2013P1Q1(I)

(i) $\underline{\nabla}\cdot(\underline{A}\times\underline{B}) = \partial_i(\epsilon_{ijk}A_j B_k) = \epsilon_{ijk}(\partial_i A_j)B_k +$

$+\epsilon_{ijk}(\partial_i B_k)A_j = \epsilon_{kij}(\partial_i A_j)B_k - \epsilon_{jik}(\partial_i B_k)A_j =$

$= \underline{B}\cdot(\underline{\nabla}\times\underline{A}) - \underline{A}\cdot(\underline{\nabla}\times\underline{B})$

(ii) DIVERGENCE THM: $\int_V (\underline{\nabla}\cdot\underline{F})\,dV = \oint_S \underline{F}\cdot\underline{dS}$ WHERE V IS A VOLUME BOUNDED BY THE CLOSED SURFACE S, $\underline{F}$ IS A VECTOR FIELD.

USING (i): $\underline{\nabla}\cdot(\underline{\nabla}a\times\underline{B}) = \underline{B}\cdot(\underline{\nabla}\times\nabla a) - \underline{\nabla}a\cdot(\underline{\nabla}\times\underline{B})$

USING: $\underline{\nabla}\times\underline{\nabla}a = \underline{0}$:

$$\underline{\nabla}\cdot(\underline{\nabla}a\times\underline{B}) = -\underline{\nabla}a\cdot(\underline{\nabla}\times\underline{B})$$

USING DIVERGENCE THM, W/ $\underline{F} = \nabla a\times\underline{B}$:

$$\int_V \underline{\nabla}\cdot(\underline{\nabla}a\times B)\,dV = \int -\nabla a\cdot(\underline{\nabla}\times\underline{B})\,dV =$$

$$= \oint_S (\underline{\nabla}a\times\underline{B})\cdot\underline{dS}$$

RESTYLE:

$$\iiint_V \nabla a\cdot(\underline{\nabla}\times\underline{B})\,dV = -\iint_S (\underline{\nabla}a\times\underline{B})\cdot\underline{\hat{n}}\,dS$$

(iii) $a = xy + z^2$

$\underline{B} = y\,\hat{\underline{i}} - yz\,\hat{\underline{j}} + x\,\hat{\underline{k}}$

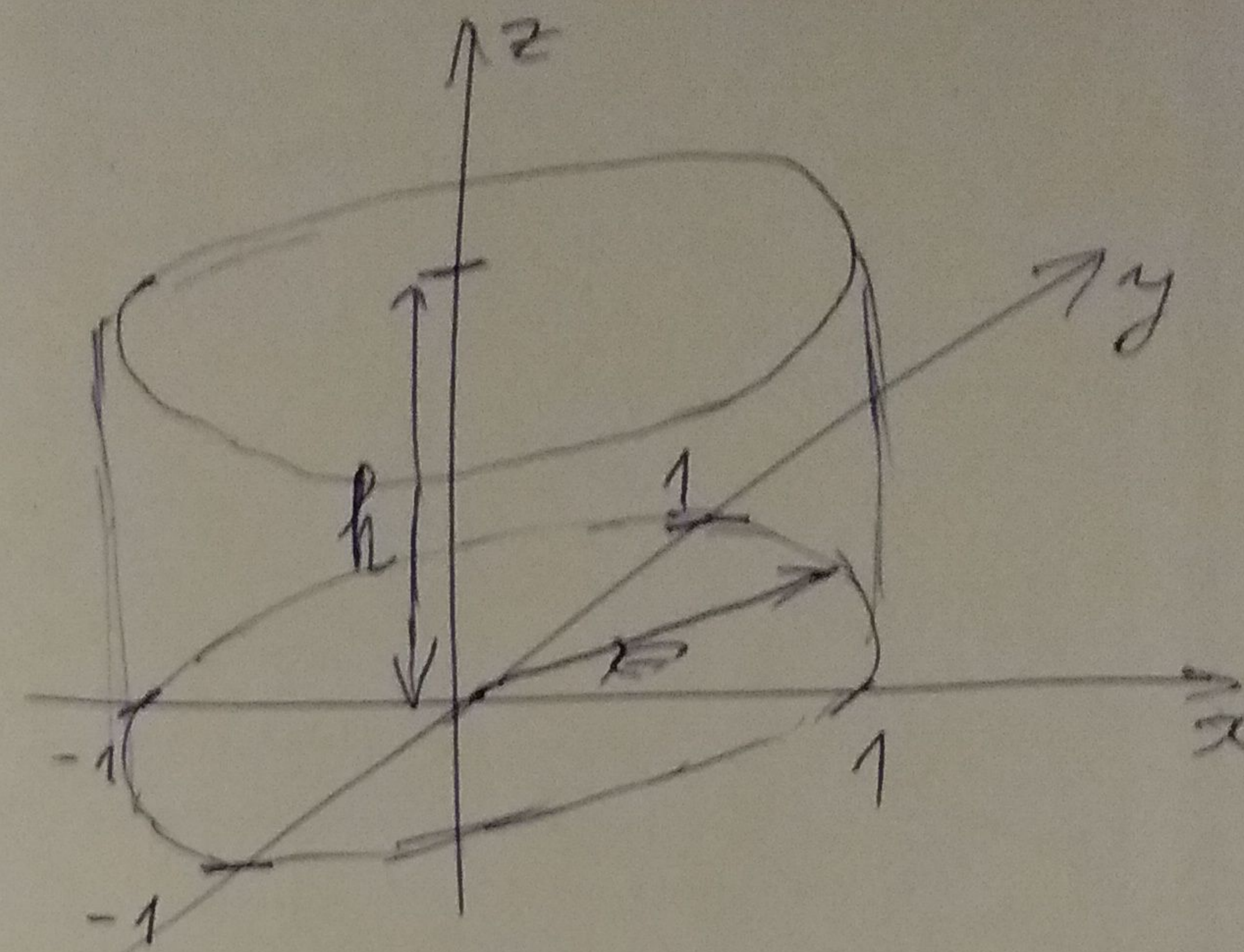
$\underline{\nabla}a = (\partial_x, \partial_y, \partial_z)(xy+z^2) =$
$= (y, x, 2z)$

$$\underline{\nabla}\times\underline{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & -yz & x \end{vmatrix} = (y, -1, -1)$$

