2016P2Q6(I)

(a) A (CARTESIAN) TENSOR T OF ORDER 2 IS A SET OF COEFFICIENTS $T_{i_1 i_2}$, DEFINED WITH RESPECT TO A SET OF ORTHONORMAL BASIS VECTORS $\underline{e}_i$, SUCH THAT THE COEFFICIENTS WITH RESPECT TO ANOTHER ORTHONORMAL BASIS $\underline{e}'_i = L_{ij}\underline{e}_j$ ARE GIVEN BY THE TRANSFORMATION LAW:

$$T'_{i_1 i_2} = L_{i_1 j_1} L_{i_2 j_2} T_{j_1 j_2}$$

WHERE L IS GIVEN BY:

$$L_{ij} \equiv \underline{e}'_i \cdot e_j$$

[FROM NOTES]

(b) IN AN UNPRIMED BASIS:

$$J = C_{ij}A_{ij} = C_{lk}A_{lk}$$

IN ANOTHER, PRIMED BASIS:

$$J = C'_{ij}\underbrace{A'_{ij}}_{\text{REWRITE}} = C'_{ij}L_{il}L_{jk}A_{lk}$$

$\ominus$

$$\left(C'_{ij}L_{il}L_{jk} - C_{lk}\right)A_{lk} = 0$$

$$A_{lk} \neq 0$$

$$\Rightarrow C'_{ij}L_{il}L_{jk} = C_{lk}$$

$$L_{jk}C'_{ij}L_{il} = C_{lk}$$

LEAVE EINSTEIN NOTATION

$$L^T C' L = C$$

~~REARRANGE~~ $L\cdot \quad \cdot L^T$ USING ORTHOGONALITY OF L

$$C' = LCL^T$$

$$C_{i_1 i_2} = L_{i_1 j_1} C_{j_1 j_2} L_{i_2 j_2}$$

SO BY THE DEF. GIVEN ABOVE, C_{ij} IS AN ORDER 2 TENSOR.

2016 P2 Q6 (II)

ROTATION MATRIX AROUND x_3 AXIS, BY θ ANGLE:

$$R_{x_3}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} C & -S & 0 \\ S & C & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

ROTATION MATRIX OF $\frac{\pi}{2}$ ABOUT x_3 AXIS IS THEN:

$$R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T' = R^T T R = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T_{12} & -T_{11} & T_{13} \\ T_{22} & -T_{21} & T_{23} \\ T_{32} & -T_{31} & T_{33} \end{pmatrix} =$$

$$= \begin{pmatrix} T_{22} & -T_{21} & T_{23} \\ -T_{12} & T_{11} & -T_{13} \\ T_{32} & -T_{31} & T_{33} \end{pmatrix}$$

$$T' = T \Rightarrow \begin{cases} T_{11} = T_{22} \\ T_{12} = -T_{21} \\ T_{21} = -T_{12} \\ \left.\begin{matrix} T_{13} = T_{23} \\ T_{13} = T_{23} \end{matrix}\right\} \longrightarrow T_{13} = T_{23} = 0 \\ \left.\begin{matrix} T_{32} = T_{31} \\ -T_{31} = T_{32} \end{matrix}\right\} \longrightarrow T_{31} = T_{32} = 0 \end{cases}$$

T IS IN THE FORM:

$$T = \begin{pmatrix} \alpha & \omega & 0 \\ -\omega & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$$

INVARIANCE UNDER GENERAL ROTATION:

$$T' = R^T T R = \begin{pmatrix} C & S & 0 \\ -S & C & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \omega & 0 \\ -\omega & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} \begin{pmatrix} C & -S & 0 \\ S & C & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} C & S & 0 \\ -S & C & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C\alpha + S\omega & -S\alpha + C\omega & 0 \\ -C\omega + S\alpha & S\omega + C\alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} =$$

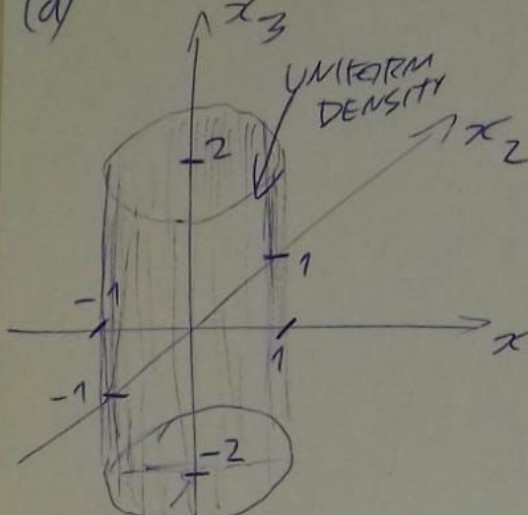
$$= \begin{pmatrix} C(C\alpha + S\omega) + S(-C\omega + S\alpha) & C(-S\alpha + C\omega) + S(S\omega + C\alpha) & 0 \\ -S(C\alpha + S\omega) + C(-C\omega + S\alpha) & -S(-S\alpha + C\omega) + C(S\omega + C\alpha) & 0 \\ 0 & 0 & \beta \end{pmatrix} =$$

2016 P2 Q6 (III)

$$= \begin{pmatrix} c^2\alpha + s^2\alpha & c^2\omega + s^2\omega & 0 \\ -s^2\omega - c^2\omega & s^2\alpha + c^2\alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} = \begin{pmatrix} \alpha & \omega & 0 \\ -\omega & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$$

SO T_{ij} IS INVARIANT UNDER A GENERAL ROTATION ABOUT THE x_3-AXIS.

