SF2705 Fourier Analysis - Course Summary

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All integrals in this course are Lebesgue integrals.

Fourier series in $L^1(\mathbb{T})$

Basic setup

Notation: ($L^1(\mathbb{T})$)

- $\mathbb{T}$ denotes the unit cirle in $\mathbb{C}$. We abuse notation and identify $f(t) \leftrightarrow f(e^{it})$ whenever f is a function on $\mathbb{T}$.
- $L^1(\mathbb{T})$ is the space of all integrable functions $f : \mathbb{T} \to \mathbb{C}$ with the norm

$$\|f\|_{L^1} := \frac{1}{2\pi} \int_0^{2\pi} \left|f(t)\right| \, \mathrm{d}t.$$

Definition: (**Trigonometric polynomial/series**)

A trigonometric polynomial on $\mathbb{T}$ is a function of t of the form

$$\sum_{n=-N}^{N} a_n e^{int},$$

while a trigonometric series is a formal series in t of the form

$$\sum_{n=-\infty}^{\infty} a_n e^{int}.$$

Definition: (**Fourier coefficients/series**)

If $f \in L^1(\mathbb{T})$, then its Fourier coefficients are

$$\hat{f}(n) := \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} \, \mathrm{d}t, \qquad n \in \mathbb{Z}.$$

Its Fourier series is the (formal) trigonometric series

$$S[f] \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int}.$$

Proposition: (**Fourier coefficients are bounded**)

If $f \in L^1(\mathbb{T})$, then

$$\left|\hat{f}(n)\right| \leq \|f\|_{L^1}, \qquad n \in \mathbb{Z}.$$

↳ Comment: This result will be strengthened by the Riemann-Lebesgue lemma.

Notation/Proposition: (**Dirichlet kernel**)

The Dirichlet kernel for $n \in \mathbb{N}$ is

$$D_n(t) := \sum_{k=-n}^{n} e^{ikt} = \frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}.$$

Some properties are that

$$\frac{1}{2\pi}\int_0^{2\pi} D_n(t)\ \mathrm{d}t = 1, \quad \text{while} \quad \frac{1}{2\pi}\int_0^{2\pi} \left|D_n(t)\right|\ \mathrm{d}t = O(\log n) \quad \text{as } n \to \infty.$$

Definition: (**Convolution**)

If $f, g \in L^1(\mathbb{T})$, then their convolution is

$$(f * g)(t) := \frac{1}{2\pi}\int_0^{2\pi} f(t-s)g(s)\ \mathrm{d}s \in L^1(\mathbb{T}).$$

Proposition: (**Properties of convolutions**)

- $f * g = g * f$
- $(f * g) * h = f * (g * h)$
- $f * (g + h) = f * g + f * h$
- $\|f * g\|_{L^1} \leq \|f\|_{L^1}\|g\|_{L^1}$
- $\widehat{(f * g)}(n) = \hat{f}(n)\hat{g}(n)$

Notation/Proposition: (**Partial sums**)

If $f \in L^1(\mathbb{T})$, then its nth partial sum is

$$S_n(f)(t) := \sum_{k=-n}^{n} \hat{f}(k)e^{int} = (D_n * f)(t)$$

Cesàro means and summability kernels

Definition/Proposition: (**Fejér kernel**)

The Fejér kernel for $n \in \mathbb{N}$ is

$$K_n(t) := \frac{1}{n+1}\sum_{k=0}^{n} D_k(t) = \sum_{k=-n}^{n}\left(1 - \frac{|k|}{n+1}\right)e^{ikt} = \frac{1}{n+1}\left(\frac{\sin\frac{n+1}{2}t}{\sin\frac{t}{2}}\right)^2.$$

Definition/Proposition: (**Cesàro mean**)

If $f \in L^1(\mathbb{T})$, then its nth Cesàro mean for $n \in \mathbb{N}$ is

$$\sigma_n(f)(t) := \frac{1}{n+1}\sum_{k=0}^{n} S_k(f)(t) = (K_n * f)(t).$$

Definition: (**Summability kernel**)

