2010P1Q9(I) EULER EQUATION: $f_y(x,y,y') - \frac{d}{dx} f_{y'}(x,y,y') = 0$

EXPANDED FORM:

$$f_y - f_{y'x} - f_{y'y}\,y' - f_{y'y'}\,y'' = 0$$

IF $f = f(y, y')$ THEN $f_{y'x} = 0$

SO EULER EQUATION BECOMES: $f_y - f_{y'y}\,y' - f_{y'y'}\,y'' = 0$

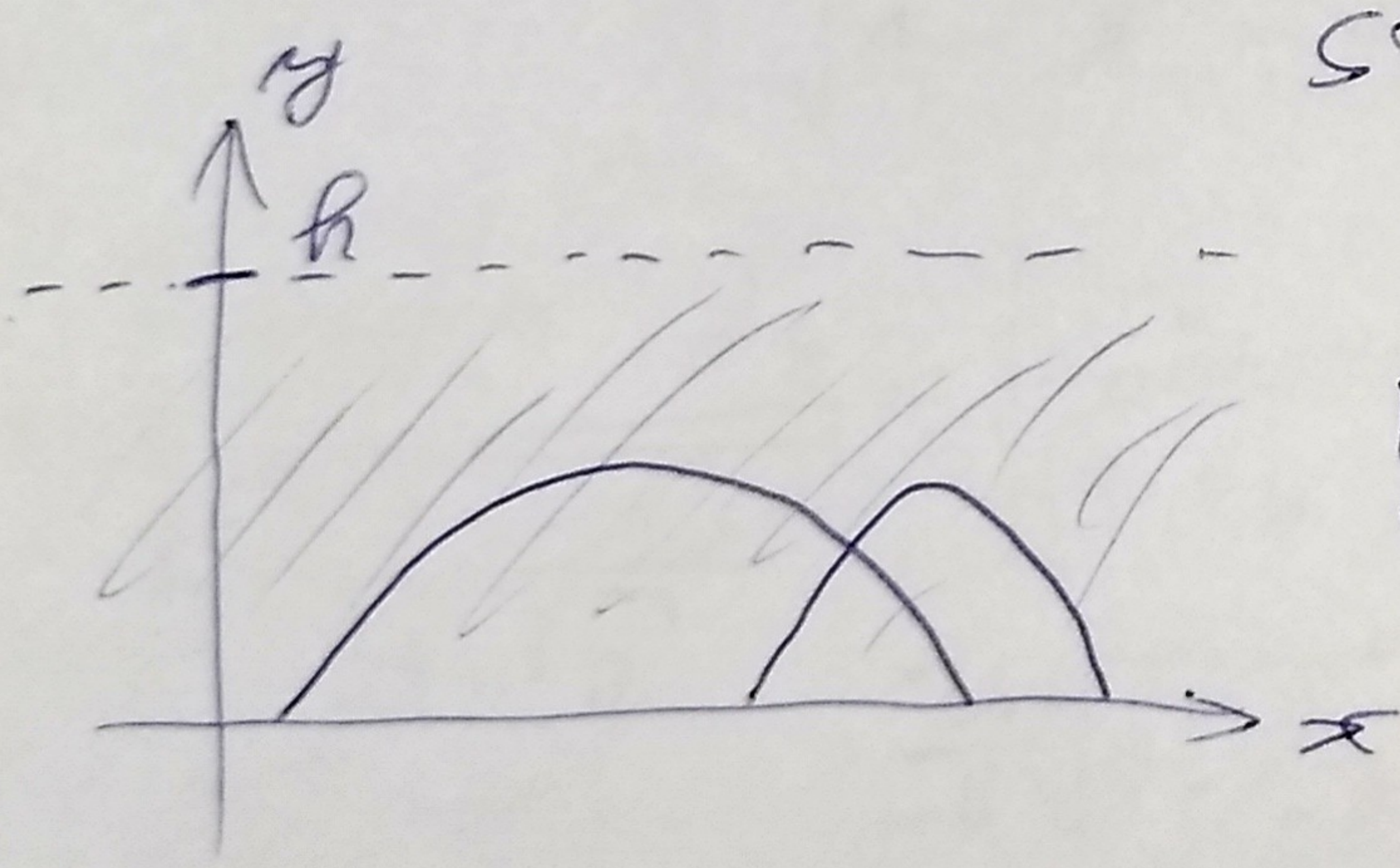
ALSO, WE FORM: $\frac{d}{dx}(f - y' f_{y'}) = f_y y' + f_{y'} y'' - \frac{d}{dx}(y' f_{y'}) =$

$$= f_y y' + \cancel{f_{y'} y''} - \cancel{y'' f_{y'}} - y' f_{y'y} y' - y' f_{y'y'} y'' =$$

$$= y' \underbrace{\left(f_y - f_{y'y} y' - f_{y'y'} y''\right)}_{=0,\ \text{BY EULER EQ.}} = 0$$

THUS: $f - y' f_{y'} =$ CONSTANT

SO: $y' f_{y'} - f = A$



PATH LENGTH ELEMENT OF LIGHT RAY: $\sqrt{(dx)^2 + (dy)^2} = ds$

TIME NEEDED FOR LIGHT TO GET THROUGH THIS LENGTH ELEMENT: $\frac{ds}{c(y)} = \frac{\sqrt{(dx)^2 + (dy)^2}}{c(y)} = \frac{\sqrt{1 + y'^2}\,dx}{c(y)}$

