$\boxed{\begin{array}{c} \text{P II} \\ 2009\,Q10 \end{array}}$ (1)

$a^2=b^2=c^2=1$

$ab=c$

$abb=cb=a$

V_4

$\boxed{\begin{array}{cccc} 1 & a & b & c \\ a & 1 & c & b \\ b & c & 1 & a \\ c & b & a & 1 \end{array}}$ ✓

✓

$ac\,cb = aca$

$ab = c$

$acbc = c$

$acb = 1$

$ac = b^{-1} = b$

g_1, g_2 conjugate

$g_2 g = g g_1 \overset{\text{as Abelian}}{=} g_1 g \Rightarrow g_1 = g_2$ each elt in own conjugacy class.

CONJUGACY CLASSES OF V: ✓

$\{1\}\ \{a\}\ \{b\}\ \{c\}$

explain why.

AN IRREDUCIBLE REP OF V:

~~$1=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ $a=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ $b=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$ $c=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$~~

$\{1,a,b,c\} \to \{1,1,1,1\}$ ✓

NUMBER OF CONJUGACY CLASSES AND INEQUIVALENT IRREPS ARE EQUAL.

V HAS 4 INEQ. IRREP ✓

REP1: $\{1,a,b,c\} \to \{1,1,1,1\}$ ✓

REP2: $\{1,a,b,c\} \to \{1,-1,1,-1\}$ ✓

REP3: $\{1,a,b,c\} \to \{1,1,-1,-1\}$ ✓

REP4: $\{1,a,b,c\} \to \{1,-1,-1,1\}$ ✓

$$\begin{array}{cccc} D(1) & D(a) & D(b) & D(c) \\ D(a) & D(1) & D(c) & D(b) \\ D(b) & D(c) & D(1) & D(a) \\ D(c) & D(b) & D(a) & D(1) \end{array}$$

$\to$ THIS TABLE ISOMORPHIC TO V TABLE $\Rightarrow$ D IS A REP.

2009P2Q10(II)

GROUP MULTIPLICATION TABLE OF V:

1	a	b	c
a	1	c	b
b	c	1	a
c	b	a	1

$ab = c = ba$
$abb = cb$
$a = cb$
✓ $ca = ccb = b$

VERY GOOD WORK WELL DONE.

CONJUGACY CLASSES:

$g q_1 g^{-1} = q_2 \ / \cdot g \quad g, q_1, q_2 \in V$

$g q_1 = q_2 g = g q_2$
↳ GROUP ABELIAN ✓

$\Rightarrow q_1 = q_2 \Rightarrow$ EVERY $g \in V$ IS ALONE ITS OWN CONJUGACY CLASS.

CONJUGACY CLASSES OF V:
$\{1\}, \{a\}, \{b\}, \{c\}$

surely $gqg^{-1} = q_2$ abelian $gg^{-1}q = q_2$

An irreducible representation, D, of group V is a set of matrices M(S) acting on vector space S, which S has no non-trivial proper subspaces under the action of matrices forming D. ✓

Number of conjugacy classes and the number of inequivalent irreducible representations of a finite group are equal. ✓

$\Rightarrow$ V HAS 4 INEQUIVALENT IRREP. THESE ARE:

So write — — = — — —
this is to include a scalar

$\{1, a, b, c\} \to \{1, 1, 1, 1\} = d^{(1)}$
$\to \{1, -1, -1, 1\} = d^{(2)}$
$\to \{1, 1, -1, -1\} = d^{(3)}$
$\to \{1, -1, 1, -1\} = d^{(4)}$ good

PTO.

Also $|G| = \sum n_i^2$ of dim of irrep

MULTIPLICATION TABLE FOR MATRICES LISTED:

D(1)	D(a)	D(b)	D(c)
D(a)	D(1)	D(c)	D(b)
D(b)	D(c)	D(1)	D(a)
D(c)	D(b)	D(a)	D(1)

THIS TABLE IS ISOMORPHIC TO THE TABLE OF V PROVIDED ABOVE, SO D IS INDEED A REPRESENTATON OF V. ✓

CHARACTER TABLE:

	1	a	b	c
$d^{(1)}$	1	1	1	1
$d^{(2)}$	1	-1	-1	1
$d^{(3)}$	1	1	-1	-1
$d^{(4)}$	1	-1	1	-1

✓

DECOMPOSITION OF D:

$\{D(1)_{11}, D(a)_{11}, D(b)_{11}, D(c)_{11}\} = d^{(3)}$
$\{D(1)_{22}, D(a)_{22}, D(b)_{22}, D(c)_{22}\} = d^{(4)}$
$\{D(1)_{33}, D(a)_{33}, D(b)_{33}, D(c)_{33}\} = d^{(2)}$ ✓

EVERY VECTOR FORMED BY COLUMN OF CHARACTER TABLE ORTHOGONAL TO ANY COLUMN BELONGING TO DIFFERENT CONJUGACY CLASS (HERE: TO EVERY OTHER COLUMN)

P(II) 2007 Q 10(II) DECOMP OF D:

$$D'(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad D'(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad D'(b) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad D'(c) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$D''(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad D''(a) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad D''(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad D''(c) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$D'''(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad D'''(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad D'''(b) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad D'''(c) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

CHARACTER TABLE FOR V:

	1	a	b	c
REP1	1	1	1	1
REP2	1	-1	1	-1
REP3	1	1	-1	-1
REP4	1	-1	-1	1
orgird rep.	3	-1	-1	-1

ORTHOGONALITY RELATION:
EVERY COL. IN DIFFERENT CONJUGACY CLASS (HERE: EVERY COL.) ARE ORTHOGONAL TO EACH OTHERS.

VERIFICATION:

$$(1\ 1\ 1\ 1)\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = (1\ 1\ 1\ 1)\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = (1\ 1\ 1\ 1)\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} =$$

$$= (1\ -1\ 1\ -1)\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = (1\ -1\ 1\ -1)\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = (1\ 1\ -1\ -1)\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = 0$$

we want. here
no of copies of irrep $\times |G| = \Sigma$ char. elt in irrep $\times$ char. elt in rep.

ie $4 \times m_1 = 3\times1 + -1\times1 + -1\times1 + -1\times1 \quad \therefore m_1 = 0$
$4 \times m_2 = 3\times1 + -1\times-1 + -1\times1 + -1\times-1 \quad m_2 = 1$
$4 \times m_3 = 3\times1 + -1\times1 + -1\times-1 + -1\times-1 \quad m_3 = 1$
$4 \times m_4 = 3\times1 + -1\times-1 + -1\times-1 + -1\times1 \quad m_4 = 1.$

not v. surprising.