2015P2Q6 (I) (a)

$x_i' = M_{ij} x_j$

$$\begin{pmatrix} \underline{e}_1' \\ \underline{e}_2' \\ \underline{e}_3' \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \underline{e}_3 \end{pmatrix}$$

WHERE: $M_{ij} = \underline{e}_i' \cdot \underline{e}_j$

(IS THIS A "RESTRICTION" THOUGH?)

M is orthogonal

if $\det M < 0$: M IS LEFT HANDED.

if $\det M > 0$: M IS RIGHT HANDED.

$T'_{ij} = M_{ip} M_{jq} T_{pq}$

(b)

$$T = \tfrac{1}{2}(T + T^T) + \tfrac{1}{2}(T - T^T) = S + A$$ ✓

$$M_{ip} M_{jq} S_{pq} = M_{ip} M_{jq} \tfrac{1}{2}\left(T_{pq} + (T^T)_{pq}\right) =$$

$$= M_{ip} M_{jq} \tfrac{1}{2}(T_{pq} + T_{qp}) = \tfrac{1}{2}(T'_{ij} + T'_{ji}) = \tfrac{1}{2}(T + T^T)'_{ij} =$$ ✓

$= S'_{ij}$ WE FIND: $M_{ip} M_{jq} S_{pq} = S'_{ij}$ ✓

$\Rightarrow$ S IS A SECOND ORDER TENSOR.

$$M_{ip} M_{jq} A_{pq} = M_{ip} M_{jq} \tfrac{1}{2}\left(T_{pq} - (T^T)_{pq}\right) =$$

$$= M_{ip} M_{jq} \tfrac{1}{2}(T_{pq} - T_{qp}) = \tfrac{1}{2}(T'_{ij} - T'_{ji}) = \tfrac{1}{2}(T - T^T)'_{ij} =$$ ✓

$= A'_{ij}$ ✓

$M_{ip} M_{jq} A_{pq} = A'_{ij} \Rightarrow$ A IS A SECOND ORDER TENSOR

2015P2Q6 (II)

(c)

$$S = \frac{1}{2}(F + F^T) = \frac{1}{2}\left[\begin{pmatrix} x_1^2 & -x_1^2 + x_1x_2 - x_2^2 & x_1 - x_2 \\ x_1^2 + x_1x_2 + x_2^2 & x_2^2 & -x_1 - x_2 \\ -x_1 + x_2 & x_1 + x_2 & 3(x_1^2 + x_2^2) \end{pmatrix} + \right.$$

$$\left. + \begin{pmatrix} x_1^2 & x_1^2 + x_1x_2 + x_2^2 & -x_1 + x_2 \\ -x_1^2 + x_1x_2 - x_2^2 & x_2^2 & x_1 + x_2 \\ x_1 - x_2 & -x_1 - x_2 & 3(x_1^2 + x_2^2) \end{pmatrix}\right] =$$

$$= \begin{pmatrix} x_1^2 & x_1x_2 & 0 \\ x_1x_2 & x_2^2 & 0 \\ 0 & 0 & 3(x_1^2 + x_2^2) \end{pmatrix} = S$$ (SYMMETRIC PART OF F THIS IS) ✓

PRINCIPAL AXES & VALUES:

$$\underline{\underline{S}}\begin{pmatrix}0\\0\\1\end{pmatrix} = 3(x_1^2 + x_2^2)\begin{pmatrix}0\\0\\1\end{pmatrix} \Rightarrow \underline{e}_1 = \begin{pmatrix}0\\0\\1\end{pmatrix} \checkmark \quad \lambda_1 = 3(x_1^2 + x_2^2)$$

good

$$\det S = 3(x_1^2 + x_2^2)(x_1^2 x_2^2 - x_1x_2x_1x_2) = 0$$

$$\mathrm{Tr}\, S = 4(x_1^2 + x_2^2) \Rightarrow \sum_{i=1}^{3} \lambda_i = 4(x_1^2 + x_2^2)$$

$$\left.\begin{aligned} &\prod_{i=1}^{3} = 0 \\ &\lambda_1 = 3(x_1^2 + x_2^2) \end{aligned}\right\} \Rightarrow \lambda_2 = 0 \Rightarrow \lambda_3 = x_1^2 + x_2^2$$

✓ ✓

$$S e_2 = \begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix} x_1^2 & x_1x_2 & 0 \\ x_1x_2 & x_2^2 & 0 \\ 0 & 0 & 3(x_1^2 + x_2^2) \end{pmatrix}\begin{pmatrix}p\\q\\r\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow r = 0$$

$\boxed{\cancel{\Rightarrow r \neq 0}}$ $\qquad x_1^2 p + x_1x_2 q = 0 \quad / : x_1$

$$x_1 p + x_2 q = 0$$

$\begin{pmatrix}x_2\\-x_1\\0\end{pmatrix}$ is nicer.

$$e_2 = \begin{pmatrix} -\frac{x_2}{x_1} \\ 1 \\ 0 \end{pmatrix} \Leftarrow p = -\frac{x_2}{x_1} q$$

2015 P2Q6(III)

$$S e_3 = \lambda_3 e_3$$

$$\begin{pmatrix} x_1^2 & x_1x_2 & 0 \\ x_1x_2 & x_2^2 & 0 \\ 0 & 0 & 3(x_1^2+x_2^2) \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = (x_1^2+x_2^2)\begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

use $A-\lambda I$ directly.

$$3(x_1^2+x_2^2)r = (x_1^2+x_2^2)r$$
$$\Rightarrow r = 0$$

$$x_1^2 p + x_1x_2 q = (x_1^2+x_2^2)p \qquad | -x_1^2 p$$

$$x_1x_2 q = x_2^2 p$$

$$\frac{x_1}{x_2} q = p \longrightarrow e_3 = \begin{pmatrix} \frac{x_1}{x_2} \\ 1 \\ 0 \end{pmatrix} \checkmark \quad \propto \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$$

