2016P2Q4 (I)

(a)(i) $|I| =$

$$z = r e^{i\theta}, \quad dz = r\, i\theta\, e^{i\theta}\, d\theta$$

$$= \left| \int_C \frac{f(z)}{\sqrt{z}}\, dz \right| \leq \int_C \frac{|f(z)|}{|\sqrt{z}|}\, |dz| = \int_{\theta=\frac{3}{2}\pi}^{\theta=\frac{1}{2}\pi} \frac{|f(z)|}{\sqrt{r}}\, r\, d\theta = \int_{\theta=\frac{3}{2}\pi}^{\theta=\frac{1}{2}\pi} |f(z)| \sqrt{r}\, d\theta =$$

$$= \sqrt{r} \int_{\theta=\frac{3}{2}\pi}^{\theta=\frac{1}{2}\pi} |f(z)|\, d\theta$$ ✓

$$\lim_{R\to 0} |I| = \lim_{r\to 0} \sqrt{r} \int_{\theta=\frac{3}{2}\pi}^{\theta=\frac{1}{2}\pi} |f(z)|\, d\theta = 0 \Rightarrow \lim_{R\to 0} I = \underline{0}$$

so what does that say about the behaviour of f?

(ii) INTEGRAL VANISHES AS $R \to \infty$ IF IT GOES TO 0 FASTER THAN ARCLENGTH OF C INCREASES:

INTEGRAND SHOULD APPROACH 0 FASTER THAN $\frac{1}{r}$

~~why ??? $\frac{1}{r}$~~

$\Updownarrow$

~~why not eg $\frac{1}{r^2}$?~~

CONSTRAINT:

$f(z)$ SHOULD APPROACH 0 FASTER THAN $\frac{1}{r}\sqrt{r} = \frac{1}{\sqrt{r}}$. ✓

(iii)

ABOVE: $r \to x,\ \theta \to 0,\ z = r e^{i\theta}$

$$\frac{f(z)}{\sqrt{z}} = \frac{f(x+i\epsilon)}{\sqrt{x+i\epsilon}} = \frac{f(re^{i\theta})}{\sqrt{re^{i\theta}}} = \frac{f(re^{i\theta})}{\sqrt{r}\, e^{\frac{i\theta}{2}}} \longrightarrow \underline{\frac{f(x)}{\sqrt{x}}}$$ ✓

BELOW: $r \to x,\ \theta \to 2\pi,\ z = r e^{i\theta}$ ✓

$$\frac{f(z)}{\sqrt{z}} = \frac{f(re^{i\theta})}{\sqrt{re^{i\theta}}} = \frac{f(re^{i\theta})}{\sqrt{r}\, e^{\frac{i\theta}{2}}} \longrightarrow \frac{f(x)}{\sqrt{x}(-1)} = \underline{-\frac{f(x)}{\sqrt{x}}}$$

(b)(i)

$$g(z) = \frac{z(z^2+3)}{(z^2+1)(z^2+4)\sqrt{z^2-1}} = \frac{z(z^2+3)}{(z-i)(z+i)(z-2i)(z+2i)\sqrt{z^2-1}}$$

AT $z = 2i, i, -i, -2i$, $g(z)$ IS NOT ANALYTIC, AS IT HAVE SIMPLE POLES THERE. ✓

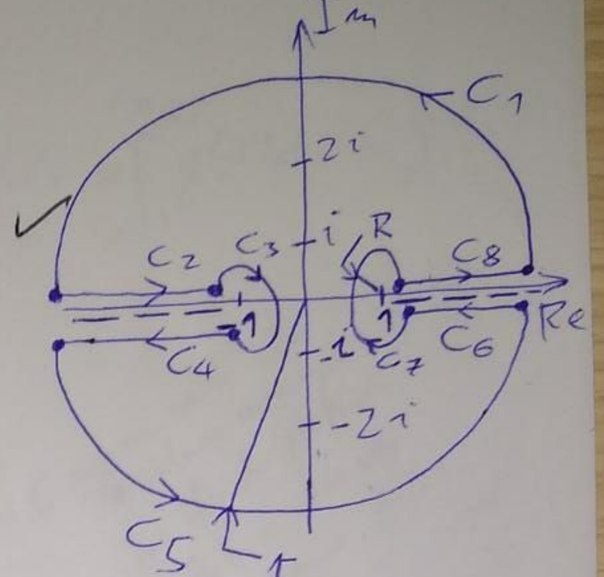
$$g(z) = \frac{z(z^2+3)}{(z^2+1)(z^2+4)} \frac{1}{\sqrt{(z-1)(z+1)}} = \frac{z(z^2+3)}{(z^2+1)(z^2+4)} \frac{\sqrt{(z-1)(z+1)}}{(z-1)(z+1)}$$

$\to g(z)$ ALSO NOT ANALYTIC AT $z = \pm 1$ AS IT HAVE SIMPLE POLES ~~ZEROS~~ THERE.

no, these are branch points.