FROM FERMAT'S PRINCIPLE, LIGHT RAY FOLLOWS THE CURVE WHICH MINIMISES: $T = \int \frac{ds}{c(y)} = \int \frac{\sqrt{1+y'^2}}{c(y)}\,dx$

INTEGRAND DOES NOT CONTAIN x EXPLICITLY, SO WE CAN USE THE PREVIOUSLY DERIVED RESULT:

$$\underline{y' \frac{\partial f}{\partial y'} - f = A}$$

$$y' \frac{1}{c(y)} \frac{1}{2} (1 + y'^2)^{-\frac{1}{2}} 2y' - \frac{1}{c(y)} (1 + y'^2)^{\frac{1}{2}} = A$$

MAKE COMMON DENOMINATORS:

$$\frac{y'^2}{c(y)(1+y'^2)^{\frac{1}{2}}} - \frac{1 \cdot (1 + y'^2)}{c(y)(1+y'^2)^{\frac{1}{2}}} = A$$

2010P1Q9(II) SIMPLIFY:

$$\frac{-1}{C(y)(1+y'^2)^{\frac{1}{2}}} = A$$

REARRANGE, WITH THE AIM OF FINDING WHAT y' EQUALS TO:

$$+1 = A^2 C^2(y)(1+y'^2)$$

$$\frac{1-A^2C^2(y)}{A^2C^2(y)} = y'^2$$

$$y' = \pm\frac{\sqrt{1-A^2C^2(y)}}{A\,C(y)}$$

SEPARATE VARIABLES, THEN INTEGRATE:

$$\int\frac{C(y)\,dy}{\sqrt{1-A^2C^2(y)}} = \pm\int\frac{dx}{A}$$

PLUG IN FORMULA FOR $C(y)$:

$$\int\frac{\frac{C_0}{\sqrt{1-ky}}\,dy}{\sqrt{1-A^2\frac{C_0^2}{1-ky}}} = \int\frac{\frac{C_0}{\sqrt{1-ky}}\,dy}{\sqrt{\frac{1-ky}{1-ky}-A^2\frac{C_0^2}{1-ky}}} =$$

$$= \int\frac{\frac{C_0}{\sqrt{1-ky}}\,dy}{\frac{\sqrt{1-ky-A^2C_0^2}}{\sqrt{1-ky}}} = \int\frac{C_0\,dy}{\sqrt{1-ky-A^2C_0^2}} = \pm\int\frac{dx}{A}$$

$$\frac{2C_0}{-k}\sqrt{1-ky-A^2C_0^2} = \pm\frac{x}{A}+B$$ (B IS A CONSTANT)

$$\frac{4C_0^2}{k^2}\left(1-ky-A^2C_0^2\right) = \left(\pm\frac{x}{A}+B\right)^2$$

$$-ky = \frac{k^2}{4C_0^2}\left(\pm\frac{x}{A}+B\right)^2 + A^2C_0^2 - 1$$

$$y = -\frac{k}{4C_0^2}\left(\pm\frac{x}{A}+B\right)^2 - \frac{A^2C_0^2}{k} + \frac{1}{k}$$

THIS IS A PARABOLA, AS REQUIRED.

IF $y(x=\pm x_0)=0 \Rightarrow B=0$

SO IN THIS CASE WE HAVE:

$$y = -\frac{k}{4C_0^2}\left(\frac{x}{A}\right)^2 - \frac{A^2C_0^2}{k} - \frac{1}{k} = -\frac{k}{4C_0^2}\left(\frac{x}{A}\right)^2 - \frac{A^2C_0^2-1}{k}$$

2010 P1 Q 9 (III)

RECALL: $1 = A^2 C^2(y)(1+y'^2)$

IF $y' = 0 \Rightarrow y = y_0$

$$1 = A^2 C^2(y_0) = A^2 \frac{C_0^2}{1-ky_0}$$

$$\Rightarrow 1 - ky_0 = A^2 C_0^2$$

$$\hookrightarrow A^2 = \frac{1-ky_0}{C_0^2}$$

PLUG THIS IN TO OUR EQUATION:

$$y = -\frac{k}{4C_0^2}\left(\frac{x}{A}\right)^2 - \frac{1-ky_0-1}{k}$$

$$y = -\frac{k}{4C_0^2}\left(\frac{x}{A}\right)^2 + \frac{ky_0}{k} = -\frac{k}{4}\left(\frac{x}{A}\right)^2 + y_0$$

$y(\pm x_0) = 0$

$$\Rightarrow 0 = -\frac{k}{4}\left(\frac{x_0}{C_0A}\right)^2 + y_0$$

$$\frac{k}{4}\frac{x_0^2}{(C_0A)^2} = y_0$$

$$\frac{k}{4}\frac{x_0^2}{C_0^2\underbrace{\frac{1-ky_0}{C_0^2}}} = y_0$$

$$\frac{kx_0^2}{4(1-ky_0)} = y_0$$

$$kx_0^2 = 4y_0(1-ky_0) \qquad \cdot k$$

$$(kx_0)^2 = 4ky_0(1-ky_0)$$

AS REQUIRED.