MM7052 Topology - Theory Questions

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14/12-2025

Formulations only

Question 1 - Formulation

(Closed map lemma) Suppose F is a continuous map from a compact space to a Hausdorff space. Then F is a closed map.

Question 2 - Formulation

(Homotopy Invariance of π_1) If $\varphi : X \to Y$ is a homotopy equivalence, then for any point $p \in X$, $\varphi_* : \pi_1(X, p) \to \pi_1(Y, \varphi(p))$ is an isomorphism.

Question 3 - Formulation

(Attaching a Disk) Let X be a path-connected topological space, and let $\widetilde{X}$ be the space obtained by attaching a closed 2-cell D to X along an attaching map $\varphi : \partial D \to X$. Let $v \in \partial D$, $\widetilde{v} = \varphi(v) \in X$ and $\gamma = \varphi_*(\alpha) \in \pi_1(X, \widetilde{v})$, where α is a generator of the infinite cyclic group $\pi_1(\partial D, v)$. Then the homomorphism $\pi_1(X, \widetilde{v}) \to \pi_1(\widetilde{X}, \widetilde{v})$ induced by the inclusion $X \hookrightarrow \widetilde{X}$ is surjective, and its kernel is the smallest normal subgroup containing γ. If $\pi_1(X, \widetilde{v})$ has a finite presentation

$$\pi_1(X, \widetilde{v}) \cong \langle \beta_1, \ldots, \beta_n \mid \sigma_1, \ldots, \sigma_s \rangle,$$

then $\pi_1(\widetilde{X}, \widetilde{v})$ has the presentation

$$\pi_1(\widetilde{X}, \widetilde{v}) \cong \langle \beta_1, \ldots, \beta_n \mid \sigma_1, \ldots, \sigma_s, \tau \rangle,$$

where τ is an expression for $\gamma \in \pi_1(X, \widetilde{v})$ in terms of $\{\beta_1, \ldots, \beta_n\}$.

Question 4 - Formulation

> **(Existence of the Universal Covering Space)** Every connected and locally simply connected topologcal space (in particular, every connected manifold) has a universal covering space.

Question 5 - Formulation

> Let X be a topological space that has a universal covering space, and let $x_0 \in X$. There is a one-to-one correspondence between isomorphism classes of coverings of X and conjugacy classes of subgroups of $\pi_1(X, x_0)$. The correspondence associates each covering $\widehat{q} : \widehat{E} \to X$ with the conjugacy class of its induced subgroup.

[We may skip showing that $\widehat{q}$ is a covering map.]

Ideas only

Question 1 - Ideas

(Closed map lemma) Suppose F is a continuous map from a compact space to a Hausdorff space. Then F is a closed map.

Lemma 1. [...]

↳ Proof:

Cover X using a cover of A.

□

Lemma 2. [...]

↳ Proof:

Create separation of A with points in $X \setminus A$.

□

Use (1) and then (2).

Question 2 - Ideas

(Homotopy Invariance of π_1) If $\varphi : X \to Y$ is a homotopy equivalence, then for any point $p \in X$, $\varphi_* : \pi_1(X, p) \to \pi_1(Y, \varphi(p))$ is an isomorphism.

Lemma: Prove that induced homomorphisms φ_*, ψ_* are compatible with the isomorphism Φ_h from a change of basepoint.

Lemma. [...]

↳ Proof:

Define $F(s,t) = H(f(s),t)$ and set s or t to 0 or 1.

□

Apply Lemma to $\psi \circ \varphi$ and id_X.

Apply Lemma to $\varphi \circ \psi$ and id_Y.