WE FOUND:

$$\lambda_1 = 3(x_1^2+x_2^2) \qquad \lambda_2 = 0 \qquad \lambda_3 = x_1^2+x_2^2$$

$$e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad e_2 = \begin{pmatrix} -\frac{x_2}{x_1} \\ 1 \\ 0 \end{pmatrix} \qquad e_3 = \begin{pmatrix} \frac{x_1}{x_2} \\ 1 \\ 0 \end{pmatrix}$$

GEOMETRICAL DESCRIPTION OF THE PRINCIPAL AXES:

e_1 HAS A CONSTANT COMPONENT IN THE x_3 DIRECTION AND HAS NO COMPONENT IN OTHER DIRECTIONS THROUGHOUT THE WHOLE SPACE.

$e_1 \cdot e_2 = e_1 \cdot e_3 = 0 \Rightarrow e_1$ IS PERPENDICULAR TO BOTH e_2 & e_3.

$e_2 \cdot e_3 = 0 \Rightarrow e_2$ IS ORTHOGONAL TO e_3 THROUGHOUT THE WHOLE SPACE. e_2 & e_3 HAVE A COMPONENT (CONSTANT LENGTH) IN THE x_3 DIRECTION. waffle

~~THE COMPONENT OF~~ e_2 & e_3 ~~IN THE x_1 DIRECTION~~ ~~MAGNITUDE OF THE~~

~~GOES TO ∞ AS WE APPROA~~

PTO.

~~THE x~~ e_2 & e_3 HAVE A VARIABLE LENGTH COMPONENT IN THE x_1 DIRECTION, ITS LENGTH IS DEPENDENT ON WHERE WE ARE IN SPACE. SEE e_2 & e_3 DEFINITION ABOVE.

This matrix is orthogonal
some theory is missing in your
mind.

$$\begin{pmatrix} \boxed{\begin{matrix} x_1\alpha & -x_2\alpha \\ x_2\alpha & x_1\alpha \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} \qquad \alpha = \frac{1}{\sqrt{x_1^2 + x_2^2}}$$

the evectors need to be
normalised.

Then it is a rotation about the z axis
of the form $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

This tensor question
also uses some skills from
Michaelmas matrices which you
do not seem to know.

2015P2Q6 IV

TRANSFORMATION ~~MATR~~ MATRIX M_{ij}?

$$\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}-\frac{x_2}{x_1}\\1\\0\end{pmatrix}\begin{pmatrix}\frac{x_1}{x_2}\\1\\0\end{pmatrix}\right)=\begin{pmatrix}M_{11}&M_{12}&M_{13}\\M_{21}&M_{22}&M_{23}\\M_{31}&M_{32}&M_{33}\end{pmatrix}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$$

order

SO IS M JUST $\begin{pmatrix}0&-\frac{x_2}{x_1}&\frac{x_1}{x_2}\\0&1&1\\1&0&0\end{pmatrix}$?

PROBABLY NOT. I AM LIKELY TO BE MISTAKEN HERE

(d) $F_{ij}=P\delta_{ij}+\hat{S}_{ij}+\hat{A}_{ij}$

$F_{ij}=S_{ij}+\hat{A}_{ij}$ $\quad \hat{A}_{ij}=\frac{1}{2}(F-F^T)_{ij}$

$$\hat{S}=S-\frac{\mathrm{Tr}\,S}{3}I=\begin{pmatrix}x_1^2&x_1x_2&0\\x_1x_2&x_2^2&0\\0&0&3(x_1^2+x_2^2)\end{pmatrix}-\frac{4(x_1^2+x_2^2)}{3}I=$$

$$=\begin{pmatrix}-\frac{1}{3}x_1^2-\frac{4}{3}x_2^2&x_1x_2&0\\x_1x_2&-\frac{4}{3}x_1^2-\frac{1}{3}x_2^2&0\\0&0&\frac{5}{3}(x_1^2+x_2^2)\end{pmatrix}=\hat{S}\ ✓$$

$$P=\frac{4(x_1^2+x_2^2)}{3}\qquad \hat{A}_{ij}=\frac{1}{2}(F_{ij}-F_{ji})=\begin{pmatrix}0&-x_1^2-x_2^2&x_1-x_2\\x_1^2+x_2^2&0&-x_1-x_2\\-x_1+x_2&x_1+x_2&0\end{pmatrix}$$

CHECKING IF $\hat{S}$ HAS THE SAME PRINCIP. AXES AS FOUND PREVIOUSLY:

$$\hat{S}\begin{pmatrix}0\\0\\1\end{pmatrix}=\frac{5}{3}(x_1^2+x_2^2)\begin{pmatrix}0\\0\\1\end{pmatrix}\ ✓$$

discussed in [illegible]

$$\hat{S}\begin{pmatrix}-\frac{x_2}{x_1}\\1\\0\end{pmatrix}=\begin{pmatrix}\left(-\frac{1}{3}x_1^2-\frac{4}{3}x_2^2\right)\left(-\frac{x_2}{x_1}\right)+x_1x_2\\x_1x_2\cdot\left(-\frac{x_2}{x_1}\right)+\left(-\frac{4}{3}x_1^2-\frac{1}{3}x_2^2\right)\\0\end{pmatrix}\neq\begin{pmatrix}-\frac{x_1}{x_1}\\1\\0\end{pmatrix}$$

$\Rightarrow$ SO THE ANSWER IS NO (PROBABLY THE ANSWER IS YES AND IM WRONG SOMEWHERE)