2017 P1Q1(I)

(a) $\int_V (\nabla \cdot \underline{G})\, dV = \oint_S \underline{G} \cdot d\underline{S}$

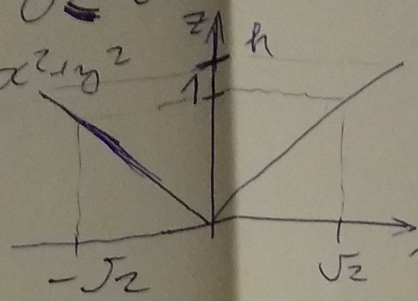
WHERE V IS A CLOSED VOLUME BOUNDED BY CLOSED SURFACE S, $d\underline{S}$ IS OUTWARD POINTING NORMAL FROM THIS SURFACE.

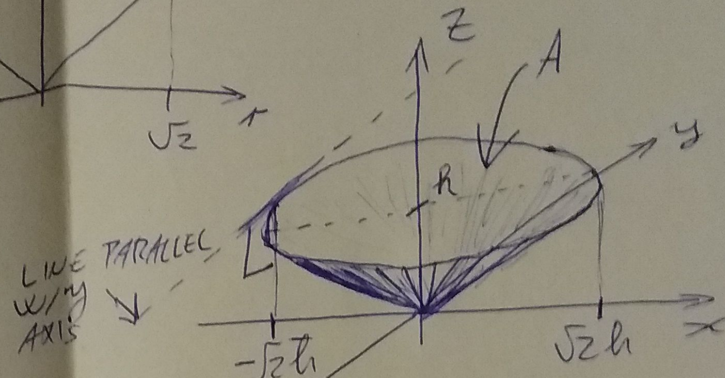
(b) $x^2 + y^2 = 2z^2 \quad 0 \leq z < h$

LET: $r^2 = x^2 + y^2$

$r^2 = 2z^2$

$z = \frac{1}{\sqrt{2}} r$





(c) LETS DEFINE A CLOSED CURVE S: $S = A \cup D$, WHERE D IS A DISC ~~CYL~~ LIEING PARALLEL WITH THE x & y AXIS, AT HEIGHT h, CENTERED ON THE Z AXIS

$$\nabla \cdot \underline{G} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \cdot \begin{pmatrix} x^3 + 2xy \\ y^3 + \sin x \\ z \end{pmatrix} =$$

$$= 3x^2 + 2y + 3y^2 + 1 = 3r^2 + 1 + 2y$$

$$\int_V (\nabla \cdot \underline{G})\, dV = \int_V 3r^2 + 1 + \underbrace{2y}_{\substack{\text{ODD FUNCTION ON} \\ \text{A SYMMETRIC INTEGRATION} \\ \text{DOMAIN} \Rightarrow \text{WE OMIT IT.} \\ \text{(WITHOUT CHANGING THE RESULT)}}} dV = \int_V 3r^2 + 1\, dV =$$

$$= \int_{z=0}^{h} \int_{r=0}^{\sqrt{2}z} \int_{\theta=0}^{2\pi} (3r^2 + 1)\, r\, dr\, d\theta\, dz = 2\pi \int_{z=0}^{h} \int_{r=0}^{\sqrt{2}z} 3r^3 + r\, dr\, dz =$$

$$= 2\pi \int_{z=0}^{h} 3\left[\frac{r^4}{4}\right]_0^{\sqrt{2}z} + \left[\frac{r^2}{2}\right]_0^{\sqrt{2}z} dz = 2\pi \int_{z=0}^{h} 3z^4 + z^2\, dz = 2\pi\left(3\left[\frac{z^5}{5}\right]_0^h + \left[\frac{z^3}{3}\right]_0^h\right) =$$

$$= 2\pi\left(\frac{3}{5}h^5 + \frac{1}{3}h^3\right)$$

2017P1Q1(II)

BY DIVERGENCE THM

$$\oint_S \underline{G}\cdot d\underline{S} = \int_V (\nabla\cdot G)\, dV$$

$$= \int_D \underline{G}\cdot d\underline{S} + \int_A \underline{G}\cdot dS$$

BY DEFINITION OF S.

REARRANGE:

$$\int_A \underline{G}\cdot d\underline{A} = \oint_S \underline{G}\cdot d\underline{S} - \int_D \underline{G}\cdot dS = \underbrace{\int_V (\nabla\cdot G)\, dV}\ - \int_D \underline{G}\cdot d\underline{S}$$

LET'S FIND: $\int_D \underline{G}\cdot dS$

$$\int_D \underline{G}\cdot d\underline{S} = \int_D \begin{pmatrix} x^3+2xy \\ y^3+\sin x \\ z \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dS = \int_D z\, dS = \int_D h\, dS =$$

$$= (\sqrt{2}h)^2 \pi\ h = 2\pi h^3$$

RETURN to $\int_A \underline{G}\cdot d\underline{A}$:

$$\int_A \underline{G}\cdot d\underline{A} = \overbrace{2\pi\left(\frac{3}{5}h^5 + \frac{1}{3}h^3\right)} - 2\pi h^3 =$$

$$= 2\pi\left(\frac{3}{5}h^5 - \frac{2}{3}h^3\right)$$

A BIT WEIRD BUT NOT ENTIRELY UNBELIEVABLE.