2016P2Q4(II) (b)(ii)

$$\int_C g(z)\,dz = \sum_{i=1}^{8} \int_{C_i} g(z)\,dz$$



$$h(z) = \frac{z(z^2+3)}{(z^2+1)(z^2+4)}$$

$$f(z) = \frac{1}{\sqrt{z^2-1}}$$

I do not think it helps to use notation not in the question.

$$g(z) = h(z) f(z)$$

$$\lim_{|z|\to\pm1} g(z) = \underbrace{\lim_{|z|\to\pm1} h(z)}\ \lim_{|z|\to\pm1} f(z)$$

$$\lim_{|z|\to\pm1} h(z) = \lim_{|z|\to\pm1} \frac{z(z^2+3)}{(z^2+1)(z^2+4)} = \frac{\pm4}{2\cdot5} = \pm\frac{2}{5}$$

$$\lim_{R\to0}\int_{C_7,C_3} g(z)\,dz = \int_{C_3,C_7} \pm\frac{2}{5}\cdot\lim_{|z|\to\pm1} f(z)\,dz \le \left|\int_{C_3,C_7} f(z)\,dz\right| =$$

C_7 CORRESPONDS TO +
C_3 TO − SIGN ON THE RHS.

$C_7: \theta = 2\pi$
$C_3: \theta = -\pi$

$$= \int_{C_3,C_7} \frac{1}{\sqrt{(z-1)(z+1)}}\,dz = \int_{C_3,C_7} \frac{r i\theta\,d\theta}{\sqrt{|re^{i\theta}|\,|re^{i\theta}\pm2|}} \le \int \frac{r\theta\,d\theta}{\sqrt{(r)(r\pm2)}}$$

$$z = \pm1 + re^{i\theta}$$

$C_7: \theta = 0$
$C_3: \theta = +\pi$

$$\frac{dz}{d\theta} = ri\theta\,d\theta$$

$$\lim_{R\to0}\int_{C_7,C_3} g(z)\,dz = \lim_{r\to0}\sqrt{r}\int_{C_7,C_3}\frac{\theta\,d\theta}{\sqrt{r\pm2}} = 0$$

This is very hard to follow.

AS $|z|\to\infty$, $g(z) = \frac{z^3}{z^2 z^2 z} = z^{-2}$, SO BY ANSWER TO (a)(i),

$$\int_{C_1,C_5} g(z)\,dz = 0$$

2016P2Q4(III)

$$\lim_{R\to 0,\, r\to\infty}\int_{C8} g(z)\,dz = \int_1^\infty g(x)\,dx = J$$

$$\lim_{R\to 0,\, r\to\infty}\int_{C6} g(z)\,dz = \int_{r=\infty}^{r=1} -g(z)\,dz = \int_{r=1}^{r=\infty} g(z)\,dz = \int_{r=1}^{r=\infty} g(x)\,dx = J$$

BY ANSWER TO (a)(ii)

$$\lim_{R\to 0,\, r\to\infty}\int_{C4} g(z)\,dz = \int_{r=1}^{r=\infty} g(z)\,dz = J$$

$$\lim_{R\to 0,\, r\to\infty}\int_{C2} g(z)\,dz = \int_{r=\infty}^{r=1} -g(z)\,dz = \int_{r=1}^{\infty} g(z)\,dz = J$$

I agree with the residues except for the definition of g in each region from $\sqrt{z^2-1}$. So nice overall work.

COMBINING ABOVE RESULTS

$$\int_C g(z)\,dz = \sum_{i=1}^{\infty}\int_{C_i} g(z)\,dz = 2\pi i \sum_{\text{RESIDUES WITHIN } C} \text{RES}\, f(z_i) = 4J$$

I agree with this.

$$\text{RES}\, f(2i) = \left.\frac{z(z^2+3)}{(z^2+1)(z+2i)\sqrt{z^2-1}}\right|_{z=2i} = \frac{2i(-4+3)}{(-4+1)(4i)\sqrt{-5}} = \frac{\cancel{2i}(-1)}{(-3)(4\cancel{i})i\sqrt{5}} =$$

$$= \frac{1}{6}\,\frac{1}{i}\cdot\frac{1}{\sqrt{5}}$$

$$\text{RES}\, f(i) = \left.\frac{z(z^2+3)}{(z+i)(z^2+4)\sqrt{z^2-1}}\right|_{i} = \frac{\cancel{i}(-1+3)}{\cancel{2i}\cdot 3\cdot\sqrt{-2}} = \frac{1}{3i\sqrt{2}}$$

$$\text{RES}\, f(-i) = \left.\frac{z(z^2+3)}{(z-i)(z^2+4)\sqrt{z^2-1}}\right|_{-i} = \frac{\cancel{-i2}}{(\cancel{-2i})(3)\sqrt{-2}} = \frac{1}{3i\sqrt{2}}$$

$$\text{RES}\, f(-2i) = \left.\frac{z(z^2+3)}{(z^2+1)(z-2i)\sqrt{z^2-1}}\right|_{z=-2i} = \frac{\cancel{-}2i(-4+3)}{(-3)(-4i)\sqrt{-5}} = \frac{2i}{12\,i\,i\sqrt{5}} =$$

$$= \frac{1}{6}\,\frac{1}{i}\,\frac{1}{\sqrt{5}}$$

$$J = \frac{1}{4}\,2\pi i\sum_i \text{RES}\, f(z_i) = \frac{\pi i}{2}\left(\frac{1}{6}\,\frac{1}{i}\,\frac{1}{\sqrt{5}} + \frac{1}{3i\sqrt{2}} + \frac{1}{3i\sqrt{2}} + \frac{1}{6i\sqrt{5}}\right) =$$

$$= \frac{\pi}{2}\left(\frac{1}{3}\,\frac{1}{\sqrt{5}} + \frac{2}{3}\,\frac{1}{\sqrt{2}}\right) = \pi\left(\frac{1}{6}\,\frac{1}{\sqrt{5}} + \frac{1}{3}\,\frac{1}{\sqrt{2}}\right)$$