2014P2Q6(I)

(i) TRANSFORMATION LAW FOR A TENSOR OF ORDER n:

$$T'_{i_1 \dots i_n} = L_{i_1 j_1} \dots L_{i_n j_n} T_{j_1 \dots j_n}$$

✓

WHERE $L_{ij} = \underline{e}'_i \cdot \underline{e}_j$

WHERE $\underline{e}'_i$ IS THE i-TH BASIS VECTOR OF THE FRAME WE ARE TRANSFORMING TO & $\underline{e}_j$ IS THE j-TH BASIS VECTOR OF THE FRAME WE ARE TRANSFORMING FROM.

✓

v bad grammar!

ISOTROPIC TENSORS ARE TENSORS WITH THE SAME COMPONENTS IN ALL FRAMES: ✓

$$T'_{ijk\dots} = T_{ijk}$$

✓

(ii) CONSIDER: $\dfrac{\partial x_j}{\partial x'_i} = \dfrac{\partial (L_{kj} x'_k)}{\partial x'_i} = L_{kj} \dfrac{\partial x'_k}{\partial x'_i} =$

✓

$$= L_{kj} \delta_{ki} = L_{ij}$$

$$x'_i = L_{ij} x_j \quad / \cdot L_{ji}$$

$$L_{ji} x'_i = L_{ji} L_{ij} x_j$$

$$L_{ji} x'_i = x_j \quad \text{SINCE } L \text{ IS ORTHOGONAL.}$$

no.

BY CHAIN RULE:

$$\frac{\partial}{\partial x'_i} = \underbrace{\frac{\partial x_j}{\partial x'_i}} \frac{\partial}{\partial x_j} = \underbrace{L_{ij}} \frac{\partial}{\partial x_j}$$

TRANSFORMATION LAW SATISFIED SO $\dfrac{\partial}{\partial x_i}$ IS A TENSOR (OF ORDER 1).

~~Stick in your def of orthog vectors?~~

~✓

Is all this needed for 4 marks?

$$\left(\frac{\partial u_i}{\partial x_j}\right)' = \left(\frac{\partial}{\partial x_j}\right)' (u_i)' = L_{jq} \frac{\partial}{\partial x_q} L_{ip} u_p = L_{jq} L_{ip} \frac{\partial u_p}{\partial x_q}$$

✓

SO $\dfrac{\partial u_p}{\partial x_q}$ TRANSFORMS LIKE A TENSOR, SO IT IS A TENSOR.
(WE HAVE 2 INDICES, SO ITS AN ORDER 2 ONE.)

2014P2Q6(II)

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(iii) TRANSFORMATION LAW FOR AN AXIAL VECTOR: $a'_i = \det(L) L_{ij} a_j$

is this the question?

OBEYS THE SAME TRANSFORMATION LAW AS A VECTOR IF: $\det(L) = 1$ ✓

$$(\nabla \times u)'_i = \underbrace{(\epsilon_{ijk})'}_{\epsilon \text{ IS A PSEUDOTENSOR, THATS WHY IT TRANSFORMS LIKE THIS.}} \underbrace{(\partial_j x_k)}_{} =$$

$$= \det(L) L_{ip} L_{jq} L_{kr} \epsilon_{pqr} L_{j\alpha} L_{k\beta} \partial_\alpha x_\beta =$$

$$= \det(L) L_{ip} \underbrace{L_{jq} L_{j\alpha}}_{} \underbrace{L_{kr} L_{k\beta}}_{} \epsilon_{pqr} \partial_\alpha x_\beta =$$

SINCE L IS ORTHOGONAL

$$= \det(L) L_{ip} \delta_{q\alpha} \delta_{r\beta} \epsilon_{pqr} \partial_\alpha x_\beta =$$

$$= \det(L) L_{ip} \epsilon_{pqr} \partial_q x_r =$$

✓ good

$$= \det(L) L_{ip} (\nabla \times u)_p$$

$\nabla \times u$ TRANSFORMS AS AN AXIAL-VECTOR, SO IT CONSTITUTES AN AXIAL-VECTOR (FIELD).

(iv) $$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{SYMMETRIC SECOND ORDER TENSOR}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\text{ANTISYMMETRIC SECOND ORDER TENSOR}}$$ ✓

$$\epsilon_{ijk} w_k = \begin{pmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & 0 \\ -3 & 0 & 0 \end{pmatrix}$$

THIS IS THE GENERAL FORM OF ANTISYMMETRIC SECOND ORDER TENSORS, SO ANY OF SUCH CAN BE EQUAL TO IT.

