UNDERSTANDING FRESNEL DIFFRACTION

diffraction: passage of wave past some obstruction (ie NOT Michaelson-Morley experiment)

Huygens' principle: each point on a wavefront acts as a source of secondary wavelets which propogate, overlap, interfere, and thus carry the wavefront forward.

Diffraction Integral Derivation

consider planar aperture, Σ. Consider an element of it:

monochromatic waves:

$$\underline{\Psi}(\underline{r},t)=\mathrm{Re}\left[\Psi(\underline{r})e^{-i\omega t}\right]$$

wave arriving at the aperture:

$$\Psi_1(\underline{r})=\frac{a_s e^{iks}}{s}$$

(a_s: "strength" of produced waves at S)

Aperture: can change amplitude or phase.
↳ this is described by aperture function.

Element of aperture, as source of secondary wavelets with a strength & phase given by: $a_\Sigma = A\Psi_1(x,y)\,h(x,y)\,dx\,dy$

where: $A=-\frac{i}{\lambda}$ FOR SOME COMPLICATED REASON.

Secondary wavelet creates disturbance at P, dist. r away from aperture element:

$$d\Psi_P=\underbrace{-\frac{i}{\lambda}\frac{a_s e^{iks}}{s}h(x,y)\,dx\,dy}_{a_\Sigma}\,K(\Theta)\frac{e^{ikr}}{r}$$

$$=-\frac{i}{\lambda}h(x,y)K(\Theta)\frac{a_s e^{ik(s+r)}}{sr}dx\,dy$$

To calculate total amplitude at point P, sum over all elements of aperture:

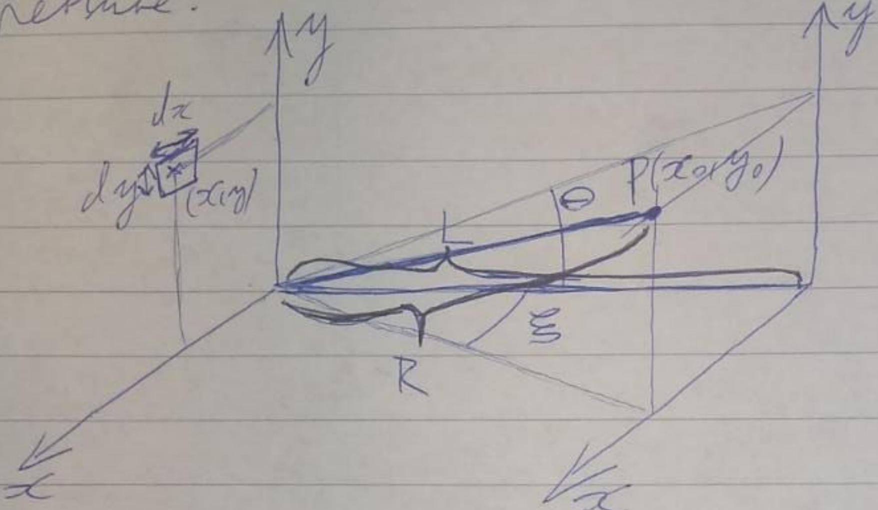
FOR SOME COMPLICATED REASON: $K(\Theta)=\frac{\cos\Theta_S+\cos\Theta_P}{2}$

at small angles, $K(\Theta)\approx 1$

$$\Psi_P=\iint_\Sigma -\frac{i}{\lambda}h(x,y)K(\Theta_S,\Theta_P)\frac{a_s e^{ik(s+r)}}{sr}dx\,dy$$

THIS IS THE FRESNEL–KIRCHOFF DIFFRACTION INT.

Assume source is well behind aperture (& on axis) so aperture is hit by plane wave at normal incidence. Observe diffraction pattern in a plane distance L from aperture.



Choose P sufficiently close to axis that we can say: $K \approx 1$.

distance r from element of aperture to point P:

$$r^2 = L^2 + (x_0 - x)^2 + (y_0 - y)^2$$

$$= \underbrace{L^2 + x_0^2 + y_0^2}_{R^2} - 2(x_0 x + y_0 y) + x^2 + y^2$$

$$= R^2\left(1 - 2\,\frac{x_0 x + y_0 y}{R^2} + \frac{x^2 + y^2}{R^2}\right)$$

remember: 1 plusminus something small square rooted is 1 plusminus half times that something small $\Rightarrow$

$$r \simeq R\left(1 - \frac{x_0 x + y_0 y}{R^2} + \frac{1}{2}\,\frac{x^2 + y^2}{R^2}\right)$$

$$= R - \frac{x_0 x + y_0 y}{R} + \frac{x^2 + y^2}{2R}$$

Phase of each wavelet $\propto k r$.

