2014P2Q6(I)

(i) TRANSFORMATION LAW FOR A TENSOR OF ORDER n:

$$T'_{i_1 \dots i_n} = L_{i_1 j_1} \dots L_{i_n j_n} T_{j_1 \dots j_n}$$

WHERE $L_{ij} = \underline{e}'_i \cdot \underline{e}_j$

WHERE $\underline{e}'_i$ IS THE i-TH BASIS VECTOR OF THE FRAME WE ARE TRANSFORMING TO & $\underline{e}_j$ IS THE j-TH BASIS VECTOR OF THE FRAME WE ARE TRANSFORMING FROM.

ISOTROPIC TENSORS ARE TENSORS WITH THE SAME COMPONENTS IN ALL FRAMES:

$$T'_{ijk\dots} = T_{ijk}$$

(ii) CONSIDER: $\dfrac{\partial x_j}{\partial x'_i} = \dfrac{\partial (L_{kj} x'_k)}{\partial x'_i} = L_{kj} \dfrac{\partial x'_k}{\partial x'_i} =$

$= L_{kj} \delta_{ki} = L_{ij}$

$x'_i = L_{ij} x_j \quad / \cdot L_{ji}$

$L_{ji} x'_i = L_{ji} L_{ij} x_j$

$L_{ji} x'_i = x_j$ SINCE L IS ORTHOGONAL.

BY CHAIN RULE:

$$\frac{\partial}{\partial x'_i} = \underbrace{\frac{\partial x_j}{\partial x'_i}} \frac{\partial}{\partial x_j} = \underbrace{L_{ij}} \frac{\partial}{\partial x_j}$$

TRANSFORMATION LAW SATISFIED SO $\dfrac{\partial}{\partial x_i}$ IS A TENSOR (OF ORDER 1).

$$\left(\frac{\partial u_i}{\partial x_j}\right)' = \left(\frac{\partial}{\partial x_j}\right)' (u_i)' = L_{jq} \frac{\partial}{\partial x_q} L_{ip} u_p = L_{jq} L_{ip} \frac{\partial u_p}{\partial x_q}$$

SO $\dfrac{\partial u_p}{\partial x_q}$ TRANSFORMS LIKE A TENSOR, SO IT IS A TENSOR (WE HAVE 2 INDICES, SO ITS AN ORDER 2 ONE.)

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(iii) TRANSFORMATION LAW FOR AN AXIAL VECTOR: $a'_i = \det(L) L_{ij} a_j$

OBEYS THE SAME TRANSFORMATION LAW AS A VECTOR IF: $\det(L) = 1$

$$(\nabla \times u)'_i = \underbrace{(\epsilon_{ijk})'}_{\text{ϵ IS A PSEUDOTENSOR. THATS WHY IT TRANSFORMS LIKE THIS.}} (\partial_j x_k) =$$

$$= \det(L) L_{ip} L_{jq} L_{kr} \epsilon_{pqr} L_{j\alpha} L_{k\beta} \partial_\alpha x_\beta =$$

$$= \det(L) L_{ip} \underbrace{L_{jq} L_{j\alpha}} \underbrace{L_{kr} L_{k\beta}} \epsilon_{pqr} \partial_\alpha x_\beta =$$

SINCE L IS ORTHOGONAL

$$= \det(L) L_{ip} \delta_{q\alpha} \delta_{r\beta} \epsilon_{pqr} \partial_\alpha x_\beta =$$

$$= \det(L) L_{ip} \epsilon_{pqr} \partial_q x_r =$$

$$= \det(L) L_{ip} (\nabla \times u)_p$$

$\nabla \times u$ TRANSFORMS AS AN AXIAL-VECTOR, SO IT CONSTITUTES AN AXIAL-VECTOR (FIELD).

(iv) $$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{SYMMETRIC SECOND ORDER TENSOR}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\text{ANTISYMMETRIC SECOND ORDER TENSOR}}$$

$$\epsilon_{ijk} w_k = \begin{pmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{pmatrix}$$

THIS IS THE GENERAL FORM OF ANTISYMMETRIC SECOND ORDER TENSORS, SO ANY OF SUCH CAN BE EQUAL TO IT.

