2010 P1 Q9 (III)

RECALL: $1 = A^2 C^2(y)(1 + y'^2)$

IF $y' = 0 \Rightarrow y = y_0$

$1 = A^2 C^2(y_0) = A^2 \frac{C_0^2}{1 - k y_0}$

$\Rightarrow 1 - k y_0 = A^2 C_0^2$

$\hookrightarrow A^2 = \frac{1 - k y_0}{C_0^2}$

PLUG THIS IN TO OUR EQUATION:

$$y = -\frac{k}{4C_0^2}\left(\frac{x}{A}\right)^2 - \frac{1 - k y_0 - 1}{k}$$

$$y = -\frac{k}{4C_0^2}\left(\frac{x}{A}\right)^2 + \frac{k y_0}{k} = -\frac{k}{4}\left(\frac{x}{A}\right)^2 + y_0$$

$y(\pm x_0) = 0$

$$\Rightarrow 0 = -\frac{k}{4}\left(\frac{x_0}{C_0 A}\right)^2 + y_0$$

$$\frac{k}{4}\frac{x_0^2}{(C_0 A)^2} = y_0$$

$$\frac{k}{4}\frac{x_0^2}{C_0^2 \underbrace{\frac{1 - k y_0}{C_0^2}}} = y_0$$

✓

$$\frac{k x_0^2}{4(1 - k y_0)} = y_0$$

$/k$

$$k x_0^2 = 4 y_0 (1 - k y_0)$$

$$(k x_0)^2 = 4 k y_0 (1 - k y_0)$$

AS REQUIRED.

I'm sure this is right but it is hard to follow.

2010P1Q9(I) EULER EQUATION: $f_y(x,y,y') - \frac{d}{dx} f_{y'}(x,y,y') = 0$

EXPANDED FORM:

$$f_y - f_{y'x} - f_{y'y}\,y' - f_{y'y'}\,y'' = 0$$

IF $f = f(y,y')$ THEN $f_{y'x} = 0$

SO EULER EQUATION BECOMES: $f_y - f_{y'y}\,y' - f_{y'y'}\,y'' = 0$

ALSO, WE FORM: $\frac{d}{dx}(f - y' f_{y'}) = f_y y' + f_{y'} y'' - \frac{d}{dx}(y' f_{y'}) =$

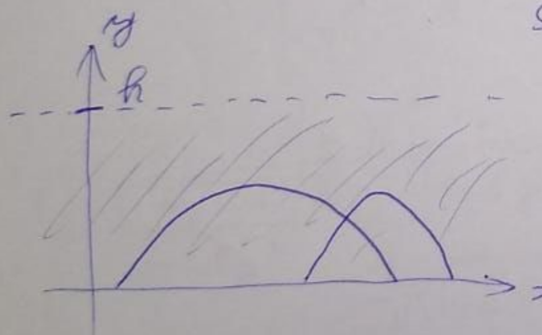
normally $\frac{\partial f}{\partial y'}$, etc.

$$= f_y y' + \cancel{f_{y'} y''} - \cancel{y'' f_{y'}} - y' f_{y'y}\, y' - y' f_{y'y'}\, y'' =$$

$$= y' \underbrace{\left(f_y - f_{y'y}\, y' - f_{y'y'}\, y''\right)}_{=0,\ \text{BY EULER EQ.}} = 0$$

THUS: $f - y' f_{y'} = \text{CONSTANT}$

SO: $y' f_{y'} - f = A$



PATH LENGTH ELEMENT OF LIGHT RAY: $\sqrt{(dx)^2 + (dy)^2} = ds$.

TIME NEEDED FOR LIGHT TO GET THROUGH THIS LENGTH ELEMENT: $\frac{ds}{C(y)} = \frac{\sqrt{(dx)^2+(dy)^2}}{C(y)} = \frac{\sqrt{1+y'^2}\,dx}{C(y)}$

FROM FERMAT'S PRINCIPLE, LIGHT RAY FOLLOWS THE CURVE WHICH MINIMISES: $T = \int \frac{ds}{C(y)} = \int \frac{\sqrt{1+y'^2}}{C(y)}\,dx$

INTEGRAND DOES NOT CONTAIN x EXPLICITLY, SO WE CAN USE THE PREVIOUSLY DERIVED RESULT:

$$y' \frac{\partial f}{\partial y'} - f = A$$

$$y' \frac{1}{C(y)} \frac{1}{2}(1+y'^2)^{-\frac{1}{2}} 2y' - \frac{1}{C(y)}(1+y'^2)^{\frac{1}{2}} = A$$

MAKE COMMON DENOMINATORS:

$$\frac{y'^2}{C(y)(1+y'^2)^{\frac{1}{2}}} - \frac{1\cdot(1+y'^2)}{C(y)(1+y'^2)^{\frac{1}{2}}} = A$$

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$$\Rightarrow 1 - ky_0 = A^2 C_0^2$$

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PLUG THIS IN TO OUR EQUATION:

$$y = -\frac{k}{4C_0^2}\left(\frac{x}{A}\right)^2 - \frac{1-ky_0-1}{k}$$

$$y = -\frac{k}{4C_0^2}\left(\frac{x}{A}\right)^2 + \frac{ky_0}{k} = -\frac{k}{4}\left(\frac{x}{A}\right)^2 + y_0$$

$y(\pm x_0) = 0$

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✓

$$\frac{kx_0^2}{4(1-ky_0)} = y_0$$

$\cdot k$

$$kx_0^2 = 4y_0(1-ky_0)$$

$$(kx_0)^2 = 4ky_0(1-ky_0)$$

AS REQUIRED.

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