2013P1Q4 (I)

(i) $\tilde{h}(k) = \int_{-\infty}^{\infty} e^{-ikx} h(x)\,dx =$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y)^* g(x+y)\,dy\, e^{-ikx}\,dx =$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y)^* g(x+y)\, e^{-ikx}\,dy\,dx =$$

INTRODUCE CHANGE OF VARIABLE: $z = x + y \longrightarrow dz = dx$

$x = z - y$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y)^* g(z)\, e^{-ikz} e^{iky}\,dy\,dx =$$

$$= \int_{-\infty}^{\infty} f(y)^* e^{iky}\,dy \int_{-\infty}^{\infty} g(z) e^{-ikz}\,dz =$$

$$= \tilde{f}(k)^* \tilde{g}(k)$$ ✓

✓

(ii) LET $f = g$, $x = 0$

$$h(x) = \int_{-\infty}^{\infty} f(y)^* f(x+y)\,dy = \int_{-\infty}^{\infty} |f(y)|^2\,dy$$

$$h(x) = \mathrm{IFT}[\tilde{h}(k)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)^* \tilde{g}(k)\, \underbrace{e^{ikx}}_{1 \text{ SINCE } x=0}\,dk =$$

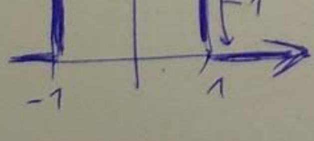
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{f}(k)|^2\,dk$$

~~CHANGE OF VARIABLE $y \to z$~~

CHANGE OF VARIABLE: relabelling $y \to x$

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{f}(k)|^2\,dk$$

(iii) $f(x)$



$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-1}^{1} 1\,dx = 2$$ ✓

$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx = \int_{-1}^{1} 1 \cdot e^{-ikx}\,dx = \frac{i}{k}\left[e^{-ikx}\right]_{-1}^{1} =$$

$$= \frac{i}{k}\left(e^{-ik} - e^{ik}\right) = \frac{2\sin k}{k}$$ ✓

~~$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4\sin^2 k}{k^2}\,dk$~~

2013 P1 Q4 (II)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\tilde{f}(k)|^2 dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4\sin^2 k}{k^2} dk =$$

$$= \frac{2}{\pi}\left(\underbrace{\left[-k^{-1}\sin^2 k\right]_{-\infty}^{\infty}}_{\to 0} - \int_{-\infty}^{\infty} -\frac{1}{k} 2\sin k\cos k \, dk\right) =$$

$$= \frac{2}{\pi}\cdot 2\cdot \underbrace{\frac{\pi}{4}}\cdot 2 = 2 \quad ✓$$

USING HINT: $\int_0^\infty \underbrace{\frac{\sin x\cos x}{x}} dx = \frac{\pi}{4}$

EVEN FUNCTION, SO

$$\int_{-\infty}^{\infty} \frac{\sin x\cos x}{x} = \frac{\pi}{4}\cdot 2$$

$$\int_{-\infty}^{\infty}|f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\tilde{f}(k)|^2 dk = 2$$

$\Longrightarrow$ PARSEVAL'S THEOREM IS VERIFIED FOR THE GIVEN FUNCTION.