Appendix A - Recursion

A.1 Introduction

Recursive function is no different than a normal function. The motivation to use recursive functions vs non-recursive is that the recursive solutions are usually easier to read and comprehend. Certain applications, like **tree search, directory traversing** etc. are very well suited for recursion. The drawbacks are that you may need a little time to learn how to use it and you may need a little longer to debug errors. It takes more processing time and more memory. But there can be cases when recursion is the best way.

For example if you need to get the full tree of a directory and store it somewhere. You can write loops but it will be very complicated. And it will be very simple if you use recursion. You'll only get files of root directory, store them and call the same function for each of the subdirectories in root.

Recursion is a way of thinking about problems, and a method for solving problems. The basic idea behind recursion is the following: it's a method that solves a problem by solving smaller (in size) versions of the same problem by breaking it down into smaller subproblems. Recursion is very closely related to mathematical induction.
We'll start with recursive definitions, which will lay the groundwork for recursive programming. We'll then look at a few prototypical examples of using recursion to solve problems. We'll finish by looking at problems and issues with recursion.

A.2 Recursive Definitions

We'll start thinking recursively with a few recursive definitions:
1. Factorial
2. Fibonacci numbers

A.2.1 Factorial:

The factorial of a non-negative integer n is defined to be the product of all positive integers less than or equal to n. For example, 5! = 5 * 4 * 3 * 2 * 1 = 120.

```
1! = 1
2! = 2 * 1
3! = 3 * 2 * 1
4! = 4 * 3 * 2 * 1
5! = 5 * 4 * 3 * 2 * 1
…..
….. And so on
```

We can easily evaluate n! for any valid value of n by multiplying the values iteratively. However, there is a much more interesting recursive definition quite easily seen from the factorial expressions: 5! is nothing other than 5 * 4!. If we know 4!, we can trivially compute 5!.
4! on the other hand is 4 * 3!, and so on until we have n! = n * (n - 1)!, with 1! = 1 as the base case. The mathematicians have however added the special case of 0! = 1 to make it easier (yes, it does, believe it or not). For the purposes of this discussion, we'll use both 0! = 1 and 1! = 1 as the two base cases.

$$n! = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n=0 \\ n*(n-1)! & \text{if } n>1 \end{cases}$$

Now we can expand 5! recursively, stopping at the base condition.

$$\begin{array}{l} 5! = 5 * 4! \\ \qquad\quad 4 * 3! \\ \qquad\qquad 3 * 2! \\ \qquad\qquad\quad 2 * 1! \\ \qquad\qquad\qquad 1 \end{array}$$

The recursion tree for 5! shows the values as well.

$$\begin{array}{l} \qquad \textbf{120 returned} \\ 5! = 5 * 4! \quad \textbf{24 returned} \\ \qquad\quad 4 * 3! \quad \textbf{6 returned} \\ \qquad\qquad 3 * 2! \quad \textbf{2 returned} \\ \qquad\qquad\quad 2 * 1! \quad \textbf{1 returned} \\ \qquad\qquad\qquad 1 \end{array}$$

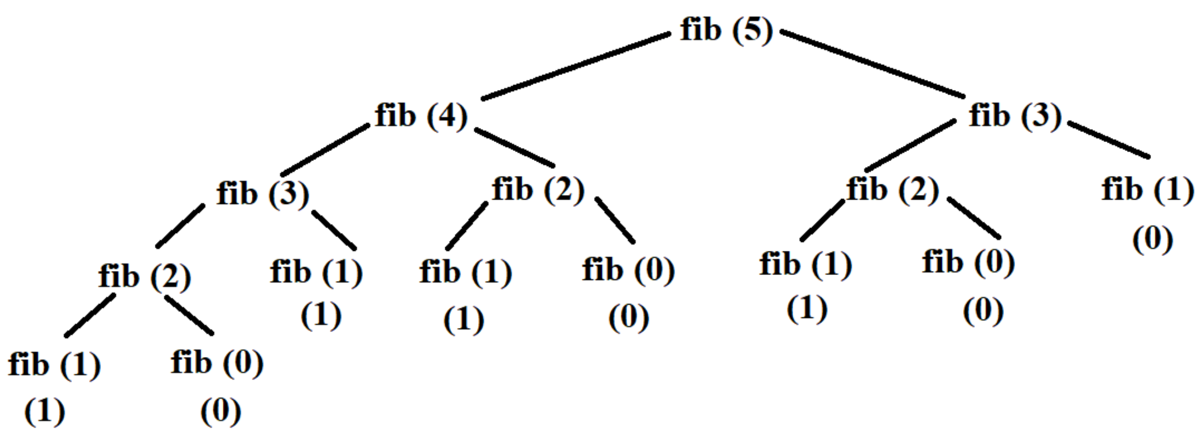
A.2.2 Fibonacci numbers:

The Fibonacci numbers are ‹0, 1, 1, 2, 3, 5, 8, 13, ...› (some define it without the leading 0, which is ok too). If we pay closer attention, we see that each number, except for the first two, is nothing but the sum of the previous two numbers. We can easily compute this iteratively, but let's stick with the recursive method. We already have the recursive part of the recursive definition, so all we need is the non-recursive part or the base case. The mathematicians have decided that the first two numbers in the sequence are 0 and 1, which give us the base cases (notice the two base cases).

