2013P1Q4(I)

(i)
$$\tilde{h}(k)=\int_{-\infty}^{\infty} e^{-ikx} h(x)\,dx=$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)^* g(x+y)\,dy\, e^{-ikx}\,dx=$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)^* g(x+y)\, e^{-ikx}\,dy\,dx=$$

INTRODUCE CHANGE OF VARIABLE: $z=x+y \longrightarrow dz=dx$, $x=z-y$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)^* g(z)\, e^{-ikz} e^{iky}\,dy\,dx=$$

$$=\int_{-\infty}^{\infty} f(y)^* e^{iky}\,dy \int_{-\infty}^{\infty} g(z) e^{-ikz}\,dz=$$

$$=\tilde{f}(k)^*\tilde{g}(k)$$

(ii) LET $f=g$, $x=0$

$$h(x)=\int_{-\infty}^{\infty} f(y)^* f(x+y)\,dy=\int_{-\infty}^{\infty}|f(y)|^2\,dy$$

$$h(x)=\mathrm{IFT}[\tilde{h}(k)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(k)^*\tilde{g}(k)\underbrace{e^{ikx}}_{1 \text{ SINCE } x=0}\,dk=$$

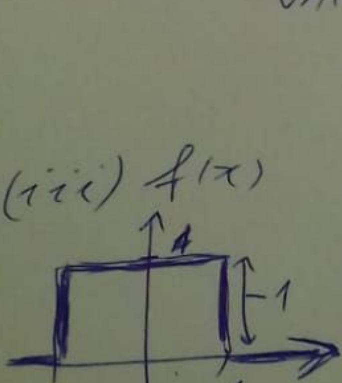
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}|\tilde{f}(k)|^2\,dk$$

~~CHANGE OF VARIABLE k →~~

CHANGE OF VARIABLE: $y \Rightarrow x$

$$\int_{-\infty}^{\infty}|f(x)|^2\,dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}|\tilde{f}(k)|^2\,dk$$

(iii) $f(x)$



$$\int_{-\infty}^{\infty}|f(x)|^2\,dx=\int_{-1}^{1}1\,dx=2$$

$$\tilde{f}(k)=\int_{-\infty}^{\infty}e^{-ikx}f(x)\,dx=\int_{-1}^{1}1\cdot e^{-ikx}\,dx=\frac{i}{k}\left[e^{-ikx}\right]_{-1}^{1}=$$

$$=\frac{i}{k}\left(e^{-ik}-e^{ik}\right)=\frac{2\sin k}{k}$$

$\cancel{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4\sin^2 k}{k^2}\,dk}$

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$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\tilde{f}(k)\right|^2 dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} 4\frac{\sin^2 k}{k^2}\,dk =$$

$$= \frac{2}{\pi}\left(\underbrace{\left[-k^{-1}\sin^2 k\right]_{-\infty}^{\infty}}_{\to 0} - \int_{-\infty}^{\infty} -\frac{1}{k}\,2\sin k\cos k\,dk\right) =$$

$$= \frac{2}{\pi}\cdot 2\cdot \underbrace{\frac{\pi}{4}\cdot 2}_{} = 2$$

└ USING HINT: $\int_0^{\infty} \underbrace{\frac{\sin x\cos x}{x}}_{} dx = \frac{\pi}{4}$

└ EVEN FUNCTION, SO

$$\int_{-\infty}^{\infty}\frac{\sin x\cos x}{x} = \frac{\pi}{4}\cdot 2$$

$$\int_{-\infty}^{\infty}|f(x)|^2 dx = 2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\tilde{f}(k)\right|^2 dk = 2$$

$\Longrightarrow$ PERSEVAL'S THEOREM IS VERIFIED FOR THE GIVEN FUNCTION.