HOW TO DO LINE INTEGRALS

$\underline{E}$ OF A POINT CHARGE: $\underline{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{\underline{r}}$

CALCULATE V FIELD IN THE PRESENCE OF A POINT CHARGE:

$$V \equiv -\int_{\text{POINT AT }\infty}^{\text{POINT AT DISTANCE P FROM } q} \underline{E}\cdot d\underline{r} = -\int_{\infty}^{P} \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\underbrace{\hat{\underline{r}}\cdot d\underline{r}}_{-dr}$$

BECAUSE $\hat{\underline{r}}$ IS OUTWARDS, $d\underline{r}$ IS INWARDS.

$$= \int_{\infty}^{P} \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}dr = -\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r}\right]_{\infty}^{P} = -\frac{q}{4\pi\varepsilon_0}\frac{1}{P}$$

RIGHT? NO.

$\boxed{\text{DON'T OMIT PARAMETRIZATION}}$

$$V \equiv -\int_{\text{POINT AT }\infty}^{\text{POINT AT P}} \underline{E}\cdot d\underline{r} = -\int_{\text{POINT AT }\infty}^{\text{POINT AT P}} \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{\underline{r}}\cdot d\underline{r} = -\int_{\text{POINT AT }\infty}^{\text{POINT AT P}} \frac{q}{4\pi\varepsilon_0}\frac{\underline{r}}{r^3}\cdot d\underline{r}$$

LETS APPROACH FROM POSITIVE x AXIS.

ON THIS PATH, PARAMETRIZE $\underline{r}$:

$$\underline{r}(t) = \left(\frac{P}{t}, 0, 0\right),$$

WITH t RUNNING FROM 0 to 1.

(WHEN $t=0$, $\underline{r}$ IS AT ∞, $t=1$, $\underline{r}$ IS AT POINT P.)

REWRITE INTEGRAL: V

$$= -\int_{t=0}^{t=1} \frac{q}{4\pi\varepsilon_0}\frac{\underline{r}(t)}{r^3}\cdot\frac{d\underline{r}}{dt}\,dt$$

$$\frac{d\underline{r}}{dt} = \frac{d}{dt}\left(\frac{P}{t}, 0, 0\right) = \left(-\frac{P}{t^2}, 0, 0\right)$$

COMPUTE THE DOT PRODUCT:

$$\underline{r}(t)\cdot\frac{d\underline{r}}{dt} = \left(\frac{P}{t}, 0, 0\right)\cdot\left(-\frac{P}{t^2}, 0, 0\right) = -\frac{P^2}{t^3}$$

CONTINUE WITH INTEGRAL:

$$= -\int_{t=0}^{t=1} \frac{q}{4\pi\varepsilon_0} \frac{1}{r^3}\left(-\frac{P^2}{t^3}\right)dt$$

$$= -\int_{t=0}^{t=1} \frac{q}{4\pi\varepsilon_0} \left(\frac{P}{t}\right)^{-3} \left(-\frac{P^2}{t^3}\right)dt = -\int_{t=0}^{t=1} \frac{q}{4\pi\varepsilon_0}\left(-\frac{1}{P}\right)dt =$$

$$= \int_{t=0}^{t=1} \frac{q}{4\pi\varepsilon_0} \frac{1}{P} dt = \frac{q}{4\pi\varepsilon_0}\frac{1}{P}\left[t\right]_0^1 = \frac{q}{4\pi\varepsilon_0}\frac{1}{P}$$

RIGHT? YES.

ANOTHER PARAMETRIZATION (NOT PRACTICAL, JUST TO MAKE SURE WE SEE WHAT'S GOING ON)

$$\underline{r}(t) = \left(\frac{-P}{t}, 0, 0\right)$$

t RUNNING FROM 0 TO -1
$t=0$ CORRESPONDS TO POINT AT ∞
$t=-1$ CORRESP. TO POINT AT P.

$$V \equiv -\int_{\text{POINT AT } \infty}^{\text{POINT AT } P} \underline{E}\cdot d\underline{r} = -\int_{\text{POINT AT } \infty}^{\text{POINT AT } P} \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{\underline{r}}\cdot d\underline{r} = -\int_{\text{POINT AT } \infty}^{\text{POINT AT } P} \frac{q}{4\pi\varepsilon_0}\frac{\underline{r}}{r^3}\cdot d\underline{r}$$

APPROACHING P FROM ⊕ x AXIS.

$$= -\int_{t=0}^{t=-1} \frac{q}{4\pi\varepsilon_0} \frac{\left(-\frac{P}{t}, 0, 0\right)}{r^3} \frac{d\left(-\frac{P}{t}, 0, 0\right)}{dt} dt = -\int_{t=0}^{t=-1} \frac{q}{4\pi\varepsilon_0} \frac{\left(-\frac{P}{t}, 0, 0\right)}{r^3}\cdot\left(\frac{P}{t^2}, 0, 0\right)dt$$

$$= -\int_{t=0}^{t=-1} \frac{q}{4\pi\varepsilon_0}\left(-\frac{P^2}{t^3}\right)\frac{1}{r^3}dt = -\int_{t=0}^{t=-1} \frac{q}{4\pi\varepsilon_0}\left(-\frac{P^2}{t^3}\right)\left(\frac{-P}{t}\right)^{-3} dt$$