Now we can write the recursive definition for any Fibonacci number n >= 0.

$$\mathrm{fib}\,(n) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n=0 \\ \mathrm{fib}\,(n-1) + \mathrm{fib}(n-2) & \text{if } n>1 \end{cases}$$

We can now compute fib(5) using this definition.



Before moving on, you should note how many times we're computing Fibonacci of 3 (fib(3) above), and Fibonacci of 2 (fib(2) above) and so on. This redundancy in computation leads to gross inefficiency, but something we can easily address is Memoization, which is the later topic we study.

A.3 Recursive programming

A recursive function is one that calls itself, since it needs to solve the same problem, but on a smaller sized input. In essence, a recursive function is a function that is defined in terms of itself.

Let's start with computing the factorial. We already have the recursive definition, so the question is how do we convert that to a recursive method that actually computes a factorial of a given non-negative integer. This is an example of functional recursion.

```
function factorial(n)
  if n == 0 or n == 1
    return 1
  else
    return n * factorial(n - 1)
  end if
end function
```

Once you have formulated the recursive definition, the rest is usually quite trivial. This is indeed one of the greatest advantages of recursive programming.

Let's now look at an example of structural recursion, one that uses a recursive data structure. We want to compute the sum of the list of numbers. Linked list = 3-> 8-> 2-> 1-> 13-> 4->None.

The "easiest" way to find the sum of numbers of a linked list iteratively, as shown below.

```
function sumList(head)
  sum = 0 // initialize the sum variable
  ptr = head // initialize a pointer to the head of the list
  while ptr is not null // loop until the end of the list
    sum = sum + ptr.data // add the current node's data to the sum
    ptr = ptr.next // move the pointer to the next node
  end while
  return sum // return the sum of the list
end function
```

But a linked list is a recursive data structure. Hence, by thinking recursively, we note the following:

1. The sum of the numbers in a list is nothing but the sum of the first number plus the sum of the rest of the list. The problem of summing the rest of the list is the same problem as the original,

except that it's smaller by one element! Ah, recursion at play.
2. The shortest list has a single number, which has the sum equal to the number itself. Now we have a base case as well. Note that if we allow our list of numbers to be empty, then the base case will need to be adjusted as well: the sum of an empty list is simply 0.
Note the key difference between the iterative and recursive approaches: for each number in the list vs the rest of the list.

Now we can write the recursive definition of this problem:

$$\text{sum} = \begin{cases} \text{n.elem} & \text{When n is the only node} \\ 0 & \text{When the linkedlist is empty} \\ \text{n.elem} + \text{sum(n.next)} & \text{Otherwise} \end{cases}$$

and using recursive definition, we can write the recursive method as shown below:

```
function sumList(head)
  if head is null
    return 0
  else
    return head.data + sumList(head.next)
```

Examples:
Now we will look at the following examples which can be solved by recursive programming:

1. Length of a String
2. Length of a linked list
3. Sequential search in a sequence
4. Binary search in a sorted array
5. Finding the maximum in a sequence (linear version)
6. Finding the maximum in an array (binary version)
7. Selection sort
8. Insertion sort
9. Exponentiation – a^n

A.3.1 Length of a linked list:

A linked list is also a recursive structure: a linked list is either empty, or a node followed by the rest of the list.

We can also compute the length of a list recursively as follows: the length of a linked list is 1 longer than the rest of the list! The empty list has a length of 0, which is our base case.

Recursive definition is given below:

$$\text{length(list\_a)} = \begin{cases} 0 & \text{When the node is Null} \\ 1 + \text{length(list\_a.next)} & \text{Otherwise} \end{cases}$$

Recursive process is shown below:

```
function countList(head)
  if head is null
    return 0
  else
    return 1 + countList(head.next)
```

A.3.2 Sequential search in a sequence:

How would you find something in a linked list? Well, look at the first node and check if the key is in that node. If so, done. Otherwise, check the rest of the linked list for the given key. If you search an empty list for any key, the answer is false, so that's our base case.

This is almost exactly the same, at least in form, as finding the length of a linked list, and also an example of structural recursion.

Recursive definition is given below:

$$\text{contains (n, k)} = \begin{cases} \text{false} & \text{if n in null} \\ \text{true} & \text{if n.elem == k} \\ \text{contains(n.next, k)} & \text{otherwise} \end{cases}$$

Recursive process is shown below:

```
function sequentialSearch(head, key)
  if head is null
    return false
  else if head.data == key
    return true
  else
    return sequentialSearch(head.next, key)
```

What if the sequence is an array? How do we deal with the rest of the array part then? We can handle it like this:

We can maintain a left index, along with a reference to the array, that is used to indicate where the beginning of the array is.

Initially, left = 0, meaning that the array begins at the expected index of 0. Eventually, a value of left = len(array) - 1 means that the rest of the array is simply the last element, and then left = len(array) means that it's an 0-sized array.

Recursive definition:

$$\textbf{contains}(a, l, k) = \begin{cases} \mathbf{-1} & \textbf{if length of a is == l} \\ \mathbf{l} & \textbf{if } \mathbf{a[l] == k} \\ \textbf{contains}(\mathbf{a, l+1, k}) & \textbf{Otherwise} \end{cases}$$

Recursive method:

```
function sequentialSearch(array, index, key)
  if index >= array.length
    return -1
  else if array[index] == key
    return index
  else
    return sequentialSearch(array, index + 1, key)
```

We start the search with **contains(arr, 0, key)**, and then at each step, the rest of the array is given by advancing the left boundary(index).