Question 3 - Ideas

(Attaching a Disk) Let X be a path-connected topological space, and let $\widetilde{X}$ be the space obtained by attaching a closed 2-cell D to X along an attaching map $\varphi : \partial D \to X$. Let $v \in \partial D$, $\widetilde{v} = \varphi(v) \in X$ and $\gamma = \varphi_*(\alpha) \in \pi_1(X, \widetilde{v})$, where α is a generator of the infinite cyclic group $\pi_1(\partial D, v)$. Then the homomorphism $\pi_1(X, \widetilde{v}) \to \pi_1(\widetilde{X}, \widetilde{v})$ induced by the inclusion $X \hookrightarrow \widetilde{X}$ is surjective, and its kernel is the smallest normal subgroup containing γ. If $\pi_1(X, \widetilde{v})$ has a finite presentation

$$\pi_1(X, \widetilde{v}) \cong \langle \beta_1, \dots, \beta_n \mid \sigma_1, \dots, \sigma_s \rangle,$$

then $\pi_1(\widetilde{X}, \widetilde{v})$ has the presentation

$$\pi_1(\widetilde{X}, \widetilde{v}) \cong \langle \beta_1, \dots, \beta_n \mid \sigma_1, \dots, \sigma_s, \tau \rangle,$$

where τ is an expression for $\gamma \in \pi_1(X, \widetilde{v})$ in terms of $\{\beta_1, \dots, \beta_n\}$.

Split the space as the interior of D & everything except a point in D.

Check requirements for Seifert-Van Kampen on $\widetilde{U}$, $\widetilde{V}$.

Choose basepoint and loop in $\widetilde{U} \cap \widetilde{V}$ and compute $\pi_1(\widetilde{U} \cap \widetilde{V}) \cong \mathbb{Z}$.

Use Seifert-Van Kampen in the case of one simply connected set.

Relate this to basepoint $\widetilde{v}$ and loop γ using change-of-basepoint isomorphism.

Relate $\pi_1(\widetilde{V})$ to $\pi_1(X)$ using strong deformation retractions.

Question 4 - Ideas

(Existence of the Universal Covering Space) Every connected and locally simply connected topologcal space (in particular, every connected manifold) has a universal covering space.

Define $\widetilde{X}$ as set of path classes.

STEP 1: *Topologize* $\widetilde{X}$.

Define basis sets: Concatenate on paths lying inside a simply connected open set.

STEP 2: $\widetilde{X}$ *is path-connected*. [This step may be skipped]

STEP 3: q *is a covering map*.

Fixing x_1, show that $q^{-1}(U)$ is union of sets $[f * U]$.

Conclude continuous and surjective.

Show that q restricts to homeomorphisms. Also conclude $\widetilde{X}$ is locally-path connected.

Show that the union from before is disjoint.

STEP 4: $\widetilde{X}$ is simply connected.

View F as a lift of f, and construct another lift using f_t, going t far along f.

Question 5 - Ideas

Let X be a topological space that has a universal covering space, and let $x_0 \in X$. There is a one-to-one correspondence between isomorphism classes of coverings of X and conjugacy classes of subgroups of $\pi_1(X, x_0)$. The correspondence associates each covering $\widehat{q} : \widehat{E} \to X$ with the conjugacy class of its induced subgroup.

Reduce to finding a covering inducing the subgroup H using covering isomorphism theorem.

Theorem says that $\pi_1(X, x_0) \cong \mathrm{Aut}_q(E)$.

Use covering space action to construct a quotient space $\widehat{E}$ and $\widehat{q}$.

[We may skip showing that $\widehat{q}$ is a covering map.]

Show that induced subgroup is H using the isotropy group.

Solutions

Question 1 - Solution

(Closed map lemma) Suppose F is a continuous map from a compact space to a Hausdorff space. Then F is a closed map.

We first prove two lemmas.

Lemma 1. Every closed subset of a compact space is compact.

↳ Proof:
Suppose X is compact and $A \subseteq X$ is closed.

Cover X using a cover of A.

Take an open cover $\mathcal{U}$ of A. Then, $\mathcal{U} \cup \{X \setminus A\}$ is an open cover of X. By compactness, there is a finite subcover, which also covers A. □

Lemma 2. Every compact subset of a Hausdorff space is closed.

↳ Proof:
Suppose X is Hausdorff and $A \subseteq X$ is compact.

Create separation of A with points in $X \setminus A$.

Take a point $p \in X \setminus A$. We know from earlier that there are disjoint open subsets $U \supseteq A, V \ni p$. Note that V is a neighborhood of p contained in $X \setminus A$. Since p was arbitrary, we conclude that $X \setminus A$ is open and therefore A is closed. □

We continue with the theorem. Let $F : X \to Y$ where X is compact and Y is Hausdorff. Take a closed subset $A \subseteq X$.

