2015P1Q5(I) [SEEN THIS IN A PHYSICS WRITING]

(a) $AB=BA$

$Av_i = \lambda_i \cdot v_i$ v_i: EIGENVECTOR

λ_i: CORRESP. EIGENVALUE

$$\underbrace{BA}v_i = B\lambda_i v_i = \lambda_i B v_i = \underbrace{AB}_{\text{SINCE: } AB=BA} v_i$$

$AB v_i = \lambda_i B v_i$

$\Rightarrow$ Bv_i IS AN EIGENVECTOR OF A, WITH EIGENVALUE λ_i.

$\Rightarrow$ Bv_i MUST BE IN THE SAME DIRECTION AS v_i.

$\Rightarrow$ $Bv_i = \mu_i v_i$, SO v_i IS ALSO AN EIGENVECTOR OF B.

THIS CAN BE DONE FOR ANY v_i OF A, SO A & B MUST HAVE THE SAME EIGENVECTORS.

$$\exp A \exp B = \left(I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \ldots\right)\left(I + B + \frac{1}{2}B^2 + \frac{1}{3!}B^3 + \ldots\right) =$$

$$= I + A + B + \frac{1}{2}A^2 + \frac{1}{2}B^2 + AB + \ldots$$

$$\exp(A+B) = I + A + B + \frac{1}{2}(A+B)^2 + \ldots = I + A + B + \frac{1}{2}A^2 + \frac{1}{2}B^2 +$$

$$+ \frac{1}{2}AB + \frac{1}{2}BA + \ldots =$$

$$= I + A + B + \frac{1}{2}A^2 + \frac{1}{2}B^2 + AB + \ldots =$$

$$= \exp(A)\exp(B)$$

HOPEFULLY.

TRY ANOTHER METHOD:

$A = U\Lambda_A U^+$

$B = U\Lambda_B U^+$

$\exp A = I + U\Lambda_A U^+ + \frac{1}{2!}U\Lambda_A^2 U^+ + \frac{1}{3!}U\Lambda_A^3 U^+ + \ldots$

$\exp B = I + U\Lambda_B U^+ + \frac{1}{2!}U\Lambda_B^2 U^+ + \frac{1}{3!}U\Lambda_B^3 U^+ + \ldots$

$\boxed{\text{2015 P1 Q5(II)}}$

$$\exp A \exp B = I + U\Lambda_A U^+ + U\Lambda_B U^+ + U\Lambda_A\Lambda_B U^+ +$$
$$+ \frac{1}{2!}U\Lambda_A^2 U^+ + \frac{1}{2!}U\Lambda_B^2 U^+ +$$
$$+ \frac{1}{2!}U\Lambda_A^2\Lambda_B U^+ + \frac{1}{2!}U\Lambda_B^2\Lambda_A U^+ + \frac{1}{3!}U\Lambda_A^3 U^+ + \frac{1}{3!}U\Lambda_B^3 U^+ +$$
$$+ \ldots =$$

$$= U\left(I + \Lambda_A + \Lambda_B + \Lambda_A\Lambda_B + \frac{1}{2!}\Lambda_A^2 + \frac{1}{2!}\Lambda_B^2 + \frac{1}{2!}\Lambda_A^2\Lambda_B + \frac{1}{2!}\Lambda_B^2\Lambda_A + \right.$$
$$\left. + \frac{1}{3!}\Lambda_A^3 + \frac{1}{3!}\Lambda_B^3 + \ldots\right)U^+ =$$

$$= U\left(\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{k!}\Lambda_A^k\frac{1}{(n-k)!}\Lambda_B^{n-k}\right)U^+$$

$$\exp(A+B) = \exp\left(U(\Lambda_A+\Lambda_B)U^+\right) = I + U(\Lambda_A+\Lambda_B)U^+ + \frac{1}{2!}\left[U(\Lambda_A+\Lambda_B)U^+\right]^2 +$$
$$+ \frac{1}{3!}\left[U(\Lambda_A+\Lambda_B)U^+\right]^3 + \ldots = I + U(\Lambda_A+\Lambda_B)U^+ + \frac{1}{2!}U(\Lambda_A+\Lambda_B)^2U^+ +$$
$$+ \frac{1}{3!}U(\Lambda_A+\Lambda_B)^3U^+ + \ldots = U\left(I + \Lambda_A\Lambda_B + \frac{1}{2!}(\Lambda_A+\Lambda_B)^2 + \frac{1}{3!}(\Lambda_A+\Lambda_B)^3 + \right.$$
$$\left. \ldots\right)U^+ = U\left(\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\Lambda_A^k\Lambda_B^{n-k}\binom{n}{k}\right)U^+ =$$

$$= U\left(\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\frac{n!}{(n-k)!\,k!}\Lambda_A^k\Lambda_B^{n-k}\right)U^+ =$$

$$= U\left(\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{k!}\frac{1}{(n-k)!}\Lambda_A^k\Lambda_B^{n-k}\right)U^+$$

