2012 P1Q9(I)

(i) (a) $f_y(x,y,y') - \frac{d}{dx} f_{y'}(x,y,y') = 0$

(b) EXPANDED FORM OF E-L EQS:

$$f_y - f_{y'x} - f_{y'y}y' - f_{y'y'}y'' = 0$$

$f_y = 0$, SO: $\quad -f_{y'x} - f_{y'y'}y'' = 0$

(AND THEREFORE: $f_{y'y} = 0$)

$$y'' f_{y'y'} + f_{y'x} = 0$$

LET'S FORM: $\quad \frac{d}{dx}(f_{y'}) = f_{y'x} + f_{y'y'}y'' = 0$

THUS: $f_{y'}(x,y') = c$ (THIS IS THE FIRST INTEGRAL)

(ii) (a) INTEGRAND DOES NOT DEPEND ON y EXPLICITLY, SO:

$$\frac{\partial}{\partial y'}\left(n(x)\sqrt{1+y'^2}\right) = c$$

$$n(x)\frac{1}{2}(1+y'^2)^{-\frac{1}{2}}2y' = c$$

$\cancel{n(x)(1+y'^2)^{-\frac{1}{2}} = c}$

$\cancel{n^2(x)(1+y'^2)^{-1} = c^2}$

$\cancel{y'^2 = c^2 - n^2(x)}$

$\cancel{n^2(x) = c^2(1+y'^2)}$

$$n(x)(1+y'^2)^{-\frac{1}{2}}y' = c$$

$$n(x)y' = c(1+y'^2)^{\frac{1}{2}}$$

$$n^2(x)y'^2 = c^2(1+y'^2)$$

$$(n^2(x) - c^2)y'^2 = c^2$$

$$y' = \frac{c}{\sqrt{n^2(x) - c^2}}$$

AS REQUIRED.

(b)

$$\frac{dy}{dx} = \frac{c}{\sqrt{n^2(x) - c^2}}$$

$n(x) = 1 + \beta x$

$$\frac{dy}{dx} = \frac{c}{\sqrt{(1+\beta x)^2 - c^2}} \Rightarrow \int dy = y = \int \frac{c\,dx}{\sqrt{(1+\beta x)^2 - c^2}} =$$

USE SUBSTITUTION:

$u = 1 + \beta x \Rightarrow du = \beta dx$

$$= \int \frac{\frac{c}{\beta}du}{\sqrt{u^2 - c^2}} = \quad = \int \frac{\frac{c}{\beta} c \sinh\theta d\theta}{\sqrt{c^2\cosh^2\theta - c^2}} =$$

USE SUBSTITUTION: $u = c\cosh\theta$

$du = c\sinh\theta d\theta$

2012 P1Q9(II)

$$= \int \frac{\frac{C}{B}\,\cancel{d\theta}\; C\sinh\theta\, d\theta}{C\,\sinh\theta} = \int \frac{C}{B}\, d\theta = \frac{C}{B}\,\theta + K = y$$

~~(B & C ABSORBED TO K)~~

$$\theta + K = y\frac{B}{C}$$

$$\theta = y\frac{B}{C} - K\frac{B}{C}$$

$$\cancel{C\cosh\theta = C\cosh\left(y\frac{B}{C} - K\right)}$$

$$C\cosh(\theta) = C\cosh\left(\frac{B}{C}(y-K)\right)$$

RECALL:

$$C\cosh\theta = u = 1 + Bx$$

THUS

$$1 + Bx = C\cosh\left(\frac{B}{C}(y-K)\right)$$

$$x = -\frac{1}{B} + \frac{C}{B}\cosh\left(\frac{B}{C}(y-K)\right)$$

SEEMS ENCOURAGING.

$$y_0 = y(x_0) = K$$

$$x_0 = \frac{-1+C}{B}$$

$$x_0 = -\frac{1}{B} + \frac{C}{B}\cosh\left(\frac{B}{C}\left(y(x_0) - K\right)\right)$$

$$x_0 = -\frac{1}{B} + \frac{C}{B}\cosh\left(\frac{B}{C}\left(\frac{-1+C}{B} - K\right)\right)$$

$$x_0 = -\frac{1}{B} + \frac{C}{B}\cosh\left(\frac{-1+C}{C} - \frac{B}{C}K\right)$$

$$= \frac{-1+C}{B} \qquad /\cdot B$$

$$\cancel{-1} + C = \cancel{-1} + C\cosh\left(\frac{-1+C}{C} - \frac{B}{C}K\right)$$

$$1 = \cosh\left(\frac{-1+C}{C} - \frac{B}{C}K\right)$$

$$\Rightarrow \frac{-1+C}{C} - \frac{B}{C}K = 0$$

$$+\frac{-1+C}{C} = \frac{B}{C}K$$

$$-1 + C = +BK$$

$$\cancel{K = \frac{C-1}{-B} = \frac{1-C}{B}}$$

$$K = \frac{-1+C}{B} = \frac{C-1}{B}$$

I DON'T THINK I GRASPED THE ENTIRETY OF WHAT'S GOING ON HERE SO THIS ABOVE MIGHT HAVE BEEN WISHFUL MATH.

REWRITE:

$$x = -\frac{1}{B} + \frac{C}{B}\cosh\left(\frac{B}{C}(y - y_0)\right)$$

(c)