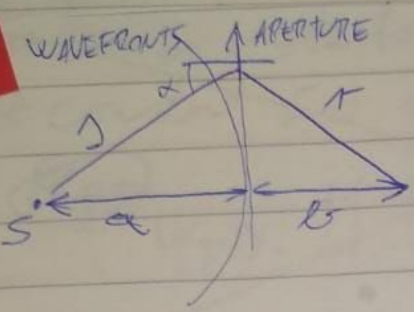
if L is large enough, phase from quadratic term: $\frac{k(x^2+y^2)}{2R} \ll \pi$ i.e. ap. w/ max extent D: $R \gg \frac{D^2}{\lambda}$

$\Longrightarrow$ FRAUNHOFER CONDITIONS: $\psi_P \propto \iint_\Sigma \psi_\Sigma\, h(x,y) \exp\left[-ik\left(\frac{x_0 x + y_0 y}{R}\right)\right] dx\,dy$

ψ_Σ: CONSTANT IF PLANE WAVE & NORMAL INCIDENCE.

$\exp(-ikR)$: constant also, so we don't care.

Fresnel diffraction when $R \gg \frac{D^2}{\lambda}$ no longer holds.

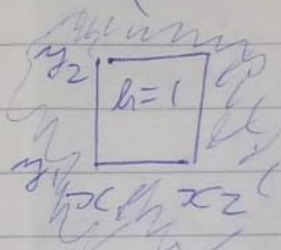


$$r_1 + r_2 = \sqrt{a^2+x^2+y^2} + \sqrt{b^2+x^2+y^2}$$
$$= a\sqrt{1+\frac{x^2+y^2}{a^2}} + b\sqrt{1+\frac{x^2+y^2}{b^2}}$$
$$= a\left(1+\frac{1}{2}\frac{x^2+y^2}{a^2}\right) + b\left(1+\frac{1}{2}\frac{x^2+y^2}{b^2}\right) + \text{HIGHER ORDER TERMS.}$$
$$= a+b + \frac{x^2+y^2}{2a} + \frac{x^2+y^2}{2a}$$

WRITE: $\frac{1}{R} = \frac{1}{a} + \frac{1}{b} \Rightarrow$ OPTICAL PATH $=$ CONST $+ (x^2+y^2)/(2R)$

$$\psi_P \propto \iint_\Sigma h(x,y) \exp\left(ik\frac{x^2+y^2}{2R}\right) dx\,dy$$

Rectangular aperture:



SEP. OF INTEGRALS.

$$\psi_P \propto \iint_\Sigma h(x,y) e^{ik\frac{x^2+y^2}{2R}} dx\,dy$$

$$\propto \int_{x_1}^{x_2} \exp\left(ik\frac{x^2}{2R}\right) dx \int_{y_1}^{y_2} \exp\left(ik\frac{y^2}{2R}\right) dy$$

introduce dimensionless new variables: $u = x\sqrt{\frac{2}{\lambda R}}$, $v = y\sqrt{\frac{2}{\lambda R}}$

$$\psi_P \propto \int_{x_1}^{x_2} \exp\left(\frac{i\pi x^2}{\lambda R}\right) dx \int_{y_1}^{y_2} \exp\left(\frac{i\pi y^2}{\lambda R}\right) dy \propto \int_{u_1}^{u_2} \exp\left(\frac{i\pi u^2}{2}\right) du \int_{v_1}^{v_2} \exp\left(\frac{i\pi v^2}{2}\right) dv$$

Define Fresnel integrals:

$$C(w) = \int_0^w \cos\left(\frac{\pi u^2}{2}\right) du \,;\quad S(w) = \int_0^w \sin\left(\frac{\pi u^2}{2}\right) du$$

$\rightarrow$ evaluate numerically w/ Cornu spiral.

Particular value of w determines a point $C + iS$ in complex plane.

<u>Example</u>

slit or edge extending in y dir. Diffraction integral:

$$\psi_P \propto \int_{w_1}^{w_2} \exp\left(\frac{i\pi u^2}{2}\right) du = C(w_2) + iS(w_2) - C(w_1) - iS(w_1)$$
$$= [C(w_2) - C(w_1)] + i[S(w_2) - S(w_1)]$$

$$w_1 = x_1\sqrt{\frac{2}{\lambda R}} \quad w_2 = x_2\sqrt{\frac{2}{\lambda R}}$$

Normalise by amplitude of unobstructed wavefront, which has length $\sqrt{2}$.