> A summability kernel is a sequence $\{k_n\}$ (for a discrete or real parameter n) of continuous functions in $L^1(\mathbb{T})$ satisfying:
> (i) $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} k_n(t)\ \mathrm{d}t = 1$
> (ii) There is a constant C such that $\|k_n\| \leq C$ for all n
> (iii) For any $\delta \in (0, \pi)$ we have $\displaystyle\lim_{n\to\infty} \int_\delta^{2\pi-\delta} \big|k_n(t)\big|\ \mathrm{d}t = 0$

↳ Comment:
The Fejér kernel K_n is a summability kernel. The Dirichlet kernel D_n is not a summability kernel.

Theorem:

> If $f \in L^1(\mathbb{T})$ and $\{k_n\}$ is a summability kernel, then
> $$\lim_{n\to\infty} \|k_n * f - f\|_{L^1} = 0.$$

Corollary: (**Cesàro means converge in L^1**)

> If $f \in L^1(\mathbb{T})$, then $\displaystyle\lim_{n\to\infty} \big\|\sigma_n(f) - f\big\|_{L^1} = 0.$

Corollary:

> The trigonometric polynomials are dense in $L^1(\mathbb{T})$.

Theorem:

> If $f \in C(\mathbb{T})$ and $\{k_n\}$ is a summability kernel, then
> $$\lim_{n\to\infty} \|k_n * f - f\|_\infty = 0. \qquad (\text{Recall } \|g\|_\infty = \max\big|g(t)\big|)$$

Corollary: (**Weierstrass' approximation theorem**)

> Every continuous 2π-periodic function can be approximated uniformly by trigonometric polynomials.

Consequences on Fourier coefficients

Theorem: (**Uniqueness of Fourier coefficients**)

> If $f, g \in L^1(\mathbb{T})$ and $\hat{f}(n) = \hat{g}(n)\ \forall n \in \mathbb{Z}$, then $f = g$ a.e.

Theorem: (**Riemann-Lebesgue lemma**)

> If $f \in L^1(\mathbb{T})$, then $\displaystyle\lim_{n\to\infty} \hat{f}(n) = 0.$

Pointwise convergence of Cesàro means

Theorem: (**Fejér**)

> Let $f \in L^1(\mathbb{T})$. If
> $$f^*(t_0) := \lim_{h \to 0} \frac{f(t_0+h) + f(t_0-h)}{2}$$
> exists, then $\lim_{n\to\infty} \sigma_n(f)(t_0) = f^*(t_0)$. In particular, $\sigma_n(f)$ converges to f at every point of continuity of f.

Corollary:

> If $f \in L^1(\mathbb{T})$ has an absolutely convergent Fourier series, then $S_n(f)$ converges to f at every point of continuity of f.

Order of magnitude of Fourier coefficients

Theorem:

> Let $f \in L^1(\mathbb{T})$ such that $\hat{f}(0) = 0$ (if not, we can shift f up/down). Define
> $$F(t) := \int_0^t f(s) \, \mathrm{d}s.$$
> Then, F is continuous, 2π-periodic and $\hat{F}(n) = \frac{1}{in}\hat{f}(n)$ for $n \neq 0$.

Theorem:

> If f is any k times differentiable function such that $f^{(k)} \in L^1(\mathbb{T})$, then
> $$\left|\hat{f}(n)\right| \leq \frac{\left\|f^{(k)}\right\|_{L^1}}{|n|^k}.$$

Pointwise convergence of partial sums

Theorem:

> Let $f \in L^1(\mathbb{T})$ and assume that there exists C such that $\left|\hat{f}(n)\right| \leq \dfrac{C}{|n|}$ (for example if f is differentiable). Then, $S_n(f)(t)$ and $\sigma_n(f)(t)$ converge for the same values of t and to the same limit.

Corollary:

> If $f \in C^1(\mathbb{T})$, then $S_n(f)(t)$ converges uniformly in $t \in \mathbb{T}$ to $f(t)$.

