2012P1Q4(I)

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = \int_{-\infty}^{\infty} e^{-\lambda x^2} e^{-ikx} dx =$$

$$z = \sqrt{\lambda} x + \frac{ik}{\sqrt{\lambda}\, 2} \longrightarrow dz = \sqrt{\lambda}\, dx$$

$$z^2 = \lambda x^2 + ikx - \frac{k^2}{4\lambda}$$

$$-z^2 = -\lambda x^2 - ikx + \frac{k^2}{4\lambda}$$

$$= \int_{-\infty}^{\infty} e^{-z^2} e^{-\frac{k^2}{4\lambda}} \frac{dz}{\sqrt{\lambda}} = \frac{1}{\sqrt{\lambda}} e^{-\frac{k^2}{4\lambda}} \int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\frac{\pi}{\lambda}}\, e^{-\frac{k^2}{4\lambda}}$$ ✓

by hint.

LET'S HAVE: $g(x) = f'(x)$

for what purpose?

$$\tilde{g}(k) = \int_{-\infty}^{\infty} f'(x) e^{-ikx} dx =$$

$$= \left[f(x) e^{-ikx} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-ik) e^{-ikx} dx =$$ ✓

$$= ik \int_{-\infty}^{\infty} f(x) \exp(-ikx) dx = ik \tilde{f}(k)$$

✓

$h(x) = f''(x)$

Apply done twice

$$\tilde{h}(k) = ik \tilde{f'}(k) = (ik)^2 \tilde{f}(k) = -k^2 \tilde{f}(k)$$

$$\psi'' = \dot{\psi} \xrightarrow{\text{FT WRT } x} \boxed{-k^2 \tilde{\psi}(k,t)} = \frac{\partial \tilde{\psi}(k,t)}{\partial t}$$

$\cancel{\text{FT WRT } x}$

$\cancel{-k^2 \tilde{\psi} = \dot{\tilde{\psi}}}$

SOLUTION

$$\tilde{\psi}(k,t) = \tilde{\psi}(k, t=0)\, e^{-k^2 t}$$

$$\tilde{\psi}(k, t=0) = \int_{-\infty}^{\infty} \psi(x, t=0) e^{-ikx} dx = \int_{-\infty}^{\infty} e^{-\lambda x^2} e^{-ikx} dx =$$ ✓

$$= \sqrt{\frac{\pi}{\lambda}} \exp\left(-\frac{k^2}{4\lambda}\right)$$ (USING PREVIOUS FINDING(S) ? we only ?

✓

2012P1Q4(II)

$$\tilde{\psi}(k,t=0)=\int_{-\infty}^{\infty}\psi(x,t=0)e^{-ikx}dx=\int_{-\infty}^{\infty}e^{-\lambda x^2}e^{-ikx}dx=$$

$$=\sqrt{\frac{\pi}{\lambda}}\exp\left(-\frac{k^2}{4\lambda}\right)$$

$$\tilde{\psi}(k,t)=\sqrt{\frac{\pi}{\lambda}}\exp\left(-\frac{k^2}{4\lambda}\right)\exp\left(-k^2t\right)=\sqrt{\frac{\pi}{\lambda}}e^{-k^2\left(\frac{1}{4\lambda}+t\right)}$$ ✓

$$\psi(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\psi}(k,t)e^{ikx}dk=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{\frac{\pi}{\lambda}}e^{-k^2\left(\frac{1}{4\lambda}+t\right)}e^{ikx}dk=$$

WHAT WE ARE DOING HERE BACK TO REAL SPACE HAS BEEN DONE ONWARDS TO FOURIER SPACE WITH DIFFERENT PARAMETER

$$\frac{1}{4\lambda}+t=\beta$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{\frac{\pi}{\lambda}}e^{-\left(k\sqrt{\beta}-\frac{ix}{2\sqrt{\beta}}\right)^2}e^{-\frac{x^2}{4\beta}}dk=$$

$$=\frac{1}{2\pi}e^{-\frac{x^2}{4\beta}}\sqrt{\frac{\pi}{\lambda}}\int_{-\infty}^{\infty}e^{-\left(k\sqrt{\beta}-\frac{ix}{2\sqrt{\beta}}\right)^2}dk=\frac{1}{2\pi}e^{-\frac{x^2}{4\beta}}\sqrt{\frac{\pi}{\lambda}}\int e^{-\left(y-\frac{ix}{2\sqrt{\beta}}\right)^2}\frac{dy}{\sqrt{\beta}}=$$

$$y=k\sqrt{\beta} \qquad \lambda=+1$$
$$dy=\sqrt{\beta}\,dk$$

HMM, WOULD BE GOOD THOUGH I DIDN'T FIND AN EASIER ONE

$$=\frac{1}{2\pi}e^{-\frac{x^2}{4\beta}}\sqrt{\frac{\pi}{\lambda}}\frac{1}{\sqrt{\beta}}\sqrt{\frac{\pi}{1}}=e^{-\frac{x^2}{4\beta}}\frac{1}{2}\sqrt{\frac{1}{\lambda}}\frac{1}{\sqrt{\beta}}=$$

??

easier method?

$$=\cancel{\frac{1}{2}\sqrt{\frac{1}{\lambda}}\frac{1}{4\lambda t}}=\frac{1}{2}\sqrt{\frac{1}{\lambda}}\frac{1}{\sqrt{\frac{1}{4\lambda}+t}}=\frac{1}{2}\frac{1}{\sqrt{\frac{1}{4}+\lambda t}}=\frac{1}{\sqrt{1+4\lambda t}}$$ ✓

final answer.