$\boxed{\text{P II } 2009 \text{ Q}10}$ (1)

$a^2 = b^2 = c^2 = 1$

$ab = c$

$abb = cb = a$

V ⟹

$$\boxed{\begin{matrix} 1 & a & b & c \\ a & 1 & c & b \\ b & c & 1 & a \\ c & b & a & 1 \end{matrix}}$$

✓

✓

$ac \quad cb = aca$

$ab = c$

$acbc = c$

$acb = 1$

$ac = b^{-1} = b$

g_1, g_2 conjugate

$g_2 g = g g_1 = g_1 g \Rightarrow g_1 = g_2$ as Abelian each elt in own conjugacy class.

CONJUGACY CLASSES OF V: ✓

$\{1\} \; \{a\} \; \{b\} \; \{c\}$

explain why.

AN IRREDUCIBLE REP OF V:

~~$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $a = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $b = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$~~

$\{1, a, b, c\} \rightarrow \{1, 1, 1, 1\}$ ✓

NUMBER OF CONJUGACY CLASSES AND INEQUIVALENT IRREPS ARE EQUAL.

V HAS 4 INEQ. IRREP ✓

REP1: $\{1, a, b, c\} \rightarrow \{1, 1, 1, 1\}$ ✓

REP2: $\{1, a, b, c\} \rightarrow \{1, -1, 1, -1\}$ ✓

REP3: $\{1, a, b, c\} \rightarrow \{1, 1, -1, -1\}$ ✓

REP4: $\{1, a, b, c\} \rightarrow \{1, -1, -1, 1\}$ ✓

$$\begin{matrix} D(1) & D(a) & D(b) & D(c) \\ D(a) & D(1) & D(c) & D(b) \\ D(b) & D(c) & D(1) & D(a) \\ D(c) & D(b) & D(a) & D(1) \end{matrix}$$

→ THIS TABLE ISOMORPHIC TO V TABLE ⟹ D IS A REP. ✓

2009 P2 Q10 (II)

GROUP MULTIPLICATION TABLE OF V:

1	a	b	c
a	1	c	b
b	c	1	a
c	b	a	1

$ab = c = ba$

$abb = cb$

$a = cb$

✓ $ca = ccb = b$

VERY GOOD WORK WELL DONE.

CONJUGACY CLASSES:

$g q_1 g^{-1} = q_2 \quad / \cdot g \qquad g, q_1, q_2 \in V$

$g q_1 = q_2 g = g q_2$

→ GROUP ABELIAN ✓

$\Rightarrow q_1 = q_2 \Rightarrow$ EVERY $g \in V$ IS ALONE ITS OWN CONJUGACY CLASS.

CONJUGACY CLASSES OF V:

$\{1\}, \{a\}, \{b\}, \{c\}$

surely $g q_1 g^{-1} = q_2$ abelian $g g^{-1} q_1 = q_2$, $q_1 = q_2$.

An irreducible representation, D, of group V is a set of matrices M(S) acting on vector space S, which S has no non-trivial proper subspaces under the action of matrices forming D. ✓

Number of conjugacy classes and the number of inequivalent irreducible representations of a finite group are equal. ✓

$\Rightarrow$ V HAS 4 INEQUIVALENT IRREP. THESE ARE:

So write — = — —

this is too involved a sentence

$$\{1, a, b, c\} \;\Big|\; \begin{aligned} &\to \{1, 1, 1, 1\} = d^{(1)} \\ &\to \{1, -1, -1, 1\} = d^{(2)} \\ &\to \{1, 1, -1, -1\} = d^{(3)} \\ &\to \{1, -1, 1, -1\} = d^{(4)} \end{aligned}$$

good

PTO.

Also $|G| = \sum n_i^2$ ← dim of irrep.

MULTIPLICATION TABLE FOR MATRICES LISTED:

D(1)	D(a)	D(b)	D(c)
D(a)	D(1)	D(c)	D(b)
D(b)	D(c)	D(1)	D(a)
D(c)	D(b)	D(a)	D(1)

THIS TABLE IS ISOMORPHIC TO THE TABLE OF V PROVIDED ABOVE, SO D IS INDEED A REPRESENTATION OF V. ✓

CHARACTER TABLE:

	1	a	b	c
$d^{(1)}$	1	1	1	1
$d^{(2)}$	1	-1	-1	1
$d^{(3)}$	1	1	-1	-1
$d^{(4)}$	1	-1	1	-1

✓

DECOMPOSITION OF D:

$\{D(1)_{11}, D(a)_{11}, D(b)_{11}, D(c)_{11}\} = d^{(3)}$

$\{D(1)_{22}, D(a)_{22}, D(b)_{22}, D(c)_{22}\} = d^{(4)}$

$\{D(1)_{33}, D(a)_{33}, D(b)_{33}, D(c)_{33}\} = d^{(2)}$ ✓

EVERY VECTOR FORMED BY COLUMN OF CHARACTER TABLE ORTHOGONAL TO ANY COLUMN BELONGING TO DIFFERENT CONJUGACY CLASS (HERE: TO EVERY OTHER COLUMN)

P(II) 2007 Q 10(III) DECOMP OF D:

$$D'(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad D'(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad D'(b) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad D'(c) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$D''(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad D''(a) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad D''(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad D''(c) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$D'''(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad D'''(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad D'''(b) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad D'''(c) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

CHARACTER TABLE FOR V:

	1	a	b	c
REP1	1	1	1	1
REP2	1	-1	1	-1
REP3	1	1	-1	-1
REP4	1	-1	-1	1
orig'l rep.	3	-1	-1	-1

ORTHOGONALITY RELATION: EVERY COL. IN DIFFERENT CONJUGACY CLASS (HERE: EVERY COL.) ARE ORTHOGONAL TO EACH OTHER.

VERIFICATION:

$$(1\ 1\ 1\ 1)\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = (1\ 1\ 1\ 1)\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = (1\ 1\ 1\ 1)\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} =$$

$$= (1\ -1\ 1\ -1)\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = (1\ -1\ 1\ -1)\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = (1\ 1\ -1\ -1)\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = 0$$

we work here

no of copies of irrep $\times |G| = \Sigma$ char. elt in irrep $\times$ char. elt in rep.

ie $4 \times m_1 = 3\times 1 + -1\times 1 + -1\times 1 + -1\times 1$ $\quad \therefore m_1 = 0$

$4 \times m_2 = 3\times 1 + -1\times -1 + -1\times 1 + -1\times -1$ $\quad m_2 = 1$

$4 \times m_3 = 3\times 1 + -1\times 1 + -1\times -1 + -1\times -1$ $\quad m_3 = 1$

$4 \times m_4 = 3\times 1 + -1\times -1 + -1\times -1 + -1\times 1$ $\quad m_4 = 1.$

not v. surprising.