EULER-LAGRANGE

LET $g \in C^2[a, b]$ (g IS TWICE DIFFERENTIABLE & SECOND DERIVATIVE IS CONTINUOUS)

WITH $g(a) = g(b) = 0$

ASSUME THAT THE FUNCTIONAL F REACHES/ATTAINS A LOCAL MINIMUM AT $y = f$, IE:

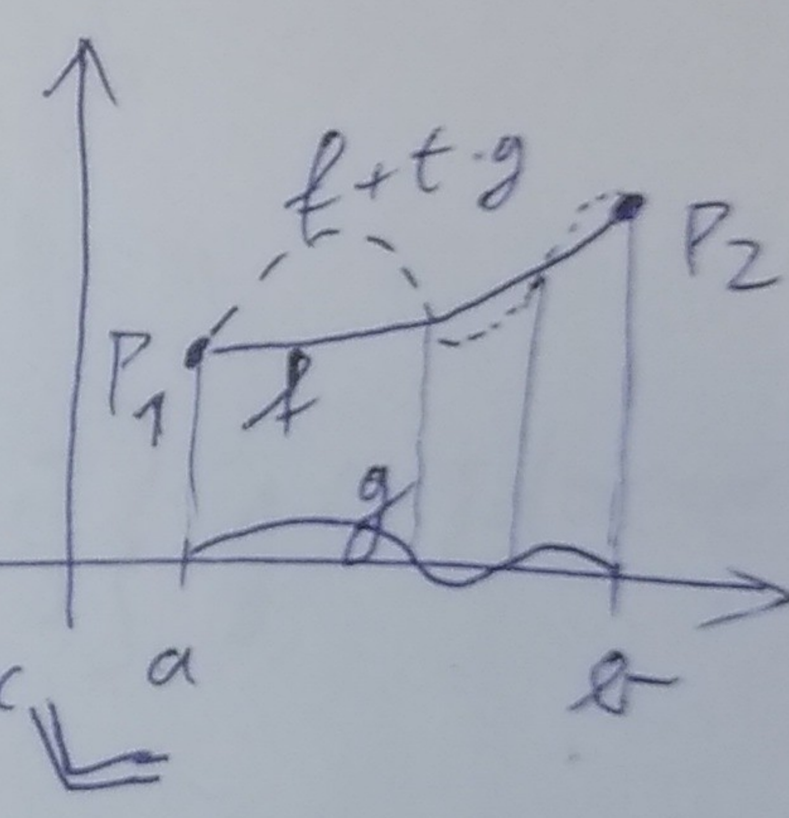
$$F(f) \leq F(f + t\cdot g)$$ [F IS REAL VALUED]

FOR „SMALL" VALUES OF t, FOR $\forall g$.

$$J[y(x)] = \int_a^b F[x, y(x), y'(x)]\,dx$$

$$\Phi(t) = J[y(x) + t\cdot g(x)] = \int_a^b F[x, y(x) + t\cdot g(x), y'(x) + t\cdot g'(x)]\,dx$$

THEN, $\Phi(t)$ HAS A LOCAL MINIMUM AT $t=0 \Longrightarrow \Phi'(t)\Big|_{t=0} = \Phi'(0) = 0$



$$\Phi'(t) = \frac{d}{dt}\int_a^b F[x, y(x) + t\cdot g(x), y'(x) + t\cdot g'(x)]\,dx$$

$$= \int_a^b \frac{d}{dt} F[x, y + t\cdot g, y' + t\cdot g']\,dx$$

$$F_x \underbrace{\frac{dx}{dt}}_{0} + F_y[x, y + t\cdot g, y' + t g']\frac{d}{dt}\overbrace{(y + t\cdot g)}^{g(x)} +$$

0 (x HAS NO t DEPENDENCE)

EZ ITT AZT JELENTI, HOGY F A MÁSODIK ARGUMENTJE, VAGYIS $y + t\cdot g$ SZERINT VAN DERIVÁLVA

$$\frac{\partial F}{\partial\,\text{MÁSODIK ARGUMENT}}\;\frac{\partial\,\text{MÁSODIK ARGUMENT}}{\partial t}$$

No t DEPENDENCE

$$\frac{\partial F}{\partial\,\text{HARMADIK ARGUMENT}}\;\frac{\partial\,\text{HARMADIK ARGUMENT}}{\partial t}$$

$$+ F_{y'}[x, y + t\cdot g, y' + t\cdot g'] \cdot \frac{d}{dt}\underbrace{(y' + t\cdot g')}_{g'(x)} =$$

HARMADIK ARGUMENT SZERINTI DERIVÁLÁS

I

(WE ARE WRITING $\varphi'(t)$ HERE)

$$= \int_a^b \{ F_y[x, y + t\cdot g, y' + t\cdot g'] \cdot g +$$

$$+ F_{y'}[x, y + t\cdot g, y' + t\cdot g'] \cdot g' \} dx$$

$$\varphi'(0) = \varphi'(t)\Big|_{t=0} = \int_a^b \{ F_y[x, y, y'] \cdot g + F_{y'}[x, y, y'] \cdot g' \} dx =$$

$$= \underbrace{\int_a^b F_y[x, y, y'] \cdot g \, dx}_{I_1} + \underbrace{\int_a^b F_{y'}[x, y, y'] \cdot g' \, dx}_{I_2} = I_1 + I_2$$

