2014P1Q1(I)

(i) $\nabla\times(\nabla\times\underline{u}) = \nabla(\nabla\cdot u) - \nabla^2\underline{u}$

$\nabla\times(\nabla\times\underline{u}) = \epsilon_{ijk}\partial_j(\nabla\times\underline{u})_k =$

$= \epsilon_{ijk}\partial_j(\epsilon_{klm}\partial_l u_m) = \epsilon_{ijk}\epsilon_{klm}\partial_j\partial_l u_m =$

$= \epsilon_{kij}\epsilon_{klm}\partial_j\partial_l u_m = (\delta_{il}\delta_{jm} - \delta_{jl}\delta_{im})\partial_j\partial_l u_m =$

$= \partial_j\partial_i u_j - \partial_l\partial_l u_i = \nabla(\nabla\cdot\underline{u}) - \nabla^2\underline{u}$ AS REQUIRED.

✓

$\nabla\times(\underline{u}\times\underline{v}) = \underline{u}(\nabla\cdot\underline{v}) - \underline{v}(\nabla\cdot\underline{u}) + (\underline{v}\cdot\nabla)\underline{u} - (\underline{u}\cdot\nabla)v$

$\nabla\times(\underline{u}\times\underline{v}) = \epsilon_{ijk}\partial_j(\underline{u}\times\underline{v})_k = \epsilon_{ijk}\partial_j(\epsilon_{klm}u_l v_m) =$

$= \epsilon_{kij}\epsilon_{klm}\partial_j(u_l v_m) = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(v_m\partial_j u_l +$
$+ u_l\partial_j v_m) =$

$= v_j\partial_j u_i - v_i\partial_l u_l + u_i\partial_m v_m - u_j\partial_j v_i =$

✓

$= (\underline{v}\cdot\nabla)\underline{u} - \underline{v}(\nabla\cdot\underline{u}) + \underline{u}(\nabla\cdot\underline{v}) - (\underline{u}\cdot\nabla)v =$

REARRANGE:

$= \underline{u}(\nabla\cdot\underline{v}) - \underline{v}(\nabla\cdot\underline{u}) + (\underline{v}\cdot\nabla)\underline{u} - (\underline{u}\cdot\nabla)\underline{v}$ AS DESIRED.

(ii) DIVERGENCE THEOREM: $\int_V(\nabla\cdot\underline{F})dV = \oint_S \underline{F}\cdot d\underline{S}$ WHERE $\underline{F}$ IS A VECTOR FIELD, V IS A VOLUME BOUNDED BY CLOSED SURFACE S, $d\underline{S}$ IS OUTWARD POINTING NORMAL ON SURFACE (I.E. $\hat{\underline{n}}dS$)

2014 P1Q1(ii)

$$\int_V \nabla\cdot(\underline{F}\times\underline{G})\,dV = \oint_S (\underline{F}\times\underline{G})\cdot d\underline{s}$$ BY DIVERGENCE THM.

$$\nabla\cdot(\underline{F}\times\underline{G}) = \partial_i(\epsilon_{ijk}F_j G_k) =$$

$$= \epsilon_{ijk}\left[(\partial_i F_j)G_k + F_j(\partial_i G_k)\right] =$$

$$= G_k\epsilon_{kij}\partial_i F_j + \epsilon_{jki}F_j\partial_i G_k =$$

$$= \underline{G}\cdot(\underline{\nabla}\times\underline{F}) - \underline{F}\cdot(\underline{\nabla}\times\underline{G})$$

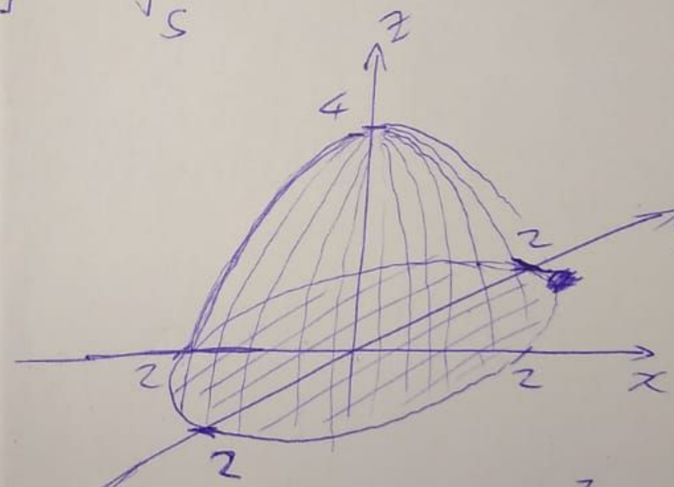
say what properties of ε are used.

CYCLIC PROPERTY

PLUG BACK IN:

$$\int_V \left[\underline{G}\cdot(\underline{\nabla}\times\underline{F}) - \underline{F}\cdot(\underline{\nabla}\times\underline{G})\right]dV = \oint_S (\underline{F}\times\underline{G})\cdot\hat{\underline{n}}\,dS$$

(iii)



$$\underline{F} = (xz\sin(yz)+x^3,\ \cos(yz),\ 3zy^2 - e^{x^2+y^2})$$

$$\underline{\nabla}\cdot\underline{F} = \left(\frac{\partial}{\partial x}(xz\sin(yz)+x^3) + \frac{\partial}{\partial y}(\cos(yz)) + \frac{\partial}{\partial z}(3zy^2 - e^{x^2+y^2})\right) =$$

$$= z\sin(yz) + 3x^2 + -z\sin(yz) + 3y^2 =$$

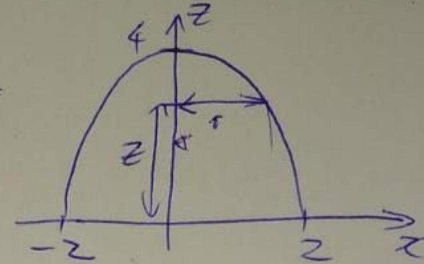
$$= 3x^2 + 3y^2 = 3(x^2+y^2)$$ ✓

$$\int_S \underline{F}\cdot\hat{\underline{n}}\,dS = \int_V (\nabla\cdot F)\,dV = \int_V 3(x^2+y^2)\,dV =$$

2014P1Q1(III)

$$\cancel{\iiint_{z=0}^{4}} = \int_{z=0}^{4}\int_{r=0}^{\sqrt{4-z}}\int_{\theta=0}^{2\pi} 3r^2\, r\,dr\,d\theta\,dz =$$

✓



$$z = 4 - x^2 - y^2$$

$$4 - z = x^2 + y^2 = r^2$$

$$r = \sqrt{4-z}$$

$$= 2\pi\int_{z=0}^{4}\int_{r=0}^{\sqrt{4-z}} 3r^3\,dr\,dz =$$

$$= 2\pi\int_{z=0}^{4} 3\left[\frac{r^4}{4}\right]_0^{\sqrt{4-z}} dz = 2\pi\int_{z=0}^{4} 3\frac{(4-z)^2}{4}\,dz = \frac{3}{2}\pi\int_{z=0}^{4}(4-z)^2\,dz =$$

$$= \frac{3}{2}\pi\int_{z=0}^{4} 16 - 8z + z^2\,dz = \frac{3}{2}\pi\left(\left[16z\right]_0^4 - 8\left[\frac{z^2}{2}\right]_0^4 + \left[\frac{z^3}{3}\right]_0^4\right) =$$

$$= \frac{3}{2}\pi\left(64 - 64 + \frac{1}{3}64\right) = \underline{32\pi}$$

✓