Theorem: (**Dini's test**)

> Let $f \in L^1(\mathbb{T})$. If for some $t_0 \in \mathbb{T}$ we have
> $$\int_{-1}^1 \left| \frac{f(t+t_0) - f(t_0)}{t} \right| \, \mathrm{d}t < \infty,$$
> then $S_n(f)(t_0) \to f(t_0)$ as $n \to \infty$.

Negative convergence results

Theorem:

> There exists a continuous function whose Fourier series diverges at a point.

Theorem:

> There exists a function $f \in L^1(\mathbb{T})$ such that $\lim_{n\to\infty} \left\|S_n(f) - f\right\|_{L^1} \neq 0$.

Theorem: (**Kolmogorov**)

> There exists a function $f \in L^1(\mathbb{T})$ whose Fourier series diverges everywhere.

Fourier series in $L^2(\mathbb{T})$

General results in Hilbert spaces

Definition: (**ON-system**)

> Let $\mathcal{H}$ be a complex Hilbert space. A subset $E \subseteq \mathcal{H}$ is orthogonal if for all $\varphi, \psi \in E$ with $\varphi \neq \psi$ we have $\langle \varphi, \psi \rangle = 0$. Further, E is an orthonormal system/ON-system if also $\|\varphi\| = 1$ for all $\varphi \in E$.

Lemma:

> Let $\{\varphi_n\}_{n=1}^N$ be a finite orthnormal system and let $a_1, \dots, a_N \in \mathbb{C}$. Then,
>
> $$\left\| \sum_{n=1}^N a_n \varphi_n \right\|^2 = \sum_{n=1}^N |a_n|^2 .$$
>
> (Pythagoras with coordinates given by the finite ON-system)

Corollary:

> Let $\{\varphi_n\}_{n\geq 1}$ be a (possibly infinite) ON-system and let $a_1, a_2, \dots \in \mathbb{C}$ such that $\sum_{n=1}^\infty |a_n|^2 < \infty$. Then $\sum_{n=1}^\infty a_n \varphi_n$ converges in $\mathcal{H}$.

Theorem: (**Bessel's inequality**)

> Let $\{\varphi_n\}$ be an ON-system. Then, for any $f \in \mathcal{H}$,
>
> $$\sum_{n=1}^\infty \left|\langle f, \varphi_n \rangle\right|^2 \leq \|f\|^2 .$$
>
> (Pythagoras, but possibly omitting some dimensions in the left hand side)

Definition/Proposition: (**Complete ON-system**)

> Let $\{\varphi_n\}$ be an ON-system. It is a complete ON-system if any of the following equivalent statements hold:
> (i) If $f \in \mathcal{H}$ is orthogonal to all φ_n, then $\psi = 0$. (Gram-Schmidt can terminate)
> (ii) For every $f \in \mathcal{H}$ we have $\|f\|^2 = \sum_{n=1}^{\infty} \left|\langle f, \varphi_n \rangle\right|^2$. (Parseval's identity)
> (iii) For every $f \in \mathcal{H}$ we have $f = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n$. ($\{\varphi_n\}$ is a basis)

Theorem: (**Parseval's identity**)

> Let $\{\varphi_n\}$ be a complete ON-system. Then, for any $f, g \in \mathcal{H}$,
> $$\langle f, g \rangle = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \overline{\langle g, \varphi_n \rangle}.$$

Fourier series in L^2

Definition: ($L^2(\mathbb{T})$)

> $L^2(\mathbb{T})$ is the space of all square-integrable functions $f : \mathbb{T} \to \mathbb{C}$ with the inner product
> $$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}\, \mathrm{d}t,$$
> which makes it a Hilbert space.

↳ Comment:
$L^2(\mathbb{T}) \subset L^1(\mathbb{T})$.

Theorem:

> $\{e^{int}\}_{n \in \mathbb{Z}}$ is a complete ON-system in $L^2(\mathbb{T})$.

