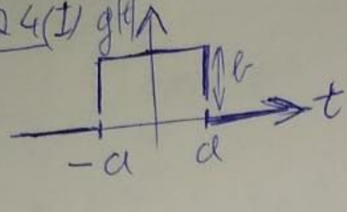
2015P1Q4(I)



$$\tilde{g}(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-i\omega t}\, dt =$$

$$= \int_{-a}^{a} b\, e^{-i\omega t}\, dt = b\,\frac{i}{\omega}\left[e^{-i\omega t}\right]_{-a}^{a} =$$

$$= b\,\frac{i}{\omega}\left[e^{-i\omega a} - e^{i\omega a}\right] =$$

$$= \frac{b}{\omega}\, 2\sin(a\omega)$$

so why do we need a and b?

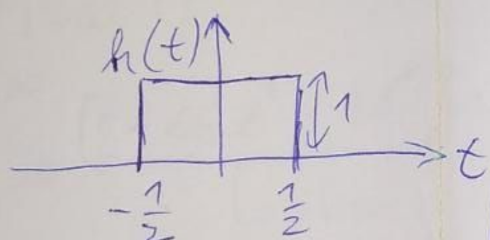
(a)

FOR $a(t)$: $b = 1$, $a = 1$

$$\tilde{a}(\omega) = \frac{2}{\omega}\sin(\omega)$$ ✓

(b) $\tilde{f}(\omega) = \tilde{h}(\omega)\cdot\tilde{h}(\omega)$

$f(t) = h(t) * h(t)$



FOR $h(t)$: $b = 1$, $a = \frac{1}{2}$

$$\tilde{h}(\omega) = \frac{1}{\omega}\, 2\sin\left(\frac{\omega}{2}\right)$$

$$\Rightarrow \tilde{f}(\omega) = \frac{4}{\omega^2}\sin^2\left(\frac{\omega}{2}\right)$$ ✓

$$\tilde{b}(\omega) = \tilde{a}(\omega)\cdot\tilde{f}(\omega)$$

$$= \frac{2}{\omega}\sin(\omega)\frac{4}{\omega^2}\sin^2\left(\frac{\omega}{2}\right) = \frac{8}{\omega^3}\sin(\omega)\sin^2\left(\frac{\omega}{2}\right)$$ ✓

(c) $\tilde{r}(\omega) = \tilde{b}(\omega) + \alpha\,\widetilde{b(t-\tau)}$

FROM LECTURE NOTES: $g(x) = f(x-\alpha) \Longleftrightarrow \tilde{g}(k) = e^{-ik\alpha}\tilde{f}(k)$ ✓

$$\widetilde{b(t-\tau)} = e^{-i\omega\tau}\,\tilde{b}(\omega)$$

$$\tilde{r}(\omega) = \frac{8}{\omega^3}\sin(\omega)\sin^2\left(\frac{\omega}{2}\right) + \alpha e^{-i\omega\tau}\frac{8}{\omega^3}\sin(\omega)\sin^2\left(\frac{\omega}{2}\right) =$$ ✓

$$= \left(1 + \alpha e^{-i\omega\tau}\right)\cdot\frac{8}{\omega^3}\sin(\omega)\sin^2\left(\frac{\omega}{2}\right)$$

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(d) $\Delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{J}(\omega) e^{i\omega t} d\omega =$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2(1+\epsilon\omega^2) e^{-\frac{\omega^2}{4}} e^{i\omega t} d\omega =$$

$$z = \frac{\omega}{2} - it \xrightarrow{\checkmark} \frac{dz}{d\omega} = \frac{1}{2}$$

$$d\omega = 2dz$$

$$2(z+it) = \omega \qquad z^2 = \frac{\omega^2}{4} - i\omega t - t^2$$

$$4(z+it)^2 = \omega^2 \qquad -z^2 = -\frac{\omega^2}{4} + i\omega t + t^2$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2(1+\epsilon 4(z+it)^2) e^{-z^2} e^{-t^2} 2dz =$$

$$= \frac{2}{\pi} e^{-t^2} \int_{-\infty}^{\infty} (\underbrace{1+\epsilon 4z^2 - \epsilon 4t^2}_{\text{EVEN}} + \underbrace{\epsilon 8zit}_{\text{ODD}}) \underbrace{e^{-z^2}}_{\text{EVEN}} dz =$$

$$= \frac{2}{\pi} e^{-t^2} \left(\int_{-\infty}^{\infty} e^{-z^2} dz + \int_{-\infty}^{\infty} \epsilon 4z^2 e^{-z^2} dz - \epsilon 4t^2 \int_{-\infty}^{\infty} e^{-z^2} dz \right) =$$

$$= \frac{2}{\pi} e^{-t^2} \left((1-4\epsilon t^2)\sqrt{\pi} + \int_{-\infty}^{\infty} \epsilon 4z^2 e^{-z^2} dz \right) =$$

$$\int_{-\infty}^{\infty} z^2 e^{-z^2} dz = \int_{-\infty}^{\infty} z\, z e^{-z^2} dz =$$

$$= \underbrace{\left[-\frac{1}{2} e^{-z^2} z\right]_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} -\frac{1}{2} e^{-z^2} dz = \frac{1}{2}\sqrt{\pi}$$

$$= \frac{2}{\pi} e^{-t^2} \left((1-4\epsilon t^2)\sqrt{\pi} + \epsilon 4 \frac{1}{2}\sqrt{\pi} \right) =$$

$$= \frac{2}{\sqrt{\pi}} e^{-t^2} (1 - 4\epsilon t^2 + 2\epsilon)$$