2014P1Q4(I)

(i) $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk$

(ii) $\tilde{g}(k) = \int_{-\infty}^{\infty} x^n f(x) e^{-ikx} dx = i^n \int_{-\infty}^{\infty} (-ix)^n f(x) e^{-ikx} dx =$

$= i^n \int_{-\infty}^{\infty} \frac{d^n}{dk^n}\left[f(x) e^{-ikx}\right] dx = i^n \frac{d^n}{dk^n} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx =$

$= i^n \frac{d^n}{dk^n} \tilde{f}(k)$

(iii) $\tilde{f}(k) = i \frac{d}{dk} \widetilde{e^{-x^2}}$

$\widetilde{e^{-x^2}} = i \frac{d}{dk} \int_{-\infty}^{\infty} e^{-x^2} e^{-ikx} dx =$

$z = x + \frac{ik}{2} \longrightarrow dz = dx$

$= i \frac{d}{dk} \int_{-\infty}^{\infty} e^{-z^2} e^{-\frac{k^2}{4}} dz =$

$z^2 = x^2 + ikx - \frac{k^2}{4}$

$= i \frac{d}{dk}\left(\sqrt{\pi}\, e^{-\frac{k^2}{4}}\right) =$

$= \sqrt{\pi}\, i \frac{-1}{2} k e^{-\frac{k^2}{4}} = -\frac{1}{2}\sqrt{\pi}\, i e^{-\frac{k^2}{4}}$

(iv) CORRELATION OF f & g: $h = f \otimes g$

$h(x) = \int_{-\infty}^{\infty} f(y)^* g(x+y) dy$

$\tilde{h}(k) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)^* g(x+y) dy\, e^{-ikx} dx$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^*(y) g(x+y) e^{-ikx} dy\, dx =$

$z = x+y$

$x = z - y$

2014P1Q4(II)

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^*(y) g(z) e^{-ikz} e^{iky} dy\, dz =$$

$$= \int_{-\infty}^{\infty} f(y)^* e^{iky} dy \int_{-\infty}^{\infty} g(z) e^{-ikz} dz =$$

$$= \left(\int_{-\infty}^{\infty} f(y) e^{-iky} dy\right)^* \int_{-\infty}^{\infty} g(z) e^{-ikz} dz =$$

$$= \tilde{f}(k)^* \tilde{g}(k)$$

$$h(x) = \mathrm{IFT}[\tilde{h}(k)]$$

$$\int_{-\infty}^{\infty} f(y)^* g(x+y) dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)^* \tilde{g}(k) e^{iky} dk$$

SET $y=0$, RELABEL: $y \to x$

$$\int_{-\infty}^{\infty} f(x)^* g(x) dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)^* \tilde{g}(k) dk$$

(v)

$$E = \int_{-\infty}^{\infty} |x e^{-x^2}|^2 dx = \int_{-\infty}^{\infty} x^2 e^{-2x^2} dx =$$

$$= \int_{-\infty}^{\infty} -4x\left(-\frac{1}{4}\right) x e^{-2x^2} dx = \left[\cancel{e^{-2x^2}\frac{-1}{4}x}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-2x^2}\frac{-1}{4} dx =$$

$\to 0$

$$= \frac{1}{4}\int_{-\infty}^{\infty} e^{-2x^2} dx = \frac{1}{4}\int_{-\infty}^{\infty} e^{-z^2}\frac{dz}{\sqrt{2}} = \frac{1}{4}\sqrt{\frac{\pi}{2}}$$

$z = \sqrt{2}x$

$dz = \sqrt{2}dx$

~~THIS SHOULD BE EQUAL TO:~~

$$\cancel{\frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{f}(k)|^2 dk =}$$

$$\cancel{= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2}\sqrt{\pi}\exp\left(-\frac{k^2}{2}\right) dk = \frac{1}{4}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \exp(-z^2)\sqrt{2}\, dz =}$$

$$\cancel{= \frac{1}{4}\sqrt{2}}$$

$z = \frac{k}{\sqrt{2}}$

$dz = \frac{1}{\sqrt{2}}dk$

2014 P1 Q4 (III)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\tilde{f}(k)\right|^2 dk = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{2}\sqrt{\pi}\; e^{-\frac{k^2}{4}}\right)^2 dk =$$

$$= \frac{1}{8}\int_{-\infty}^{\infty} e^{-\frac{k^2}{2}}\, dk = \frac{1}{8}\int_{-\infty}^{\infty} e^{-z^2}\sqrt{2}\, dz = \frac{\sqrt{2}\sqrt{\pi}}{8} =$$

$$z = \frac{k}{\sqrt{2}}$$

$$dz = \frac{1}{\sqrt{2}}\, dk$$

$$= \frac{\sqrt{2}}{2}\,\frac{\sqrt{\pi}}{4} = \frac{1}{4}\sqrt{\frac{\pi}{2}}$$

YES, THERE IS CONSISTENCY WITH PARSEVAL'S THEOREM.