2016P2Q6(III)

$$\epsilon_{ijk}\omega_k = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \underbrace{\hat{S}_{ij}}_{\text{SYMMETRIC ORDER 2 TENSOR.}}$$

✓

$$\hat{S}_{ij} = p\,\delta_{ij} + d_{ij}$$

$$p = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_i} + \frac{\partial u_j}{\partial x_j}\right) = \frac{\partial u_i}{\partial x_i}$$

are you sure?

$$d_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{\partial u_i}{\partial x_i}$$

$$\omega_k = \frac{1}{\epsilon_{ijk}}\,\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

(iv) $u_i = (a x_2, b x_1, 0)$

$$\cancel{p = \frac{\partial u_i}{\partial x_i} = \frac{\partial}{\partial x_1} a x_2 + \frac{\partial}{\partial x_2} b x_1 + \frac{\partial}{\partial x_3} 0 = 0}$$

$$\omega_k = \frac{1}{\underline{\epsilon_{ijk}}}\,\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$??

IF $k=1$:

$$\omega_1 = \frac{1}{\varepsilon_{231}}\,\frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial u_2}\right) = +\frac{1}{\epsilon_{321}}\,\frac{1}{2}\left(\frac{\partial u_3}{\partial u_2} - \frac{\partial u_2}{\partial u_3}\right) =$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x_3}(b x_1) - \frac{\partial}{\partial x_2} 0\right) = 0$$ not the right method

2014P2Q6 (IV)

FOR $k=2$:

$$\omega_2 = \frac{1}{\varepsilon_{132}} \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) = +\frac{1}{\varepsilon_{312}} \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) =$$

$$= +\frac{1}{2}\left(\frac{\partial}{\partial x_1} 0 - \frac{\partial}{\partial x_3} a x_2\right) = 0$$

FOR $k=3$

$$\omega_3 = \frac{1}{\varepsilon_{123}} \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_2}(a x_2) - \frac{\partial}{\partial x_1}(b x_1)\right) =$$

$$= \frac{1}{2}(a-b)$$

SO WE HAVE: $\omega_k = \delta_{k3} \frac{1}{2}(a-b)$

$$\frac{\partial u_i}{\partial x_j} = \begin{pmatrix} \frac{\partial a x_2}{\partial x_1} & \frac{\partial a x_2}{\partial x_2} & \frac{\partial a x_2}{\partial x_3} \\ \frac{\partial b x_1}{\partial x_1} & \frac{\partial b x_1}{\partial x_2} & \frac{\partial b x_1}{\partial x_3} \\ \frac{\partial 0}{\partial x_1} & \frac{\partial 0}{\partial x_2} & \frac{\partial 0}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

well yes. directly.

$$\hat{S}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \begin{pmatrix} 0 & \frac{a+b}{2} & 0 \\ \frac{a+b}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

✓ THIS IS ALREADY TRACELESS, SO: $\hat{S}_{ij} = \Delta_{ij}$

$$\left.\begin{array}{l} \mathrm{Tr}\,\hat{S} = 0 \Rightarrow \sum_{i=1}^{3} \lambda_i = 0 \\ \det \hat{S} = 0 \Rightarrow \prod_{i=1}^{3} \lambda_i = 0 \end{array}\right\} \longrightarrow \begin{array}{l} \lambda_1 = 0 \\ \lambda_2 = -\lambda_3 \end{array}$$

✓ ✓ good.

$$\cancel{\begin{pmatrix} 0 & \frac{a+b}{2} & 0 \\ \frac{a+b}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{a+b}{2}\begin{pmatrix} y \\ x \\ 0 \end{pmatrix} \Rightarrow x = y = \frac{1}{\sqrt{2}}\sqrt{\frac{2}{a+b}}}$$

2014P2Q6(V)

$$\begin{pmatrix} 0 & \frac{a+b}{2} & 0 \\ \frac{a+b}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda_2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{a+b}{2} y \\ \frac{a+b}{2} x \\ 0 \end{pmatrix}$$

too quick

$$\left.\begin{aligned} \lambda_2 x &= \frac{a+b}{2} y \\ \lambda_2 y &= \frac{a+b}{2} x \\ x^2+y^2 &= 1 \end{aligned}\right\} \quad \lambda_2 x = \left(\frac{a+b}{2}\right)^2 \frac{x}{\lambda_2}$$

$$\lambda_2^2 = \left(\frac{a+b}{2}\right)^2$$

$$e_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \leftarrow \lambda_2 = \frac{a+b}{2}$$

$$\lambda_3 = -\frac{a+b}{2}$$

$$e_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\lambda_1 = 0$$

$$e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

PRINCIPAL VALUES

PRINCIPAL AXES

Very good ideas but too inefficient.