(d)



THIS CYLINDER IS INVARIANT UNDER ROTATION ABOUT THE x_3 AXIS, SO ITS MOMENT OF INERTIA TENSOR MUST BE INVARIANT AS WELL. ~~SO BY EARLIER FINDINGS~~)
SO, USING THE EARLIER FINDINGS WE CONCLUDE THAT I_{ij} HAS THE FORM.

$$I = \begin{pmatrix} \alpha & \omega & 0 \\ -\omega & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$$

$$\omega = I_{12} = \int_V \rho \left(\cancel{(x_{11}^2 + x_{22}^2 + x_{33}^2) \cdot \underbrace{\delta_{12}}_{0}} - x_1 x_2 \right) dV =$$

$$= \int_V -\rho \, x_1 x_2 \, dV = 0$$

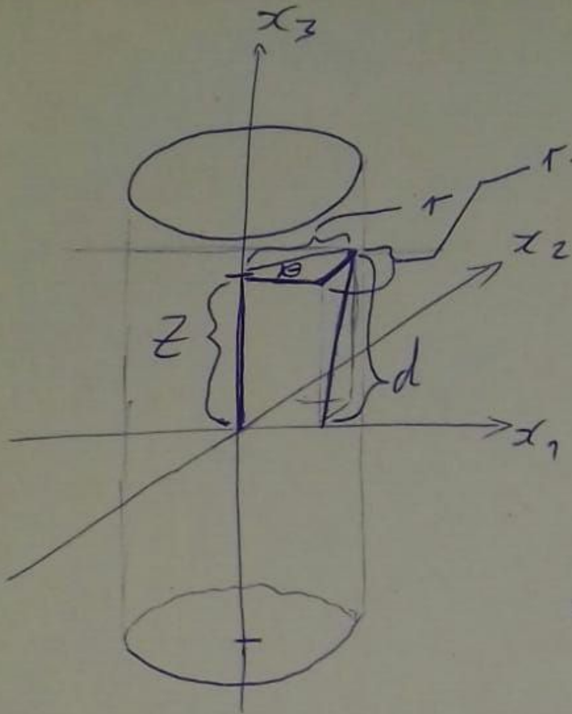
THIS MUST BE ZERO BECAUSE FOR EVERY VOLUME ELEMENT FOR WHICH $-\rho x_1 x_2$ IS POSITIVE, THERE IS A VOLUME ELEMENT WITH $-\rho x_1 x_2$ HAVING THE SAME ABS. VALUE BUT OPPOSITE SIGN. (SEE SKETCH).

IN SHORT: BY SYMMETRY CONSIDERATIONS

SO NOW WE HAVE:

$$I = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$$

2016P2Q6(IV) | LET'S CALCULATE α:



$$d^2 = r^2 \sin^2\theta + z^2$$

$$I_{11} = \int_V \rho \left(x_1x_1 + x_2x_2 + x_3x_3 - x_1x_1 \right) dV =$$

$$= \int_V \rho \underbrace{\left(x_2^2 + x_3^2 \right)}_{d^2} dV = \int_V \rho\, d^2\, dV =$$

$$= \int_V \rho \left(r^2 \sin^2\theta + z^2 \right) \underbrace{dV}_{r\,d\theta\,dr\,dz} = \rho \int_{z=-2}^{z=2} \int_{r=0}^{r=1} \int_{\theta=0}^{\theta=2\pi} \left(r^2 \sin^2\theta + z^2 \right) r\,d\theta\,dr\,dz =$$

$$= \rho \int_{z=-2}^{z=2} \int_{r=0}^{r=1} \cancel{\left(r^2\sin^2\theta + z^2\right)} \cancel{\left(r^2\pi + z^2\right) r} \left(r^2 \pi + 2\pi z^2 \right) r\,dr\,dz =$$

$$= \rho \int_{z=-2}^{z=2} \pi \left(\left[\frac{r^4}{4} \right]_0^1 + 2z^2 \left[\frac{r^2}{2} \right]_0^1 \right) dz = \rho \int_{z=-2}^{z=2} \pi \left(\frac{1}{4} + z^2 \right) dz =$$

$$= \rho\, \pi \left(\left[\frac{1}{4} z \right]_{-2}^{2} + \left[\frac{z^3}{3} \right]_{-2}^{2} \right) = \rho\pi \left(1 + \frac{16}{3} \right) = \alpha$$

WE NEED β NOW.

2016 P2 Q6(V)

$$\beta = I_{33} = \int_V \rho \left(x_1^2 + x_2^2\right) dV =$$

$$= \rho \int_{z=-2}^{z=2} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} r^2 \, r \, d\theta \, dr \, dz = \rho \int_{z=-2}^{2} \int_{0}^{2\pi} \left[\frac{r^4}{4}\right]_0^1 d\theta \, dz =$$

$$= \rho \int_{z=-2}^{z=2} \frac{1}{4} \cdot 2\pi \, dz = \rho 2\pi$$

SO WE END UP WITH:

$$I = \begin{pmatrix} \rho\pi\left(1+\frac{16}{3}\right) & 0 & 0 \\ 0 & \rho\pi\left(1+\frac{16}{3}\right) & 0 \\ 0 & 0 & \rho 2\pi \end{pmatrix}$$