GF SUMMARY $\boxed{\text{BVP}}$

SUPPOSE, WANNA SOLVE: $y''(x) + p\,y'(x) + q\,y(x) = f(x) \qquad x \geq 0$

SUBJECT TO $y(0) = y'(0) = 0$

$$GF: \frac{\partial^2 G}{\partial x^2} + p\frac{\partial G}{\partial x} + qG = \delta(x-\xi)$$

SUBJECT TO: $G(0,\xi) = \frac{\partial G}{\partial x}(0,\xi) = 0$

THEN: $y(x) = \int_0^\infty G(x,\xi) f(\xi)\, d\xi$

$x \geq 0 \quad \xi \geq 0$

VERIFY SOL.:

$$Ly(x) = L\int_0^\infty G(x,\xi) f(\xi)\,d\xi = \int_0^\infty LG(x,\xi) f(\xi)\,d\xi$$

$$L = \partial_x^2 + p\,\partial_x + q \qquad = \int_0^\infty \delta(x-\xi) f(\xi)\,d\xi = f(x)$$

AS REQUIRED.

integrate from $\xi-\epsilon$ to $\xi+\epsilon$:

$$\lim_{\epsilon\to 0}\int_{\xi-\epsilon}^{\xi+\epsilon} \partial_x^2 G + p\,\partial_x G + qG\,dx = \lim_{\epsilon\to 0}\int_{\xi-\epsilon}^{\xi+\epsilon}\delta(x-\xi)\,dx = 1$$

G MUST BE CONT & HAVE DISCONT FIRST DERIVATIVE.

$$[G]_{x=\xi} = 0 \qquad \left[\frac{\partial G}{\partial x}\right]_{x=\xi} = 1$$

REWRITE: $G(x,\xi) = \begin{cases} A(\xi)y_1(x) + B(\xi)y_2(x) & 0 \leq x < \xi \\ C(\xi)y_1(x) + D(\xi)y_2(x) & x > \xi \end{cases}$

BC AT $x=0$, MATRIX FORM: $\begin{pmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0)\end{pmatrix}\begin{pmatrix} A(\xi) \\ B(\xi)\end{pmatrix} = \begin{pmatrix} 0 \\ 0\end{pmatrix}$

CONTINUITY &
JUMP CONDITION AT $x=\xi$, MATRIX FORM:

$$\begin{pmatrix} y_1(\xi) & y_2(\xi) \\ y_1'(\xi) & y_2'(\xi)\end{pmatrix}\begin{pmatrix} C(\xi) \\ D(\xi)\end{pmatrix} = \begin{pmatrix} 0 \\ 1\end{pmatrix}$$

SOL.: $\begin{pmatrix} C(\xi) \\ D(\xi)\end{pmatrix} = \frac{1}{W(\xi)}\begin{pmatrix} y_2'(\xi) & -y_2(\xi) \\ -y_1'(\xi) & y_1(\xi)\end{pmatrix}\begin{pmatrix} 0 \\ 1\end{pmatrix}$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \frac{1}{W(\xi)}\begin{pmatrix} -y_2(\xi) \\ y_1(\xi)\end{pmatrix}$$

$\Rightarrow$ GF FOR IVP: $G(x,\xi) = \begin{cases} 0 & 0 \leq x \leq \xi \\ \frac{1}{W(\xi)}\left[y_1(\xi)y_2(x) - y_1(x)y_2(\xi)\right] & x \geq \xi \end{cases}$

GF FOR IVP

$Ly = f \qquad a \le x \le b$

TWO POINT HOMOGENEOUS BCS:

$$\alpha_1 y'(a) + \alpha_2 y(a) = 0$$

$$\beta_1 y'(b) + \beta_2 y(b) = 0$$

GF: SOL OF:

$$LG = \delta(x-\xi)$$

SUBJECT TO BCS:

$$\alpha_1 \frac{\partial G}{\partial x}(a,\xi) + \alpha_2 G(a,\xi) = 0$$

$$\beta_1 \frac{\partial G}{\partial x}(b,\xi) + \beta_2 G(b,\xi) = 0$$

& THIS GF IS DEFINED FOR:

$$a \le x, \xi \le b$$

SOL FOR y IS THEN: $y(x) = \int_a^b G(x,\xi) f(\xi)\, d\xi$

(BC: $Ly = L\int_a^b G(x,\xi) f(\xi)\, d\xi = \int_a^b LG(x,\xi) f(\xi)\, d\xi$

$= \int_a^b \delta(x-\xi) f(\xi)\, d\xi = f(x)$, AS REQUIRED.)

G CAN BE WRITTEN AS:

$$G(x,\xi) = \begin{cases} A(\xi)\, y_a(x) & a \le x \le \xi \\ B(\xi)\, y_b(x) & \xi \le x \le b \end{cases}$$

[COMPARE W/ IVP CASE]

JUMP & CONTINUITY CONDITIONS: $[G]_{x=\xi} = 0$, $\left[\frac{\partial G}{\partial x}\right]_{x=\xi} = 1$

MATRIX FORM:

$$\begin{pmatrix} y_a(\xi) & y_b(\xi) \\ y_a'(\xi) & y_b'(\xi) \end{pmatrix} \begin{pmatrix} -A(\xi) \\ B(\xi) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

SOL:

$$\begin{bmatrix} -A(\xi) \\ B(\xi) \end{bmatrix} = \frac{1}{W(\xi)} \begin{pmatrix} y_b'(\xi) & -y_b(\xi) \\ -y_a'(\xi) & y_a(\xi) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \cancel{\frac{1}{W(\xi)} \begin{pmatrix} -y_b(\xi) \\ y_a(\xi) \end{pmatrix}}$$

$$= \frac{1}{W(\xi)} \begin{pmatrix} -y_b(\xi) \\ y_a(\xi) \end{pmatrix}$$

Sol does not exist or not unique if wronskian vanishes