2013P2Q4(II)(i)

CAUCHY'S RESIDUE THEOREM: $\oint_C f(z)dz = 2\pi i \sum$ RESIDUES OF f INSIDE C

$$f(z) = f(z_0)(z-z_0)^0 + f'(z_0)(z-z_0)^1 + \frac{1}{2!}f''(z_0)(z-z_0)^2 + \dots$$

$$\frac{f(z)}{z-z_0} = \underbrace{f(z_0)(z-z_0)^{-1}} + f'(z_0) + \frac{1}{2!}f''(z_0)(z-z_0)^1 + \frac{1}{3!}f'''(z_0)(z-z_0)^2 + \dots$$

RESIDUE OF $\frac{f(z)}{z-z_0}$ IS THIS

SO BY CRT: $\oint_C \frac{f(z)dz}{z-z_0} = 2\pi i f(z_0)$ (THIS IS JUST CAUCHY'S INTEGRAL FORMULA)

(ii) $\frac{e^{iz}}{a^2+z^2} = \frac{e^{iz}}{(z-ia)(z+ia)}$

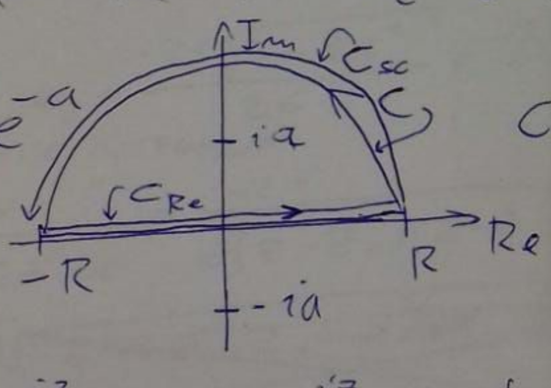
SO THE SINGULARITIES OF ~~THE~~ THE INTEGRAND: $z = \pm ia$

THESE ARE FIRST ORDER POLES.

FROM RESIDUE THEOREM WE FOUND: $\oint_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$

$$\oint_C \frac{e^{iz}\,dz}{(z-ia)(z+ia)} = \oint_C \frac{\overbrace{\frac{e^{iz}}{z+ia}}^{f(z)}}{z-\underbrace{ia}_{z_0}}dz = 2\pi i \frac{e^{i(ia)}}{(ia)+(ia)} = 2\pi i \frac{e^{-a}}{2ia} =$$

$$= \frac{\pi}{a}e^{-a}$$



$C = \underbrace{C_{sc}}_{\text{PATH ALONG SEMICIRCLE ABOVE REAL AXIS, RADIUS}=R} + \underbrace{C_{Re}}_{\text{PATH ALONG REAL LINE FROM }-R\text{ TO }R}$

$$\oint_C \frac{e^{iz}}{a^2+z^2}dz = \underbrace{\oint_{sc} \frac{e^{iz}}{a^2+z^2}dz}_{I_{sc}} + \underbrace{\int_{C_{Re}} \frac{e^{iz}}{a^2+z^2}dz}_{I_{Re}}$$

AS $R \to \infty$, $I_{sc} \to 0$, BECAUSE LENGTH OF C_{sc} IS INCREASING PROPORTIONAL TO r, WHILE THE VALUE OF INTEGRAND $\left(\frac{e^{iz}}{a^2+z^2}\right)$ GOES TO 0 WITH A RATE $\frac{1}{r^2}$ AS R (AND z) $\to \infty$; $O(r)$ INCREASE MULTIPLIED BY $O\left(\frac{1}{r^2}\right)$ DECREASE RESULTS IN $O\left(\frac{1}{r}\right)$ DECREASE, SO $I_{sc} \to 0$.

$$\lim_{R\to\infty} I_{Re} = \int_{-\infty}^{\infty} \frac{e^{ix}}{a^2+x^2}dx = \oint_{C,\,R\to\infty} \frac{e^{iz}}{a^2+z^2}dz = \frac{\pi}{a}e^{-a}$$

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$$\frac{\pi}{a}e^{-a} = \int_{-\infty}^{\infty}\frac{e^{ix}}{a^2+x^2}dx = \int_{-\infty}^{\infty}\frac{\cos x}{a^2+x^2}dx + i\underbrace{\int_{-\infty}^{\infty}\underbrace{\frac{\sin x}{a^2+x^2}}_{\text{ODD}}dz}_{=0}$$

$$\int_{-\infty}^{\infty}\frac{\cos x}{a^2+x^2}dx = \frac{\pi}{a}e^{-a}$$

(iii)

$$I = \int_0^{2\pi}\frac{1}{1+\epsilon\cos\theta}d\theta \qquad -1<\epsilon<1$$

$$z = e^{i\theta} \rightarrow \cos\theta = \frac{z+\frac{1}{z}}{2}$$

$$dz = i\underbrace{e^{i\theta}}_{z}d\theta \rightarrow d\theta = \frac{1}{iz}dz$$

$$I = \oint_{|z|=1}\frac{1}{1+\epsilon\frac{z+1/z}{2}}\frac{1}{iz}dz = \oint_{|z|=1}\frac{1(-i)}{z+\epsilon\left(\frac{z^2+1}{2}\right)}dz = \underbrace{\oint_{|z|=1}\frac{-2i\ dz}{2z+\epsilon z^2+\epsilon}}_{\text{CLOSED CONTOUR INTEGRAL.}}$$

INTEGRATION PATH: UNIT CIRCLE CENTERED AT ORIGIN, CCW.

