2016P2Q5(I) $\tilde{y}(\omega) = \dfrac{-\omega \tilde{f}(\omega)}{\omega^3 - i\omega^2 + 4\omega - 4i}$

(a)

$$f(t) = y'(t) \iff \tilde{f}(\omega) = i\omega \tilde{y}(\omega)$$

$$(\omega^3 - i\omega^2 + 4\omega - 4i)\tilde{y}(\omega) = -\omega \tilde{f}(\omega)$$

$$(iy''' + iy'' + 4(-i)y' - 4iy) = -if$$

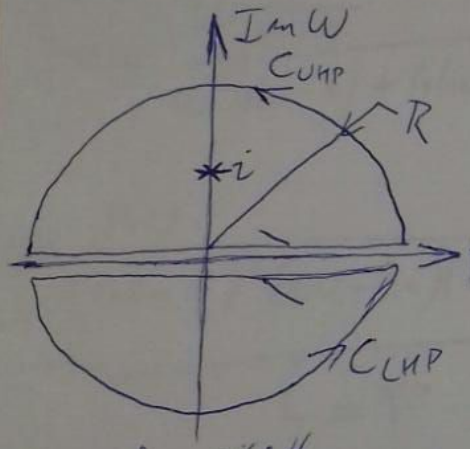
$$y''' + y'' - 4y' - 4y = f$$

THIRD ORDER ODE

(b)

$$f(t) = \mathrm{IFT}\left[\tilde{f}(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-i}{\omega - i} e^{i\omega t} d\omega =$$

POLES: $\omega = i$



$$= \frac{-i}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\omega - i} e^{i\omega t} d\omega$$

$$\int_{-\infty}^{\infty} \frac{1}{\omega - i} e^{i\omega t} d\omega = \lim_{R\to\infty}\left(\oint_{C_{UHP}} \frac{e^{i\omega t}}{\omega - i} d\omega - \int_{\text{SEMICIRCLE PART OF } C_{UHP}} \frac{e^{i\omega t}}{\omega - i} d\omega\right)$$

$\lim_{R\to\infty} \int_{\text{SEMICIRCLE PART OF } C_{UHP}} \frac{e^{i\omega t}}{\omega - i} d\omega = 0$ FOR $t > 0$ BY JORDAN'S LEMMA.

$$\int_{\infty}^{-\infty} \frac{1}{\omega - i} e^{i\omega t} d\omega = \lim_{R\to\infty}\left(\oint_{C_{LHP}} \frac{e^{i\omega t}}{\omega - i} d\omega - \int_{\text{SEMICIRCLE PART OF } C_{LHP}} \frac{e^{i\omega t}}{\omega - i} d\omega\right)$$

$\lim_{R\to\infty} \int_{\text{SEMICIRCLE PART OF } C_{LHP}} \frac{e^{i\omega t}}{\omega - i} d\omega = 0$ FOR $t < 0$ BY JORDAN'S LEMMA

FOR $t > 0$:

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{\omega - i} e^{i\omega t} d\omega = \lim_{R\to\infty} \oint_{C_{UHP}} \frac{e^{i\omega t}}{\omega - i} d\omega = 2\pi i \left(\mathrm{RES}\ \frac{e^{i\omega t}}{\omega - i}\right)\Big|_{\omega = i}$$

FOR $t < 0$:

$$\int_{\infty}^{-\infty} \frac{1}{\omega - i} e^{i\omega t} d\omega = \lim_{R\to\infty} \oint_{C_{LHP}} \frac{e^{i\omega t}}{\omega - i} d\omega = 0 \quad (\text{NO POLES IN } C_{LHP})$$

$$\hookrightarrow \int_{-\infty}^{\infty} \frac{1}{\omega - i} e^{i\omega t} d\omega = 0$$

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$$2\pi i \left(\text{RES} \frac{e^{iwt}}{w-i} \right)\Big|_{w=i} = 2\pi i\, e^{it} = 2\pi i e^{-t}$$

$$f(t) = \frac{-i}{2\pi} 2\pi i e^{-t} = e^{-t} \quad \text{FOR } t>0$$

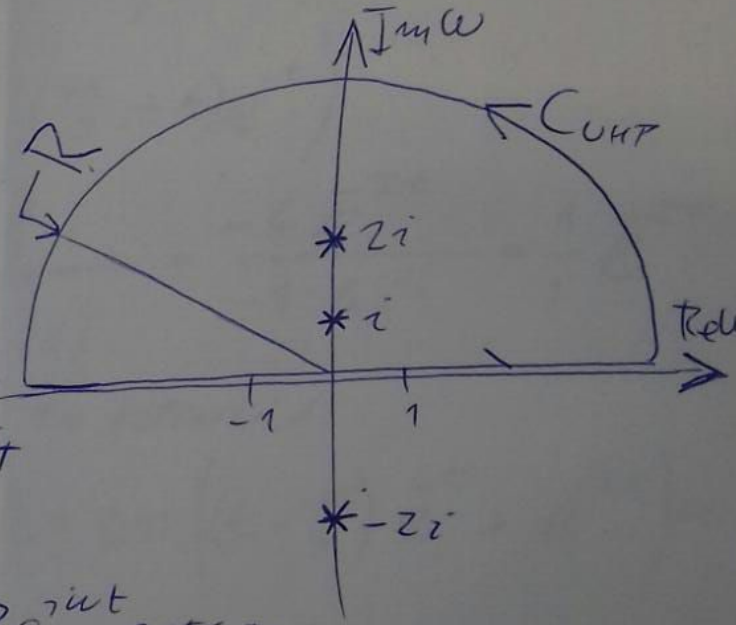
$$f(t) = 0 \quad \text{FOR } t<0 \qquad \text{FOR } t<0: \ \frac{1}{2\pi}\int_{-\infty}^{\infty} 0\, e^{iwt}\, dw = 0$$

