

↳ Comment:
An example of such a p is a norm.

Definition: (**Dual space**)

> Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$ with a topology. The dual space of X is
> $$X^* := \{f : X \to \mathbb{R} \text{ or } \mathbb{C} : f \text{ is continuous, linear}\}.$$
> If the topology of X comes from a norm, then we give X^* the operator norm.

Theorem:

> Let X be a normed space over $\mathbb{R}$ or $\mathbb{C}$, and $S \subseteq X$ be a closed linear subspace. Then, for any vector $x \in X \setminus S$, there exists $f \in X^*$ such that
> (i) $f|_S = 0$
> (ii) $\|f\|_{\text{op}} = 1$
> (iii) $f(x) = \inf_{v \in S} \|x - v\|_X \neq 0$

Corollary:

> Let X be a normed space and $x \in X$ be a vector $x \neq 0$. Then, there exists $f \in X^*$ such that $\|f\|_{\text{op}} = 1$ and $f(x) = \|x\|_X$.

Corollary:

> Let X be a normed space and $x, y \in X$ be vectors $x \neq y$. Then, there exists $f \in X^*$ with $f(x) \neq f(y)$.
>
> (Continuous linear functionals separate points)

Theorem:

> Let X be a normed space. Then,
> $$X \hookrightarrow X^{**}$$
> $$x \mapsto \text{ev}_x$$
> is a linear isometry. Here, $\text{ev}_x(f) := f(x)$ for $f \in X^*$.

Theorem:

> If $X = L^p[0,1]$ for $0 < p < 1$, then $X^* = \{0\}$.
>
> (This is a metric space without a norm)

Hilbert spaces and duality

Theorem:

> Let X be Hilbert space and $C \subseteq X$ be a nonempty, closed, convex subset. Then, C has a unique element of smallest norm.

Theorem:

> Let X be a Hilbert space and $S \subseteq X$ be a closed subspace. Then, X can be realized as the direct sum $X = S \oplus S^\perp$.

Theorem: (**Riesz**)

> Let X be a Hilbert space. Then, every continuous linear functional f is of the form
>
> $$f(x) = \langle x, y \rangle \quad \text{for some } y \in X.$$
>
> In particular, $X \cong X^*$.