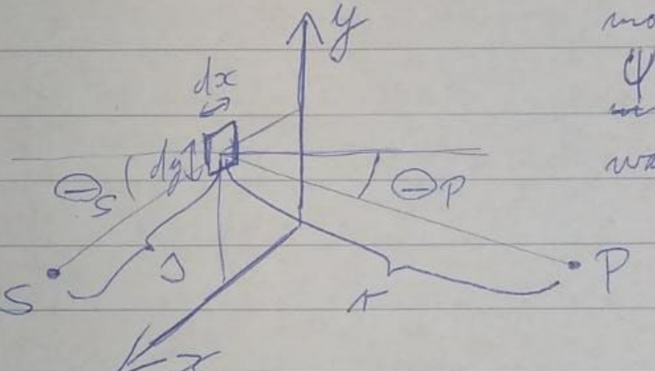
UNDERSTANDING FRESNEL DIFFRACTION

diffraction: passage of wave past some obstruction (ie NOT Michelson-Morley experiment)

Huygens' principle: each point on a wavefront acts as a source of secondary wavelets which propagate, overlap, interfere, and thus carry the wavefront forward.

Diffraction Integral Derivation

consider planar aperture, Σ. Consider an element of it:



monochromatic waves:

$$\underline{\Psi}(\underline{r},t) = \mathrm{Re}\left[\psi(\underline{r})e^{-i\omega t}\right]$$

wave arriving at the aperture:

$$\psi_1(\underline{r}) = \frac{a_s e^{iks}}{s}$$

(a_s: "strength" of produced waves at S)

Aperture: can change amplitude or phase.
↳ this is described by aperture function.

Element of aperture, as source of secondary wavelets with a strength & phase given by: $a_\Sigma = A\psi_1(x,y)\,h(x,y)\,dx\,dy$

where: $A = -\frac{i}{\lambda}$ FOR SOME COMPLICATED REASON.

Secondary wavelet creates disturbance at P, dist. r away from aperture element:

$$d\psi_P = \underbrace{-\frac{i}{\lambda}\frac{a_s e^{iks}}{s}h(x,y)\,dx\,dy}_{a_\Sigma}\,K(\theta)\frac{e^{ikr}}{r}$$

$$= -\frac{i}{\lambda}h(x,y)K(\theta)\frac{a_s e^{ik(s+r)}}{sr}\,dx\,dy$$

FOR SOME COMPLICATED REASON: $K(\theta) = \frac{\cos\theta_S + \cos\theta_P}{2}$

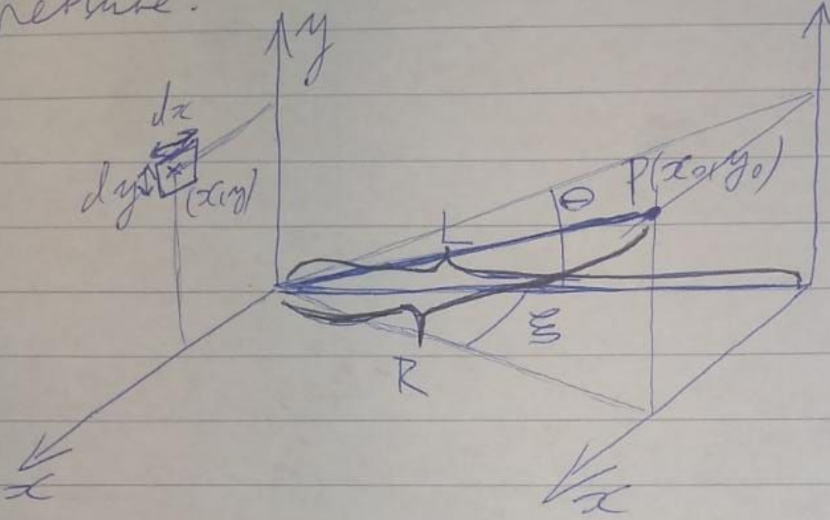
at small angles, $K(\theta) \approx 1$

To calculate total amplitude at point P, sum over all elements of aperture:

$$\psi_P = \iint_\Sigma -\frac{i}{\lambda}h(x,y)K(\theta_S,\theta_P)\frac{a_s e^{ik(s+r)}}{sr}\,dx\,dy$$

THIS IS THE FRESNEL-KIRCHOFF DIFFRACTION INT.

Assume source is well behind aperture (& on axis) so aperture is hit by plane wave at normal incidence. Observe diffraction pattern in a plane distance L from aperture.



