2016P2Q4 (I)

(a)(i) $|I| =$

$$\left\{\begin{array}{l} z = re^{i\theta} \\ dz = r i \theta e^{i\theta} d\theta \end{array}\right.$$

$$= \left| \int_C \frac{f(z)}{\sqrt{z}} dz \right| \leq \int_C \frac{|f(z)|}{|\sqrt{z}|} |dz| = \int_{\theta=\frac{3}{2}\pi}^{\theta=\frac{1}{2}\pi} \frac{|f(z)|}{\sqrt{r}} r \, d\theta = \int_{\theta=\frac{3}{2}\pi}^{\theta=\frac{1}{2}\pi} |f(z)| \sqrt{r} \, d\theta =$$

$$= \sqrt{r} \int_{\theta=\frac{3}{2}\pi}^{\theta=\frac{1}{2}\pi} |f(z)| d\theta$$

$$\lim_{R\to 0} |I| = \lim_{r \to 0} \sqrt{r} \int_{\theta=\frac{3}{2}\pi}^{\theta=\frac{1}{2}\pi} |f(z)| d\theta = 0 \Rightarrow \lim_{R\to 0} I = \underline{0}$$

(ii) INTEGRAL VANISHES AS $R \to \infty$ IF IT GOES TO 0 FASTER THAN ARCLENGTH OF C INCREASES:

INTEGRAND SHOULD APPROACH 0 FASTER THAN $\frac{1}{r}$

$\Updownarrow$

CONSTRAINT:

$f(z)$ SHOULD APPROACH 0 FASTER THAN $\frac{1}{r}\sqrt{r} = \frac{1}{\sqrt{r}}$.

(ii)

ABOVE:

$r \to x, \; \theta \to 0, \; z = re^{i\theta}$

$$\frac{f(z)}{\sqrt{z}} = \frac{f(x+i\epsilon)}{\sqrt{x+i\epsilon}} = \frac{f(re^{i\theta})}{\sqrt{re^{i\theta}}} = \frac{f(re^{i\theta})}{\sqrt{r}\, e^{\frac{i\theta}{2}}} \longrightarrow \underline{\frac{f(x)}{\sqrt{x}}}$$

BELOW:

$r \to x, \; \theta \to 2\pi, \; z = re^{i\theta}$

$$\frac{f(z)}{\sqrt{z}} = \frac{f(re^{i\theta})}{\sqrt{re^{i\theta}}} = \frac{f(re^{i\theta})}{\sqrt{r}\, e^{i\frac{\theta}{2}}} \longrightarrow \frac{f(x)}{\sqrt{x}(-1)} = \underline{-\frac{f(x)}{\sqrt{x}}}$$

(b)(i)

$$g(z) = \frac{z(z^2+3)}{(z^2+1)(z^2+4)\sqrt{z^2-1}} = \frac{z(z^2+3)}{(z-i)(z+i)(z-2i)(z+2i)\sqrt{z^2-1}}$$

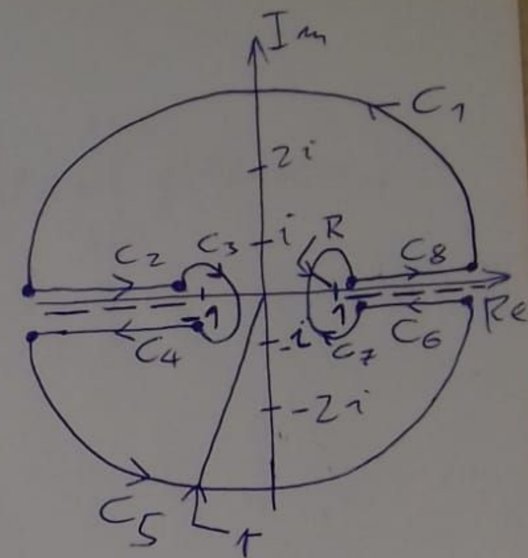
AT $z = 2i, i, -i, -2i$, $g(z)$ IS NOT ANALYTIC, AS IT HAVE SIMPLE POLES THERE.

$$g(z) = \frac{z(z^2+3)}{(z^2+1)(z^2+4)} \frac{1}{\sqrt{(z-1)(z+1)}} = \frac{z(z^2+3)}{(z^2+1)(z^2+4)} \frac{\sqrt{(z-1)(z+1)}}{(z-1)(z+1)}$$

$\rightarrow$ $g(z)$ ALSO NOT ANALYTIC AT $z = \pm 1$, AS IT HAVE SIMPLE POLES THERE.

2016P2Q4(II) (b)(ii)

$$\int_C g(z)\,dz = \sum_{i=1}^{8} \int_{C_i} g(z)\,dz$$



$$h(z) = \frac{z(z^2+3)}{(z^2+1)(z^2+4)}$$

$$f(z) = \frac{1}{\sqrt{z^2-1}}$$

$$g(z) = h(z) f(z)$$

$$\lim_{|z|\to\pm 1} g(z) = \underbrace{\lim_{|z|\to\pm 1} h(z)}\ \lim_{|z|\to\pm 1} f(z)$$

$$\lim_{|z|\to\pm 1} h(z) = \lim_{|z|\to\pm 1} \frac{z(z^2+3)}{(z^2+1)(z^2+4)} = \frac{\pm 4}{2\cdot 5} = \pm\frac{2}{5}$$

$$\lim_{R\to 0}\int_{C_7, C_3} g(z)\,dz = \int_{C_3, C_7} \pm\frac{2}{5} \cdot \lim_{|z|\to\pm 1} f(z)\,dz \le \left|\int_{C_3, C_7} f(z)\,dz\right| =$$

C_7 CORRESPONDS TO +

C_3 TO − SIGN ON THE RHS.

