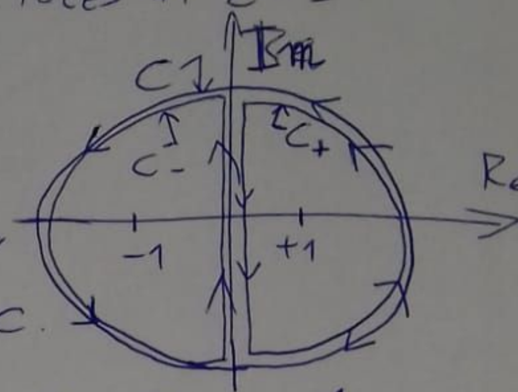
2012 P2 Q4

(i) $$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz$$

$$\frac{1}{z^2-1} = \underbrace{\frac{1}{z+1}\,\frac{1}{z-1}}$$ ⟹ INTEGRAND HAS SIMPLE POLES AT $z = \pm 1$



$$\oint_C \frac{dz}{z^2-1} = \oint_{C_-} \frac{dz}{z^2-1} + \int_{C_+} \frac{dz}{z^2-1}$$

(SINCE THE PARTS OF C_- & C_+ RUNNING ALONG THE IMAGINARY AXIS CANCEL OUT, AND THE REMAINING PARTS ARE JUST C.

$$\frac{1}{2\pi i}\oint_{C_-} \frac{f(z)}{z-z_0}\,dz = \oint_{C_-} \frac{\overbrace{\frac{1}{z-1}}^{f(z)}}{z - \underbrace{(-1)}_{z_0}}\,dz = f(z_0 = -1) = \frac{1}{-1-1} = -\frac{1}{2}$$

$$\Rightarrow \oint_{C_-} \frac{dz}{z^2-1} = -\pi i$$

$$\frac{1}{2\pi i}\oint_{C+} \frac{f(z)\,dz}{z - z_0} = \oint_{C+} \frac{\overbrace{\frac{1}{z+1}}^{f(z)}}{z - \underbrace{1}_{z_0}} = f(z_0 = 1) = \frac{1}{1+1} = \frac{1}{2}$$

$$\Rightarrow \oint_{C_+} \frac{dz}{z^2+1} = \pi i$$

$$\oint_C \frac{dz}{z^2-1} = -\pi i + \pi i = \underline{\underline{0}}$$

EXPRESSION FOR a_n:

$$f(z) = \frac{1}{z(z-1)} = \frac{1}{z}\,\frac{1}{z-1} = \frac{-1}{z}\,\frac{1}{1-z} =$$

USING THE FORMULA FOR SUM OF INFINITE GEOMETRIC SERIES, CONDITION: $|z| < 1$ → WE ARE ASKED TO EXPAND AROUND 0 SO THATS OK.

$$= \frac{-1}{z}\left(1 + z + z^2 + \ldots\right) =$$

$$= -\left(z^{-1} + 1 + z + z^2 + \ldots\right) =$$

$$= -\sum_{n=-1}^{\infty} z^n$$