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$$\epsilon_{ijk}\omega_k = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \underbrace{\hat{S}_{ij}}$$ SYMMETRIC ORDER 2 TENSOR.

$$\hat{S}_{ij} = p\delta_{ij} + \Delta_{ij}$$

$$p = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_i} + \frac{\partial u_j}{\partial x_j}\right) = \frac{\partial u_i}{\partial x_i}$$

$$\Delta_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{\partial u_i}{\partial x_i}$$

$$\omega_k = \frac{1}{\epsilon_{ijk}}\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

(v)

$$u_i = (a x_2, b x_1, 0)$$

$$\cancel{p = \frac{\partial u_i}{\partial x_i} = \frac{\partial}{\partial x_1} a x_2 + \frac{\partial}{\partial x_2} b x_1 + \frac{\partial}{\partial x_3} 0 = 0}$$

$$\omega_k = \frac{1}{\epsilon_{ijk}}\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

IF $k=1$:

$$\omega_1 = \frac{1}{\epsilon_{231}}\frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial u_2}\right) = +\frac{1}{\epsilon_{321}}\frac{1}{2}\left(\frac{\partial u_3}{\partial u_2} - \frac{\partial u_2}{\partial u_3}\right) =$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x_3}(b x_1) - \frac{\partial}{\partial x_2} 0\right) = 0$$

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FOR $k=2$:

$$\omega_2 = \frac{1}{\varepsilon_{132}} \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) = + \frac{1}{\varepsilon_{312}} \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) =$$

$$= + \frac{1}{2}\left(\frac{\partial}{\partial x_1} 0 - \frac{\partial}{\partial x_3} a x_2\right) = 0$$

FOR $k=3$

$$\omega_3 = \frac{1}{\varepsilon_{123}} \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_2}(a x_2) - \frac{\partial}{\partial x_1}(b x_1)\right) =$$

$$= \frac{1}{2}(a-b)$$

SO WE HAVE: $\omega_k = \delta_{k3} \frac{1}{2}(a-b)$

$$\frac{\partial u_i}{\partial x_j} = \begin{pmatrix} \frac{\partial a x_2}{\partial x_1} & \frac{\partial a x_2}{\partial x_2} & \frac{\partial a x_2}{\partial x_3} \\ \frac{\partial b x_1}{\partial x_1} & \frac{\partial b x_1}{\partial x_2} & \frac{\partial b x_1}{\partial x_3} \\ \frac{\partial 0}{\partial x_1} & \frac{\partial 0}{\partial x_2} & \frac{\partial 0}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \begin{pmatrix} 0 & \frac{a+b}{2} & 0 \\ \frac{a+b}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

THIS IS ALREADY TRACELESS, SO: $\hat{S}_{ij} = S_{ij}$

$$\left.\begin{array}{l} \mathrm{Tr}\,\hat{S} = 0 \Rightarrow \sum_{i=1}^{3} \lambda_i = 0 \\ \det \hat{S} = 0 \Rightarrow \prod_{i=1}^{3} \lambda_i = 0 \end{array}\right\} \longrightarrow \begin{array}{l} \lambda_1 = 0 \\ \lambda_2 = -\lambda_3 \end{array}$$

$$\cancel{\begin{pmatrix} 0 & \frac{a+b}{2} & 0 \\ \frac{a+b}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{a+b}{2}\begin{pmatrix} y \\ x \\ 0 \end{pmatrix} \Rightarrow x = y = \frac{1}{\sqrt{2}}\sqrt{\frac{2}{a+b}}}$$

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$$\begin{pmatrix} 0 & \frac{a+b}{2} & 0 \\ \frac{a+b}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda_2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{a+b}{2} y \\ \frac{a+b}{2} x \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} \lambda_2 x &= \frac{a+b}{2} y \\ \lambda_2 y &= \frac{a+b}{2} x \\ x^2 + y^2 &= 1 \end{aligned}\right\}$$

$\cancel{x = \lambda y =}$

$$\lambda_2 x = \left(\frac{a+b}{2}\right)^2 \frac{x}{\lambda_2}$$

$$\lambda_2^2 = \left(\frac{a+b}{2}\right)^2$$

$$e_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \longleftarrow \lambda_2 = \frac{a+b}{2}$$

$$e_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \longleftarrow \lambda_3 = -\frac{a+b}{2}$$

$$e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \longleftarrow \lambda_1 = 0$$

PRINCIPAL VALUES

PRINCIPAL AXES