2011P1Q4(I)

IFT: $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(k)e^{ikx}\,dk$

AUTOCORRELATION: $h(x)=f(x)\otimes f(x)=\int_{-\infty}^{\infty}f(y)^{*}f(x+y)\,dy$

$$\tilde{h}(k)=FT[h(x)]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(y)^{*}f(x+y)\,dy\,e^{-ikx}\,dx=$$

CHANGE OF VARIABLE: $z=x+y \longrightarrow \begin{matrix} dz=dx \\ x=z-y \end{matrix}$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(y)^{*}f(z)\,e^{-ikz}\,e^{iky}\,dy\,dz=$$

$$=\int_{-\infty}^{\infty}f(y)^{*}e^{iky}\,dy\int_{-\infty}^{\infty}f(z)e^{-ikz}\,dz=\tilde{f}(k)^{*}f(k)=|f(k)|^{2}$$

$$\tilde{f}(k)=\int_{-\infty}^{\infty}f(x)e^{-ikx}\,dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos x)\,e^{-ikx}\,dx=\underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos x)(\cos(kx))\,dx}=$$

$$=\left[\sin x\cos kx\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin x(-k)\sin(kx)\,dx=$$

$$=2\cos\frac{k\pi}{2}+k\Big(\underbrace{\left[-\cos x\sin(kx)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}}_{\to 0}-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}-\cos(x)\,k\cos(kx)\,dx=$$

$$=2\cos\frac{k\pi}{2}+k^{2}\underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(x)\cos(kx)\,dx}$$

$$\tilde{f}(k)=2\cos\frac{k\pi}{2}+k^{2}\tilde{f}(k)$$

$$\to \tilde{f}(k)=\frac{2\cos\frac{k\pi}{2}}{1-k^{2}}$$

$$\tilde{f}(k)=\frac{2\cos\frac{k\pi}{2}}{1-k^{2}}\Longrightarrow |f(k)|^{2}=\frac{4\cos^{2}\left(\frac{k\pi}{2}\right)}{(1-k^{2})^{2}}$$

$$\int_{-\infty}^{\infty}|f(x)|^{2}\,dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^{2}x\,dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4\cos^{2}\left(\frac{k\pi}{2}\right)}{(1-k^{2})^{2}}\,dk=$$

SUBSTITUTION: $u=\frac{k\pi}{2}$ $\quad \frac{2}{\pi}u=k$

$du=\frac{\pi}{2}dk$

2011P1Q4(II)

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{2\cos^2(u)}{\left(1-\left(\frac{2}{\pi}u\right)^2\right)^2} \frac{2}{\pi}du = \frac{2}{\pi^2}\int_{-\infty}^{\infty} \frac{2\cos^2 u \left(\frac{\pi}{2}\right)^4 du}{\left(1-\left(\frac{2}{\pi}u\right)^2\right)^2\left(\frac{\pi}{2}\right)^4} =$$

$$= \int_{-\infty}^{\infty} \frac{\cos^2 u \frac{\pi^2}{4}}{\left(\frac{\pi^2}{4}-u^2\right)^2} du = \frac{\pi}{2}^2 \int_0^{\infty} \frac{\cos^2 u}{\left(\frac{\pi^2}{4}-u^2\right)^2} du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x\, dx =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1+\cos(2x))dx = \left[\frac{1}{2}x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left[\frac{1}{4}\sin(2x)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$

$$= \frac{\pi}{2} + 0$$

CHANGE OF VAR: $u \to t$

$$\frac{\pi}{2}^2 \int_0^{\infty} \frac{\cos^2 t}{\left(\frac{\pi^2}{4}-t^2\right)^2} dt = \frac{\pi}{2}$$

$$\int_0^{\infty} \frac{\cos^2 t}{\left(\frac{\pi^2}{4}-t^2\right)^2} dt = \frac{1}{\pi}$$