2016 P2 Q8(I)

(a) G/N: SET OF COSETS OF N IN G, IE SET OF COSETS Ng, $g \in G$.

SHOW THAT IT IS INDEED A GROUP?

N normal $gN = Ng \; \forall g \in G$. L=R cosets.

LAGRANGE: ORDER OF GROUP IS DIVISIBLE BY ORDER OF ALL SUBGROUPS.

Partition G into cosets. Each elt in exactly one coset.

Show cosets incl. id, inv. + 2 cosets gives another ie closed.

(b)

$$\sigma_x \sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \sigma_z \cdot i$$

✓

$$\sigma_z \cdot i \notin \{\sigma_x, \sigma_y, \sigma_z, I\}$$

$\Rightarrow$ SET NOT CLOSED, SO NOT A GROUP FOR SURE. ✓

$$\sigma_y \sigma_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i \cdot \sigma_z$$

$$\sigma_x \sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i \cdot \sigma_y$$

$$\sigma_z \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i \cdot \sigma_y$$

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$$\sigma_x \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\sigma_y \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -I$$

✓

$$\sigma_z \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = I$$

$I \in G \Rightarrow$ PRESENCE OF IDENTITY REQUIREMENT SATISFIED.

2016 P2 Q 8(II)

IN GENERAL:

$q_1, q_2 \in \{0, 1, 2, 3\}$

better to do one explicit case + then generalise.

$$(i)^{q_1}\sigma_i \cdot (i)^{q_2}\sigma_j = i\,\epsilon_{ijk}\,\sigma_k\,(i)^{q_1+q_2} \pm$$

$$\pm\,\delta_{ij}\,I\,(i)^{q_1+q_2} = \tau$$

I'm not sure about your signs here.

$(+ \text{ IF } i, j \neq y)$

$\forall\, \sigma, \tau \in G \Rightarrow$ GROUP IS CLOSED UNDER MULTIPLICATION

$\sigma_x^{-1} = \sigma_x,\ \sigma_y^{-1} = -\sigma_y,\ \sigma_z^{-1} = \sigma_z$ ✓ $\quad I^{-1} = I$

$(-\sigma_x)^{-1} = -\sigma_x,\ (-\sigma_y)^{-1} = \sigma_y,\ (-\sigma_z)^{-1} = -\sigma_z \quad (iI)^{-1} = -iI$

$\quad (-iI)^{-1} = iI$

$(i\sigma_x)^{-1} = -i\sigma_x,\ (i\sigma_y)^{-1} = i\sigma_y,\ (i\sigma_z)^{-1} = -i\sigma_z \quad (-I)^{-1} = -I$

$(-i\sigma_x)^{-1} = i\sigma_x,\ (-i\sigma_y)^{-1} = -i\sigma_y,\ (-i\sigma_z)^{-1} = i\sigma_z$

$\Rightarrow$ INVERSE PRESENT FOR EVERY $g \in G$

PRESENCE OF IDENTITY, CLOSEDNESS (closure), PRES. OF INVERSES

$\Longrightarrow$ GROUP CONDITIONS SATISFIED.

associativity?? — IS THIS NEEDED HERE?

(C) $g^{|g|} = I,\ (g^{-1})^{|g^{-1}|} = I \Rightarrow g^{|g|}(g^{-1})^{|g^{-1}|} = II = I$

LET $|g| - |g^{-1}| = a$

$$g^{|g|}(g^{-1})^{|g^{-1}|} = g^{a+|g^{-1}|}(g^{-1})^{|g^{-1}|} = g^a\, g^{|g^{-1}|}(g^{-1})^{|g^{-1}|} =$$

$$= g^a\left(g\, g^{-1}\right)^{|g^{-1}|} = g^a\, I^{|g^{-1}|} = I$$

You are missing the idea that the order is the least $|g|$ st $g^{|g|} = e$.

$\Rightarrow a = 0 \Rightarrow |g| = |g^{-1}|$

eg. C_{12}. $g^{24} = e$. but $g^{12} = e$.