$$[J_\pm, J_z] = [J_x, J_z] \pm i[J_y, J_z] = -iJ_y \mp J_x =$$

$$= \mp J_x \pm iJ_y = \mp J_\pm$$

$$J_z J_\pm |\beta, m\rangle = \left(J_\pm J_z + [J_z, J_\pm]\right)|\beta, m\rangle =$$

$$= \left(J_\pm J_z \pm J_\pm\right)|\beta, m\rangle = (m \pm 1) J_\pm |\beta, m\rangle$$

$\Rightarrow J_\pm |\beta, m\rangle$ IS EIGENSTATE OF J_z

$$J_\pm |\beta, m\rangle = \alpha_\pm |\beta, m \pm 1\rangle$$

$$|\alpha_+|^2 = \langle \beta m | J_+^\dagger J_+ |\beta m\rangle = \langle \beta m | J_- J_+ |\beta m\rangle =$$

$$= \langle \beta m | (J_x - iJ_y)(J_x + iJ_y) |\beta m\rangle =$$

$$= \langle \beta m | J^2 - J_z^2 + i[J_x, J_y] |\beta m\rangle =$$

$$= \langle \beta m | J^2 - J_z^2 - J_z |\beta m\rangle = \beta - m^2 - m =$$

$$= \beta - m(m+1)$$

$$|\alpha_-|^2 = \langle \beta m | J_-^\dagger J_- |\beta m\rangle = \langle \beta m | J_+ J_- |\beta m\rangle =$$

$$= \langle \beta m | (J_x + iJ_y)(J_x - iJ_y) |\beta m\rangle =$$

$$= \langle \beta m | J^2 - J_z^2 + i[J_y, J_x] |\beta m\rangle =$$

$$= \langle \beta m | J^2 - J_z^2 + J_z |\beta m\rangle = \beta - m^2 + m =$$

$$= \beta - m(m-1)$$

$$\Rightarrow \alpha_\pm = \sqrt{\beta - m(m \pm 1)}$$

J_i ARE HERMETIAN $\Rightarrow \langle\psi|J_i^2|\psi\rangle = \left|J_i|\psi\rangle\right|^2 \geqslant 0$

$$\beta = \langle\beta,m|J^2|\beta,m\rangle = \langle\beta m|J_x^2 + J_y^2 + J_z^2|\beta m\rangle \geqslant m^2$$

(GIVEN THAT $\langle\beta m|J_z^2|\beta m\rangle = m^2$)

FOR α_+: $\beta - m_{MAX}(m_{MAX}+1) = 0$ (FROM EQ ON PREVIOUS PAGE)

FOR α_-: $\beta - m_{MIN}(m_{MIN}-1) = 0$

$$m_{MAX}(m_{MAX}+1) = m_{MIN}(m_{MIN}-1)$$

$$m_{MAX}^2 + m_{MAX} = m_{MIN}^2 - m_{MIN}$$

$$m_{MIN}^2 - m_{MIN} - m_{MAX}^2 - m_{MAX} = 0$$

$$m_{MIN} = \frac{1 \pm \sqrt{1 - 4\cdot 1\cdot(-m_{MAX}^2 - m_{MAX})}}{2} =$$

$$= \frac{1 \pm \sqrt{1 + 4(m_{MAX}^2 + m_{MAX})}}{2} =$$

$$= \frac{1}{2}\left(1 \pm \sqrt{(2m_{MAX}+1)^2}\right) =$$

$$= \frac{1}{2}\left(1 \pm (2m_{MAX}+1)\right)$$

$m_{MAX} > m_{MIN}$ SO ONLY ~~+~~ $-$ COUNT FROM $\pm$.

$$\frac{1}{2}\left(1 - (2m_{MAX}+1)\right) = -m_{MAX}$$

$\beta = m_{MAX}(m_{MAX}+1)$

DEFINE: $j = m_{MAX}$

$\beta = j(j+1)$

$-j \leqslant m \leqslant j$

$|\beta j\rangle \xrightarrow[\text{APPLICATION}]{\text{INTEGER NUMBER OF } J_-} |\beta -j\rangle$

$\Downarrow$

$2j$ MUST BE INTEGER.

RELABEL KETS:

$|\beta, m\rangle = |j(j+1), m\rangle \xrightarrow{\text{RELABEL}}$

$\rightarrow |j, m\rangle$

$J_\pm|j,m\rangle = \alpha_\pm(m)|j,m\pm 1\rangle =$

$= \sqrt{j(j+1) - m(m\pm 1)}\,|j, m\pm 1\rangle$