Instead of just a yes/no answer, what if we wanted the position of the key in the array? We can simply **return left** instead of true as the position if found, or use a sentinel -1 instead of false if not.

A.3.3 Binary search in a sorted array:

Given the abysmal performance of sequential search, we obviously want to use binary search whenever possible. Of course, the pre-conditions must be met first:

1. The sequence must support random access (an array that is)
2. The data must be sorted

Now we can write the recursive definition:.

$$\textbf{binarySearch(array, key, left, right)} = \begin{cases} \textbf{false} & \textbf{if } l > r \\ \textbf{true} & \textbf{if key = array[mid]} \\ \textbf{binarySearch(array, key, left, mid-1)} & \textbf{if key< array[mid]} \\ \textbf{binarySearch(array, key, mid+1, right)} & \textbf{if key> array[mid]} \end{cases}$$

Recursive process:

```
function binarySearch(array, key, left, right)
  if left > right
    return -1
  else
    mid = (left + right) / 2
    if array[mid] == key
      return mid
    else if array[mid] > key
      return binarySearch(array, key, left, mid - 1)
    else
      return binarySearch(array, key, mid + 1, right)
```

We start the search with **binarySearch(arr, key, 0, len(arr) - 1)**, and then at each step, the rest of the array is given by one half of the array – left or right, depending on the comparison of the key with the middle element.
Instead of just a yes/no answer, what if we wanted the position of the key in the array? We can simply **return mid** as the position instead of true if found, or use a sentinel -1 instead of false if not.

A.3.4 Finding the maximum in a sequence (linear version):

Given a sequence of keys, our task is to find the maximum key in the sequence. This is of course trivially done iteratively (for a non-empty sequence): take the 1st one as maximum, and

then iterate from the 2nd to the end, exchanging the current with the maximum if the current is larger than the maximum.
Formulating this recursively: the maximum key in a sequence is the larger of the following two:
1. the 1st key in the sequence
2. the maximum key in the rest of the sequence

Once we have (recursively) computed the maximum key in the rest of the sequence, we just have to compare the 1st key with that, and we have our answer! The base case is also trivial (for a non-empty sequence): the maximum key in a single-element sequence is the element itself.
Since the rest of the sequence does not need random access, we can easily do this for a linked list or an array. Let's write it for a linked list first.

```
function findMax(head)
  if head is null
    return -infinity
  else
    return max(head.data, findMax(head.next))
```

We start to find the maximum with **findMax(head)** (where head is the reference to the first node of the list), and then at each step, the rest of the array is given by advancing the head reference.

What if the sequence is an array? Well, then we use the same technique we've used before — use a left (and optionally right) boundary to window into the array.

Recursive process:

```
function findMax(array, n)
  if n == 1
    return array[0]
  else
    return max(array[n-1], findMax(array, n-1))
```

We start to find the maximum with **findMax(arr, 0)**, and then at each step, the rest of the array is given by advancing the left boundary(index).

A.3.5 Finding the maximum in an array (binary version):

If our sequence is an array, we can also find the maximum by formulating the following recursive definition: the maximum key in an array is the larger of the following two:

1. the maximum key in the left half of the array
2. the maximum key in the right half of the array

```
function findMax(array, left, right)
  if left == right
    return array[left]
  else
    mid = (left + right) / 2
    leftMax = findMax(array, left, mid)
    rightMax = findMax(array, mid + 1, right)
    return max(leftMax, rightMax)
```

We start to find the maximum with **findMax(arr, 0, len(array)-1)**, and then at each step, the array is divided into two halves.

A.4 Advance Recursion Part 1

A.4.1 Selection sort:

How about sorting a sequence recursively? Since it does not require random access, we'll look at recursive versions for both linked lists and arrays.
The basic idea behind selection sort is the following: put the 1st minimum in the 1st position, the 2nd minimum in the 2nd position, the 3rd minimum in the 3rd position, and so on until each key is placed in its position according to its rank. To come up with a recursive formulation, the following observation is the key:

Once the 1st minimum in the 1st position, it will never change its position. Now all we have to do is to sort the rest of the sequence (from 2nd position onwards), and we'll have a sorted sequence.

Now we can write the recursive definition for a linked list, and see the pseudocode.

```
function selectionSort(head)
  if head is null or head.next is null
    return head
  else
    min = findMin(head) // find the minimum node in the list
    swap(head, min) // swap the head node with the minimum node
    head.next = selectionSort(head.next) // recursively sort the rest of the list
    return head
```

Here, be careful about one thing: the swap method means we are not exchanging the nodes rather than we are exchanging the element of nodes.

```
Function swap(a, b)
  temp = a.element // store a.element in temp
  a.element = b.element // assign b.element to a.element
  b.element = temp // assign temp to b.element
end Function
```

We sort a list headed by head reference by calling **selectSort(head)**. Note that we're not finding the minimum key, but rather the node that contains the minimum key since we need to exchange the left key with the minimum one. We can write that iterative of course, but a recursive one is simply more fun. This is of course almost identical to finding the maximum in a sequence, with two differences: we find the minimum, and we return the node that contains the minimum, not the actual minimum key.