Length^2 of spanning vector is proportional to intensity.

<u>Edge</u>

Define origin to be at O_b.
Fresnel conditions satisfied bec. S, O_b, b are in a straight line.
(So source is not off-axis)

For observation point c, for example, move origin to O_c, so Fresnel cond. are still satisfied.
integrate from $x = -x_c$ to $x = \infty$

For max $|\psi_P|$, $x_2 = \infty$ & $x_1 \simeq x_d$
(CAN READ THIS OFF FROM CORNU SPIRAL)
$\to w_d \approx 1.22$

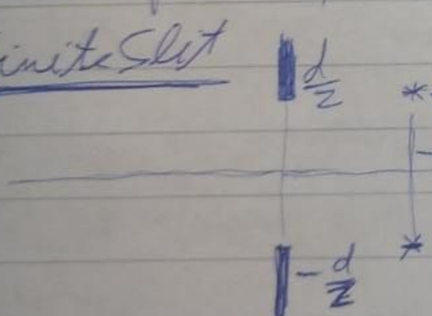
Diffracted wave at b: integrate from $x = 0$ to $x = \infty$
ie $w_1 = 0$ to $w_2 = \infty$

$\to \psi_P = [C(\infty) - C(0)] + i[S(\infty) - S(0)]$
$= 0.5 + i0.5$

$\to$ half of ampli of unobstructed wavefront $\Rightarrow$ quarter of intensity.

for obs. point a, integrate from $x_1 = x_a > 0$ to ∞

<u>Finite Slit</u>



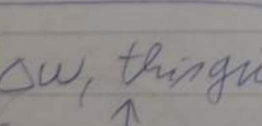
$*\to$ OBS POINT AROUND HERE: $w_1 = -d\sqrt{\frac{2}{\lambda R}}$; $w_2 = 0$

$\to$ OBS POINT IN BETWEEN: INT FROM w_1 TO w_2, where $w_2 - w_1 = d\sqrt{\frac{2}{\lambda R}}$ $(= \Delta w)$

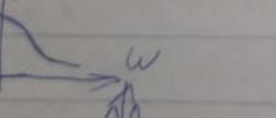
$*\to$ FOR OBS. POINT AROUND HERE: $w_1 = 0$, $w_2 = d\sqrt{\frac{2}{\lambda R}}$
(ie $x_1 = 0$, $x_2 = d$)

Spanning vector on Cornu spiral is between two points separated by a fixed length along the curve.

For small Δw, this gives th like this:

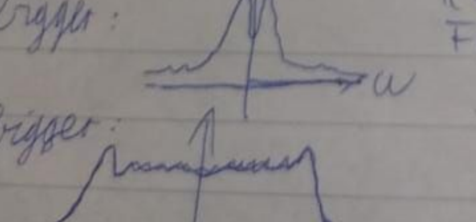


As it gets bigger:



READ OFF THESE FROM CORNU SPIRAL.

even bigger:



CIRCULAR APERTURE

Recall diffraction integral for Fresnel case:

$$\psi_P \propto \iint \frac{h(x,y)\,K(x,y)\exp\left(ik\frac{x^2+y^2}{2R}\right)}{\Delta r}\,dx\,dy$$

Keep obliquity factor and r & s variation, but consider ~~only~~ pattern only on-axis.

$$\psi_P \propto \int_{S=0}^{S=r_a} \frac{K}{\sqrt{a^2+S^2}\,\sqrt{b^2+S^2}}\exp\left(\frac{ikS^2}{2R}\right)2\pi S\,dS$$

SOURCE AP. DIST — AP-SCREEN DIST

~~SOURCE-AP. PATH~~ ~~ELEMENT~~ APERTURE

Make substitution: $S^2 = \Delta$ (NOT THE SAME Δ AS BEFORE), $2S\,dS = d\Delta$

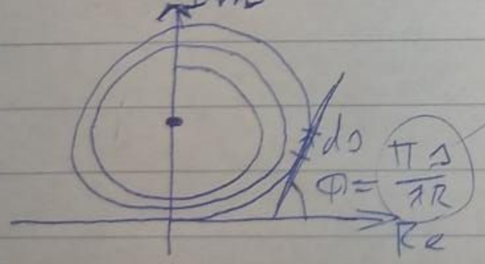
$\rightarrow \frac{d\Delta}{dS} = 2S$

$k = \frac{2\pi}{\lambda}$

REWRITE:

$$\psi_P \propto \int_{\Delta=0}^{\Delta=r_a^2} \frac{K(\Delta)}{\sqrt{a^2+\Delta}\,\sqrt{b^2+\Delta}}\exp\left(\frac{i\pi\Delta}{\lambda R}\right)\pi\,d\Delta$$

