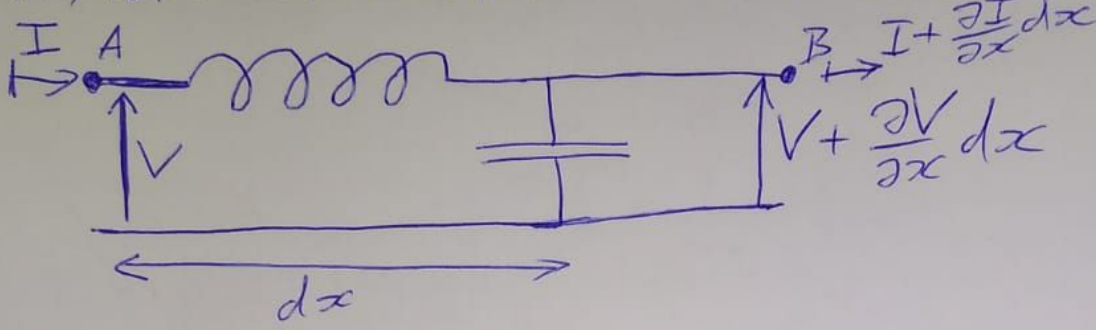
TRANSMISSION LINES

A SHORT ELEMENT OF ZERO RESISTANCE IDEAL TRANSMISSION LINE LENGTH dx, WHERE $dx \ll \lambda$ (λ BEING VOLTAGE OR CURRENT WAVELENGTH.)



POTENTIAL DIFFERENCE BETWEEN POINT A & B =

$$= - \text{THE RATE OF CHANGE OF FLUX OVER TIME} = -\frac{d}{dt}(LI) = -L\frac{\partial I}{\partial t}$$

THIS DIFFERENCE IS ALSO EQUAL TO $\frac{\partial V}{\partial x} dx$, AS SEEN ON THE DRAWING ↑.

INDUCTANCE PER UNIT LENGTH

$$\frac{\partial V}{\partial x} dx = -\underbrace{(L_0 dx)}_{= L} \frac{\partial I}{\partial t} \Rightarrow \boxed{\frac{\partial V}{\partial x} = -L_0 \frac{\partial I}{\partial t}}$$

THE CURRENT CHARGES UP THE CAPACITOR. POTENTIAL DIFF. BETWEEN CAPACITOR PLATES: $V + \frac{\partial V}{\partial x} dx$

CAPACITANCE PER UNIT LENGTH

CHARGE PILING UP ON PLATES: $q = \underbrace{(C_0 dx)}_{\text{CAPACITANCE}}\left(V + \frac{\partial V}{\partial x} dx\right) = C_0 V dx$ TO FIRST ORDER.

THIS IS WHERE THE CURRENT GOES. SO:

CURRENT IN THE BEGINNING $\cdot\, dt$ = CURRENT IN THE END $\cdot\, dt$ + + CHARGE ON CAPACITOR

$\rightarrow I$ $\quad - \left(I + \frac{\partial I}{\partial x} dx\right)$

dt(CURRENT IN THE BEGINNING − CURRENT IN THE END) = = CHARGE ON CAPACITOR.

$$dt\left(-\frac{\partial I}{\partial x} dx\right) = C_0 V dx$$

$$-\frac{\partial I}{\partial x} \cancel{dx} = \frac{\partial}{\partial t}(C_0 V \cancel{dx}) \Rightarrow \boxed{-\frac{\partial I}{\partial x} = C_0 \frac{\partial V}{\partial t}}$$

<u>OBTAINED:</u>

$$\frac{\partial V}{\partial x} = -L_0 \frac{\partial I}{\partial t} \qquad -\frac{\partial I}{\partial x} = C_0 \frac{\partial V}{\partial t}$$

$\frac{\partial}{\partial t}$

$$\frac{\partial^2 V}{\partial x \partial t} = -L_0 \frac{\partial^2 I}{\partial t^2}$$

$\frac{\partial}{\partial x}$

$$\frac{\partial^2 V}{\partial t \partial x} = \frac{1}{C_0}\,\frac{-\partial^2 I}{\partial x^2}$$

$$-L_0 \frac{\partial^2 I}{\partial t^2} = \frac{1}{C_0}\,\frac{-\partial^2 I}{\partial x^2}$$

<u>WAVE EQUATION FOR CURRENT:</u>

$$\boxed{\frac{\partial^2 I}{\partial x^2} = L_0 C_0 \frac{\partial^2 I}{\partial t^2}}$$

$\frac{\partial}{\partial x}$

$$-\frac{1}{L_0}\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 I}{\partial x \partial t}$$