$\underline{\nabla}a\cdot(\underline{\nabla}\times\underline{B}) = (y, x, 2z)\cdot(y, -1, -1) = y^2 - x - 2z.$

$$\underline{\nabla}a\times\underline{B} = (y, x, 2z)\times(y, -yz, x) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ y & x & 2z \\ y & -yz & x \end{vmatrix} = (x^2 - 2z^2y,\ 2zy - yx,\ -y^2z - xy)$$

2013 P1 Q1 (II)



$\iiint_V \underline{\nabla} a \cdot (\underline{\nabla} \times \underline{B}) \, dV =$

$= \iiint_V y^2 - x - 2z \, dV =$

$= \iiint_V y^2 \, dV - \iiint_V x \, dV - \iiint_V 2z \, dV =$

$= \int_{r=0}^{1}\int_{z=0}^{h}\int_{\theta=0}^{2\pi} (r\sin\theta)^2 \, r \, dr \, d\theta \, dz - \underbrace{\int_{r=0}^{1}\int_{z=0}^{h}\int_{\theta=0}^{2\pi} r \underbrace{\cos\theta}_{ODD} \, r \, dr \, d\theta \, dz}_{0} -$

$- \int_{r=0}^{1}\int_{\theta=0}^{2\pi}\int_{z=0}^{h} 2z \, r \, dr \, d\theta \, dz = \left[\frac{r^4}{4}\right]_0^1 \frac{2\pi}{2} \left[z\right]_0^h - \left[\frac{r^2}{\cancel{2}}\right]_0^1 2\pi \, \cancel{2} \left[\frac{z^2}{\cancel{2}}\right]_0^h =$

$= \frac{h\pi}{4} - h^2\pi$

$- \iint_S (\underline{\nabla} a \times \underline{B}) \cdot \hat{\underline{n}} \, dS = - \int_{\text{LATERAL PART}} (\underline{\nabla} a \times \underline{B}) \cdot \hat{\underline{n}} \, dS - \int_{\text{TOP \& BOTTOM}} (\underline{\nabla} a \times \underline{B}) \cdot \hat{\underline{n}} \, dS$

$\int_{\text{LATERAL PART}} (x^2 - 2z^2 y, \; 2zy - yx, \; -y^2 z - xy) \cdot \underbrace{(x, y, 0)}_{\hat{\underline{n}}} \, dS =$

$= \int_{\text{LATERAL PART}} x^3 - 2z^2 yx + 2zy^2 - y^2 x \, dS = \int_{\text{LATERAL PART}} 2zy^2 \, dS =$

(OTHER TERMS CANCEL OUT ON THE DOMAIN OF INTEGRATION (THEY ARE ODD))

2013 PMQ 1(iii)

$$= \int_{z=0}^{h} \int_{\theta=0}^{2\pi} 2z\,(1\sin\theta)^2 \, 1\,d\theta\,dz = 2\left[\frac{z^2}{2}\right]_0^h \frac{2\pi}{2} = h^2\pi$$

$$\int_{TOP} (\underline{\nabla} a \times \underline{B}) \cdot \underline{\hat{n}} \, dS = \int_{TOP} (x^2 - 2z^2y,\ 2zy - yx,\ -y^2z - xy) \cdot (0,0,1)\, dS =$$

$$= \int_{TOP} -y^2 z \underbrace{- xy}_{\substack{\text{ANTISYMMETRIC} \\ \text{FUNCTION ON} \\ \text{A SYMMETRIC SURFACE}}} dS = \cancel{\int_{r=0}^{1}\int_{\theta=0}^{2\pi} - (r\sin\theta)^2 z \, d\theta \Big|_{z=h} =}$$

$$= - \int_{r=0}^{1} \int_{\theta=0}^{2\pi} (r\sin\theta)^2 z \; r\,dr\,d\theta \Bigg|_{z=h} = -h\left[\frac{r^4}{4}\right]_0^1 \frac{2\pi}{2} = -\frac{1}{4}\pi h$$

$$\int_{BOTTOM} (\underline{\nabla} a \times \underline{B}) \cdot \underline{\hat{n}} \, dS = \int_{BOTTOM} (x^2 - 2z^2y,\ 2zy - yx,\ -y^2z - xy) \cdot (0,0,-1)\, dS =$$

$$= \int_{\substack{BOTTOM \\ (z=0)}} y^2 z + xy \, dS = 0$$

PUTTING THESE TOGETHER:

$$-\iint_S (\underline{\nabla} a \times \underline{B}) \cdot \underline{\hat{n}} \, dS = -h^2\pi - \left(-\frac{1}{4}\pi h + 0\right) = \frac{h\pi}{4} - h^2\pi$$

AS FOUND FOR THE LHS

$\Rightarrow$ BOTH SIDES VERIFIED.