Choose P sufficiently close to axis that we can say: $K \approx 1$.

distance r from element of aperture to point P:

$$r^2 = L^2 + (x_0 - x)^2 + (y_0 - y)^2$$

$$= \underbrace{L^2 + x_0^2 + y_0^2}_{R^2} - 2(x_0 x + y_0 y) + x^2 + y^2$$

$$= R^2\left(1 - 2\,\frac{x_0 x + y_0 y}{R^2} + \frac{x^2 + y^2}{R^2}\right)$$

remember: 1 plusminus something small square rooted is 1 plusminus half times that something small $\Rightarrow$

$$r \cong R\left(1 - \frac{x_0 x + y_0 y}{R^2} + \frac{1}{2}\,\frac{x^2 + y^2}{R^2}\right)$$

$$= R - \frac{x_0 x + y_0 y}{R} + \frac{x^2 + y^2}{2R}$$

Phase of each wavelet $\propto k r$.

i.e. ap. w/ max extent D: $R \gg \frac{D^2}{\lambda}$

if L is large enough, phase from quadratic term: $\frac{k(x^2 + y^2)}{2R} \ll \pi$

$\Longrightarrow$ FRAUNHOFER CONDITIONS: $$\psi_P \propto \iint_\Sigma \psi_\Sigma\, h(x,y) \exp\left[-ik\left(\frac{x_0 x + y_0 y}{R}\right)\right] dx\,dy$$

ψ_Σ: CONSTANT IF PLANE WAVE & NORMAL INCIDENCE.

$\exp(-ikR)$: constant also, so we don't care.

Fresnel diffraction when $R \gg \frac{D^2}{\lambda}$ no longer holds.

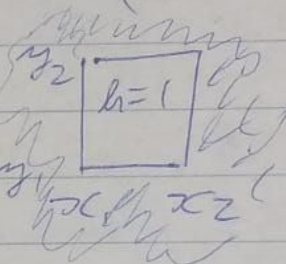
Other term in r starts to dominate, other terms are approximated to be constant. So we'll have:

$$\Psi_P \propto \iint_{\Sigma} \frac{h(\overbrace{x, y}^{\text{APERTURE COORD.}}) K(\theta_S, \theta_P) \exp\left(ik \frac{x^2+y^2}{2R}\right)}{s r} dx\, dy$$

Approximate even more: (ie s & r variations $\approx$ CONST, $K \approx 1$)

$$\Psi_P \propto \iint_{\Sigma} h(x,y) \exp\left(ik \frac{x^2+y^2}{2R}\right) dx\, dy$$

Rectangular aperture:



SEP. OF INTEGRALS.

$$\Psi_P \propto \iint_{\Sigma} h(x,y) e^{ik\frac{x^2+y^2}{2R}} dx\, dy$$

$$\propto \int_{x_1}^{x_2} \exp\left(ik \frac{x^2}{2R}\right) dx \int_{y_1}^{y_2} \exp\left(ik \frac{y^2}{2R}\right) dy$$

introduce dimensionless new variables: $u = x\sqrt{\frac{2}{\lambda R}}$, $v = y\sqrt{\frac{2}{\lambda R}}$

$$\Psi_P \propto \int_{x_1}^{x_2} \exp\left(\frac{i\pi x^2}{\lambda R}\right) dx \int_{y_1}^{y_2} \exp\left(\frac{i\pi y^2}{\lambda R}\right) dy \propto \int_{u_1}^{u_2} \exp\left(\frac{i\pi u^2}{2}\right) du \int_{v_1}^{v_2} \exp\left(\frac{i\pi v^2}{2}\right) dv$$

Define Fresnel integrals:

$$C(w) = \int_0^w \cos\left(\frac{\pi u^2}{2}\right) du \; ; \; S(w) = \int_0^w \sin\left(\frac{\pi u^2}{2}\right) du$$

$\hookrightarrow$ evaluate numerically w/ Cornu spiral.

Particular value of w determines a point $C + iS$ in complex w plane.

<u>Example</u>

slit or edge extending in y dir. Diffraction integral:

$$\Psi_P \propto \int_{w_1}^{w_2} \exp\left(\frac{i\pi u^2}{2}\right) du = C(w_2) + iS(w_2) - C(w_1) - iS(w_1)$$

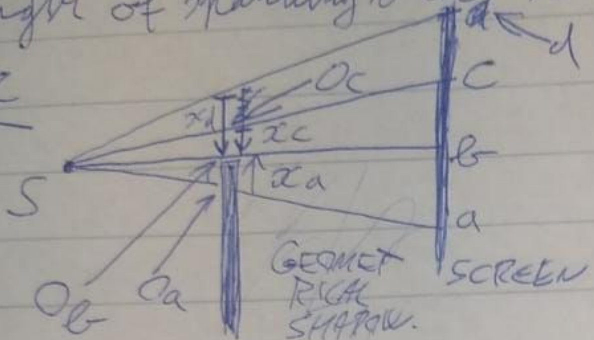
$$= [C(w_2) - C(w_1)] + i[S(w_2) - S(w_1)]$$

$$w_1 = x_1\sqrt{\frac{2}{\lambda R}} \quad w_2 = x_2\sqrt{\frac{2}{\lambda R}}$$

Normalise by amplitude of unobstructed wavefront, which has length $\sqrt{2}$.

