

2013P1Q1(I)

$$(i) \nabla \cdot (\underline{A} \times \underline{B}) = \partial_i (\epsilon_{ijk} A_j B_k) = \epsilon_{ijk} (\partial_i A_j) B_k + \epsilon_{ijk} (\partial_i B_k) A_j = \epsilon_{kij} (\partial_i A_j) B_k - \epsilon_{jki} (\partial_i B_k) A_j = \underline{B} \cdot (\nabla \times \underline{A}) - \underline{A} \cdot (\nabla \times \underline{B})$$

(cyclic)

$$(ii) \text{DIVERGENCE THM: } \int_V (\nabla \cdot \underline{F}) dV = \oint_S \underline{F} \cdot \underline{dS}$$

WHERE V IS
A VOLUME BOUNDED
BY THE CLOSED
SURFACE S ,
 \underline{F} IS A VECTOR
FIELD.

$$\text{USING (i): } \nabla \cdot (\nabla a \times \underline{B}) = \underline{B} \cdot (\nabla \times \nabla a) - \nabla a \cdot (\nabla \times \underline{B})$$

$$\text{USING: } \nabla \times \nabla a = \underline{0} \quad \text{proof}$$

$$\nabla \cdot (\nabla a \times \underline{B}) = -\nabla a \cdot (\nabla \times \underline{B}) \quad \checkmark$$

$$\text{USING DIVERGENCE THM, w/ } \underline{F} = \nabla a \times \underline{B}:$$

$$\int_V \nabla \cdot (\nabla a \times \underline{B}) dV = \int -\nabla a \cdot (\nabla \times \underline{B}) dV = \oint_S (\nabla a \times \underline{B}) \cdot \underline{dS}$$

RESTYLE:

$$\iiint_V \nabla a \cdot (\nabla \times \underline{B}) dV = - \iint_S (\nabla a \times \underline{B}) \cdot \underline{\hat{n}} dS$$

$$(iii) a = xy + z^2$$

$$\underline{B} = y \hat{i} - yz \hat{j} + x \hat{k}$$

$$\nabla a = (\partial_x, \partial_y, \partial_z)(xy + z^2) = (y, x, 2z)$$

$$\nabla \times \underline{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & -yz & x \end{vmatrix} = (y, -1, -1) \quad \checkmark$$

$$\nabla a \cdot (\nabla \times \underline{B}) = (y, x, 2z) \cdot (y, -1, -1) = y^2 - x - 2z$$

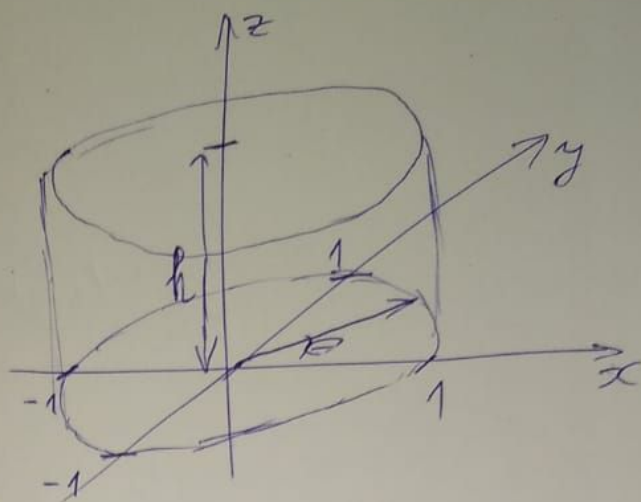
$$\nabla a \times \underline{B} = (y, x, 2z) \times (y, -yz, x) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ y & x & 2z \\ y & -yz & x \end{vmatrix} = (x^2 - 2z^2y, 2zy - yx, -y^2z - xy)$$