POLES OF INTEGRAND WHERE DENOMINATOR = 0

$$2z+\epsilon z^2+\epsilon = 0$$

$$\epsilon z^2 + 2z + \epsilon = 0$$

$$z_\pm = \frac{-2\pm\sqrt{4-4\epsilon^2}}{2\epsilon} = \frac{-2\pm2\sqrt{1-\epsilon^2}}{2\epsilon} = \frac{-1\pm\sqrt{1-\epsilon^2}}{\epsilon}$$

REWRITE INTEGRAND:

$$\frac{-2i}{2z+\epsilon z^2+\epsilon} = \frac{-2i}{\epsilon(z-z_+)(z-z_-)}$$

BOTH TERMS INVOLVING z ARE TO THE POWER 1 $\Rightarrow$ THE POLES ARE SIMPLE.

~~AS $\epsilon\rightarrow0^+$, $z_+\rightarrow 0$, $z_-\rightarrow -\infty$~~

~~$\epsilon\rightarrow0^-$, $z_+\rightarrow0^+$, $z_-\rightarrow+\infty$~~

BY VIETA'S: $z_+z_- = \frac{\epsilon}{\epsilon} = 1 \Rightarrow |z_+|$ OR $|z_-|$ IS ALWAYS LESS OR EQUAL TO 1.

$z_+z_- = 1$ IF $\epsilon = \pm1$

$\rightarrow$ THIS CANNOT BE (NOT ALLOWED BY PROBLEM),

SO:

$$|z_+||z_-|<1$$

$\rightarrow$ ONE POLE IS ALWAYS INSIDE UNIT CIRCLE.

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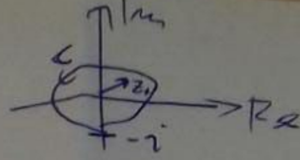
$$\oint_{|z|=1} \frac{-2i \quad dz}{\epsilon(z-z_+)(z-z_-)} = I$$



IF $|z_+|<1$:

$$I = \oint_{|z|=1} \frac{\overbrace{\frac{-2i}{\epsilon(z-z_-)}}^{f} \, dz}{z-\underbrace{z_+}_{z_0}} = f(z_0)\cdot 2\pi i = 2\pi i \, \frac{-2i}{\epsilon(z_+-z_-)} =$$

$$= \frac{4\pi}{\epsilon(z_+-z_-)} = \frac{4\pi}{\epsilon\left(\frac{-1+\sqrt{1-\epsilon^2}}{\epsilon} - \frac{-1-\sqrt{1-\epsilon^2}}{\epsilon}\right)} =$$

$$= \frac{4\pi}{2\sqrt{1-\epsilon^2}} = \frac{2\pi}{\sqrt{1-\epsilon^2}}$$

IF $|z_-|<1$:

$$I = \oint_{|z|=1} \frac{\overbrace{\frac{-2i}{\epsilon(z-z_+)}}^{f} \, dz}{z-\underbrace{z_-}_{z_0}} = 2\pi i \, f(z_0) = 2\pi i \, \frac{-2i}{\epsilon(z_--z_+)} =$$

$$= \frac{4\pi}{\epsilon(z_--z_+)} = \frac{4\pi}{\epsilon\left(\frac{-1-\sqrt{1-\epsilon^2}}{\epsilon} - \frac{-1+\sqrt{1-\epsilon^2}}{\epsilon}\right)} =$$

$$= \frac{2\pi}{-\sqrt{1-\epsilon^2}}$$

THE RESULTS DISAGREE, SO WE SHOULD FIND IF $|z_+|<1$ OR $|z_-|<1$.

$$\left.\begin{array}{l} z_- = \frac{-1-\sqrt{1-\epsilon^2}}{\epsilon} \qquad -1-\sqrt{1-\epsilon^2} < -1 \\ \qquad\qquad\qquad\qquad 1 > \epsilon > -1 \end{array}\right] \Rightarrow |z_-| > 1 \Rightarrow |z_+| < 1, \text{ so:}$$

$$\oint_{|z|=1} \frac{-2i \quad dz}{\epsilon(z-z_+)(z-z_-)} = \int_0^{2\pi} \frac{1 \quad d\theta}{1+\epsilon\cos\theta} = \frac{2\pi}{\sqrt{1-\epsilon^2}}$$