Recursive process of findMin:

```
function findMin(head)
  if head is null
    return
  else
    return min(head.data, findMin(head.next))
```

If the sequence is an array, then we have to use the left (and optionally right) boundary to window into the array.

```
function selectionSort(array,  i = 0)
  if i == array length - 1 // base case: the array is sorted
    return
  else
    minIndex = i // assume the first element is the minimum
    for j = i + 1 to  array length - 1 // loop through the rest of the array
      if array[j] < array[minIndex] // find the actual minimum
        minIndex = j

    swap(array[i], array[minIndex]) // swap the minimum with the first element
    selectionSort(array,  i + 1) // recursively sort the remaining array
end function
```

Here, be careful about one thing: the swap method means we are not exchanging the indices rather than we are exchanging the element of these indices in the array.

```
Function swap(a, b)
  temp = a // store a in temp
  a = b // assign b to a
```

```
  b = temp // assign temp to b
end function
```

We sort an array(arr) by calling **selectSort(arr, 0)**.

A.4.2 Insertion Sort:

Insertion works by inserting each new key in a sorted array so that it is placed in its rightful position, which now extends the sorted array by the new key. In the beginning, there is a single key in the array, which by definition is sorted. Then the second key arrives, which is then inserted into the already sorted array (of one key at this point), and now the sorted array has two keys. Then the third key arrives, which is inserted into the already sorted array, creating a sorted array of 3 keys. And so on. Iteratively, it's a fairly simple operation. The question is how can we formulate this recursively. The following observation is the key to this recursive formulation:
Given an array of n keys, sort the first n-1 keys and then insert the nth key in the sorted array such that all n keys are now sorted.
Note how the recursive part comes first, and then the nth key is inserted (iteratively) into the sorted array.
Unlike recursive selection sort, we're going from right to left in the recursive version of insertion sort. Note that we don't need random access, but need to be able to iterate in both directions (reverse direction to insert the new key in the sorted partition). So, if we're sorting a linked list using the insertion sort algorithm, the list must be doubly-linked.

Recursive process of insertion sort:

```
function insertionSort(array, n, i = 1)
  if i == n // base case: the array is sorted
    return
  else
    key = array[i] // store the current element
    j = i - 1 // start from the previous element
    while j >= 0 and array[j] > key // loop through the sorted subarray
      array[j + 1] = array[j] // shift the larger elements to the right
      j = j - 1 // move to the next element

    array[j + 1] = key // insert the current element in the correct position
    insertionSort(array, n, i + 1) // recursively sort the remaining array

end function
```

We sort an array(arr) by calling **insertionSort(arr, len(arr)- 1, 1)**.

A.4.3 Exponentiation – a^n:

This is another example of functional recursion. To compute a^n, we can iteratively multiply a n times, and that's that. Thinking recursively, a^n = a * a^{n-1}, and a^{n-1} = a * a^{n-2}, and so on. The recursion stops when the exponent n = 0, since by definition a^0 = 1.

Recursive definition:

$$a^n = \begin{cases} 1 & \text{if } n = 0 \\ a * a^{n-1} & \text{if } n > 0 \end{cases}$$

Recursive approach:

```
function power(base, exponent)
  if exponent == 0 // base case: any number raised to 0 is 1
    return 1
  else
    return base * power(base, exponent - 1) // recursive case: multiply the base by itself
exponent times
end function
```

As it turns out, there is actually a much more efficient recursive formulation for the exponentiation of a number. We start by noting that $2^8 = 2^4 * 2^4$, and that $2^7 = 2^3 * 2^3 * 2$. We can generalize that with the following recursive definition, and its implementation.

$$a^n = \begin{cases} 1 & \text{if } n = 0 \\ a^{n/2} * a^{n/2} & \text{if } n \text{ is even} \\ a^{(n-1)/2} * a^{(n-1)/2} * a & \text{if } n \text{ is odd} \end{cases}$$

Recursive approach:

```
function power(base, exponent)
  if exponent == 0 // base case: any number raised to 0 is 1
    return 1
```

```
  else
    If exponent is even
      return power(base, exponent/2)*power(base, exponent/2)
    else
      return power(base, (exponent-1)/2)*power(base, (exponent-1)/2) * a
```

But why would we care about this formulation over the more familiar one? If we solve for the running time, both solutions take the same time, so what is the benefit of this approach? Notice how we're computing the following expressions twice:
1. exp(a, n/2)
2. exp(a, (n - 1)/2)

Why not compute it once, and then use the result twice (or as many times as needed)? We can, and as we will find out, that will give us a huge boost when we compute the running time of this algorithm. Here is the modified version.

```
function power(base, exponent)
  if exponent == 0 // base case: any number raised to 0 is 1
    return 1
  else
    If exponent is even
      temp = power(base, exponent/2)
      return temp * temp
    else
      temp = power(base, (exponent-1)/2)
      return temp * temp * base
```

All we are doing is removing the redundancy in computations by saving the intermediate results in temporary variables. This is a simple case of a technique known as **Memoization**, which is the next topic we study. Remember that we have already seen such redundancy in recursive computation — when computing the Fibonacci numbers.

A.5 Issues/problems to watch out for

1. **Inefficient recursion:**
The recursive solution for Fibonacci numbers outlined in these notes shows massive redundancy, leading to very inefficient computation. The 1st recursive solution for exponentiation also shows how redundancy shows up in recursive programs. There are ways to avoid computing the same value more than once by caching the intermediate results, either using Memoization (a top-down technique — see next topic), or Dynamic Programming (a bottom-up technique — survive this semester to enjoy it in the next one).

2. **Space for activation frames:**

Each recursive method call requires that it's activation record be put on the system call stack. As the depth of recursion gets larger and larger, it puts pressure on the system stack, and the stack may potentially run out of space.