Theorem: (**Conclusions from Hilbert theory**)

> Let $f, g \in L^2(\mathbb{T})$.
> (a) $\frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}\, \mathrm{d}t = \sum_{n \in \mathbb{Z}} \hat{f}(n)\overline{\hat{g}(n)}$ and $\frac{1}{2\pi} \int_0^{2\pi} \left|f(t)\right|^2 \mathrm{d}t = \sum_{n \in \mathbb{Z}} \left|\hat{f}(n)\right|^2$ (Parseval's identity)
> (b) $\left\|S_n(f) - f\right\|_{L^2} \to 0$ as $n \to \infty$
> (c) Given a sequence $\{a_n\} \in \ell^2(\mathbb{Z})$, there is a unique $h \in L^2(\mathbb{T})$ such that $\hat{h}(n) = a_n$.

Theorem: (**Carleson**)

> If $f \in L^2(\mathbb{T})$, then $S_n(f)$ converges almost everywhere to f.

Mixed topics on Fourier series

Lacunary Fourier series

Definition: (**Lacunary**)

> A sequence $\{\lambda_n\}_{n\geq 1}$ with $\lambda_n \in \mathbb{Z}_+$ is lacunary if there exists a constant $q > 1$ such that
>
> $$\lambda_{n+1} > q\lambda_n, \quad n \geq 1.$$
>
> A Fourier series is lacunary if it has the form $\sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t}$ where $\{\lambda_n\}$ is lacunary. (Many missing frequencies)

↳ Example: $\lambda_n = 2^n$ is lacunary.

Theorem: (**Weierstrass**)

> The function $\sum_{n=1}^{\infty} 2^{-n} \cos 2^n t$ is continuous but nowhere differentiable.

[Skipped: Jacobi identity, Heat equation]

Weyl equidistribution theorem

Theorem: (**Weyl**)

> Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. The following are equivalent:
>
> (i) For all 1-periodic continuous functions f,
>
> $$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(x_k \bmod 1) = \int_0^1 f(x)\, \mathrm{d}x$$
>
> (ii) For every integer $m \neq 0$,
>
> $$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i m x_k} = 0 \quad \text{(Weyl's condition)}$$

Fourier transforms

The Fourier transform on $L^1(\mathbb{R})$

Definition: ($L^1(\mathbb{R})$)

> $L^1(\mathbb{R})$ is the space of all integrable functions on $\mathbb{R}$ with the norm
>
> $$\|f\|_{L^1} = \int_{-\infty}^{\infty} |f(x)|\, \mathrm{d}x.$$

Definition: (**Fourier transform**)

> Let $f \in L^1(\mathbb{R})$. Its Fourier transform $\hat{f} : \mathbb{R} \to \mathbb{C}$ is
>
> $$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\, \mathrm{d}x.$$

Theorem:

Let $f \in L^1(\mathbb{R})$. Then,

$$\left|\hat{f}(\xi)\right| \leq \|f\|_{L^1}, \quad \xi \in \mathbb{R},$$

and $\hat{f}$ is uniformly continuous.

Definition: (**Convolution**)

Let $f, g \in L^1(\mathbb{R})$. Their convolution is

$$(f * g)(x) := \int_{-\infty}^{\infty} f(x-y)g(y)\ \mathrm{d}y.$$

↳ Comment:
The same properties as for convolutions in $L^1(\mathbb{T})$ hold.

Theorem:

Let $f \in L^1(\mathbb{R})$ and define

$$F(x) := \int_{-\infty}^{x} f(y)\ \mathrm{d}y.$$

If $F \in L^1(\mathbb{R})$, then $\hat{F}(\xi) = \frac{1}{i\xi}\hat{f}(\xi)$, for $\xi \in \mathbb{R} \setminus \{0\}$.

Theorem: (**Riemann-Lebesgue lemma**)

Let $f \in L^1(\mathbb{R})$. Then, $\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0$.

Definition: (**Summability kernel**)

A summability kernel on $\mathbb{R}$ is a family of continuous functions $\{k_\lambda\}$ on $\mathbb{R}$ such that

(i) $\int_{\mathbb{R}} k_\lambda(x)\ \mathrm{d}x = 1$

(ii) There is a constant C such that $\|k_\lambda\|_{L^1} \leq C$ for all λ

(iii) For any $\delta > 0$ we have $\lim_{\lambda \to \infty} \int_{|x|>\delta} \left|k_\lambda(x)\right|\ \mathrm{d}x = 0$

Proposition: (**Construction of a summability kernel**)

Let $f \in L^1(\mathbb{R})$ and suppose $\int_{\mathbb{R}} f = 1$. Then, $k_\lambda(x) := \lambda f(\lambda x)$ is a summability kernel.