Length^2 of spanning vector is proportional to intensity.

<u>Edge</u>



Define origin to be at O_b. Fresnel conditions satisfied bec. S, O_b, b are in a straight line. (So source is not off-axis)

For observation point c, for example, move origin to O_c, so Fresnel cond. are still satisfied. integrate from $x = -x_c$ to $x = \infty$

For max $|\psi_P|$, $x_2 = \infty$ & $x_1 \approx x_d$ (CAN READ THIS OFF FROM CORNU SPIRAL) $\to w_d \approx 1.22$

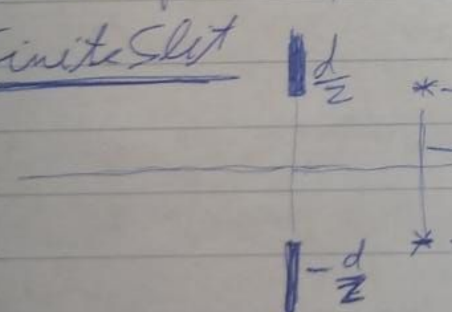
Diffracted wave at b: integrate from $x=0$ to $x=\infty$ ie $w_1 = 0$ to $w_2 = \infty$

$$\to \psi_P = [C(\infty) - C(0)] + i[S(\infty) - S(0)] = 0.5 + i\,0.5$$

$\to$ half of ampli of unobstructed wavefront $\Rightarrow$ quarter of intensity.

for obs. point a, integrate from $x_1 = x_a > 0$ to ∞

<u>Finite Slit</u>



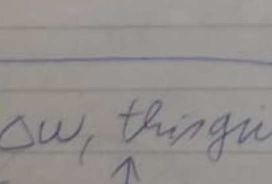
$\to$ OBS POINT AROUND HERE: $w_1 = -d\sqrt{\frac{2}{\lambda R}}$; $w_2 = 0$

$\to$ OBS POINT IN BETWEEN: INT FROM w_1 TO w_2, where $w_2 - w_1 = d\sqrt{\frac{2}{\lambda R}}$ $(= \Delta w)$

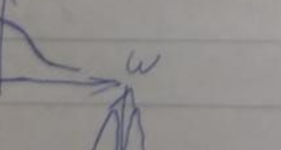
$\to$ FOR OBS. POINT AROUND HERE: $w_1 = 0$, $w_2 = d\sqrt{\frac{2}{\lambda R}}$ (ie $x_1 = 0$; $x_2 = d$)

Spanning vector on Cornu spiral is between two points separated by a fixed length along the curve.

For small Δw, this gives sth like this:

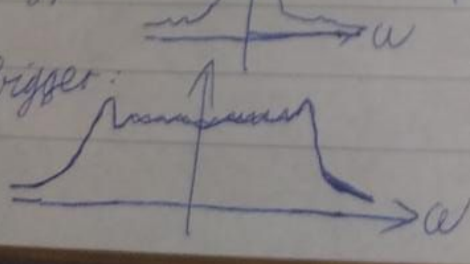


As it gets bigger:



READ OFF THESE FROM CORNU SPIRAL.

even bigger:



CIRCULAR APERTURE

Recall diffraction integral for Fresnel case:

$$\psi_P \propto \iint \frac{h(x,y)\,K(x,y)\exp\left(ik\frac{x^2+y^2}{2R}\right)}{\sigma r}\,dx\,dy$$

Keep obliquity factor and r & σ variation, but consider ~~only~~ pattern only on-axis

$$\psi_P \propto \int_{S=0}^{S=r_a} \frac{K}{\sqrt{a^2+S^2}\sqrt{b^2+S^2}} \exp\left(\frac{ikS^2}{2R}\right) 2\pi S\,dS$$