3. **Infinite recursion:**
Ever forgot a base case? Or miss one of the base cases? You end up with infinite recursion, since there is nothing stopping your recursion! Whenever you see a Stack Overflow error, check your recursion!

A.6 Advanced Recursion Part 2: Optimizing Recursive Program Memoization

A.6.1 Introduction:

To develop a recursive algorithm or solution, we first have to define the problem (and the recursive definition, often the hard part), and then implement it (often the easy part). This is called the **top-down** solution, since the recursion opens up from the top until it reaches the base case(s) at the bottom.
Once we have a recursive algorithm, the next step is to see if there are **redundancies** in the computation— that is, if the same values are being computed multiple times. If so, we can benefit from **memoizing** the recursion. And in that case, we can create a memoized version and see what savings we get in terms of the running time.
Recursion has certain overhead costs that may be minimized by transforming the memoized recursion into an iterative solution. And, finally, we see if there are further improvements that we can make to improve the time and space complexity.

The steps are as follows:
1. write down the recursion,
2. implement the recursive solution,
3. memoize it,
4. transform into an iterative solution, and finally
5. make further improvements.

A.6.2 Example using the Fibonacci sequence:

To see this in action, let's take Fibonacci numbers as an example. Fortunately for us, the mathematicians have already defined the problem for us – the Fibonacci numbers are ‹ 0, 1, 1, 2, 3, 5, 8, 13,... › (some define it without the leading 0, which is ok too). Each number, except for the first two, is nothing but the sum of the previous two numbers. The first two are by definition 0 and 1. These two facts give us the recursive definition to compute the n^{th} Fibonacci number for some n >= 0.

Let's go through the 5 steps below.

Step 1: Write or formulate the recursive definition of the nth Fibonacci number (defined only for n>=0)

$$fib(n) = \begin{cases} n & \text{if } n<2 \\ fib(n-1) + fib(n-2) & \text{if } n>=2 \end{cases}$$

Step 2: Write the recursive implementation. This usually follows directly from the recursive definition

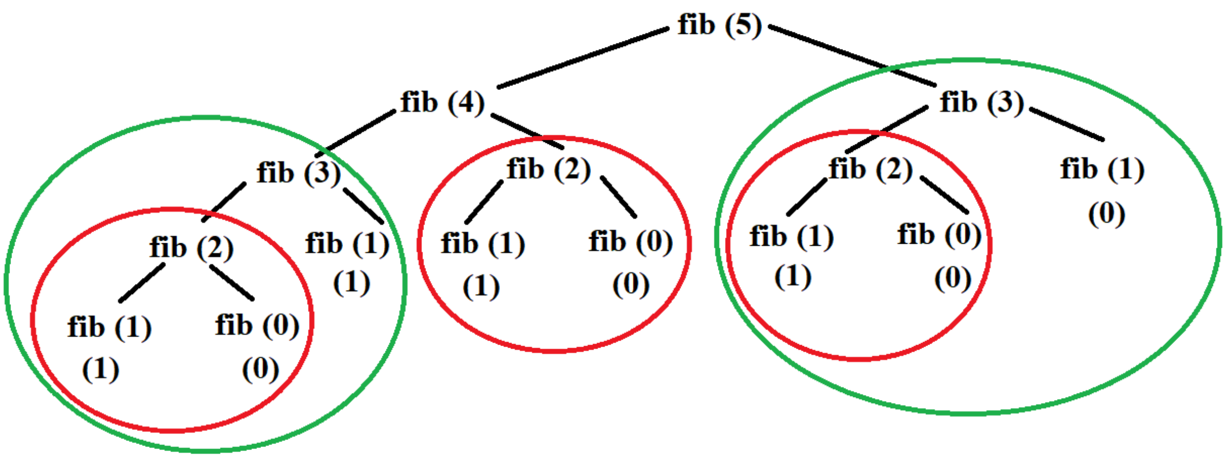
```
function fib(n)
  if n == 0 or n == 1 // base case: the first two Fibonacci numbers are 0 and 1
    return n
  else // recursive case: use the Fibonacci formula
    return fib(n - 1) + fib(n - 2)
end function
```

Step 3: Memoize the recursion

Would this recursion benefit from memoization? Well, let's see by "unrolling" the recursion **fib(5)** a few levels:

Now you should notice something very interesting — we're computing the same fibonacci number quite a few times. We're computing fib(2) 3 times and fib(3) 2 times. Is there any reason why we couldn't simply save the result after computing it the first time, and re-using it each time it's needed afterward?

This is the primary motivation for **memoization** – to avoid re-computing overlapping subproblems. In this example, fib(2) and fib(3) are overlapping subproblems that occur independently in different contexts.



Memoization certainly looks like a good candidate for this particular recursion, so we'll go ahead and memoize it. Of course, the first question is how we save the results of these overlapping subproblems that we want to reuse later on.

The basic idea is very simple — before computing the i^{th} fibonacci number, first check if that has already been solved; if so, just look up the answer and return; if not, compute it and save it before returning the value. We can modify our fib method accordingly (using some pseudocode for now). Since fibonacci is a function of 1 integer parameter, the easiest is to use a 1-dimensional array or table with a capacity of **n + 1** (we need to store fib(0) ... fib(n), which requires n + 1 slots) to store the intermediate results.