2013 P1 Q1 (II)

$$\iiint_V \nabla a \cdot (\nabla \times \underline{B}) dV =$$

$$= \iiint_V y^2 - x - 2z dV =$$

$$= \iiint_V y^2 dV - \iiint_V x dV - \iiint_V 2z dV =$$



$$= \int_{r=0}^1 \int_{\theta=0}^{2\pi} \int_{z=0}^h (r \sin \theta)^2 r dr d\theta dz - \underbrace{\int_{r=0}^1 \int_{\theta=0}^{2\pi} \int_{z=0}^h r \cos \theta r dr d\theta dz}_{=0} - \underbrace{\int_{r=0}^1 \int_{\theta=0}^{2\pi} \int_{z=0}^h 2z r dr d\theta dz}_{=0}$$

$$= \int_{r=0}^1 \int_{\theta=0}^{2\pi} \int_{z=0}^h 2z r dr d\theta dz = \left[\frac{r^4}{4} \right]_0^1 \frac{2\pi}{2} \left[\frac{z^2}{2} \right]_0^h - \left[\frac{r^2}{2} \right]_0^1 \frac{2\pi}{2} \left[\frac{z^2}{2} \right]_0^h =$$

$$= \frac{h^4}{4} - h^2 \pi$$

$$- \iint_S (\nabla a \times \underline{B}) \cdot \underline{\hat{n}} dS = - \underbrace{\int_{\text{LATERAL PART}} (\nabla a \times \underline{B}) \cdot \underline{\hat{n}} dS}_{\text{Top & Bottom}} - \int_{\text{Top & Bottom}} (\nabla a \times \underline{B}) \cdot \underline{\hat{n}} dS$$

$$\underbrace{\int_{\text{LATERAL PART}} (x^2 - 2z^2y, 2zy - yx, -y^2z - xz) \cdot (x, y, 0) dS}_{\underline{\hat{n}}}$$

$$= \underbrace{\int_{\text{LATERAL PART}} x^3 - 2z^2yx + 2zy^2 - y^2xz dS}_{\text{LATERAL PART}} = \int 2zy^2 dS =$$

(OTHER TERMS CANCEL OUT ON THE DOMAIN OF INTEGRATION (THEY ARE ODD))

2013 P1 Q1 (III)

$$= \int_{z=0}^h \int_{\theta=0}^{2\pi} z z (1 \sin \theta)^2 d\theta dz = z \left[\frac{z^2}{2} \right]_0^h \frac{2\pi}{2} = h^2 \pi$$

$$\int_{\text{TOP}} (\nabla \underline{a} \times \underline{B}) \cdot \underline{\hat{n}} dS = \int_{\text{TOP}} (x^2 - z^2 y, 2zy - y^2 z - xy, -y^2 z - xy) \cdot (0, 0, 1) dS$$

$$= \int_{\text{TOP}} -y^2 z - xy dS = \int_{\theta=0}^{2\pi} \int_{r=0}^h (r \sin \theta)^2 z dr d\theta = 0$$

ANTISYMMETRIC FUNCTION ON A SYMMETRIC SURFACE.

$$= - \int_{\theta=0}^{2\pi} \int_{r=0}^h (r \sin \theta)^2 z r dr d\theta = -h \left[\frac{r^4}{4} \right]_0^h \frac{2\pi}{2} = -\frac{1}{4} \pi h$$

use with polar eq

$$\int_{\text{BOTTOM}} (\nabla \underline{a} \times \underline{B}) \cdot \underline{\hat{n}} dS = \int_{\text{BOTTOM}} (x^2 - z^2 y, 2zy - y^2 z - xy, -y^2 z - xy) \cdot (0, 0, -1) dS =$$

$$= \int_{\text{BOTTOM}} (y^2 z + xy) dS = 0$$

PUTTING THESE TOGETHER:

$$- \iint_S (\nabla \underline{a} \times \underline{B}) \cdot \underline{\hat{n}} dS = -h^2 \pi - \left(-\frac{1}{4} \pi h + 0 \right) = \frac{h^2 \pi}{4} - h^2 \pi$$

AS FOUND FOR THE LHS.
 \Rightarrow BOTH SIDES VERIFIED.