$\frac{\partial}{\partial t}$

$$\frac{\partial^2 I}{\partial x \partial t} = -C_0 \frac{\partial^2 V}{\partial t^2}$$

<u>WAVE EQUATION FOR VOLTAGE:</u>

$$\boxed{\frac{\partial^2 V}{\partial x^2} = L_0 C_0 \frac{\partial^2 V}{\partial t^2}}$$

SO, COMPARING WITH STANDARD FORM OF WAVE EQ.,
CURRENT & VOLTAGE WAVES HAVE SPEED:

$$v = \pm \frac{1}{\sqrt{L_0 C_0}}$$

SOLUTIONS IN THE FORM:

$$V = V_0 \exp\left[-j(kx - \omega t)\right]$$ FOR WAVES TRAVELLING IN THE $\oplus$ DIRECTION.

$$I = I_0 \exp\left[-j(kx - \omega t)\right]$$

SUBSTITUTE <u>THESE</u>↑ INTO $\partial_x V = -L_0 \dot{I}$ & $-\partial_x I = C_0 \dot{V}$:

$$\cancel{-jkV = -L_0 j\omega I} \qquad \cancel{\frac{V}{I} = \frac{L_0\omega}{jk}}$$

$$\cancel{jkI = C_0\omega V} \qquad \cancel{\frac{V}{I} = \frac{jk}{C_0\omega}}$$

$$(-j)kV = -L_0(-j)(-\omega)I \qquad -(-j)kI = C_0(-j)(-\omega)V$$

$$kV = L_0\omega I \qquad kI = C_0\omega V$$

$$\frac{V}{I} = \frac{L_0\omega}{k} = L_0\frac{I}{C_0 V} \qquad \frac{\omega}{k} = \frac{I}{C_0 V}$$

$$\frac{V}{I} = L_0 \frac{I}{C_0 V} \Rightarrow \frac{V^2}{I^2} = \frac{L_0}{C_0} \Rightarrow Z = \frac{V}{I} = \sqrt{\frac{L_0}{C_0}}$$

CHARACTERISTIC IMPEDANCE FOR FORWARD & BACKWARD PROPAGATING WAVE

FOR WAVES TRAVELLING IN THE NEGATIVE DIRECTION:

$$V = V_0 \exp[-j(kx + \omega t)]$$

$$I = I_0 \exp[-j(kx + \omega t)]$$

$$\partial_x V = -L_0 \dot{I} \qquad -\partial_x I = C_0 \dot{V}$$

$$-jkV = -L_0(-j)\omega I \qquad -jkI = C_0 - j\omega V$$

$$kI = C_0 \omega V$$

$$kV = -L_0 \omega I \qquad \Rightarrow \frac{\omega}{k} = \frac{I}{C_0 V}$$

$$\frac{V}{I} = -\frac{L_0 \omega}{k} = -L_0 \frac{I}{C_0 V} \Rightarrow \frac{V^2}{I^2} = -\frac{L_0}{C_0}$$

$$= -\sqrt{\frac{V^2}{I^2}} = \sqrt{\frac{L_0}{C_0}} =$$

WHY NOT: $\frac{V}{I} = \sqrt{-\frac{L_0}{I_0}}$

$$Z = \frac{V}{I} = i\sqrt{\frac{L_0}{I_0}}$$

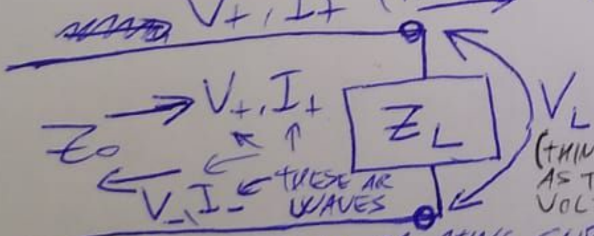
$$-Z = -\frac{V}{I} = \sqrt{\frac{L_0}{C_0}}$$

$$Z = -\sqrt{\frac{L_0}{C_0}}$$

DIMENSION OF CHARACTERISTIC IMPEDANCE IS PURE RESISTANCE.