Evaluate graphically using phasor diag. (FIG 141)



$\phi = 2n\pi \Rightarrow 2n\pi = \frac{\pi\Delta}{\lambda R}$

$\Downarrow$

$(S^2 =)\ \Delta = 2n\lambda R$ $\Rightarrow \psi \approx 0$

$\phi = (2n+1)\pi$, $\Delta = (2n+1)\lambda R\ (= S^2)$ $\Rightarrow \psi \approx 2\psi_U$ (UNOBSTRUCTED)

Fresnel half-period zones

define first zone: $0 \le \phi(S) \le \pi$

i.e. $S^2 \le \lambda R$

n^{TH} zone: $(n-1)\pi \le \phi(S) \le n\pi$

$$\sqrt{(n-1)\lambda R} \le S \le \sqrt{n\lambda R}$$

Note that with approx, each zone has the same area: $\pi(S_n^2 - S_{n-1}^2) = \pi\lambda R$

CIRCULAR APERTURE

Recall diffraction integral for Fresnel case:

$$\psi_P \propto \iint \frac{h(x,y)\,K(x,y)\exp\left(ik\frac{x^2+y^2}{2R}\right)}{\Delta r}\,dx\,dy$$

Keep obliquity factor and r & Δ variation, but consider ~~only~~ pattern only on-axis.

$$\psi_P \propto \int_{\rho=0}^{\rho=r_a} \frac{K}{\sqrt{a^2+\rho^2}\sqrt{b^2+\rho^2}}\exp\left(\frac{ik\rho^2}{2R}\right) 2\pi\rho\,d\rho$$

SOURCE AP. DIST — AP-SCREEN DIST

~~SOURCE-AP PATH ELEMENT~~ APERTURE

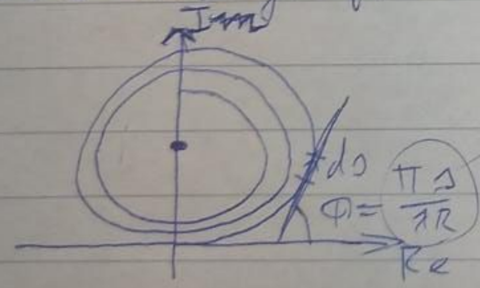
Make substitution: $\rho^2 = \Delta$ (NOT THE SAME Δ AS BEFORE), $2\rho\,d\rho = d\Delta$

$\hookrightarrow \frac{d\Delta}{d\rho} = 2\rho$ $k = \frac{2\pi}{\lambda}$

REWRITE:

$$\psi_P \propto \int_{\Delta=0}^{\Delta=r_a^2} \frac{K(\Delta)}{\sqrt{a^2+\Delta}\sqrt{b^2+\Delta}}\exp\left(\frac{i\pi\Delta}{\lambda R}\right)\pi\,d\Delta$$

Evaluate graphically using phasor diag. (FIG 141)



$\phi = 2n\pi \Rightarrow 2n\pi = \frac{\pi\Delta}{\lambda R}$

$\Downarrow$

$(\rho^2 =)\,\Delta = 2n\lambda R$ $\Rightarrow \psi \approx 0$

$\phi = (2n+1)\pi,\ \Delta = (2n+1)\lambda R\ (=\rho^2)$

$\Rightarrow \psi \approx 2\underbrace{\psi_U}_{\text{UNOBSTRUCTED}}$

Fresnel half-period zones

define first zone: $0 \le \phi(\rho) \le \pi$

i.e. $\rho^2 \le \lambda R$

n^{TH} zone: $(n-1)\pi \le \phi(\rho) \le n\pi$

$$\sqrt{(n-1)\lambda R} \le \rho \le \sqrt{n\lambda R}$$

Note that with approx, each zone has the same area: $\pi(\rho_n^2 - \rho_{n-1}^2) = \pi\lambda R$

when $R' = \frac{l}{2m}$: each open area lets through even number of Fresnel zones (with opposite phase pairwise)

$\Rightarrow \psi_P \sim 0$. $\Rightarrow$ POINTS w/ ZERO INT. ON AXIS AT $R' = \frac{l}{2m}$

$R' = \frac{l}{2m+1}$: each open zone : odd number of Fresnel zones.

So: $\psi_P \Rightarrow 2N\psi_u$ (not counting k)

Expect maxima at $R = \frac{l}{2m+1}$ on axis.

See fig. 150 for illustration