Use (1) and then (2).

By (1), A is compact. Recall that by continuity, $f(A)$ is compact. By (2), $f(A)$ is then closed.

Question 2 - Solution

(Homotopy Invariance of π_1) If $\varphi : X \to Y$ is a homotopy equivalence, then for any point $p \in X$, $\varphi_* : \pi_1(X,p) \to \pi_1(Y, \varphi(p))$ is an isomorphism.

First we prove a lemma.

Lemma: Prove that induced homomorphisms φ_*, ψ_* are compatible with the isomorphism Φ_h from a change of basepoint.

Lemma. Suppose $\varphi, \psi : X \to Y$ are continuous and H is a homotopy from φ to ψ. For any $p \in X$, let h be the path in Y from $\varphi(p)$ to $\psi(p)$ defined by $h(t) = H(p,t)$ and let $\Phi_h : \pi_1(Y, \varphi(p)) \to \pi_1(Y, \psi(p))$ be the isomorphism $[\alpha] \mapsto \left[\overline{h}\right] \cdot [\alpha] \cdot [h]$. Then the following diagram commutes:

$$\begin{tikzcd}
& \pi_1(Y,\varphi(p)) \arrow[dd, "\Phi_h"] \\
\pi_1(X,p) \arrow[ur, "\varphi_*"] \arrow[dr, "\psi_*"'] & \\
& \pi_1(Y,\psi(p))
\end{tikzcd}$$

↳ Proof:
Let f be a loop in X based at p.

Define $F(s,t) = H(f(s),t)$ and set s or t to 0 or 1.

We need to show

$$\begin{aligned}
&\psi_*[f] = \Phi_h\left(\varphi_*(f)\right) \\
\iff\ &\psi \circ f \sim \overline{h} * (\varphi \circ f) * h \\
\iff\ &h * (\psi \circ f) \sim (\varphi \circ f) * h
\end{aligned}$$

Let $F : I \times I \to Y$ be defined by $F(s,t) = H(f(s),t)$. Then F is continuous as a composition of H with the product map $f \times \mathrm{id}$, and

$$\begin{aligned}
F(s,0) &= H(f(s),0) = \varphi \circ f(s) \\
F(1,t) &= H(p,t) = h(t) \\
F(0,t) &= H(p,t) = h(t) \\
F(s,1) &= H(f(s),1) = \psi \circ f(s)
\end{aligned}$$

from which the result follows by the square lemma. $\square$

We continue with the theorem. Let $\varphi : X \to Y$ be a homotopy equivalence and let $\psi : Y \to X$ be a homotopy inverse. Pick a point $p \in X$.

Apply Lemma to $\psi \circ \varphi$ and id_X.

Since $\psi \circ \varphi, \mathrm{id}_X : X \to X$ are homotopic, Lemma gives a path h in X such that the following commutes:

$$\begin{tikzcd}
 & \pi_1(X,p) \arrow[d, "\Phi_h"] \\
\pi_1(X,p) \arrow[ur, "\mathrm{id}_X"] \arrow[dr, "(\psi\circ\varphi)_*"'] & \\
 & \pi_1(X,\psi(\varphi(p)))
\end{tikzcd}$$

Therefore $\psi_* \circ \varphi_* = (\psi \circ \varphi)_* = \mathrm{id}_X \circ \Phi_h = \Phi_h$ which is an isomorphism. In particular, $\psi_* : \pi_1(Y, \varphi(p)) \to \pi_1(Y, \psi(\varphi(p)))$ is surjective.

Apply Lemma to $\varphi \circ \psi$ and id_Y.

Since $\varphi \circ \psi, \mathrm{id}_Y : Y \to Y$ are homotopic, Lemma gives a path k in Y such that the following commutes:

$$\begin{tikzcd}
 & \pi_1(Y,\varphi(p)) \arrow[d, "\Phi_k"] \\
\pi_1(Y,\varphi(p)) \arrow[ur, "\mathrm{id}_Y"] \arrow[dr, "(\varphi\circ\psi)_*"'] & \\
 & \pi_1(Y,\varphi(\psi(\varphi(p))))
\end{tikzcd}$$

Therefore $\varphi_* \circ \psi_* = (\varphi \circ \psi_*) = \mathrm{id}_Y \circ \Phi_k = \Phi_k$ which is an isomorphism. In particular, ψ_* is injective.