REFLECTIONS FROM THE END OF A TRANSMISSION LINE

V_+, I_+ (THIS WAY PROPAGATING CURRENT & VOLTAGE WAVES)



(THIS WAY PROPAGATING CURRENT & VOLTAGE WAVES)

$V_\pm$

INCOMING + REFLECTED = TRANSMITTED AT BOUNDARY

BCs: $V_+ + V_- = V_L \qquad \frac{V_+}{I_+} = Z_0$

$I_+ + I_- = I_L \qquad \frac{V_-}{I_-} = -Z_0$

$$\frac{V_L}{I_L} = Z_L$$

$$Z_L = \frac{V_L}{I_L} = \frac{V_+ + V_-}{I_+ + I_-} = \frac{V_+ + V_-}{\frac{V_+}{Z_0} + \frac{V_-}{-Z_0}} = \frac{Z_0 V_+ + Z_0 V_-}{V_+ - V_-}$$

IMPEDANCE OF BACKWARD PROPAGATING WAVE.

$$Z_L = \frac{Z_0 V_+ + Z_0 V_-}{V_+ - V_-}$$

$$Z_L (V_+ - V_-) = Z_0 V_+ + Z_0 V_-$$

$$V_+ (Z_L - Z_0) = (Z_0 + Z_L) V_-$$

$$\cancel{\frac{V_-}{V_+} = \frac{Z_L - Z_0}{Z_L + Z_0}}$$

$$\boxed{\frac{V_-}{V_+} = \frac{Z_L - Z_0}{Z_L + Z_0} = r}$$

(VOLTAGE AMPLITUDE REFLECTION COEFFICIENT)

$$Z_L = \frac{Z_0 V_+ + Z_0 V_-}{V_+ - V_-} = \frac{Z_0 \left(\overbrace{\cancel{Z_0} I_+}^{V_+} + \overbrace{\cancel{Z_0} I_-}^{V_-}\right)}{Z_0 I_+ - Z_0 I_-} = \frac{Z_0 I_+ + Z_0 I_-}{I_+ - I_-}$$

$$\Rightarrow Z_L (I_+ - I_-) = Z_0 (I_+ + I_-)$$

$$(Z_L - Z_0) I_+ = (Z_0 + Z_L) I_-$$

$$\boxed{\frac{I_-}{I_+} = \frac{Z_L - Z_0}{Z_L + Z_0} = r}$$

(CURRENT AMPLITUDE REFLECTION COEFFS)

$$Z_L = \frac{Z_0 (V_+ + V_-)}{V_+ - V_-} = \frac{Z_0 V_L}{V_+ - (V_L - V_+)} = \frac{Z_0 V_L}{2V_+ - V_L} \Rightarrow (2V_+ - V_L) Z_L = Z_0 V_L$$

$$\Rightarrow 2V_+ Z_L = (Z_0 + Z_L) V_L \Rightarrow \boxed{\frac{V_L}{V_+} = \frac{2Z_L}{Z_L + Z_0} = t_V}$$

VOLTAGE AMPLITUDE TRANSMISSION ~~REFLECTION~~ COEFFICIENT

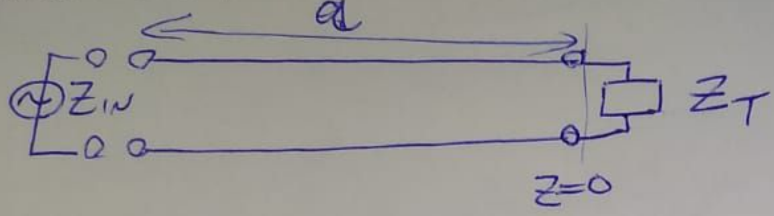
$$\frac{I_L}{I_+} = \frac{\frac{V_L}{Z_L}}{\frac{V_+}{Z_0}} = \frac{V_L}{V_+} \frac{Z_0}{Z_L} = \boxed{\frac{2Z_0}{Z_L + Z_0} = t_I}$$

CURRENT AMPLITUDE TRANSMISSION COEFFICIENT

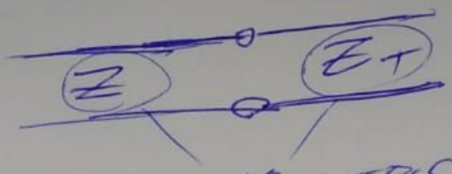
INPUT IMPEDANCE

[CHANGE TO NOTES NOTATION]

⊖ DIRECTION ⊕



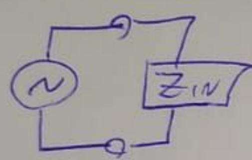
OR IMAGINE IT LIKE THIS:



CHARACTERISTIC IMPEDANCE.

