$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

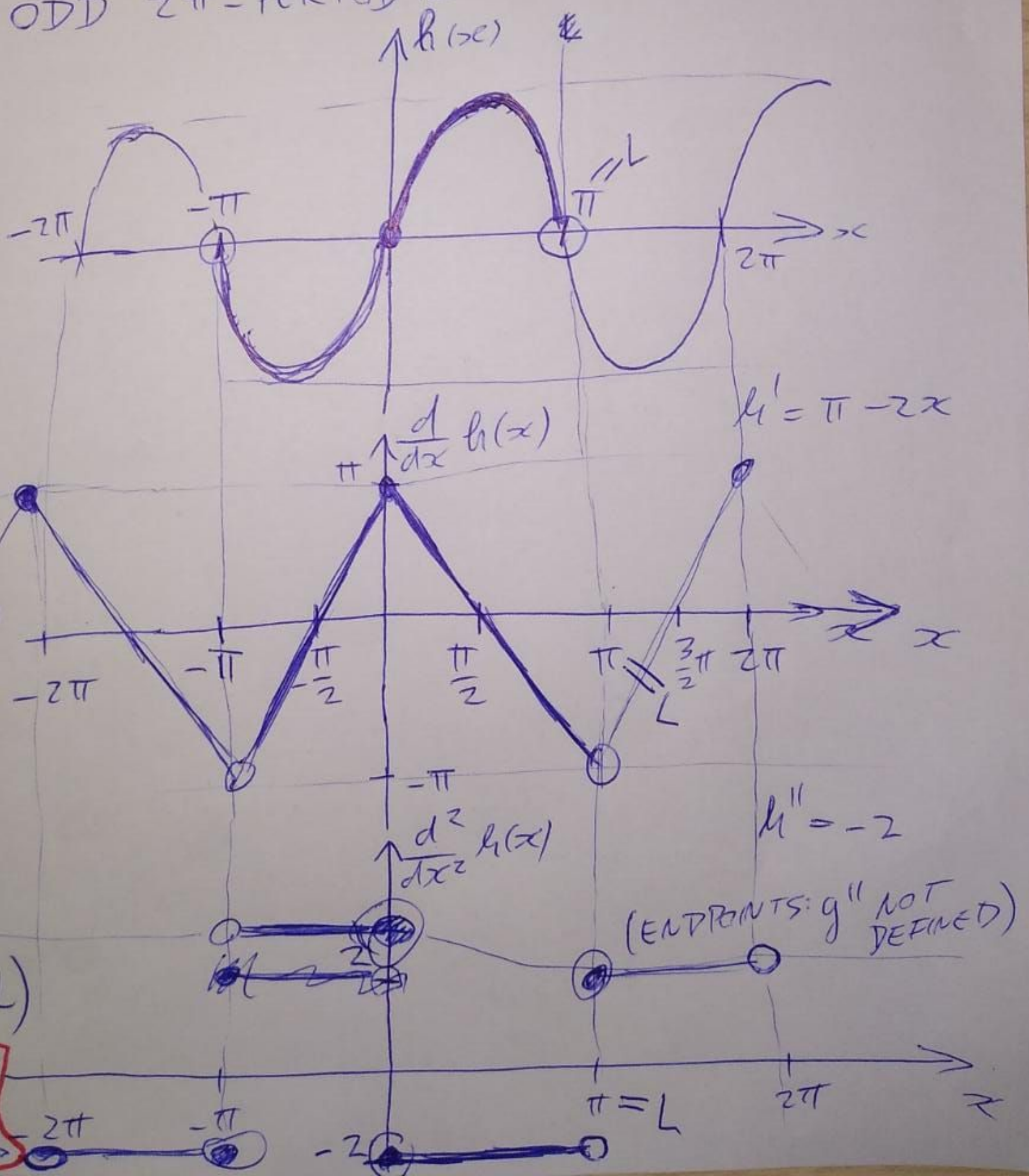
$$= \sum_{n=-\infty}^{\infty} c_n e^{\frac{i n \pi x}{L}}$$

$$c_n = \frac{1}{L} \int_{-L}^{L} f(x) e^{\frac{i n \pi x}{L}} dx$$

EXAMPLE: function $h(x) = x(\pi - x)$

defined on the ~~integral~~ interval: $0 \leq x < \pi$

ODD 2π-PERIODIC CONTINUATION:



$h(L) = h(-L)$

$h'(L) = h'(-L)$

$h''(L) \neq h''(-L)$

$\boxed{\text{THE TERM WITH } n^{-3} \text{ IS THE FIRST NONZERO}}$

FUNCTION IS ODD & AVERAGES AT 0 $\Longrightarrow$ ONLY SIN TERMS NEEDED.