In total, ψ_* is an isomorphism. Then, so is

$$\varphi_* = (\psi_*)^{-1} \circ \Phi_h : \pi_1(X, p) \to \pi_1(X, \varphi(p)).$$

Question 3 - Solution

(Attaching a Disk) Let X be a path-connected topological space, and let $\widetilde{X}$ be the space obtained by attaching a closed 2-cell D to X along an attaching map $\varphi : \partial D \to X$. Let $v \in \partial D$, $\widetilde{v} = \varphi(v) \in X$ and $\gamma = \varphi_*(\alpha) \in \pi_1(X, \widetilde{v})$, where α is a generator of the infinite cyclic group $\pi_1(\partial D, v)$. Then the homomorphism $\pi_1(X, \widetilde{v}) \to \pi_1(\widetilde{X}, \widetilde{v})$ induced by the inclusion $X \hookrightarrow \widetilde{X}$ is surjective, and its kernel is the smallest normal subgroup containing γ. If $\pi_1(X, \widetilde{v})$ has a finite presentation

$$\pi_1(X, \widetilde{v}) \cong \langle \beta_1, \ldots, \beta_n \mid \sigma_1, \ldots, \sigma_s \rangle,$$

then $\pi_1(\widetilde{X}, \widetilde{v})$ has the presentation

$$\pi_1(\widetilde{X}, \widetilde{v}) \cong \langle \beta_1, \ldots, \beta_n \mid \sigma_1, \ldots, \sigma_s, \tau \rangle,$$

where τ is an expression for $\gamma \in \pi_1(X, \widetilde{v})$ in terms of $\{\beta_1, \ldots, \beta_n\}$.

Denote $q : X \amalg D \to \widetilde{X}$ as the quotient map. We view X as a subspace of $\widetilde{X}$.

Split the space as the interior of D & everything except a point in D.

Setting up notation:

- $U := \operatorname{Int} D$
- Choose $z \in U$
- $V := X \amalg (D \setminus \{z\})$
- $\widetilde{U} := q(U)$
- $\widetilde{V} := q(V)$

Check requirements for Seifert-Van Kampen on $\widetilde{U}$, $\widetilde{V}$.

Notice U and V are saturated open. Therefore $q|_U \colon U \to \widetilde{U}$ and $q|_V \colon V \to \widetilde{V}$ are quotient maps and the images $\widetilde{U}$ and $\widetilde{V}$ are open in $\widetilde{X}$. Notice $\widetilde{U}$ and $\widetilde{U} \cap \widetilde{V}$ are path-connected because they are the continuous images of the path-connected sets U and $U \cap V$ respectively. Also $\widetilde{V}$ is path-connected because it is the union of the path-connected sets X and $q(D \setminus \{z\})$ which share the point $\tilde{v}$.

Choose basepoint and loop in $\widetilde{U} \cap \widetilde{V}$ and compute $\pi_1(\widetilde{U} \cap \widetilde{V}) \cong \mathbb{Z}$.

Choose $p \in \operatorname{Int} D \setminus \{z\}$. Let $c : I \to \operatorname{Int} D \setminus \{z\}$ be a loop based at p whose path class generates $\pi_1(\operatorname{Int} D \setminus \{z\}, p) \cong \mathbb{Z}$. Denote

- $\widetilde{p} := q(p)$
- $\widetilde{c} := q \circ c$

Note that $q|_U$ is injective and therefore a homeomorphism $U \cong \widetilde{U}$. Since U is simply connected, so is $\widetilde{U}$. Also $\operatorname{Int} D \setminus \{z\}$ is saturated open so $q|_{\operatorname{Int} D \setminus \{z\}} \colon \operatorname{Int} D \setminus \{z\} \to q(\operatorname{Int} D \setminus \{z\}) = \widetilde{U} \cap \widetilde{V}$ is a quotient map. This map is also injective so $\operatorname{Int} D \setminus \{z\} \cong \widetilde{U} \cap \widetilde{V}$. Therefore $\pi_1(\widetilde{U} \cap \widetilde{V}, \widetilde{p})$ is the infinite cyclic group generated by $[\widetilde{c}]$.