What is the cost of memoizing a recursion? It's the space needed to store the intermediate results. Unless there are overlapping subproblems, memoizing a recursion will not buy you anything at all, and in fact, cost you more space for nothing!
Remember this — **memoization trades space for time.**

Let's try out our first memoized version. (M_fib below stands for **"memoized fibonacci"**).

```
function fib(n)
  // assume that we have a “global” array (also called a table) with n+1 capacity


  // check if the nth Fibonacci number is already computed
  if memo[n] is not null
    return memo[n]
  else
    // base case: the first two Fibonacci numbers are 0 and 1
    if n == 0 or n == 1
      memo[n] = n
    else
      // recursive case: use the Fibonacci formula and store the result
```

```
    memo[n] = fib(n - 1) + fib(n - 2)

  return memo[n]

end function
```

Think about why we used a global array instead of declaring arrays inside the function!

This is all that we need to avoid redundant computations of overlapping subproblems. The first time fib(3) is called, it will compute the value and save the answer in memo[3], and then subsequent calls would simply return memo[3] without doing any work at all!

There are a few details we have left out at this point:
1. Where would we create this "table" to store the results?
2. How can we initialize each element of memo to indicate that the value has not been computed/saved yet?(Note the **"is empty"** in the code above).

Let's take these one at a time.

1. The "fib" method is a function of a single parameter — n, so if we wanted to save the intermediate results, all we need is an array that goes from 0 ... n (i.e., of n + 1 capacity). Since the local variables within a method are created afresh each time the method is called, F cannot be a local array. We can either use an instance variable within an object, or create an array in the caller of fib(n), and then pass the array to fib (in which case we will have to modify fib to have another parameter).
2. We need to use a sentinel which will indicate that the value has not been computed. Since it's an array of integers, we can't use null (which is the sentinel used to indicate the absence of an object).
However, we know that the n^{th} fibonacci number is a non-negative integer, so we can use any negative number as the sentinel. **Let us choose -1**. So, let's have an array F of n+1 capacity that holds all the values of the intermediate results we need to compute the n^{th} Fibonacci number.
We can have a wrapper method, which creates this array or table, initializes the table and passes it onto M_fib as a parameter.

First, the wrapper method called **fib**, which basically sets up the table for **M_fib**, and calls it on the user's behalf.

```
function fib(n)
   Create an array (also called a table) with n+1 capacity named memo
   Call M_fib(n,memo)


function M_fib(n, memo)

  // check if the nth Fibonacci number is already computed
  if memo[n] is not null
```

```
    return memo[n]
  else
    // base case: the first two Fibonacci numbers are 0 and 1
    if n == 0 or n == 1
      memo[n] = n
    else
      // recursive case: use the Fibonacci formula and store the result
      memo[n] = M_fib(n - 1) + M_fib(n - 2)

    return memo[n]

end function
```

To compute the 5th fibonacci number, we simply call **fib(5)**, which in turn calls **M_fib(5, F)** to compute and return the value.
Now that we have a memoized fibonacci, the next question is to see if we can improve the space overhead of memoization.

Step 4: Convert the recursion to iteration – the bottom-up solution.

To compute the 5th fibonacci number, we wait for 4th and 3rd, which in turn wait for 2nd, and so on until the base cases of n = 0 and n = 1 return the values which move up the recursion stack. Other than the n = 0 and n = 1 base cases, the first value that is actually computed and saved is n = 2, and then n= 3, and then n = 4 and finally n = 5. Then why not simply compute the solutions for
n = 2, 3, 4, 5 by iterating (using the base cases of course), and fill in the table from left to right? This is called the **bottom-up** solution since the recursion tree starts at the bottom (the base cases) and works its way up to the top of the tree (the initial call). The bottom-up technique is more popularly known as **dynamic programming**, a topic that we will spend quite a bit of time on next semester!

```
function fib(n)
  // create an array to store the Fibonacci numbers
  array f[n + 1]

  // initialize the first two Fibonacci numbers
  f[0] = 0
  f[1] = 1

  // loop from the third Fibonacci number to the nth Fibonacci number
  for i = 2 to n
    // use the Fibonacci formula and store the result in the array
    f[i] = f[i - 1] + f[i - 2]
  end for

  // return the nth Fibonacci number
  return f[n]
```

```
end function
```

You should convince yourself that this is indeed a solution to the problem, only using iteration instead of memoized recursion. Also, that it solves each subproblem (e.g., fib(3) and fib(2)) exactly once, and re-uses the saved answer.

This one avoids the overhead of recursion by using iteration, so tends to run much faster.

Can we improve this any further?

Step 5: improving the space-requirement in the bottom-up version

The n^{th} Fibonacci number depends only on the $(n - 1)^{th}$ and $(n - 2)^{th}$ Fibonacci numbers. However, we are storing ALL the intermediate results from 2 ... n - 1 Fibonacci numbers before computing the n^{th} one. What if we simply store the last two? In that case, instead of having an array of n + 1 capacity, we need just two instance variables (or an array with 2 elements). Here's what the answer may look like.

```
function fib(n)
  // initialize the first two Fibonacci numbers
  f0 = 0
  f1 = 1

  // loop from the first Fibonacci number to the nth Fibonacci number
  for i = 0 to n - 1
    // use the Fibonacci formula and store the result in a temporary variable
    temp = f0 + f1
    // update the previous two Fibonacci numbers
    f0 = f1
    f1 = temp
  end for

  // return the nth Fibonacci number
  return f0
end function
```

Exercises

A.1: Given a non-negative int n, return the sum of its digits recursively (no loops). Note that mod (%) by 10 yields the rightmost digit (126 % 10 is 6), while divide (/) by 10 removes the rightmost digit (126 / 10 is 12).

