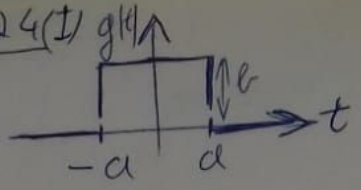
2015P1Q4(1)



$$\tilde{g}(\omega)=\int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt=$$

$$=\int_{-a}^{a} b\,e^{-i\omega t}dt=b\frac{i}{\omega}\left[e^{-i\omega t}\right]_{-a}^{a}=$$

$$=b\frac{i}{\omega}\left[e^{-i\omega a}-e^{i\omega a}\right]=$$

$$=\frac{b}{\omega}2\sin(a\omega)$$

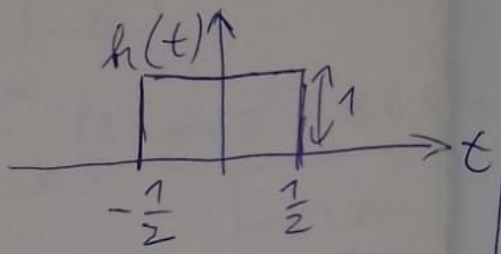
(a)

FOR $a(t)$: $b=1$, $a=1$

$$\tilde{a}(\omega)=\frac{2}{\omega}\sin(\omega)$$

(b) $\tilde{f}(\omega)=\tilde{h}(\omega)\cdot\tilde{h}(\omega)$

$$f(t)=h_1(t)*h_1(t)$$



FOR $h_1(t)$: $b=1$, $a=\frac{1}{2}$

$$\tilde{h}_1(\omega)=\frac{1}{\omega}2\sin\left(\frac{\omega}{2}\right)$$

$$\tilde{f}(\omega)=\frac{4}{\omega^2}\sin^2\left(\frac{\omega}{2}\right)$$

$$\tilde{b}(\omega)=\tilde{a}(\omega)\cdot\tilde{f}(\omega)$$

$$=\frac{2}{\omega}\sin(\omega)\frac{4}{\omega^2}\sin^2\left(\frac{\omega}{2}\right)=\frac{8}{\omega^3}\sin(\omega)\sin^2\left(\frac{\omega}{2}\right)$$

(c) $\tilde{r}(\omega)=\tilde{b}(\omega)+\alpha\widetilde{b(t-\tau)}$

FROM LECTURE NOTES: $g(x)=f(x-a)\Longleftrightarrow\tilde{g}(k)=e^{-ika}\tilde{f}(k)$

$$\widetilde{b(t-\tau)}=e^{-i\omega\tau}\tilde{b}(\omega)$$

$$\tilde{r}(\omega)=\frac{8}{\omega^3}\sin(\omega)\sin^2\left(\frac{\omega}{2}\right)+\alpha e^{-i\omega\tau}\frac{8}{\omega^3}\sin(\omega)\sin^2\left(\frac{\omega}{2}\right)=$$

$$=\left(1+\alpha e^{-i\omega\tau}\right)\cdot\frac{8}{\omega^3}\sin(\omega)\sin^2\left(\frac{\omega}{2}\right)$$

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(d) $J(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{J}(\omega)e^{i\omega t}d\omega =$

$=\frac{1}{2\pi}\int_{-\infty}^{\infty} 2(1+\epsilon\omega^2)e^{-\frac{\omega^2}{4}}e^{i\omega t}d\omega =$

$z=\frac{\omega}{2}+it \longrightarrow \frac{dz}{d\omega}=\frac{1}{2}$

$d\omega = 2dz$

$2(z+it)=\omega$

$4(z+it)^2=\omega^2$

$z^2=\frac{\omega^2}{4}-i\omega t-t^2$

$-z^2=-\frac{\omega^2}{4}+i\omega t+t^2$

$=\frac{1}{2\pi}\int_{-\infty}^{\infty} 2(1+\epsilon 4(z+it)^2)e^{-z^2}e^{-t^2}2dz =$

$=\frac{2}{\pi}e^{-t^2}\int_{-\infty}^{\infty}\Big(\underbrace{1+\epsilon 4z^2-\epsilon 4t^2}_{\text{EVEN}}+\underbrace{\epsilon 8zit}_{\text{ODD}}\Big)\underbrace{e^{-z^2}}_{\text{EVEN}}dz =$

$=\frac{2}{\pi}e^{-t^2}\left(\int_{-\infty}^{\infty}e^{-z^2}dz+\int_{-\infty}^{\infty}\epsilon 4z^2e^{-z^2}dz - \epsilon 4t^2\int_{-\infty}^{\infty}e^{-z^2}dz\right)=$

$=\frac{2}{\pi}e^{-t^2}\left((1-4\epsilon t^2)\sqrt{\pi}+\int_{-\infty}^{\infty}\epsilon 4z^2e^{-z^2}dz\right)=$

$\int_{-\infty}^{\infty}z^2e^{-z^2}dz=\int_{-\infty}^{\infty}z\,ze^{-z^2}dz=$

$=\left[-\frac{1}{2}e^{-z^2}z\right]_{-\infty}^{\infty}-\int_{-\infty}^{\infty}-\frac{1}{2}e^{-z^2}dz=\frac{1}{2}\sqrt{\pi}$

($\left[-\frac{1}{2}e^{-z^2}z\right]_{-\infty}^{\infty}\to 0$)

$=\frac{2}{\pi}e^{-t^2}\left((1-4\epsilon t^2)\sqrt{\pi}+\epsilon 4\frac{1}{2}\sqrt{\pi}\right)=$

$=\frac{2}{\sqrt{\pi}}e^{-t^2}\left(1-4\epsilon t^2+2\epsilon\right)$