APPLY INTEGRATION BY PARTS TO I_2:

$$I_2 = \int_a^b \underbrace{F_{y'}[x, y, y']}_{u} \cdot \underbrace{g'(x)}_{v'} dx = \int_a^b u \cdot v' \, dx =$$

$$= [u \cdot v]_{x=a}^{b} - \int_a^b u' v \, dx =$$

$$= \Big[F_{y'}[x, y, y'] \cdot g(x) \Big]_{x=a}^{b} - \int_a^b \frac{d}{dx} \{ F_{y'}[x, y, y'] \} \cdot g(x) \, dx =$$

$$= F_{y'}[b, y(b), y'(b)] \cdot \underbrace{g(b)}_{0} - F_{y'}[a, y(a), y'(a)] \cdot \underbrace{g(a)}_{\llcorner 0} -$$

$$- \int_a^b g(x) \frac{d}{dx} F_{y'}[x, y(x), y'(x)] dx =$$

$$= - \int_a^b g(x) \frac{d}{dx} F_{y'}[x, y, y'] dx$$

SO PLUG IN BACK TO $\varphi'(0)$:

$$\varphi'(0) = I_1 + I_2 = \int_a^b F_y[x, y, y'] \cdot g(x) \, dx - \int_a^b g(x) \frac{d}{dx} F_{y'}[x, y, y'] dx =$$

(II)

$$= \int_a^b \left\{ F_y[x,y,y'] - \frac{d}{dx} F_{y'}[x,y,y'] \right\} \cdot g(x)\,dx = 0$$

FOR ALL $g(x)$. THUS:

$$F_y[x,y,y'] - \frac{d}{dx} F_{y'}[x,y,y'] = 0$$

(EULER-LAGRANGE DIFF. EQUATION)

REMEMBER: E-L EQS. IS A NECESSARY (NOT SUFFICIENT!) CONDITION FOR A MINIMUM/EXTREMUM OF $F(y)$.
(WE STARTED WITH: <u>ASSUME</u> THAT THE FUNCTIONAL F...)

<u>NOTE I</u>: IF $y \in C^2[a,b]$, THEN:

$$\frac{d}{dx} F_{y'}[x,y,y'] = F_{y'x} + F_{y'y}\, y' + F_{y'y'}\, y''$$

SO E-L EQ BECOMES:

$$F_y - \left[F_{y'x} + F_{y'y}\, y' + F_{y'y'}\, y'' \right] = 0$$

IE:

$$y'' F_{y'y'} + y' F_{y'y} + \boxed{F_{y'x}} - F_y = 0$$

<u>NOTE II</u>: IF $F = F[y, y']$ (IE IT DOES NOT DEPEND UPON x) & $F_{y'y'} \neq 0$ THEN, FOR $\forall$ SOLUTIONS y OF E-L, $\exists C \in \mathbb{R}$ S.T. $F - y' F_{y'} = C$

PROOF: $F_{y'x} = F_{xy'} = 0$, SO E-L

$$y'' \cdot F_{y'y'} + y' F_{y'y} + 0 - F_y = 0$$

ON THE OTHER HAND, WE FORM:

$$\frac{d}{dx}\left[F - y' \cdot F_{y'} \right] = F_y\, y' + F_{y'}\, y'' - \frac{d}{dx}\left(y' F_{y'} \right) =$$

$$= F_y y' + F_{y'} y'' - y'' F_{y'} - y' \underbrace{\frac{d}{dx}(F_{y'})} =$$

$$= F_{y'y'} y'' + F_{y'y} y'$$

$$= F_y y' + \cancel{F_{y'} y''} - \cancel{y'' F_{y'}} - F_{y'y} y' - y' F_{y'y'} y'' =$$

$$= y' \big(\underbrace{F_y - y' F_{y'y} - y'' F_{y'y'}}_{0\text{, FROM E-L FOR THIS CASE.}} \big) = 0$$

THUS: $F - y' F_{y'} = C$ FOR SOME $C \in \mathbb{R}$

NOTE III

IF F DOES NOT DEPEND UPON y, (IE $F \equiv F(x, y')$) THEN, FOR $\forall$ SOLUTIONS y OF E.-L., $\exists C \in \mathbb{R}$, S.T.

$$F_{y'}(x, y') = C \quad \text{FOR } x \in [a, b]$$

PROOF: $F_y \equiv 0$, SO E.-L. HAS THE FORM:

$$y'' F_{y'y'} + 0 + F_{y'x} - 0 = 0$$

IE:

$$y'' F_{y'y'} + F_{y'x} = 0$$

LET'S FORM: $\frac{d}{dx} F_{y'}(x, y') =$

$$= F_{y'x} + F_{y'y'} y'' = y'' F_{y'y'} + F_{y'x} = 0$$

THUS: $F_{y'}(x, y') = C$

NOTE IV

IF $F = F(x, y)$ (IE NO y' DEPENDENCE), THEN FOR

$\forall$ SOLUTIONS y OF E.L., WE HAVE:

$F_y(x, y) = 0$, FOR ~~A~~ $x \in [a, b]$

(ALL OTHER TERMS FALL OUT FROM E-L EQS, SEE NOTE I.)

EVERY TERM WITH $F_{y'}$ IN THEM.

PROOF:

E-L BECOMES:

$0 + 0 + 0 - F_y = 0 \Rightarrow F_y(x, y) = 0$