2011 P1Q9(I) EULER EQUATION: $\frac{\partial}{\partial y} F - \frac{d}{dx}\left(\frac{\partial}{\partial y'} F\right) = 0$

IF $F = F(y, y')$, THEN $F_{y'x} = 0$

EXPAND EULER EQ.: $F_y - \underbrace{F_{y'x}}_{\to 0} - F_{y'y} y' - F_{y'y'} y'' = 0$

$$F_y - F_{y'y} y' - F_{y'y'} y'' = 0$$

LET'S FORM: $\frac{d}{dx}\left(F - y' F_{y'}\right) = F_y y' + \cancel{F_{y'} y''} - \cancel{y'' F_{y'}} - y' F_{y'y} y' - y' F_{y'y'} y'' =$

$$= y' \underbrace{\left(F_y - F_{y'y} y' - F_{y'y'} y''\right)}_{= 0} = 0$$

0 (FROM EULER EQ ABOVE)

✓

THUS, $F - y' F_{y'} = A$, AS REQUIRED.

LENGTH ELEMENT OF PATH OF LIGHT RAY: $\sqrt{(dx)^2 + (dy)^2} =$

$$= \sqrt{1 + y'^2}\, dx$$

TIME ELAPSED GOING THROUGH THIS ELEMENT: $\frac{\sqrt{1 + y'^2}\, dx}{c(y)}$

INTEGRAL FROM FERMAT'S PRINCIPLE:

$$T = \int_{P_1}^{P_2} \frac{\sqrt{1 + y'^2}\, dx}{c(y)}$$ ✓

INTEGRAND DOES NOT INCLUDE $x$ EXPLICITLY, SO: $F - y' F_{y'} = A$

$$\frac{\sqrt{1 + y'^2}}{c(y)} - y' \frac{1}{c(y)} \frac{1}{2} \left(1 + y'^2\right)^{-\frac{1}{2}} 2y' = A$$

2011 P1 Q9 (II)

$$\frac{(1+y'^2)}{c(y)(1+y'^2)^{\frac{3}{2}}} - \frac{y'^2}{c(y)(1+y'^2)^{\frac{3}{2}}} = A$$

$$\frac{1}{c(y)(1+y'^2)^{\frac{1}{2}}} = A \checkmark$$

SOLVE FOR $y'$:

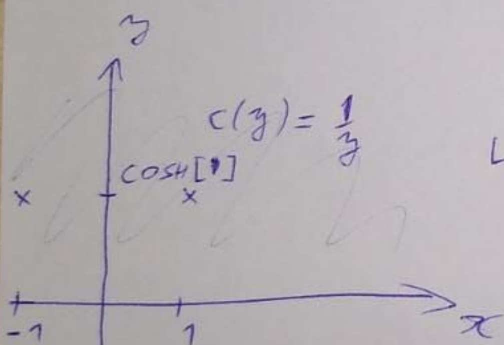
$$1 = A^2 C^2(y)(1+y'^2)$$

$$\sqrt{\frac{1-A^2C^2(y)}{A^2C^2(y)}} = \pm y' = \pm\frac{dy}{dx} = \frac{\sqrt{1-A^2C^2}}{AC}$$

SEPARATE VARIABLES & INTEGRATE:

$$\pm\int\frac{dx}{A} = \int^{y}\frac{C(\hat{y})\,d\hat{y}}{\sqrt{1-A^2C^2}}$$

$$\pm(x+B) = \int^{y}\frac{A\,C(\hat{y})\,d\hat{y}}{\sqrt{1-A^2C(\hat{y})^2}}.$$ AS REQUIRED.




LIGHT RAY PATH CALCULATION

$$\cancel{\pm(x+B) =} \int\frac{A\frac{1}{y}\,dy}{\sqrt{1-A^2\frac{1}{y^2}}} = \checkmark$$

$$= \int\frac{A\,dy}{y\sqrt{1-\frac{A^2}{y^2}}} = \int\frac{A\,dy}{\sqrt{y^2-A^2}} =$$

LET $y = A\cosh(u) \Rightarrow dy = A\sinh u\,du$

$$= \int\frac{A\sinh u\,du}{\sqrt{A^2\cosh^2(u)-A^2}} = \int\frac{A\sinh u\,du}{A\underbrace{\sqrt{\cosh^2 u-1}}_{\sinh^2 u}} =$$

$$= \int du = u + k = \pm(x+B)$$

$$u = \pm(x+B)$$ (B INCORPORATES K)

$$\cosh(u) = \cosh(x+B)$$

2011P1Q9(III) RECALL: $y = A\cosh(u) \Rightarrow \cosh(u) = \frac{y}{A}$

SO WE HAVE:

$$\frac{y}{A} = \cosh(x+B) \quad ✓$$

$$y = A\cosh(x+B)$$

$$y(\pm 1) = \cosh[1] \Rightarrow B = 0 \;✓$$

$$y(\pm 1) = A\cosh(\pm 1) = A\cosh[1] = \cosh[1] \Rightarrow A = 1$$

SO THE PATH FOLLOWED BY LIGHT RAY:

$$y = \cosh x$$

✓

MINIMAL VALUE OF $y$ ALONG THIS PATH: $\cosh 0 = 1$

TIME TAKEN TO TRAVEL:

$$T = \int_{x=1}^{x=1} \frac{\sqrt{1+((\cosh x)')^2}\,dx}{\frac{1}{y}} = \int_{x=-1}^{x=1} \cancel{\cosh x}\; y\cosh x\,dx =$$

$$= \int_{x=-1}^{x=1} \cosh^2 x\,dx = \int_{x=-1}^{x=1} \frac{1}{2}(\cosh(2x)+1)\,dx = \int_{x=-1}^{x=1} \frac{1}{2}\cosh(2x)\,dx =$$

$\frac{1}{2}\int \cosh 2x + 1\,dx$

$1 + \frac{1}{2}\sinh 2?$

$$= \left[\frac{1}{4}\sinh(2x)\right]_{x=-1}^{1} = \frac{1}{4}(\sinh(2) - \sinh(-2)) = 1 + \frac{1}{2}\sinh 2$$

NEAT ENOUGH TO BE BELIEVABLE.