PII
2009Q10 (1)

$a^2=b^2=c^2=1$

$V\Rightarrow$

$$\boxed{\begin{matrix} 1 & a & b & c \\ a & 1 & c & b \\ b & c & 1 & a \\ c & b & a & 1 \end{matrix}}$$

✓

✓

$ab=c$

$abb=cb=a$

$ac\,cb=aca$

$ab=c$

$acbc=c$

$acb=1$

$ac=b^{-1}=b$

g_1, g_2 conjugate

$g_2 g = g g_1 = g_1 g \Rightarrow g_1 = g_2$ as Abelian each elt in own conjugacy class.

CONJUGACY CLASSES OF V: ✓

$\{1\}\ \{a\}\ \{b\}\ \{c\}$

explain why.

AN IRREDUCIBLE REP OF V:

~~$1=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ $a=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ $b=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$ $c=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$~~

$\{1, a, b, c\} \to \{1, 1, 1, 1\}$ ✓

NUMBER OF CONJUGACY CLASSES AND INEQUIVALENT IRREPS ARE EQUAL.

V HAS 4 INEQ. IRREP. ✓

REP1: $\{1, a, b, c\} \to \{1, 1, 1, 1\}$ ✓

REP2: $\{1, a, b, c\} \to \{1, -1, 1, -1\}$ ✓

REP3: $\{1, a, b, c\} \to \{1, 1, -1, -1\}$ ✓

REP4: $\{1, a, b, c\} \to \{1, -1, -1, 1\}$ ✓

$$\begin{matrix} D(1) & D(a) & D(b) & D(c) \\ D(a) & D(1) & D(c) & D(b) \\ D(b) & D(c) & D(1) & D(a) \\ D(c) & D(b) & D(a) & D(1) \end{matrix}$$

$\to$ THIS TABLE ISOMORPHIC TO V TABLE $\Rightarrow$ D IS A REP.

P(II) 2007 Q 10(II) DECOMP OF D:

$$D'(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad D'(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad D'(b) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad D'(c) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$D''(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad D''(a) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad D''(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad D''(c) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$D'''(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad D'''(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad D'''(b) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad D'''(c) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

CHARACTER TABLE FOR V:

	1	a	b	c
REP1	1	1	1	1
REP2	1	-1	1	-1
REP3	1	1	-1	-1
REP4	1	-1	-1	1
orig'l rep.	3	-1	-1	-1

ORTHOGONALITY RELATION: EVERY COL. IN DIFFERENT CONJUGACY CLASS (HERE: EVERY COL.) ARE ORTHOGONAL TO EACH OTHER.

VERIFICATION:

$$(1\ 1\ 1\ 1)\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = (1\ 1\ 1\ 1)\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = (1\ 1\ 1\ 1)\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} =$$

$$= (1\ -1\ 1\ -1)\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = (1\ -1\ 1\ -1)\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = (1\ 1\ -1\ -1)\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = 0$$

we want. here

no of copies of irrep $\times |G| = \Sigma$ char. elt in irrep $\times$ char. elt in rep.

ie $4 \times m_1 = 3\times 1 + -1\times 1 + -1\times 1 + -1\times 1$ $\therefore m_1 = 0$

$4 \times m_2 = 3\times 1 + -1\times -1 + -1\times 1 + -1\times -1$ $m_2 = 1$

$4 \times m_3 = 3\times 1 + -1\times 1 + -1\times -1 + -1\times -1$ $m_3 = 1$

$4 \times m_4 = 3\times 1 + -1\times -1 + -1\times -1 + -1\times 1$ $m_4 = 1$.

not v. surprising.

P(II) 2007 Q 10(II)

DECOMP OF D:

$D'(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $D'(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $D'(b) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $D'(c) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$D''(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $D''(a) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $D''(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $D''(c) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$D'''(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $D'''(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $D'''(b) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $D'''(c) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

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	1	a	b	c
REP1	1	1	1	1
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REP4	1	-1	-1	1
orginal rep.	3	-1	-1	-1

ORTHOGONALITY RELATION:
EVERY COL. IN DIFFERENT CONJUGACY CLASS (HERE: EVERY COL.) ARE ORTHOGONAL TO EACH OTHER.

VERIFICATION:

$$(1\ 1\ 1\ 1)\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = (1\ 1\ 1\ 1)\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = (1\ 1\ 1\ 1)\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} =$$

$$= (1\ -1\ 1\ -1)\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = (1\ -1\ 1\ -1)\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = (1\ 1\ -1\ -1)\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = 0$$

we want. here
no of copies of irrep $\times |G| = \sum$ char. elt in irrep $\times$ char. elt in rep.

ie $4 \times n_1 = 3\times 1 + -1\times 1 + -1\times 1 + -1\times 1$ $\therefore n_1 = 0$
$4 \times n_2 = 3\times 1 + -1\times -1 + -1\times 1 + -1\times -1$ $n_2 = 1$
$4 \times n_3 = 3\times 1 + -1\times 1 + -1\times -1 + -1\times -1$ $n_3 = 1$
$4 \times n_4 = 3\times 1 + -1\times -1 + -1\times -1 + -1\times 1$ $n_4 = 1.$

not v. surprising.