2012P1Q4(I)

$$\tilde{f}(k)=\int_{-\infty}^{\infty} f(x)e^{-ikx}dx=\int_{-\infty}^{\infty} e^{-\lambda x^2}e^{-ikx}dx=$$

$$z=\sqrt{\lambda}x+\frac{ik}{\sqrt{\lambda}2} \longrightarrow dz=\sqrt{\lambda}dx$$

$$z^2=\lambda x^2+ikx-\frac{k^2}{4\lambda}$$

$$-z^2=-\lambda x^2-ikx+\frac{k^2}{4\lambda}$$

$$=\int_{-\infty}^{\infty} e^{-z^2}e^{-\frac{k^2}{4\lambda}}\frac{dz}{\sqrt{\lambda}}=\frac{1}{\sqrt{\lambda}}e^{-\frac{k^2}{4\lambda}}\int_{-\infty}^{\infty}e^{-z^2}dz=\sqrt{\frac{\pi}{\lambda}}e^{-\frac{k^2}{4\lambda}}$$

LET'S HAVE: $g(x)=f'(x)$

$$\tilde{g}(k)=\int_{-\infty}^{\infty}f'(x)e^{-ikx}dx=$$

$$=\left[f(x)e^{-ikx}\right]_{-\infty}^{\infty}-\int_{-\infty}^{\infty}f(x)(-ik)e^{-ikx}dx=$$

$$=ik\int_{-\infty}^{\infty}f(x)\exp(-ikx)dx=ik\tilde{f}(k)$$

- $h(x)=f''(x)$

$$\tilde{h}(k)=ik\tilde{f}'(k)=(ik)^2\tilde{f}(k)=-k^2\tilde{f}(k)$$

$$\psi''=\dot{\psi} \xrightarrow{\text{FT WRT } x} -k^2\tilde{\psi}(k,t)=\frac{\partial\tilde{\psi}(k,t)}{\partial t}$$

~~$\downarrow$ FT WRT x~~

~~$k^2\tilde{\psi}=\dot{\tilde{\psi}}$~~

SOLUTION

$$\tilde{\psi}(k,t)=\tilde{\psi}(k,t=0)e^{-k^2t}$$

$$\tilde{\psi}(k,t=0)=\int_{-\infty}^{\infty}\psi(x,t=0)e^{-ikx}dx=\int_{-\infty}^{\infty}e^{-\lambda x^2}\cdot e^{-ikx}dx=$$

$$=\sqrt{\frac{\pi}{\lambda}}\exp\left(-\frac{k^2}{4\lambda}\right)\quad\cdot(\text{USING PREVIOUS FINDINGS})$$

2012 P1 Q4 (II)

$$\tilde{\psi}(\xi, t=0) = \int_{-\infty}^{\infty} \psi(x, t=0)\, e^{-i\xi x}\, dx = \int_{-\infty}^{\infty} e^{-\lambda x^2} e^{-i\xi x}\, dx =$$

$$= \sqrt{\frac{\pi}{\lambda}} \exp\left(-\frac{\xi^2}{4\lambda}\right)$$

$$\tilde{\psi}(\xi, t) = \sqrt{\frac{\pi}{\lambda}} \exp\left(-\frac{\xi^2}{4\lambda}\right) \exp\left(-\xi^2 t\right) = \sqrt{\frac{\pi}{\lambda}}\, e^{-\xi^2\left(\frac{1}{4\lambda} + t\right)}$$

$$\psi(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\psi}(\xi, t)\, e^{i\xi x}\, d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{\lambda}}\, e^{-\xi^2\left(\frac{1}{4\lambda} + t\right)} e^{i\xi x}\, d\xi =$$

$$\frac{1}{4\lambda} + t = \beta$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{\lambda}}\, e^{-\left(\xi\sqrt{\beta} - \frac{ix}{2\sqrt{\beta}}\right)^2} e^{-\frac{x^2}{4\beta}}\, d\xi =$$

$$= \frac{1}{2\pi} e^{-\frac{x^2}{4\beta}} \sqrt{\frac{\pi}{\lambda}} \int_{-\infty}^{\infty} e^{-\left(\xi\sqrt{\beta} - \frac{ix}{2\sqrt{\beta}}\right)^2} d\xi = \frac{1}{2\pi} e^{-\frac{x^2}{4\beta}} \sqrt{\frac{\pi}{\lambda}} \int e^{-\left(y - \frac{ix}{2\sqrt{\beta}}\right)^2} \frac{dy}{\sqrt{\beta}}$$

$$y = \xi\sqrt{\beta} \qquad \lambda = +1$$

$$dy = \sqrt{\beta}\, d\xi$$

$$= \frac{1}{2\pi} e^{-\frac{x^2}{4\beta}} \sqrt{\frac{\pi}{\lambda}} \frac{1}{\sqrt{\beta}} \sqrt{\frac{\pi}{1}} = e^{-\frac{x^2}{4\beta}} \frac{1}{2} \sqrt{\frac{1}{\lambda}} \frac{1}{\sqrt{\beta}} =$$

$$= \frac{1}{2} \sqrt{\frac{1}{\lambda}} \frac{1}{\sqrt{\frac{1}{4\lambda} + t}} = \frac{1}{2} \frac{1}{\sqrt{\frac{1}{4} + \lambda t}} = \frac{1}{\sqrt{1 + 4\lambda t}}$$