2011P1Q4(I)

IFT: $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx}\, dk$

AUTOCORRELATION: $h(x) = f(x) \otimes f(x) = \int_{-\infty}^{\infty} f(y)^* f(x+y)\, dy$

$$\tilde{h}(k) = FT[h(x)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)^* f(x+y)\, dy\, e^{-ikx}\, dx =$$

CHANGE OF VARIABLE: $z = x + y \Rightarrow dz = dx$, $x = z - y$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)^* f(z)\, e^{-ikz} e^{iky}\, dy\, dz =$$

$$= \int_{-\infty}^{\infty} f(y)^* e^{iky}\, dy \int_{-\infty}^{\infty} f(z) e^{-ikz}\, dz = \tilde{f}(k)^* f(k) = |f(k)|^2$$

This does not seem finished ∫?

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x) e^{-ikx}\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)(\cos(kx))\, dx =$$

method: $= \left(\frac{e^{ix} + e^{-ix}}{2}\right) e^{-ikx}$

$$= \left[\sin x \cos kx\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x (-k) \sin(kx)\, dx =$$

$$= 2\cos\frac{k\pi}{2} + k\left(\left[-\cos x \sin(kx)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\cos(x)\, k \cos(kx)\, dx\right) =$$

$\to 0$

$$= 2\cos\frac{k\pi}{2} + k^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\cos(kx)\, dx$$

$$\tilde{f}(k) = 2\cos\frac{k\pi}{2} + k^2 \tilde{f}(k)$$

$$\tilde{f}(k) = \frac{2\cos\frac{k\pi}{2}}{1 - k^2}$$

$$\tilde{f}(k) = \frac{2\cos\frac{k\pi}{2}}{1-k^2} \Longrightarrow |f(k)|^2 = \frac{4\cos^2\left(\frac{k\pi}{2}\right)}{(1-k^2)^2}$$

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4\cos^2\left(\frac{k\pi}{2}\right)}{(1-k^2)^2}\, dk =$$

present

SUBSTITUTION: $u = \frac{k\pi}{2}$, $\frac{2}{\pi}u = k$

$du = \frac{\pi}{2}\, dk$

$$\widetilde{f \otimes f} = |f(k)|^2$$

FINISH BY: APPLY IFT TO WK:

$$\int_{-\infty}^{\infty} f(y)^* f(x+y)\, dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)^* \tilde{f}(k) e^{ikx}\, dk$$

LHS: CORRELATION

RHS: IFT OF $|f(k)|^2$

✓ SET $x = 0$, RELABEL $y \to x$ ON LHS, GET:

$$\int_{-\infty}^{\infty} f(x)^* f(x)\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)^* f(k)\, dk$$

ANOTHER FORM:

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(k)|^2\, dk$$

2011P1Q4(II)

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{2\cos^2(u)}{\left(1-\left(\frac{2}{\pi}u\right)^2\right)^2}\,\frac{2}{\pi}du = \frac{2}{\pi^2}\int_{-\infty}^{\infty} \frac{2\cos^2 u \left(\frac{\pi}{2}\right)^4 du}{\left(1-\left(\frac{2}{\pi}u\right)^2\right)^2\left(\frac{\pi}{2}\right)^4} =$$

$$= \int_{-\infty}^{\infty} \frac{\cos^2 u\,\frac{\pi^2}{4}}{\left(\frac{\pi^2}{4}-u^2\right)^2}du = \frac{\pi^2}{2}\int_0^{\infty}\frac{\cos^2 u}{\left(\frac{\pi^2}{4}-u^2\right)^2}du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 x\,dx =$$

do this separately

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1+\cos(2x))dx = \left[\frac{1}{2}x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left[\frac{1}{4}\sin(2x)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$

$$= \frac{\pi}{2} + 0 = \frac{\pi}{2} \checkmark$$

CHANGE OF VAR: $u \to t$

$$\frac{\pi^2}{2}\int_0^{\infty}\frac{\cos^2 t}{\left(\frac{\pi^2}{4}-t^2\right)^2}dt = \frac{\pi}{2}$$

where's the $\frac{1}{\sqrt{\pi}}$?

$$\int_0^{\infty}\frac{\cos^2 t}{\left(\frac{\pi^2}{4}-t^2\right)^2}dt = \frac{1}{\pi}$$

This is hard to follow all the πs etc.

Make steps clearer.