Example:
Input: sumDigits(126)
Output: 9

Input: sumDigits(49)
Output: 13

Input: sumDigits(12)
Output: 3

A.2: We have bunnies standing in a line, numbered 1, 2, ... The odd bunnies (1, 3, ..) have the normal 2 ears. The even bunnies (2, 4, ..) we'll say have 3 ears, because they each have a raised foot. Recursively return the number of "ears" in the bunny line 1, 2, ... n (without loops or multiplication).

Example:
Input: bunnyEars2(0)
Output: 0

Input: bunnyEars2(1)
Output: 2

Input: bunnyEars2(2)
Output: 5

A.3: Given a non-negative int n, return the count of the occurrences of 7 as a digit, so for example 717 yields 2. (no loops). Note that mod (%) by 10 yields the rightmost digit (126 % 10 is 6), while divide (/) by 10 removes the rightmost digit (126 / 10 is 12).

Example:
Input: count7(717)
Output: 2

Input: count7(7)
Output: 1

Input: count7(123)
Output: 0

A.4: Given a string, compute recursively (no loops) the number of lowercase 'x' chars in the string.

Example:

Input: countX("xxhixx")
Output: 4

Input: countX("xhixhix")
Output: 3

Input: countX("hi")
Output: 0

A.5: Given a string, compute recursively (no loops) a new string where all appearances of "pi" have been replaced by "3.14".

Example:
Input: changePi("xpix")
Output: "x3.14x"

Input: changePi("pipi")
Output: "3.143.14"

Input: changePi("pip")
Output: "3.14p"

A.6: Given an array of ints, compute recursively the number of times that the value 11 appears in the array. We'll use the convention of considering only the part of the array that begins at the given index. In this way, a recursive call can pass index+1 to move down the array. The initial call will pass in index as 0.

Example:
Input: array11([1, 2, 11], 0)
Output: 1

Input: array11([11, 11], 0)
Output: 2

Input: array11([1, 2, 3, 4], 0)
Output: 0

A.7: Given a string, compute recursively a new string where identical chars that are adjacent in the original string are separated from each other by a "*".

Example:
Input: pairStar("hello")
Output: "hel*lo"

Input: pairStar("xxyy")
Output: "x*xy*y"

Input: pairStar("aaaa")
Output: "a*a*a*a"

A.8: Count recursively the total number of "abc" and "aba" substrings that appear in the given string.

Example:
Input: countAbc("abc")
Output: 1

Input: countAbc("abcxxabc")
Output: 2

Input: countAbc("abaxxaba")
Output: 2

A.9: Given a string, compute recursively the number of times lowercase "hi" appears in the string, however do not count "hi" that have an 'x' immedately before them.

Example:
Input: countHi2("ahixhi")
Output: 1

Input: countHi2("ahibhi")
Output: 2

Input: countHi2("xhixhi")
Output: 0

A.10: Given a string and a non-empty substring sub, compute recursively the number of times that sub appears in the string, without the sub strings overlapping.

Example:
Input: strCount("catcowcat", "cat")
Output: 2

Input: strCount("catcowcat", "cow")
Output: 1

Input: strCount("catcowcat", "dog")
Output: 0

A.11: We have a number of bunnies and each bunny has two big floppy ears. We want to compute the total number of ears across all the bunnies recursively (without loops or multiplication).

Example:
Input: bunnyEars(0)
Output: 0

Input: bunnyEars(1)
Output: 2

Input: bunnyEars(2)
Output: 4

A.12: We have triangle made of blocks. The topmost row has 1 block, the next row down has 2 blocks, the next row has 3 blocks, and so on. Compute recursively (no loops or multiplication) the total number of blocks in such a triangle with the given number of rows.

Example:
Input: triangle(0)
Output: 0

Input: triangle(1)
Output: 1

Input: triangle(2)
Output: 3

A.13: Given a string, compute recursively a new string where all the 'x' chars have been removed.

Example:
Input: noX("xaxb")
Output: "ab"

Input: noX("abc")
Output: "abc"

Input: noX("xx")
Output: ""

A.14: Given an array of ints, compute recursively if the array contains somewhere a value followed in the array by that value times 10. We'll use the convention of considering only the part of the array that begins at the given index. In this way, a recursive call can pass index+1 to move down the array. The initial call will pass in index as 0.

Example:
Input: array220([1, 2, 20], 0)
Output: True

Input: array220([3, 30], 0)
Output: True

Input: array220([3], 0)
Output: False

A.15: Given a string, compute recursively a new string where all the lowercase 'x' chars have been moved to the end of the string.

Example:
Input: endX("xxre")

Output: "rexx"

Input: endX("xxhixx")
Output: "hixxxx"

Input: endX("xhixhix")
Output: "hihixxx"

A.16: Given a string, compute recursively (no loops) the number of "11" substrings in the string. The "11" substrings should not overlap.

Example:
Input: count11("11abc11")
Output: 2

Input: count11("abc11x11x11")
Output: 3

Input: count11("111")
Output: 1

A.17: Given a string that contains a single pair of parenthesis, compute recursively a new string made of only of the parenthesis and their contents, so "xyz(abc)123" yields "(abc)".