(c) $$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{y}(w) e^{iwt}\, dw = \begin{cases} \\ \text{FOR } t>0 \end{cases}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-w\frac{-i}{w-i}}{w^3 - iw^2 + 4w - 4i} e^{iwt}\, dw$$

$$= \frac{w\frac{i}{w-i}}{w^2(w-i)+4(w-i)} = \frac{wi}{w^2(w-i)^2+4(w-i)^2} = \frac{wi}{(w-i)^2(w^2+4)} =$$

$$= \underbrace{\frac{wi}{(w-i)^2(w-2i)(w+2i)}}_{L=P} \Rightarrow \begin{array}{l} \text{SIMPLE POLES AT: } w = \pm 2i \\ \text{DOUBLE POLE AT: } w = i \end{array}$$



$$\frac{1}{2\pi}\int_{-\infty}^{\infty} P e^{iwt}\, dw =$$

$$\lim_{R\to\infty} \left(\oint_{C_{UHP}} P e^{iwt}\, dw - \int_{\substack{\text{SEMICIRCLE} \\ \text{PART OF} \\ C_{UHP}}} P e^{iwt}\, dw \right) \cdot \frac{1}{2\pi}$$

$\lim_{R\to\infty} \int_{\text{SEMICIRCLE PART OF } C_{UHP}} P e^{iwt}\, dw = 0$ BECAUSE $P e^{iwt}$ GOES TO 0 FASTER IN THE UPPER HALF PLANE W/ $z>0$ THAN THE ARCLENGTH OF THE SEMICIRCLE GOES TO ∞ AS $R\to\infty$.

SO WE HAVE: $$\int_{-\infty}^{\infty} P e^{iwt}\, dw = \oint_{C_{UHP}} P e^{iwt}\, dw = 2\pi i \sum_{\substack{\text{POLES} \\ \text{IN } C_{UHP}}} \text{RES}\left(P e^{iwt} \right)$$ BY CAUCHY'S THEOREM

2016P2Q5(iii) • CALCULATE THE RESIDUES

$$\operatorname*{RES}_{w=i}\left(Pe^{i\omega t}\right)=\lim_{w\to i}\frac{d}{dw}\left[(w-i)^2\frac{wie^{i\omega t}}{(w-i)^2(w-2i)(w+2i)}\right]=$$

$$=\lim_{w\to i}\frac{d}{dw}\left[\frac{wie^{i\omega t}}{w^2+4}\right]=i\lim_{w\to i}\frac{(w^2+4)(we^{i\omega t})'-}{}$$

$$\frac{-wie^{i\omega t}(w^2+4)'}{(w^2+4)^2}=i\lim_{w\to i}\frac{(w^2+4)\left(w(it)e^{i\omega t}+e^{i\omega t}\right)-}{}$$

$$\frac{-wie^{i\omega t}2w}{(w^2+4)^2}=i\lim_{w\to i}\left[\left(w(it)e^{i\omega t}+e^{i\omega t}\right)-\frac{2w^2e^{i\omega t}}{(w^2+4)^2}\right]=$$

$$=i\left(i(it)e^{-t}+e^{-t}-\frac{2(i)^2e^{-t}}{3^2}\right)=$$

$$=i\left((1-t)e^{-t}+\frac{2}{9}e^{-t}\right)=i\left(\left(\frac{11}{9}-t\right)e^{-t}\right)$$

$$\operatorname*{RES}_{w=2i}\left(Pe^{i\omega t}\right)=\frac{(2i)ie^{-2t}}{(2i-i)^2(2i+2i)}=\frac{-4e^{-2t}}{-16i}=\frac{1}{i}e^{-2t}$$

• PUT THE RESIDUES BACK TO FORMULA:

$$2\pi i\left(i\left(\frac{11}{9}-t\right)e^{-t}+\frac{1}{i}e^{-2t}\right)=2\pi\left(\left(t-\frac{11}{9}\right)e^{-t}+e^{-2t}\right)=$$

$$=2\pi e^{-t}\left(\left(t-\frac{11}{9}\right)+e^{-t}\right)$$

$$y(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}Pe^{i\omega t}d\omega=e^{-t}\left(\left(t-\frac{11}{9}\right)+e^{-t}\right)\quad\text{FOR } t>0$$

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$$y(t) = \begin{cases} 0 & \text{FOR } t < 0 \\ e^{-t}\left(\left(\blacksquare\, t - \frac{11}{9}\right) + e^{-t}\right) & \text{FOR } t > 0 \end{cases}$$