(IF Z_T IS REAL SINCE CHARACTERISTIC IMPEDANCE IS ALWAYS REAL, IE RESISTIVE)

Z_{IN} IS WHAT THE POWER SUPPLY "SEES".



$$V_i = V_1 \exp(-j(kz-\omega t))$$

$$V_r = r V_1 \exp(-j(-kz-\omega t))$$

IE THE REFLECTED WAVE HAS AMPLITUDE r TIMES THE INCOMING WAVE & TRAVELS IN THE ⊖ DIR., THAT'S WHY THE EXTRA MINUS SIGN.

$$I_i = \frac{V_i}{Z} \qquad I_r = -\frac{V_r}{Z}$$

$$Z_{IN} = \left.\frac{V_i + V_r}{I_i + I_r}\right|_{z=-a} = \left.\frac{V_1 e^{-jkz} e^{j\omega t} + r V_1 e^{jkz} e^{j\omega t}}{\frac{V_1}{Z} e^{-jkz} e^{j\omega t} + \frac{V_1}{-Z} \cdot r e^{jkz} e^{j\omega t}}\right|_{z=-a}$$

$$= \frac{e^{jka} + r e^{-jka}}{e^{jka} - r e^{-jka}} Z \qquad \text{RECALL: } r = \frac{Z_t - Z}{Z_t + Z}$$

$$\frac{Z_{IN}}{Z} = \frac{e^{jka} + \frac{Z_t - Z}{Z_t + Z} e^{-jka}}{e^{jka} - \frac{Z_t - Z}{Z_t + Z} e^{-jka}} = \frac{(Z_t + Z) e^{jka} + (Z_t - Z) e^{-jka}}{(Z_t + Z) e^{jka} - (Z_t - Z) e^{-jka}} =$$

$$= \frac{2 Z_t \cos(ka) + 2 j Z \sin(ka)}{2 j Z_t \sin(ka) + 2 Z \cos(ka)} = \frac{Z_t \cos(ka) + j Z \sin(ka)}{Z \cos(ka) + j Z_t \sin(ka)}$$

SHORT-CIRCUITED LINE

$$Z_t \to 0 \Rightarrow \frac{Z_{IN}}{Z} = \frac{jZ\sin(ka)}{Z\cos(ka)} = j\tan ka$$

OPEN-CIRCUITED LINE

$$Z_t \to \infty \Rightarrow \frac{Z_{IN}}{Z} \to \frac{Z_t\cos(ka)}{jZ_t\sin(ka)} = -j\cot(ka)$$

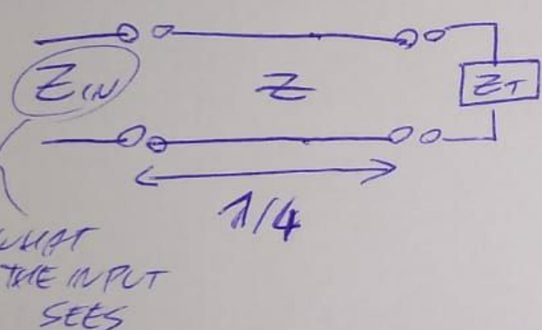
QUARTER-WAVELENGTH LINE

length of transmission line $= \frac{\lambda}{4}$

$$a = \frac{\lambda}{4} \qquad k = \frac{2\pi}{\lambda} \to \lambda = \frac{2\pi}{k}$$

$$a = \frac{\frac{2\pi}{k}}{4} = \frac{\pi}{2k} \Rightarrow ak = \frac{\pi}{2}$$

$$\cos ka = \cos\frac{\pi}{2} = 0$$



$$\Rightarrow \frac{Z_{IN}}{Z} = \frac{jZ\sin\left(\frac{\pi}{2}\right)}{jZ_t\sin\left(\frac{\pi}{2}\right)} = \frac{Z}{Z_t}$$

$$\Rightarrow \boxed{Z_{IN}Z_t = Z^2}$$

IF $Z = Z_t$, THEN, SINCE $Z_{IN} = \left.\frac{Z^2}{Z_t}\right|_{Z=Z_t} = Z$

IMPEDANCE OF THE LOAD IS THE SAME IMPEDANCE THE INPUT SEES, SO THERE ARE NO REFLECTIONS.

SAME

Another way of looking at this:
[NOTATION CHANGE: $Z = Z_0$, $Z_t = Z_L$]

recall: $r = \frac{Z_L - Z_0}{Z_L + Z_0}$ IF $Z_L - Z_0 = 0 \Rightarrow Z_L = Z_0$, $r = 0$

Z_0: CHARACTERISTIC IMPEDANCE

Z_L: IMPEDANCE OF LOAD. [BEWARE MIXED NOTATION]

No reflection when $Z = Z_{IN}$. $1 = \frac{Z_{IN}}{Z} = \frac{Z}{Z_t}$ IF $a = \frac{\lambda}{4}$ (SEE ABOVE)

↳ at load-transmission line boundary

No REFLECTION AT ALL.

if $\frac{Z}{Z_t} = 1$, there won't be reflection on the transmission line - load boundary.