• Exam 2

1) Have bordered algorithm for LU factorization. Propose bordered algo for computing Cholesky factorization of SPD matrix A, inspired by bordered LU algo (Show derivation that justifies the alg.)

- Cholesky factorization: Assuming A is SPD (or HPD), meaning $A^T = A$. There exists a lower triangular matrix L such that $\underbrace{A = LL^T}$

Cholesky factorization of A. L is known as Cholesky factor.

- Use this to solve for $Ax = y$, $L\underbrace{L^T x}_{z} = y \Rightarrow Lz = y$.

$$A = \left(\begin{array}{c|c} A_{00} & * \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) \text{ and } L = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right), \quad * \text{ isn't stored nor updated}$$

- Substituting into $A = LL^T$

$$\left(\begin{array}{c|c} A_{00} & * \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)\left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right) = \left(\begin{array}{c|c} L_{00}L_{00}^T & * \\ \hline l_{10}^T L_{00}^T & l_{10}^T l_{10} + \lambda_{11}^2 \end{array}\right)$$

Where we conclude:

$$\begin{array}{c|c} L_{00} = \text{Chol}(A_{00}) & * \\ \hline l_{10}^T = a_{10}^T L_{00}^{-T} & \lambda_{11} = \sqrt{\alpha_{11} - l_{10}^T l_{10}} \end{array}$$

Algorithm:

1. Partition $A \rightarrow \left(\begin{array}{c|c} A_{00} & * \\ \hline a_{10}^T & \alpha_{11} \end{array}\right)$.
2. Assume that $A_{00} := L_{00} = \text{Chol}(A_{00})$ has been computed by previous iterations of the loop-based algorithm.

3. Overwrite $a_{10}^T := l_{10}^T = a_{10}^T L_{00}^{-T}$

4. Overwrite $\alpha_{11} := \sqrt{\alpha_{11} - l_{10}^T l_{10}}$

b) Proof Cholesky bordered factorization is well defined for matrix A that is SPD.

l_0 - $n=1$

For this $A = \alpha_{11}$: Fact A is HPD means that α_{11} is real & positive and a Cholesky factor is then given by $\lambda_{11} = \sqrt{\alpha_{11}}$, with uniqueness if we insist that λ_{11} is positive.

2. Inductive step: Assume result is true for $n=k$. we will show that $n=k+1$.

$b = l_{10}^T$

$d = \lambda_{11}$

Let

$\lambda_{11} = \sqrt{\alpha_{11}}$

$$A = \begin{pmatrix} A_{00} & * \\ a_{10}^T & \alpha_{11} \end{pmatrix} \quad \& \quad L = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)$$

- Need to prove A is unique

- Similar to last proof in notes, assume $n=k$ is true, will prove that it holds for $n=k+1$.

$$A = \begin{pmatrix} A_{00} & a_{01} \\ a_{10}^T & \alpha_{11} \end{pmatrix} \quad \& \quad L = \begin{pmatrix} L_{00} & 0 \\ l_{10}^T & \lambda_{11} \end{pmatrix},$$

Need to choose l_{10}^T & λ_{11} such that $A = LL^T$. Substituting L into A, (in part A labeled as $L_{00} = \text{Chol}(A_{00})$), we get $A_{11} = L_{11}L_{11}^T$, $a_{12} = L_{11} l_{12}$ and $\alpha_{22} = \lambda_{22}^2 + l_{12}^T L_{11}^T L_{11} l_{12}$, solving for l_{12} and λ_{22}, we get value we used for alg. in part a: $l_{12} = L_{11}^{-1} a$ and $\lambda_{22} = \sqrt{\alpha_{22} - l_{12}^T L_{11}^T L_{11} l_{12}}$. Since A is positive definite, 1b cont.

$\alpha_{22} - l_{12}^T L_{11}^T L_{11} l_{12}$ is positive, so λ_{22} is well defined & positive.

$\therefore$ we found L of size $(K+1)(K+1)$ with non-negative diag diagonal entries. This proves Bordered Cholesky factorization by induction. by Principal of Mathematical induction this holds.