Theorem:

Let $f \in L^1(\mathbb{R})$ and let $\{k_\lambda\}$ be a summability kernel on $\mathbb{R}$. Then,

$$\lim_{\lambda \to \infty} \|k_\lambda * f - f\|_{L^1} = 0.$$

Theorem: (**Uniqueness**)

Let $f \in L^1(\mathbb{R})$ and assume that $\hat{f}(\xi) = 0$ for all $\xi \in \mathbb{R}$ (or a.e., since $\hat{f}$ is continuous). Then $f = 0$ a.e.

Theorem: (**Inversion**)

> Assume that both f and $\hat{f}$ are in $L^1(\mathbb{R})$. Then,
>
> $$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)e^{i\xi x}\,\mathrm{d}\xi = \frac{1}{2\pi}\hat{\hat{f}}(-x), \quad x \in \mathbb{R}.$$
>
> Also, f is (a.e. equal to something) uniformly continuous.

The Fourier transform on the Schwarz space

Definition: (**Schwarz space**)

> The Schwarz space $\mathcal{S}(\mathbb{R})$ on $\mathbb{R}$ consists of all functions $f : \mathbb{R} \to \mathbb{C}$ that are infinitely differentiable, and for every $k, l \geq 0$ we have
>
> $$\sup_{x\in\mathbb{R}} \left|x^k\right| \cdot \left|f^{(l)}(x)\right| < \infty.$$
>
> (All derivatives decay faster than polynomials)

↳ Comment: $\mathcal{S}(\mathbb{R}) \subseteq L^1(\mathbb{R})$

Proposition:

> If $f \in \mathcal{S}(\mathbb{R})$, then
> (a) $\widehat{(f')}(\xi) = i\xi\hat{f}(\xi)$
> (b) $(\hat{f})'(\xi) = (\widehat{-ixf(x)})(\xi)$

Theorem: (**Inversion**)

> If $f \in \mathcal{S}(\mathbb{R})$, then
>
> $$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)e^{i\xi x}\,\mathrm{d}\xi, \quad x \in \mathbb{R}.$$

Theorem:

> The Fourier transform maps $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$ bijectively.

Theorem: (**Plancherel**)

> If $f \in \mathcal{S}(\mathbb{R})$, then
>
> $$\int_{\mathbb{R}} \left|f(x)\right|^2\,\mathrm{d}x = \frac{1}{2\pi}\int_{\mathbb{R}} \left|\hat{f}(\xi)\right|^2\,\mathrm{d}\xi.$$
>
> (Like a limit of Parseval's identity)

Corollary:

> If $f, g \in \mathcal{S}(\mathbb{R})$, then
>
> $$\int_{\mathbb{R}} f(x)\overline{g(x)}\,\mathrm{d}x = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,\mathrm{d}\xi.$$

The Fourier transform on $L^2(\mathbb{R})$

Definition: (**Fourier transform**)

> Let $f \in L^2(\mathbb{R})$. By density, there is a sequence $f_n \in \mathcal{S}(\mathbb{R})$ approaching f in L^2-norm. We define the Fourier transform of f as
> $$\hat{f} := \lim_{n\to\infty} \hat{f}_n \in L^2(\mathbb{R}). \quad \text{(as a limit in } L^2\text{)}$$

Theorem: (**Plancherel**)

> Let $f, g \in L^2(\mathbb{R})$. Then,
> $$\|f\|_{L^2}^2 = \frac{1}{2\pi}\left\|\hat{f}\right\|_{L^2}^2, \qquad \int_{\mathbb{R}} f(x)\overline{g(x)}\,\mathrm{d}x = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,\mathrm{d}\xi.$$

The Fourier transform on $\mathbb{R}^n$

Definition: (**Fourier transform**)

> Let $f \in L^1(\mathbb{R}^n)$. Its Fourier transform $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ is then
> $$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-i\xi\cdot x}\,\mathrm{d}x.$$

Theorem: (**Inversion**)

> If f and $\hat{f}$ both are in $L^1(\mathbb{R}^n)$, then
> $$f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{f}(\xi)e^{i\xi\cdot x}\,\mathrm{d}\xi,$$
> and f is uniformly continuous.