Example:
Input: parenBit("xyz(abc)123")
Output: "(abc)"

Input: parenBit("x(hello)")
Output: "(hello)"

Input: parenBit("(xy)1")
Output: "(xy)"

A.18: Given a string and a non-empty substring sub, compute recursively if at least n copies of sub appear in the string somewhere, possibly with overlapping. N will be non-negative.

Example:
Input: strCopies("catcowcat", "cat", 2)
Output: True

Input: strCopies("catcowcat", "cow", 2)
Output: False

Input: strCopies("catcowcat", "cow", 1)
Output: True

A.19: Given a string, compute recursively (no loops) a new string where all the lowercase 'x' chars have been changed to 'y' chars.

Example:
Input: changeXY("codex")
Output: "codey"

Input: changeXY("xxhixx")
Output: "yyhiyy"

Input: changeXY("xhixhix")
Output: "yhiyhiy"

A.20: Given an array of ints, compute recursively if the array contains a 6. We'll use the convention of considering only the part of the array that begins at the given index. In this way, a recursive call can pass index+1 to move down the array. The initial call will pass in index as 0.

Example:
Input: array6([1, 6, 4], 0)
Output: True

Input: array6([1, 4], 0)
Output: False

Input: array6([6], 0)
Output: True

A.21: Given a string, compute recursively a new string where all the adjacent chars are now separated by a "*".

Example:
Input: allStar("hello")
Output: "h*e*l*l*o"

Input: allStar("abc")
Output: "a*b*c"

Input: allStar("ab")
Output: "a*b"

A.22: We'll say that a "pair" in a string is two instances of a char separated by a char. So "AxA" the A's make a pair. Pair's can overlap, so "AxAxA" contains 3 pairs -- 2 for A and 1 for x. Recursively compute the number of pairs in the given string.

Example:
Input: countPairs("axa")
Output: 1

Input: countPairs("axax")
Output: 2

Input: countPairs("axbx")
Output: 1

A.23: Given a string, return recursively a "cleaned" string where adjacent chars that are the same have been reduced to a single char. So "yyzzza" yields "yza".

Example:
Input: stringClean("yyzzza")
Output: "yza"

Input: stringClean("abbbcdd")
Output: "abcd"

Input: stringClean("Hello")
Output: "Helo"

A.24: Given a string, return true if it is a nesting of zero or more pairs of parenthesis, like "(())" or "((()))". Suggestion: check the first and last chars, and then recur on what's inside them.

Example:
Input: nestParen("(())")
Output: True

Input: nestParen("((()))")
Output: True

Input: nestParen("(((x))")
Output: False

A.25: Given a string and a non-empty substring sub, compute recursively the largest substring which starts and ends with sub and return its length.

Example:
Input: strDist("catcowcat", "cat")
Output: 9

Input: strDist("catcowcat", "cow")
Output: 3

Input: strDist("cccatcowcatxx", "cat")
Output: 9

A.26: Given an array of ints, is it possible to choose a group of some of the ints, such that the group sums to the given target? This is a classic backtracking recursion problem. Once you understand the recursive backtracking strategy in this problem, you can use the same pattern for many problems to search a space of choices. Rather than looking at the whole array, our convention is to consider the part of the array starting at index start and continuing to the end of the array. The caller can specify the whole array simply by passing start as 0. No loops are needed -- the recursive calls progress down the array.

Example:
Input: groupSum(0, [2, 4, 8], 10)

Output: True

Input: groupSum(0, [2, 4, 8], 14)
Output: True

Input: groupSum(0, [2, 4, 8], 9)
Output: False

A.27: Given an array of ints, is it possible to divide the ints into two groups, so that the sums of the two groups are the same. Every int must be in one group or the other. Write a recursive helper method that takes whatever arguments you like, and make the initial call to your recursive helper from splitArray(). (No loops needed.)

Example:
Input: splitArray([2, 2])
Output: True

Input: splitArray([2, 3])
Output: False

Input: splitArray([5, 2, 3])
Output: True

A.28: Given an array of ints, is it possible to divide the ints into two groups, so that the sum of one group is a multiple of 10, and the sum of the other group is odd. Every int must be in one group or the other. Write a recursive helper method that takes whatever arguments you like, and make the initial call to your recursive helper from splitOdd10(). (No loops needed.)

Example:
Input: splitOdd10([5, 5, 5])
Output: True

Input: splitOdd10([5, 5, 6])
Output: False

Input: splitOdd10([5, 5, 6, 1])
Output: True
A.29: Given an array of ints, is it possible to divide the ints into two groups, so that the sum of the two groups is the same, with these constraints: all the values that are multiple of 5 must be in one group, and all the values that are a multiple of 3 (and not a multiple of 5) must be in the other. (No loops needed.)

Example:
Input: split53([1, 1])
Output: True

Input: split53([1, 1, 1])
Output: False

Input: split53([2, 4, 2])

Output: True

A.30: Given an array of ints, is it possible to choose a group of some of the ints, such that the group sums to the given target with these additional constraints: all multiples of 5 in the array must be included in the group. If the value immediately following a multiple of 5 is 1, it must not be chosen. (No loops needed.)

Example:
Input: groupSum5(0, [2, 5, 10, 4], 19)
Output: True

Input: groupSum5(0, [2, 5, 10, 4], 17)
Output: True

Input: groupSum5(0, [2, 5, 10, 4], 12)
Output: False