Use Seifert-Van Kampen in the case of one simply connected set.

Seifert-Van Kampen in the case where $\widetilde{U}$ is simply connected gives a surjection

$$\pi_1(\widetilde{V}, \widetilde{p}) \to \pi_1(\widetilde{X}, \widetilde{p}) \tag{$*$}$$

whose kernel is the normal closure of the cyclic subgroup generated by $[\tilde{c}]$.

Relate this to basepoint $\widetilde{v}$ and loop γ using change-of-basepoint isomorphism.

Let

- Let b be a path in D from p to v.
- Let a be a loop in ∂D representing α.
- $\widetilde{b} := q \circ b$.
- $\widetilde{a} := q \circ a$ which represents γ.

The change of basepoint theorem gives an isomorphism $\Phi_{\widetilde{b}} : \pi_1(\widetilde{X}, \widetilde{p}) \to (\widetilde{X}, \widetilde{v})$ and similarly with "$\tilde{V}$" instead of "$\tilde{X}$". Notice that $\overline{b} * c * b \sim a$ in $D \setminus \{z\}$ (possibly replacing c with $\overline{c}$, depending on how c and a were chosen as generators) and therefore $\Phi_{\widetilde{b}}[\widetilde{c}] = [\widetilde{a}] = \gamma$. Applying this to $(*)$ and recalling that $\Phi_{\widetilde{b}}$ commutes with homomorphisms induced by inclusions, we get a surjection

$$\pi_1(\widetilde{V}, \widetilde{v}) \to \pi_1(\widetilde{X}, \widetilde{v}) \tag{$**$}$$

whose kernel is the smallest normal subgroup containing γ.

Relate $\pi_1(\widetilde{V})$ to $\pi_1(X)$ using strong deformation retractions.

Recall that there is a strong deformation retraction of $D \setminus \{z\}$ onto ∂D. Combined with the identity on X we get a strong deformation of V onto $X \amalg \partial D$. Say that this is given by the homotopy $H : V \times I \to V$. Recall that $q \times \mathrm{id}_I : V \times I \to \widetilde{V} \times I$ is a quotient map (since I is locally compact Hausdorff). Since H is constant on the fibers of $q \times \mathrm{id}_I$ (the only non-singleton fibers are $(q^{-1}(d), t)$ for $d \in q(\partial D)$), it descends to a homotopy $q \circ H$ giving strong deformation retraction $\widetilde{V}$ onto X. In particular, X and $\widetilde{V}$ are homotopically equivalent. By homotopy invariance of the fundamental group, $(**)$ becomes a new surjection

$$\pi_1(X, \widetilde{v}) \to \pi_1(\widetilde{X}, \widetilde{v})$$

whose kernel is the smallest normal subgroup containing γ.

The statement about presentations follows from properties of the amalgamated free product.

Question 4 - Solution

(Existence of the Universal Covering Space) Every connected and locally simply connected topologcal space (in particular, every connected manifold) has a universal covering space.

Let X be a connected and locally simply connected space. Choose a basepoint $x_0 \in X$.

Define $\widetilde{X}$ as set of path classes.

Define $\widetilde{X}$ as the set of path classes of paths in X starting at x_0. Define $q : \widetilde{X} \to X$ by $q([f]) := f(1)$.

Step 1: *Topologize* $\widetilde{X}$.

Define basis sets: Concatenate on paths lying inside a simply connected open set.

For each $[f] \in \widetilde{X}$ and each simply connected open subset $U \subseteq X$ containing $f(1)$, define the set

$$[f * U] := \{[f * a] : a \text{ is a path in } U \text{ starting at } f(1)\} \subseteq \widetilde{X}.$$

Let $\mathcal{B}$ denote the collection of all such sets $[f * U]$. We will show that $\mathcal{B}$ is a basis. First, since X is locally simply connected, for each $[f] \in \widetilde{X}$ there exists a simply connected neighborhood U of $f(1)$. Clearly $[f] \in [f * U]$. Thus the union of all sets in $\mathcal{B}$ is $\widetilde{X}$.