THE TWO EXPANSIONS ARE EQUAL SO IT MUST BE THAT:

$$\exp(A)\exp(B) = \exp(A+B)$$

2015P1Q5(III)) WE WANT TO PROVE:

$$\exp(\varepsilon X)\exp(\varepsilon Y)=\exp\left(\varepsilon X+\varepsilon Y+\tfrac{1}{2}\varepsilon^2[X,Y]\right)+O(\varepsilon^3)$$

$$\exp(\varepsilon X)=I+\varepsilon X+\tfrac{1}{2!}(\varepsilon X)^2+\tfrac{1}{3!}(\varepsilon X)^3+\dots$$

$$\exp(\varepsilon Y)=I+\varepsilon Y+\tfrac{1}{2!}(\varepsilon Y)^2+\tfrac{1}{3!}(\varepsilon Y)^3+\dots$$

$$\exp\left(\varepsilon X+\varepsilon Y+\tfrac{1}{2}\varepsilon^2[X,Y]\right)=$$

$$=I+\varepsilon X+\varepsilon Y+\tfrac{1}{2}\varepsilon^2[X,Y]+\tfrac{1}{2}\underbrace{\left(\varepsilon X+\varepsilon Y+\tfrac{1}{2}\varepsilon^2[X,Y]\right)^2}+O(\varepsilon^3)=$$

$$=\varepsilon^2XX+\varepsilon^2YY+\tfrac{1}{4}\varepsilon^4[X,Y]^2+\varepsilon^2XY+\varepsilon^2YX+O(\varepsilon^3)=$$

$$=\varepsilon^2XX+\varepsilon^2YY+\varepsilon^2XY+\varepsilon^2YX+O(\varepsilon^3)$$

$$=I+\varepsilon X+\varepsilon Y+\tfrac{1}{2}\varepsilon^2(XY-YX)+\tfrac{1}{2}\varepsilon^2(XX+YY+XY+YX)+O(\varepsilon^3)=$$

$$=I+\varepsilon X+\varepsilon Y+\tfrac{1}{2}\varepsilon^2(XX+YY)+\varepsilon^2XY+O(\varepsilon^3)$$

$$\exp(\varepsilon X)\exp(\varepsilon Y)=\left(I+\varepsilon X+\tfrac{1}{2}(\varepsilon X)^2+\tfrac{1}{3!}(\varepsilon X)^3+\dots\right)\left(I+\varepsilon Y+\tfrac{1}{2}(\varepsilon Y)^2+\tfrac{1}{3!}(\varepsilon Y)^3+\dots\right)=$$

$$=I+\varepsilon X+\varepsilon Y+\tfrac{1}{2}\varepsilon X^2+\tfrac{1}{2}\varepsilon Y^2+\varepsilon^2XY+O(\varepsilon^3)$$

THESE TWO ARE EQUAL, SO IT MUST BE THAT:

$$\exp\left(\varepsilon X+\varepsilon Y+\tfrac{1}{2}\varepsilon^2[X,Y]\right)+O(\varepsilon^3)=$$
$$=\exp(\varepsilon X)\exp(\varepsilon Y)$$

2015P1Q5(IV)

$M=\begin{pmatrix}0&a&0&0\\a&0&0&0\\0&0&0&b\\0&0&-b&0\end{pmatrix}$ $M^2=\begin{pmatrix}a^2&0&0&0\\0&a^2&0&0\\0&0&-b^2&0\\0&0&0&-b^2\end{pmatrix}$ $M^3=\begin{pmatrix}0&a^3&0&0\\a^3&0&0&0\\0&0&0&-b^3\\0&0&b^3&0\end{pmatrix}$

$M^4=\begin{pmatrix}a^4&0&0&0\\0&a^4&0&0\\0&0&b^4&0\\0&0&0&b^4\end{pmatrix}$ $M^5=\begin{pmatrix}0&a^5&0&0\\a^5&0&0&0\\0&0&0&b^5\\0&0&-b^5&0\end{pmatrix}$ $M^6=\begin{pmatrix}a^6&0&0&0\\0&a^6&0&0\\0&0&-b^6&0\\0&0&0&-b^6\end{pmatrix}$

$n\in\mathbb{Z}_0^+$

$$M^{2n}=\begin{pmatrix}(a^2)^n&0&0&0\\0&(a^2)^n&0&0\\0&0&(-b^2)^n&0\\0&0&0&(-b^2)^n\end{pmatrix}$$

$$M^{2n+1}=\begin{pmatrix}0&a^{2n+1}&0&0\\a^{2n+1}&0&0&0\\0&0&0&[(-1)^n b^{2n+1}]\\0&0&[(-1)^{n+1}b^{2n+1}]&0\end{pmatrix}$$

$$\exp(M)=I+M+\frac{1}{2!}M^2+\frac{1}{3!}M^3+\dots=\sum_{n=0}^{\infty}\frac{1}{n!}M^n=$$

$\cancel{\frac{1}{n!}a^{2n}}$ ~~=~~

2015P1 Q5(v)

$$= \begin{pmatrix} \sum_{n=0}^{\infty} \frac{1}{(2n)!} a^{2n} & \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} a^{2n+1} & 0 & 0 \\ \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} a^{2n+1} & \sum_{n=0}^{\infty} \frac{1}{(2n)!} a^{2n} & 0 & 0 \\ 0 & 0 & \sum_{n=0}^{\infty} \frac{1}{(2n)!} (-b^2)^n & \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (-1)^n b^{2n+1} \\ 0 & 0 & \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (-1)^{n+1} b^{2n+1} & \sum_{n=0}^{\infty} \frac{1}{(2n)!} (-b^2)^n \end{pmatrix} =$$

$$= \begin{pmatrix} \cosh a & \sinh a & 0 & 0 \\ \sinh a & \cosh a & 0 & 0 \\ 0 & 0 & \cos b & \sin b \\ 0 & 0 & -\sin b & \cos b \end{pmatrix}$$

AS DESIRED.