$C_7: \theta = 2\pi$

$C_3: \theta = -\pi$

$$= \int_{C_3, C_7} \frac{1}{\sqrt{(z-1)(z+1)}}\,dz = \int_{C_3, C_7} \frac{r i\theta\,d\theta}{\sqrt{|re^{i\theta}|\,|re^{i\theta} \pm 2|}} \le \int \frac{r\theta\ d\theta}{\sqrt{(r)(r\pm 2)}}$$

$$z = \pm 1 + re^{i\theta}$$

$C_7: \theta = 0$

$C_3: \theta = +\pi$

$$\frac{dz}{d\theta} = r i\theta\,d\theta$$

$$\lim_{R\to 0}\int_{C_7, C_3} g(z)\,dz = \lim_{r\to 0} \sqrt{r} \int_{C_7, C_3} \frac{\theta\,d\theta}{\sqrt{r\pm 2}} = 0$$

AS $|z|\to\infty$, $g(z) = \frac{z^3}{z^2 z^2 z} = z^{-2}$, SO BY ANSWER TO (a)(i),

$$\int_{C_1, C_5} g(z)\,dz = 0$$

2016P2Q4(III)

$$\lim_{R\to 0,\, r\to\infty} \int_{C_8} g(z)\,dz = \int_1^\infty g(x)\,dx = J$$

$$\lim_{R\to 0,\, r\to\infty} \int_{C_6} g(z)\,dz = \int_{r=\infty}^{r=1} -g(z)\,dz = \int_{r=1}^{r=\infty} g(z)\,dz = \int_{r=1}^{r=\infty} g(x)\,dx = J$$

BY ANSWER TO (a)(iii)

$$\lim_{R\to 0,\, r\to\infty} \int_{C_4} g(z)\,dz = \int_{r=\infty}^{r=1} g(z)\,dz = J$$

$$\lim_{R\to 0,\, r\to\infty} \int_{C_2} g(z)\,dz = \int_{r=\infty}^{r=1} -g(z)\,dz = \int_{r=1}^{\infty} g(z)\,dz = J$$

$$\int_C g(z)\,dz = \sum_{i=1}^{\infty} \int_{C_i} g(z)\,dz = 2\pi i \sum_{\text{RESIDUES WITHIN } C} \text{RES}\, f(z_i) = 4J$$

COMBINING ABOVE RESULTS.

$$\text{RES}\, f(2i) = \left.\frac{z(z^2+3)}{(z^2+1)(z+2i)\sqrt{z^2-1}}\right|_{z=2i} = \frac{2i(-4+3)}{(-4+1)(4i)\sqrt{5}} = \frac{2i(-1)}{(-3)(4i)\sqrt{5}} =$$

$$= \frac{1}{6}\,\frac{1}{i}\cdot\frac{1}{\sqrt{5}}$$

$$\text{RES}\, f(i) = \left.\frac{z(z^2+3)}{(z+1)(z^2+4)\sqrt{z^2-1}}\right|_{i} = \frac{i(-1+3)}{2i\cdot 3\cdot\sqrt{2}} = \frac{1}{3i\sqrt{2}}$$

$$\text{RES}\, f(-i) = \left.\frac{z(z^2+3)}{(z-i)(z^2+4)\sqrt{z^2-1}}\right|_{-i} = \frac{-i\,2}{(-2i)(3)\sqrt{2}} = \frac{1}{3i\sqrt{2}}$$

$$\text{RES}\, f(-2i) = \left.\frac{z(z^2+3)}{(z^2+1)(z-2i)\sqrt{z^2-1}}\right|_{z=-2i} = \frac{-2i(-4+3)}{(-3)(-4i)\sqrt{5}} = \frac{2i}{12i\sqrt{5}} =$$

$$= \frac{1}{6}\,\frac{1}{i}\,\frac{1}{\sqrt{5}}$$

$$J = \frac{1}{4}\,2\pi i \sum_i \text{RES}\, f(z_i) = \frac{\pi i}{2}\left(\frac{1}{6}\,\frac{1}{i}\,\frac{1}{\sqrt{5}} + \frac{1}{3i\sqrt{2}} + \frac{1}{3i\sqrt{2}} + \frac{1}{6i\sqrt{5}}\right) =$$

$$= \frac{\pi}{2}\left(\frac{1}{3}\,\frac{1}{\sqrt{5}} + \frac{2}{3}\,\frac{1}{\sqrt{2}}\right) = \pi\left(\frac{1}{6}\,\frac{1}{\sqrt{5}} + \frac{1}{3}\,\frac{1}{\sqrt{2}}\right)$$