SOURCE AP. DIST — AP–SCREEN DIST

~~SOURCE-AP PATH ELEMENT~~ APERTURE

Make substitution: $S^2 = \sigma$ (NOT THE SAME σ AS BEFORE), $2S\,dS = d\sigma$

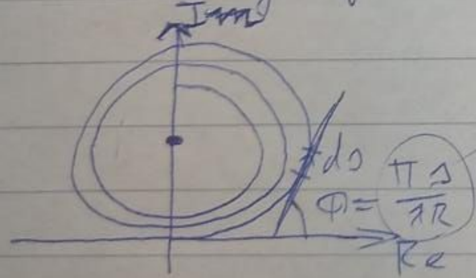
$$\frac{d\sigma}{dS} = 2S$$

$k = \frac{2\pi}{\lambda}$

REWRITE:

$$\psi_P \propto \int_{\sigma=0}^{\sigma=r_a^2} \frac{K(\sigma)}{\sqrt{a^2+\sigma}\sqrt{b^2+\sigma}} \exp\left(\frac{i\pi\sigma}{\lambda R}\right) \pi\,d\sigma$$

Evaluate graphically using phasor diag. (FIG 141)



$\varphi = 2n\pi \Rightarrow 2n\pi = \frac{\pi\sigma}{\lambda R}$

$\Downarrow$

$(S^2 =)\ \sigma = 2n\lambda R$ $\Rightarrow \psi \approx 0$

$\varphi = (2n+1)\pi$, $\sigma = (2n+1)\lambda R\ (= S^2)$

$\Rightarrow \psi \approx 2\psi_U$ — UNOBSTRUCTED

Fresnel half-period zones

define first zone: $0 \le \varphi(S) \le \pi$

i.e. $S^2 \le \lambda R$

n^{TH} zone: $(n-1)\pi \le \varphi(S) \le n\pi$

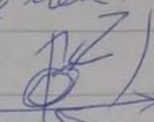
$$\sqrt{(n-1)\lambda R} \le S \le \sqrt{n\lambda R}$$

Note that with approx, each zone has the same area: $\pi(S_n^2 - S_{n-1}^2) = \pi\lambda R$

If neglect obliquity factor (& r variation → denom in diffrac int.), each zone contributes equally (& phasor diagram would be circular completely)

There would be N zones on-axis: $r_a^2 = N\lambda R$

if N is odd: bright spot at P $I \sim 4I_u$ (ie we are here:



N is even: dark spot (ie we are here) $I \sim 0$

For large apertures, spiralling in happens. (Obliquity factor, $\frac{1}{r}$ term in diffrac. integral)

Circular obstruction on-axis

obstacle radius r_a: inner zones are blocked until: $\varphi_a = \frac{\pi r_a^2}{\lambda R}$

Outer zones are clear.

So we have:

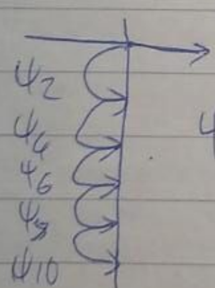
integrate from this point, ie intensity $\propto$ this length^2 squared.

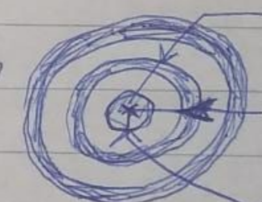
POISSON'S SPOT

Off-axis intensity: see H237 slide.

Fresnel Zone Plate

Block out zones 1, 3, 5 (ie we only going downwards on phasor diag.)





$\psi_p = \psi_2 + \psi_4 + \psi_6 + \ldots \approx 2N\psi_u$

$\Rightarrow I_p \approx 4N^2 I_u$

↳ incident plane wave is brought to focus at P, R away from aperture $\Rightarrow$ lens w/ $f = R = $ ⑧

$S_n = \sqrt{n\lambda R}$

$R = \frac{S_n^2}{n\lambda} = \frac{S_1^2}{\lambda}$

Move obs point P towards zone plate. Fresnel zones for this new P (& R ie R') is: $S_1' = \sqrt{\lambda R'}$, $S_2' = \sqrt{2\lambda R'}$, $S_3' = \sqrt{3\lambda R'}$ ...

When $R' = \frac{f}{2}$: $S_1' = \sqrt{\lambda f/2}$; $S_2' = \sqrt{\lambda f}\ (= S_1)$; $S_3' = \sqrt{\frac{3\lambda f}{2}}$, $S_4' = \sqrt{2\lambda f} = S_2$

So area btwn S_1 & S_2 lets through two zones $\Rightarrow \psi_p \sim 0$.

when $R' = \frac{f}{2m}$: each open area lets through even number of Fresnel zones (with opposite phase pairwise)

$\Rightarrow$ ~~ψ~~ $\psi_P \sim 0$. $\Rightarrow$ POINTS ~~W/~~ W/ ZERO INT. ON AXIS AT $R' = \frac{f}{2m}$

$R' = \frac{f}{2m+1}$: each open zone: odd number of Fresnel zones.

So: $\psi_P \Rightarrow 2N\psi_u$ (not counting k)

Expect maxima at $R = \frac{f}{2m+1}$ on axis.

See fig. 150 for illustration