

2012P1Q4(I)

$$\tilde{f}(z) = \int_{-\infty}^{\infty} f(x) e^{-izx} dx = \int_{-\infty}^{\infty} e^{-\lambda x^2} e^{-izx} dx =$$

$$z = \sqrt{\lambda} x + \frac{iz}{\sqrt{\lambda}} \quad \xrightarrow{\quad} \quad dz = \sqrt{\lambda} dx$$

$$z^2 = \lambda x^2 + izx - \frac{z^2}{4\lambda}$$

$$-z^2 = -\lambda x^2 - izx + \frac{z^2}{4\lambda}$$

$$= \int_{-\infty}^{\infty} e^{-z^2} e^{-\frac{z^2}{4\lambda}} \frac{dz}{\sqrt{\lambda}} = \frac{1}{\sqrt{\lambda}} e^{-\frac{z^2}{4\lambda}} \int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\frac{\pi}{\lambda}} e^{-\frac{z^2}{4\lambda}}$$

LET'S HAVE:  $g(x) = f'(x)$

$$\begin{aligned} \tilde{g}(z) &= \int_{-\infty}^{\infty} f'(x) e^{-izx} dx = \\ &= \left[ f(x) e^{-izx} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) (-iz) e^{-izx} dx = \\ &= iz \int_{-\infty}^{\infty} f(x) \exp(-izx) dx = iz \tilde{f}(z) \end{aligned}$$

~~h~~  $h(x) = f''(x)$

$$\tilde{h}(z) = iz \tilde{f}'(z) = (iz)^2 \tilde{f}(z) = -z^2 \tilde{f}(z)$$

$$\begin{aligned} \psi'' &= \psi \xrightarrow{\text{FT WRT } x} -z^2 \tilde{\psi}(z, t) = \frac{\partial \tilde{\psi}(z, t)}{\partial t} \\ \tilde{\psi}(z, t) &= \tilde{\psi}(z, t=0) e^{-z^2 t} \end{aligned}$$

SOLUTION

$$\begin{aligned} \tilde{\psi}(z, t=0) &= \int_{-\infty}^{\infty} \psi(x, t=0) e^{-izx} dx = \int_{-\infty}^{\infty} e^{-\lambda x^2} e^{-izx} dx = \\ &= \sqrt{\frac{\pi}{\lambda}} \exp\left(-\frac{z^2}{4\lambda}\right) \quad \cdot (\text{USING PREVIOUS FINDINGS}) \end{aligned}$$

2012 P1Q4(II)

$$\tilde{\psi}(k, t=0) = \int_{-\infty}^{\infty} \psi(x, t=0) e^{-ikx} dx = \int_{-\infty}^{\infty} e^{-kx^2} e^{-ikx} dx =$$

$$= \sqrt{\frac{\pi}{1}} \exp\left(-\frac{k^2}{4}\right)$$

$$\tilde{\psi}(k, t) = \sqrt{\frac{\pi}{1}} \exp\left(-\frac{k^2}{4}\right) \exp(-k^2 t) = \sqrt{\frac{\pi}{1}} e^{-k^2 \left(\frac{1}{4} + t\right)}$$

$$\psi(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\psi}(k, t) e^{ikx} dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{1}} e^{-k^2 \left(\frac{1}{4} + t\right)} e^{ikx} dk =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{1}} e^{-\left(k\sqrt{\frac{1}{4} + t} - \frac{ix}{2\sqrt{\frac{1}{4} + t}}\right)^2} e^{-\frac{x^2}{4\left(\frac{1}{4} + t\right)}} dk =$$

$$= \frac{1}{2\pi} e^{-\frac{x^2}{4\left(\frac{1}{4} + t\right)}} \sqrt{\frac{\pi}{1}} \int_{-\infty}^{\infty} e^{-\left(k\sqrt{\frac{1}{4} + t} - \frac{ix}{2\sqrt{\frac{1}{4} + t}}\right)^2} dk = \frac{1}{2\pi} e^{-\frac{x^2}{4\left(\frac{1}{4} + t\right)}} \sqrt{\frac{\pi}{1}} \int_{-\infty}^{\infty} e^{-\left(y - \frac{ix}{2\sqrt{\frac{1}{4} + t}}\right)^2} \frac{dy}{\sqrt{\frac{1}{4} + t}}$$

$$\cancel{dy} = k \sqrt{\frac{1}{4} + t} \quad \lambda = +1$$

$$dy = \sqrt{\frac{1}{4} + t} dk$$

$$= \frac{1}{2\pi} e^{-\frac{x^2}{4\left(\frac{1}{4} + t\right)}} \sqrt{\frac{\pi}{1}} \frac{1}{\sqrt{\frac{1}{4} + t}} \int_{-\infty}^{\infty} e^{-y^2} dy = e^{-\frac{x^2}{4\left(\frac{1}{4} + t\right)}} \frac{1}{2} \sqrt{\frac{1}{1}} \frac{1}{\sqrt{\frac{1}{4} + t}} =$$

$$\cancel{\frac{1}{2} \sqrt{\frac{1}{1}}} = \frac{1}{2} \sqrt{\frac{1}{1}} \frac{1}{\sqrt{\frac{1}{4} + t}} = \frac{1}{2} \frac{1}{\sqrt{\frac{1}{4} + t}} = \frac{1}{\sqrt{1 + 4t}}$$