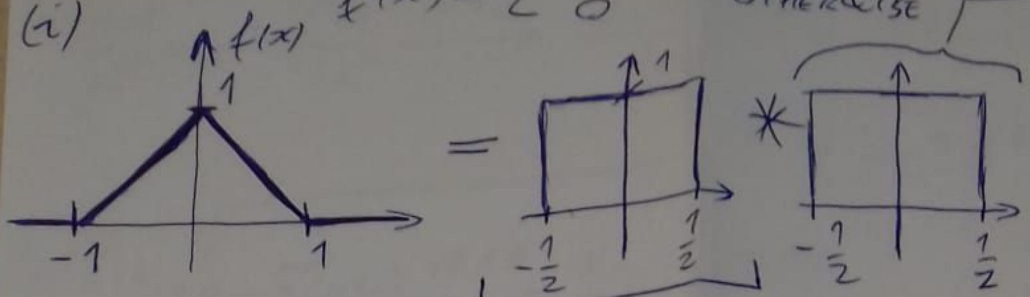
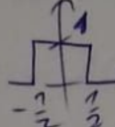
2013P2Q5 | I |

$f(x)=\begin{cases}1-|x|, & |x|<1\\ 0 & \text{OTHERWISE}\end{cases}$ $\qquad t(x)=\begin{cases}1 & \text{IF } |x|<\frac{1}{2}\\ 0 & \text{OTHERWISE}\end{cases}$

(i)



FT: $\int_{-\infty}^{\infty}$  $e^{-ikx}\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} 1\cdot e^{-ikx}\,dx = \frac{1}{k}\left[e^{-ikx}\right]_{-\frac{1}{2}}^{\frac{1}{2}} =$

$= \frac{i}{k}\left[e^{-i\frac{1}{2}k} - e^{i\frac{1}{2}k}\right] = \frac{2\sin(\frac{1}{2}k)}{k}$

BY CONVOLUTION THEOREM: $FT[f(x)] = FT[t(x)]\cdot FT[t(x)]$

$= \frac{4\sin^2(\frac{1}{2}k)}{k^2}$

(ii)

$$F_x\left[\frac{\partial u}{\partial t}\right] = \lim_{\tau\to 0} F_x\left[\frac{u(x,t+\tau)-u(x,t)}{\tau}\right] =$$

$$= \lim_{\tau\to 0}\frac{1}{\tau}\Big[F_x[u(x,t+\tau)] - F_x[u(x,t)]\Big] = \frac{\partial F_x u}{\partial t}$$

$$F_x\left[\frac{\partial u}{\partial x}\right] = \lim_{h\to 0}\int_{-\infty}^{\infty}\frac{u(x+h,t)-u(x,t)}{h}e^{-ikx}\,dx =$$

$$= \lim_{h\to 0}\int_{-\infty}^{\infty}\frac{u(x+h,t)}{h}e^{-ikx}\,dx - \lim_{h\to 0}\int_{-\infty}^{\infty}\frac{u(x,t)}{h}e^{-ikx}\,dx =$$

→ CHANGE VAR.: $x+h=z \Rightarrow x=z-h,\ dx=dz$

$$= \lim_{h\to 0}\left[\int_{-\infty}^{\infty}\frac{u(z,t)}{h}e^{-ikz}\,dz\, e^{ikh} - \int_{-\infty}^{\infty}\frac{u(x,t)}{h}e^{-ikx}\,dx =\right.$$

$$= \lim_{h\to 0}\left[\underbrace{\int_{-\infty}^{\infty}\frac{u(z,t)}{h}e^{-ikz}\,dz}\,[1+(ikh)+O(h^2)] - \underbrace{\int_{-\infty}^{\infty}\frac{u(x,t)}{h}e^{-ikx}\,dx}_{=}\right]$$

$$= \lim_{h\to 0}\left(\left[1+ik\cancel{h}+O(h^2)-1\right]\int_{-\infty}^{\infty}\frac{u(z,t)}{\cancel{h}}e^{-ikz}\,dz =\right.$$

$$= ik\int_{-\infty}^{\infty}\frac{u(z,t)}{h}e^{-ikz}\,dz = ik\,\tilde{u}(k,t)$$

2013P2Q5(II)

(iii)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \xrightarrow{FT_x} \frac{\partial^2 \tilde{u}(k,t)}{\partial t^2} = c^2(-ik)^2\, \tilde{u}(k,t)$$

$$\ddot{\tilde{u}} = -c^2 k^2 \tilde{u}$$

(iv)

$$\tilde{u}(k,t) = \tilde{A}(k)\, e^{-ikct} + \tilde{B}(k)\, e^{ikct}$$

BY CONVOLUTION THEOREM:

$$u(x,t) = A(x) * IFT\left[e^{-ikct}\right] + B(x) * IFT\left[e^{+ikct}\right]$$

$$IFT\left[e^{\pm ikct}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\pm ikct}\, e^{ikx}\, dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(ct+x)}\, dk =$$

$$= \cancel{\delta(ct+x) + \delta(ct-x)}$$

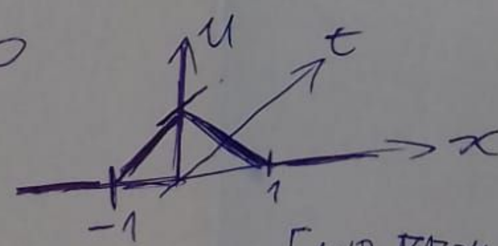
$$= \delta(ct \pm x)$$

[! SHALLOW UNDERSTANDING HERE]

$$u(x,t) = A(ct+x) + B(ct-x)$$

$$u(x,t=0) = f(x) = A(0+x) + B(0-x)$$

$$\frac{\partial u(x,t=0)}{\partial t} = 0$$



[AND FROM HERE I DON'T SEE HOW TO PROCEED]