Mixed topics on Fourier transforms

Theorem: (**Poisson summation formula**)

> Let $f \in C^2(\mathbb{R})$ and suppose there is a constant C such that
> $$\left|f(x)\right| + \left|f'(x)\right| + \left|f''(x)\right| \le \frac{C}{1+x^2}, \quad x \in \mathbb{R}.$$
> Then,
> $$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(2\pi n).$$

Theorem: (**Heisenberg inequality**)

> Let $f \in L^2(\mathbb{R})$. Then for any $x_0, \xi_0 \in \mathbb{R}$,
> $$\int_{\mathbb{R}} (x-x_0)^2\left|f(x)\right|^2\mathrm{d}x \int_{\mathbb{R}} (\xi-\xi_0)^2\left|\hat{f}(\xi)\right|^2\mathrm{d}\xi \ge \frac{\pi}{2}\|f\|_{L^2}^4,$$
> with equality if and only if $f(x) = ce^{-|k|x^2}$ for some $c, k \in \mathbb{C}$.

Theorem: (**Shannon's sampling theorem**)

Let $f \in L^1(\mathbb{R})$ be continuous, and suppose that $\hat{f}$ is supported on $[-c, c]$. Then,

$$f(t) = \sum_{n\in\mathbb{Z}} f(\frac{n\pi}{c})\frac{\sin(ct - n\pi)}{ct - n\pi}.$$

Theorem: (**Paley-Wiener**)

Let $f : \mathbb{C} \to \mathbb{C}$ be a function. The following are equivalent.

(a) f is an entire function such that $\int_{-\infty}^{\infty} \left|f(x)\right|^2 \mathrm{d}x < \infty$, and there are constants A and C such that $\left|f(z)\right| \leq Ce^{A|z|}$.

(b) There exists $F \in L^2(-A, A)$ such that

$$f(z) = \int_{-A}^{A} F(t)e^{itz} \mathrm{d}t.$$

Theorem: (**Paley-Wiener**)

Let $f : \mathbb{C} \to \mathbb{C}$. The following are equivalent.

(a) f is analytic in the upper half plane $\{\operatorname{Im} z > 0\}$, and $\sup_{y>0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|f(x + iy)\right|^2 \mathrm{d}x = C < \infty$.

(b) There exists $F \in L^2(0, \infty)$ such that

$$f(z) = \int_{0}^{\infty} F(z)e^{itz} \mathrm{d}t, \operatorname{Im} z > 0$$

and $\int_0^\infty \left|F(t)\right|^2 \mathrm{d}t = C$.

Theorem: (**Paley-Wiener**)

Let $f \in L^2(\mathbb{R})$ and $a > 0$. The following are equivalent.

(a) f can be extended to an analytic function in the strip $\{|\operatorname{Im} z| < a\}$ such that $\int_{-\infty}^{\infty} \left|f(x + iy)\right|^2 \mathrm{d}x \leq C$ for $|y| < a$.

(b) $e^{a|\xi|}\hat{f}(\xi) \in L^2(\mathbb{R})$.

Fourier analysis on groups

Abelian groups

Definition: (**Character**)

Let G be an abelian group. A continuous function $\chi : G \to \mathbb{C}$ is a character on G if

(i) $\chi(a + b) = \chi(ab)$

(ii) $\left|\chi(a)\right| = 1$

Example:

- The characters on $(\mathbb{R}, +)$ are $\chi(a) = e^{ika}$ for $k \in \mathbb{R}$.
- The characters on $(\mathbb{T}, \cdot)$ are $\chi(a) = e^{ika}$ for $k \in \mathbb{Z}$.
- The characters on $(\mathbb{Z}_N = \{\zeta^0, \zeta^1, \ldots, \zeta^{N-1}\}, \cdot)$ (where ζ is the principal Nth root of unity) are $\chi(\zeta^k) = \zeta^{mk}$ for some $m = 0, \ldots, N - 1$.

