2012P205 (I)

$$\tilde{h}(k)=\tilde{f}(k)\tilde{g}(k)=\int_{-\infty}^{\infty} f(y)e^{-iky}dy\int_{-\infty}^{\infty} g(z)e^{-ikz}dz=$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)\,g(z)\,e^{-iky}\,e^{-ikz}\,dy\,dz=$$

$$z=x-y \rightarrow dz=dx$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)\,g(x-y)\,\cancel{e^{-iky}}\,e^{-ikx}\,\cancel{e^{iky}}\,dy\,dx=$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)\,g(x-y)\,dy\,e^{-ikx}\,dx=$$

$$=\int_{-\infty}^{\infty} h(x)e^{-ikx}dx$$

$$\Rightarrow h(x)=\int_{-\infty}^{\infty} f(y)\,g(x-y)\,dy$$

$h(x)=$

$$=f(x)*f(x)=e^{-|x|}*e^{-|x|}=\int_{-\infty}^{\infty} e^{-|y|}\,e^{-|x-y|}\,dy=$$

$$=\begin{cases}\text{IF } x<0 & \int_{-\infty}^{x} e^{y}\,e^{y-x}\,dy+\int_{x}^{0} e^{y}\,e^{-(y-x)}\,dy+\int_{0}^{\infty} e^{-y}\,e^{-(y-x)}\,dy\\ \text{IF } x>0 & \int_{-\infty}^{0} e^{y}\,e^{-(x-y)}\,dy+\int_{0}^{x} e^{-y}\,e^{-(x-y)}\,dy+\int_{x}^{\infty} e^{-y}\,e^{-(y-x)}\,dy\end{cases}=$$

$$=\begin{cases}x<0 & e^{-x}\left[\frac{1}{2}e^{2y}\right]_{-\infty}^{x}+e^{x}\left[y\right]_{x}^{0}+e^{+x}\left[-\frac{1}{2}e^{-2y}\right]_{0}^{\infty}=\\ x>0 & e^{-x}\left[+\frac{1}{2}e^{+2y}\right]_{-\infty}^{0}+e^{-x}\left[y\right]_{0}^{x}+e^{x}\left[-\frac{1}{2}e^{-2y}\right]_{x}^{\infty}=\end{cases}=$$

2018 P2Q5 (II)

$$=\begin{cases} x<0 \quad e^{-x}\frac{1}{2}e^{2x} - e^{x}x + \frac{1}{2}e^{x} = (1-x)e^{x} & \text{FOR } x<0 \\ x>0 \quad e^{-x}\frac{1}{2} + e^{-x}x + e^{x}\frac{1}{2}e^{-2x} = (1+x)e^{-x} & \text{FOR } x>0 \end{cases}$$

$$h(x) = f(x) * f(x) = \mathrm{IFT}\left[\tilde{h}(k)\right] = \mathrm{IFT}\left[\tilde{f}(k)^2\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k)^2 e^{ikx}\,dk$$

BY CONVOLUTION THEOREM

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\,dx = \int_{-\infty}^{\infty} e^{-|x|} e^{-ikx}\,dx = \int_{-\infty}^{0} e^{x} e^{-ikx}\,dx +$$

$$+ \int_{0}^{\infty} e^{-x} e^{-ikx}\,dx = \int_{-\infty}^{0} e^{(1-ik)x}\,dx + \int_{0}^{\infty} e^{-(1+ik)x}\,dx =$$

$$= \frac{1}{1-ik}\left[e^{(1-ik)x}\right]_{-\infty}^{0} + \frac{-1}{1+ik}\left[e^{-(1+ik)x}\right]_{0}^{\infty} = \frac{1}{1-ik} + \frac{-1}{1+ik} =$$

$$= \frac{1+ik}{(1-ik)(1+ik)} - \frac{1-ik}{(1-ik)(1+ik)} = \frac{2}{1+k^2}$$

PUT THIS BACK TO:

$$h(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k)^2 e^{ikx}\,dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{(1+k^2)^2} e^{ikx}\,dk =$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}}{(1+k^2)^2}\,dk$$ AS REQUIRED.