Second, we check the intersection condition. Suppose $[h] \in \widetilde{X}$ is in the intersection of two basis sets $[f * U], [g * V]$. This means that there is a path a in U and a path b in V such that $h \sim f * a \sim g * b$. Let W be a simply connected neighborhood of $h(1)$ contained in $U \cap V$. We will prove that $[h * W] \subseteq [f * U] \cap [g * V]$. Consider some $[h * c] \in [h * W]$. Then $[h * c] = [f * a * c] \in [f * U]$ since $a * c$ is a path in U. Similarly $[h * c] \in [g * V]$. We have now proved that $\mathcal{B}$ is a basis.

Step 2: $\widetilde{X}$ *is path-connected.* [This step may be skipped]

Step 3: q *is a covering map.*
Let $U \subseteq X$ be any simply connected open subset. We will show that U is evenly covered.

Fixing x_1, show that $q^{-1}(U)$ is union of sets $[f * U]$.

Choose any point $x_1 \in U$. We begin by showing that $q^{-1}(U)$ is the union of the sets $[f * U]$ as $[f]$ varies over all the distinct path classes from x_0 to x_1. First, $\bigcup_{[f]}[f * U] \subseteq q^{-1}(U)$ since $q([f * U]) \subseteq U$. Conversely, let $[g] \in q^{-1}(U)$. Then $g(1) = q([g]) \in U$, so there is a path b in U from $g(1)$ to x_1. Notice $[g] = [g * b * \overline{b}] \in [(g * b) * U]$ where $g * b$ is path from x_0 to x_1. This shows the reverse inclusion.

Conclude continuous and surjective.

Since $q^{-1}(U)$ was a union of basis sets and therefore open, and the simply connected open sets U constitute a basis for X, we conclude that q is continuous. Also q is surjective because $x \in X$ can be reached via $x = q([g])$ where g is a path from x_0 to x (X is path-connected, since it is locally path-connected and connected).

Show that q restricts to homeomorphisms. Also conclude $\widetilde{X}$ is locally-path connected.

Next we show that q is homeomorphism from each set $[f * U]$ to U. It is surjective because for each $x \in U$ there is a path a from $f(1)$ to x in U, so $x = q([f * a]) \in q([f * U])$. For injectivity, let $[g], [g'] \in [f * U]$ such that $q([g]) = q([g'])$. By definition, there are paths a and a' from $f(1)$ to $g(1) = g'(1)$ in U such that $g \sim f * a$ and $g' \sim f * a'$. Since U is simply connected, $a \sim a'$ and therefore $[g] = [g']$, proving injectivity. Finally, q is an open map since it takes basis sets to open sets, and therefore its inverse is continuous. We conclude that $q : [f * U] \to U$ is a homeomorphism.

Now, $\widetilde{X}$ is locally path-connected because each basis set $[f * U]$ is path-connected as a homeomorphic copy of the path-connected set U.

Show that the union from before is disjoint.

Take two paths f and f' from x_0 to x_1. If the sets $[f*U]$ and $[f*U']$ are not disjoint, there exists $[g] \in [f*U] \cap [f'*U]$. This means that there are paths a and a' in U from x_1 to $g(1)$ such that $g \sim f * a \sim f' \sim a'$. Since U is simply connected, $a \sim a'$ so $f \sim f'$ so $[f * U] = [f' * U]$. From all this we conclude that q is a covering map.

Step 4: $\widetilde{X}$ is simply connected.
Suppose $F : I \to \widetilde{X}$ is a loop based at $[c_{x_0}]$.

View F as a lift of f, and construct another lift using f_t, going t far along f.

Let $f := q \circ F$, so F is a lift of f. For $0 \leq t \leq 1$, define $f_t : I \to X$ by $f_t(s) = f(ts)$ as a path from x_0 to $f(t)$. Let $\widetilde{f} : I \to \widetilde{X}$ be defined by $\widetilde{f}(t) := [f_t]$. Then, $q \circ \widetilde{f}(t) = q([f_t]) = f_t(1) = f(t)$, so $\widetilde{f}$ is also a lift of f starting at $[c_{x_0}]$. By the unique lifting property, $F = \widetilde{f}$. Since F is a loop,

$$[c_{x_0}] = F(1) = \widetilde{f}(1) = [f_1] = [f]$$

so f is null-homotopic. By the monodromy theorem, F is also null-homotopic.

