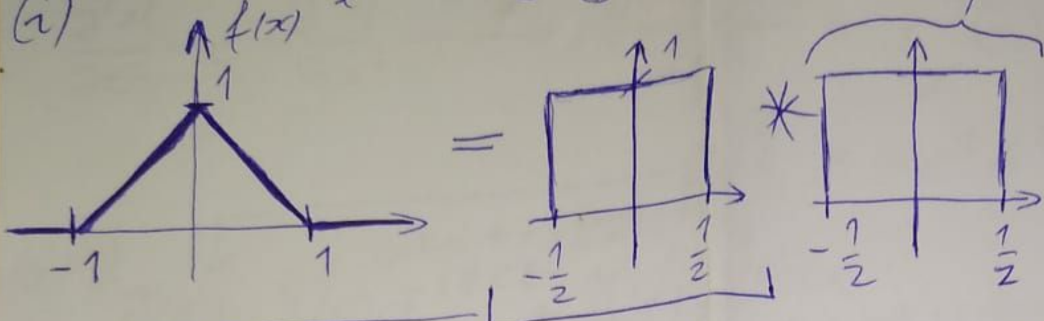
2013P2Q5 | I

$f(x)=\begin{cases}1-|x|, & |x|<1\\ 0 & \text{OTHERWISE}\end{cases}$ $\qquad t(x)=\begin{cases}1 & \text{IF } |x|<\frac{1}{2}\\ 0 & \text{OTHERWISE}\end{cases}$

(i)



✓

FT: $\int_{-\infty}^{\infty} e^{-ikx}\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} 1\cdot e^{-ikx}\,dx = \frac{i}{k}\left[e^{-ikx}\right]_{-\frac{1}{2}}^{\frac{1}{2}} =$

$= \frac{i}{k}\left[e^{-i\frac{1}{2}k} - e^{i\frac{1}{2}k}\right] = \frac{2\sin\left(\frac{1}{2}k\right)}{k}$

BY CONVOLUTION THEOREM: $FT[f(x)] = FT[t(x)]\cdot FT[t(x)]$

$= \frac{4\sin^2\left(\frac{1}{2}k\right)}{k^2}$ ✓ good

(ii)

$F_x\left[\frac{\partial u}{\partial t}\right] = \lim_{\tau\to 0} F_x\left[\frac{u(x,t+\tau)-u(x,t)}{\tau}\right] =$ ✓ good

$= \lim_{\tau\to 0}\frac{1}{\tau}\left[F_x[u(x,t+\tau)] - F_x[u(x,t)]\right] = \frac{\partial F_x u}{\partial t}$

$F_x\left[\frac{\partial u}{\partial x}\right] = \lim_{h\to 0}\int_{-\infty}^{\infty}\frac{u(x+h,t)-u(x,t)}{h}e^{-ikx}\,dx =$

$= \lim_{h\to 0}\int_{-\infty}^{\infty}\frac{u(x+h,t)}{h}e^{-ikx}\,dx - \lim_{h\to 0}\int_{-\infty}^{\infty}\frac{u(x,t)}{h}e^{-ikx}\,dx =$

CHANGE VAR.: $x+h=z \Rightarrow x=z-h,\ dx=dz$

$= \lim_{h\to 0}\left[\int_{-\infty}^{\infty}\frac{u(z,t)}{h}e^{-ikz}\,dz\, e^{ikh} - \int_{-\infty}^{\infty}\frac{u(x,t)}{h}e^{-ikx}\,dx\right] =$

$= \lim_{h\to 0}\left[\int_{-\infty}^{\infty}\frac{u(z,t)}{h}e^{-ikz}\,dz\left[1+(ikh)+O(h^2)\right] - \int_{-\infty}^{\infty}\frac{u(x,t)}{h}e^{-ikx}\,dx\right] =$ (?)

this is the standard method

STANDARD METH: $g(x)=f'(x)$
$\tilde{g}(k)=\int_{-\infty}^{\infty}f'(x)e^{-ikx}\,dx$
$=\left[f(x)e^{-ikx}\right]_{-\infty}^{\infty} -$

$= \lim_{h\to 0}\left(\left[1+ikh+O(h^2)-1\right]\int_{-\infty}^{\infty}\frac{u(z,t)}{h}e^{-ikz}\,dz =\right.$

$= ik\int_{-\infty}^{\infty}\frac{u(z,t)}{h}e^{-ikz}\,dz = ik\,\tilde{u}(k,t)$

2013 P2 Q5 (II)

(iii)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \xrightarrow{FT_x} \frac{\partial^2 \tilde{u}(k,t)}{\partial t^2} = c^2(-ik)^2\, \tilde{u}(k,t)$$

✓

$$\frac{\partial^2}{\partial t^2}\tilde{u} = -c^2 k^2 \tilde{u}$$

(iv)

cos kct + −i sin kct

$$\tilde{u}(k,t) = \tilde{A}(k)\, e^{-ikct} + \tilde{B}(k)\, e^{ikct}$$

→ cos kct + i sin kct.

BY CONVOLUTION THEOREM:

$$u(x,t) = A(x) * \mathrm{IFT}\left[e^{-ikct}\right] + B(x) * \mathrm{IFT}\left[e^{+ikct}\right]$$

$$\mathrm{IFT}\left[e^{\pm ikct}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\pm ikct}\, e^{ikx}\, dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(ct+x)}\, dk =$$

$$= \cancel{\delta(ct+x) + \delta(ct-x)}$$

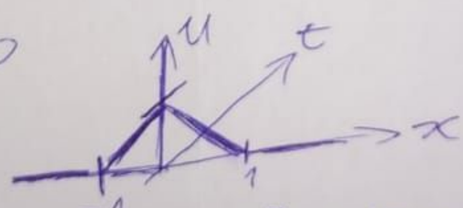
$$= \delta(ct \pm x)$$

[! SHALLOW UNDERSTANDING HERE]

$$u(x,t) = A(ct+x) + B(ct-x)$$

$$u(x,t=0) = f(x) = A(0+x) + B(0-x)$$

$$\frac{\partial u(x,t=0)}{\partial t} = 0$$



we had better go thrô this.

[AND FROM HERE I DON'T SEE HOW TO PROCEED]

#

$$-\int_{-\infty}^{\infty} f(x)(-ik)\,e^{-ikx}\,dx =$$

$$= ik\,\tilde{f}(k)$$

THIS DOES NOT USE THE "FORMAL LIMIT DEFINITION" THOUGH