Definition: (**Principal character**)

> A character χ is the principal character if $\chi(a) = 1$ for all $a \in G$.

Finite abelian groups

Here we let G be a finite abelian group.

Definition: (**Dual group**)

> If G is a finite abelian group, then its dual group $\hat{G}$ is the set of all characters of G, under multiplication.

Theorem:

> The characters of G form an orthonormal basis for the vector space V of functions $G \to \mathbb{C}$. This is with respect to the inner product
> $$\langle f, g \rangle := \frac{1}{|G|} \sum_{a \in G} f(a)\overline{g(a)}, \quad f, g \in V.$$

Definition/Theorem: (**Fourier series**)

> Any function $f : G \to \mathbb{C}$ can be expanded as
> $$f = \sum_{\chi \in \hat{G}} \hat{f}(\chi)\chi \quad \text{(Fourier series)}$$
> where $\hat{f}(\chi) := \langle f, \chi \rangle$ are Fourier coefficients.

Theorem: (**Parseval**)

> Let $f : G \to \mathbb{C}$. Then,
> $$\langle f, f \rangle = \sum_{\chi \in \hat{G}} \left| \hat{f}(\chi) \right|^2.$$

Tempered distributions

Definition: (**Convergence in $\mathcal{S}(\mathbb{R})$**)

> Let $\{\varphi_n\}$ be a sequence in $\mathcal{S}(\mathbb{R})$. It converges to 0 in $\mathcal{S}(\mathbb{R})$ as $n \to \infty$ if forall $m, k \in \mathbb{N}$ we have that
> $$|x|^m \left| \varphi_n^{(k)}(x) \right| \to 0$$
> uniformly in $x \in \mathbb{R}$ as $n \to \infty$.

Definition: (**Tempered distribution**)

> A tempered distribution u is a continuous linear functional $\mathcal{S}(\mathbb{R}) \to \mathbb{C}$.
>
> Continuity means that if $\varphi_n \to 0$ in $\mathcal{S}(\mathbb{R})$, then $u(\varphi_n) \to 0$.
>
> The set of all tempered distributions is $\mathcal{S}'(\mathbb{R})$.

Definition: (**Convergence in $\mathcal{S}'(\mathbb{R})$**)

> A sequence $\{u_n\}$ in $\mathcal{S}'$ converges in $\mathcal{S}'(\mathbb{R})$ to $u \in \mathcal{S}'(\mathbb{R})$ if for every $\varphi \in \mathcal{S}$ we have $u_n(\varphi) \to u(\varphi)$ as $n \to \infty$.

Example:

> If f is a function such that $\varphi \mapsto \int_{\mathbb{R}} f(x)\varphi(x)\ \mathrm{d}x$ is a tempered distribution, then we often identify f itself with this tempered distribution.

Definition: (**Derivative**)

> Let $u \in \mathcal{S}'$. Its derivative u' is defined by $u'(\varphi) = -u(\varphi')$.

Theorem:

> Let $f, g \in L^1(\mathbb{R})$. Then,
> $$\int_{\mathbb{R}} \hat{f}(x)g(x)\ \mathrm{d}x = \int_{R} \hat{g}(x)f(x)\ \mathrm{d}x.$$

Definition: (**Fourier transform**)

> Let $u \in \mathcal{S}'$. Its Fourier transform $\hat{u}$ is defined by $\hat{u}(\varphi) = u(\hat{\varphi})$.

Theorem: (**Uniqueness & Inversion**)

> The Fourier transform $u \mapsto \hat{u}$ is a bijection on $\mathcal{S}'$. Also,
> $$\hat{\hat{u}}(\varphi) = 2\pi u(\check{\varphi})$$
> where $\check{\varphi}(x) = \varphi(-x)$.