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

LET TERMS NOT RESULTING IN AN EXPRESSION WITH n^{-3} DEPENDENCE BE DENOTED BY EMPTY BRACKETS: [ ]

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \text{ODD CONT. OF } h(x)\ \sin\left(\frac{n\pi x}{\pi}\right)dx$$

!NOT $h(x)$ ITSELF!

$$= 2\cdot\frac{1}{\pi}\int_0^\pi h(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^\pi x(\pi-x)\sin(nx)\,dx$$

$$= \cancel{\frac{2}{\pi}\int_0^\pi x\pi\sin(nx)\,dx} \to [\ ] - \frac{2}{\pi}\int_0^\pi x^2\sin(nx)\,dx$$

$$= [\ \ ] + \frac{2}{\pi}\int_0^\pi \frac{1}{n}(\cos(nx))\,2x\,dx$$

$$= [\ ] + \frac{2}{\pi}\left([\ ] - \int_0^\pi \frac{1}{n^2}(-\sin(nx))\,2\,dx\right) = \frac{4}{\pi n^2}\int_0^\pi \sin(nx)\,dx$$

$$= \frac{4}{\pi n^3}\Big[-\cos(nx)\Big]_0^\pi = \frac{-4}{\pi n^3}\left((-1)^n-1\right)$$

SO WE HAVE: ODD EXT OF $h(x)$ $= \sum_{n=1}^{\infty} \frac{-4}{\pi n^3}\left((-1)^n-1\right)\sin(nx)$

$$= \sum_{n=1}^{\infty} \frac{\cancel{4}+8}{\pi(2n+1)^3}\sin((2n+1)x) = x(\pi-x)$$

IF $x = \frac{\pi}{2}$:

$$\sum_{n=1}^{\infty} \frac{+8}{\pi(2n+1)^3}(-1)^n = \frac{\pi^2}{4}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32} \Longleftarrow \sum_{n=1}^{\infty} \frac{32}{(2n+1)^3}(-1)^n = \pi^3$$

SOL W/ REAL FOURIER COEFFS.

LET g = ODD EXT. OF $h(x)$

$$g(x) = \sum_{-\infty}^{\infty} c_n \exp\left(\frac{i\pi x}{L} n\right)$$

!

$$c_n = \frac{1}{2L}\int_{-L}^{L} g(x)\, e^{-\frac{ix\pi}{L} n}\, dx$$

!

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g\, e^{-ixn}\, dx = \frac{1}{2\pi}\left(\left[\;\right] - \int_{-\pi}^{\pi} \frac{1}{-in} e^{-ixn} g'\, dx\right)$$

SOL. W/ COMPLEX FOURIER COEFFS.

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{(-in)^2} e^{-ixn} g''\, dx =$$

$$\cancel{= \frac{1}{2\pi}\, 2\int_{0}^{\pi} \frac{1}{(-in)^2} e}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{(-in)^2}\Big(\underbrace{\cos(-xn)}_{\text{EVEN}} + i\underbrace{\sin(-xn)}_{\text{ODD}}\Big)\underbrace{g''}_{\text{ODD}}\, dx$$

$$= \frac{1}{2\pi}\, 2\int_{0}^{\pi} \frac{i}{-n^2} \sin(-xn)\,(-2)\, dx$$

$$= \cancel{\frac{i}{2\pi n^3}} = -\frac{2}{\pi}\frac{i}{n^3}\Big[-\cos(-xn)\Big]_0^{\pi} = \frac{2}{\pi}\frac{i}{n^3}\left((-1)^n - 1\right)$$

$$b_n = i\left(c_n - c_{-n}\right) = \frac{-2}{\pi}\left(\frac{1}{n^3}\left((-1)^n - 1\right) - \frac{1}{(-n)^3}\left((-1)^{-n} - 1\right)\right) =$$

$$= -\frac{4}{\pi n^3}\left((-1)^n - 1\right)$$

AS FOUND PREVIOUSLY.