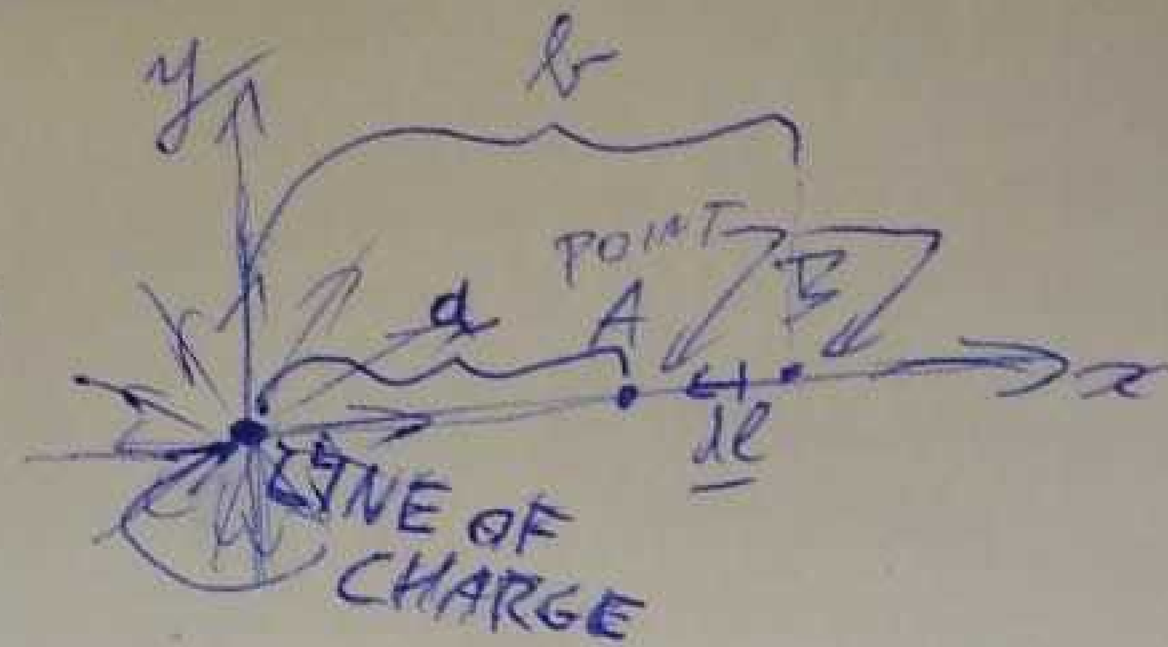
t IS ⊖, SO $\frac{-P}{t}$ IS ⊕, SO $\left(\frac{-P}{t}\right)^{-3}$ ⊕, AS r^3 IS EXPECTED TO BE. IS IT EXPECTED TO BE THOUGH, FOR SURE?

$$= -\int_{t=0}^{t=-1} \frac{q}{4\pi\varepsilon_0}\left(-\frac{P^2}{t^3}\right)\cdot -1\left(\frac{P}{t}\right)^{-3} dt$$

$$= -\int_{t=0}^{t=-1} \frac{q}{4\pi\varepsilon_0}\frac{1}{P}dt = \frac{q}{4\pi\varepsilon_0}\frac{1}{P}\cdot -\left[t\right]_0^{-1} = \frac{q}{4\pi\varepsilon_0}\frac{1}{P}$$ AS EXPECTED.

UNIFORM LINE OF CHARGE

$$\underline{E}(r) = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r}\,\hat{\underline{r}} = \frac{\lambda}{2\pi\varepsilon_0}\frac{\underline{r}}{r^2}$$



$$V_A - V_B = -\int_{\text{POINT } B}^{\text{POINT } A} \underline{E}\cdot d\underline{l}$$

$$\underline{r} = (b\cdot t, 0, 0)$$

t RUNNING FROM 1 TO $\frac{a}{b}$

$$= -\int_{\text{FROM START OF } t}^{\text{TO END OF } t} \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r^2}(bt, 0, 0)\underbrace{\frac{d(bt,0,0)}{dt}}_{=(b,0,0)} dt$$

$$= -\int_{t=1}^{t=\frac{a}{b}} \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{b^2t^2}\, b^2 t\, dt = -\int_{t=1}^{t=\frac{a}{b}} \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{t}\,dt$$

$$= -\frac{\lambda}{2\pi\varepsilon_0}\left[\ln t\right]_1^{\frac{a}{b}} = -\frac{\lambda}{2\pi\varepsilon_0}\left(\ln\frac{a}{b} - \ln 1\right) =$$

$$= -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{\frac{a}{b}}{1}\right) = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{a}{b}\right) = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right)$$

RIGHT

THIS IS RIGHT. WRONG WAY OF DOING IT:

$$-\int_B^A \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r}\underbrace{\hat{r}\cdot d\underline{l}}_{=-dl} = \int_b^a \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r}\,dr = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{a}{b}\right)$$

WRONG