Question 5 - Solution

Let X be a topological space that has a universal covering space, and let $x_0 \in X$. There is a one-to-one correspondence between isomorphism classes of coverings of X and conjugacy classes of subgroups of $\pi_1(X, x_0)$. The correspondence associates each covering $\widehat{q} : \widehat{E} \to X$ with the conjugacy class of its induced subgroup.

Reduce to finding a covering inducing the subgroup H using covering isomorphism theorem.

By the covering isomorphism theorem, coverings belong to the same isomorphism class if and only if they have the same conjugacy class of induced subgroups. Thus, it only remains to, for each conjugacy class of subgroups, find a covering whose induced subgroup lies in this conjugacy class. Let $H \subseteq \pi_1(X, x_0)$ be any subgroup. We only need to find a covering whose induced subgroup is H.

Let $q : E \to X$ be the universal covering of X and choose a basepoint $e_0 \in E$ such that $q(e_0) = x_0$.

Theorem says that $\pi_1(X, x_0) \cong \operatorname{Aut}_q(E)$.

The simply connected case of the automorphism group structure theorem says that $\pi_1(X, x_0)$ is isomorphic to the automorphism group $\operatorname{Aut}_q(E)$, under the map that sends each path class $\gamma \in \pi_1(X, x_0)$ to the unique automorphism $\varphi_\gamma \in \operatorname{Aut}_q(E)$ that satisfies $\varphi_\gamma(e_0) = e_0 \cdot \gamma$. Let $\widehat{H} \subseteq \operatorname{Aut}_q(E)$ be the image of H under this isomorphism, so $\widehat{H} = \{\varphi_\gamma : \gamma \in H\}$.

Use covering space action to construct a quotient space $\widehat{E}$ and $\widehat{q}$.

Since the action of $\operatorname{Aut}_q(E)$ on E is a covering space action, we know that the restriction of the action to $\widehat{H}$ is also a covering space action. We may then construct the quotient space $\widehat{E} := E/\widehat{H}$, denoting the quotient map as $Q : E \to \widehat{E}$. The covering space quotient theorem gives that Q is a normal covering map. Moreover, q is constant on the fibers of Q, so q descends to a continuous map $\widehat{q} : \widehat{E} \to X$:

$$
\begin{tikzcd}
E \arrow[dr, "Q"] \arrow[dd, "q"'] & \\
& \widehat{E} \arrow[dl, "\widehat{q}"] \\
X &
\end{tikzcd}
$$

[We may skip showing that $\widehat{q}$ is a covering map.]

Show that induced subgroup is H using the isotropy group.

Let $\widehat{e}_0 := Q(e_0) \in \widehat{E}$; then $\widehat{q}(\widehat{e}_0) = x_0$. By a theorem, $\widehat{q}_* \pi_1(\widehat{E}, \widehat{e}_0)$ is the isotropy group of $\widehat{e}_0$ under the monodromy action by $\pi_1(X, x_0)$ on $\widehat{E}$. Take some path class $\gamma \in \pi_1(X, x_0)$. Recall that Q restricts to a $\pi_1(X, x_0)$-equivariant map from $q^{-1}(x_0)$ to $\widehat{q}^{-1}(x_0)$ with respect to the monodromy action, we have

$$\widehat{e}_0 \cdot \gamma = Q(e_0) \cdot \gamma = Q(e_0 \cdot \gamma) = Q(\varphi_\gamma(e_0)).$$

Using all this,

$$
\begin{aligned}
& \gamma \in \widehat{q}_* \pi_1(\widehat{E}, \widehat{e}_0) \\
\iff & \text{“}\gamma \text{ is in the isotropy group of } \widehat{e}_0\text{”} \\
\iff & \widehat{e}_0 \cdot \gamma = \widehat{e}_0 \\
\iff & Q(\varphi_\gamma(e_0)) = Q(e_0) \\
\iff & \varphi_\gamma \in \widehat{H} \\
\iff & \gamma \in H.
\end{aligned}
$$

We conclude that the induced subgroup of $\widehat{q}$ is H, completing the proof.