REL. IV 2

$$L = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \left(\frac{ds}{d\lambda}\right)^2$$

$$L = c^2\left(1-\frac{2M}{r}\right)\dot{t}^2 - \left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\,\dot{\phi}^2$$

EULER-LAGRANGE FOR r:

$$\frac{d}{d\lambda}\left(\frac{\partial}{\partial \dot{r}}L\right) = \frac{d}{d\lambda}\left[-2\left(1-\frac{2M}{r}\right)^{-1}\dot{r}\right]$$

$$= 2\left(1-\frac{2M}{r}\right)^{-2\,\leftarrow !}\frac{2M}{r^2}\dot{r}^2 - 2\left(1-\frac{2M}{r}\right)^{-1}\ddot{r}$$

$$\frac{\partial L}{\partial r} = c^2\frac{2M}{r^2}\dot{t}^2 + \left(1-\frac{2M}{r}\right)^{-2}\frac{2M\dot{r}^2}{r^2} - 2r\dot{\theta}^2 - 2r\sin^2\theta\,\dot{\phi}^2$$

$$\Rightarrow 2\left(1-\frac{2M}{r}\right)^{-2}\frac{2M}{r^2}\dot{r}^2 - 2\left(1-\frac{2M}{r}\right)^{-1}\ddot{r} = c^2\frac{2M}{r^2}\dot{t}^2 + \left(1-\frac{2M}{r}\right)^{-2}\frac{2M\dot{r}^2}{r^2} - 2r\dot{\theta}^2 - 2r\sin^2\theta\,\dot{\phi}^2$$

$$\left(1-\frac{2M}{r}\right)^{-1}\frac{2M}{r^2}\dot{r}^2 - \ddot{r} = c^2\frac{M}{r^2}\left(1-\frac{2M}{r}\right)\dot{t}^2 + \left(1-\frac{2M}{r}\right)^{-1}\frac{M}{r^2}\dot{r}^2 - \left(1-\frac{2M}{r}\right)r\dot{\theta}^2 - \left(1-\frac{2M}{r}\right)r\sin^2\theta\,\dot{\phi}^2$$

C.F. $\frac{d^2r}{d\lambda^2} + \Gamma^r_{mn}\frac{dx^m}{d\lambda}\frac{dx^n}{d\lambda} = 0$

$$\Rightarrow \Gamma^r_{rr} = \left(1-\frac{2M}{r}\right)^{-1}\frac{M}{r^2} - \left(1-\frac{2M}{r}\right)^{-1}\frac{2M}{r^2} = -\left(1-\frac{2M}{r}\right)^{-1}\frac{M}{r^2}$$

$$\Gamma^r_{tt} = c^2\left(1-\frac{2M}{r}\right)\frac{M}{r^2}$$

$$\Gamma^r_{\theta\theta} = -\left(1-\frac{2M}{r}\right)r$$

$$\Gamma^r_{\phi\phi} = -\left(1-\frac{2M}{r}\right)r\sin^2\theta$$

ALL OTHER Γ^r_{XY} ARE ZERO.

EULER-LAGRANGE FOR θ:

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$$

LHS:

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{d\lambda}\left(-r^2 2\dot{\theta}\right) = -4r\dot{r}\dot{\theta} - 2r^2\ddot{\theta}$$

RHS:

$$\frac{\partial L}{\partial \theta} = -r^2 2\sin\theta\cos\theta\dot{\varphi}^2$$

$$\Rightarrow -4r\dot{r}\dot{\theta} - 2r^2\ddot{\theta} = -r^2 2\sin\theta\cos\theta\dot{\varphi}^2$$

| REARRANGE

| : $2r^2$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\varphi}^2 = 0$$

COMPARE WITH GEODESIC EQUATION, CONCLUDE:

$$\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r} \;!$$

$$\Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta$$

ALL OTHER Γ^{θ}_{XY} IS ZERO.

EULER-LAGRANGE FOR φ:

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \frac{\partial L}{\partial \varphi}$$

LHS:

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \frac{d}{d\lambda}\left(-2r^2\sin^2\theta\,\dot{\varphi}\right) =$$

$$= -4r\dot{r}\sin^2\theta\dot{\varphi} - 4r^2(\sin\theta)(\cos\theta)\dot{\theta}\dot{\varphi} - 2r^2(\sin^2\theta)\ddot{\varphi}$$

RHS:

$$\frac{\partial L}{\partial \varphi} = 0$$

LHS = RHS $\quad /: -2$

$$\Rightarrow 2 r \dot{r} \sin^2\theta \dot{\varphi} + 2 r^2 (\sin\theta)(\cos\theta) \dot{\theta}\dot{\varphi} + r^2 \sin^2\theta \ddot{\varphi} = 0 \quad \Big/ \begin{array}{l}\text{REARRANGE} \\ : r^2 \sin^2\theta\end{array}$$

$$\ddot{\varphi} + \frac{2}{r} \dot{r}\dot{\varphi} + 2 \cot\theta \, \dot{\theta}\dot{\varphi} = 0$$

COMPARE WITH GEODESIC EQUATION, CONCLUDE:

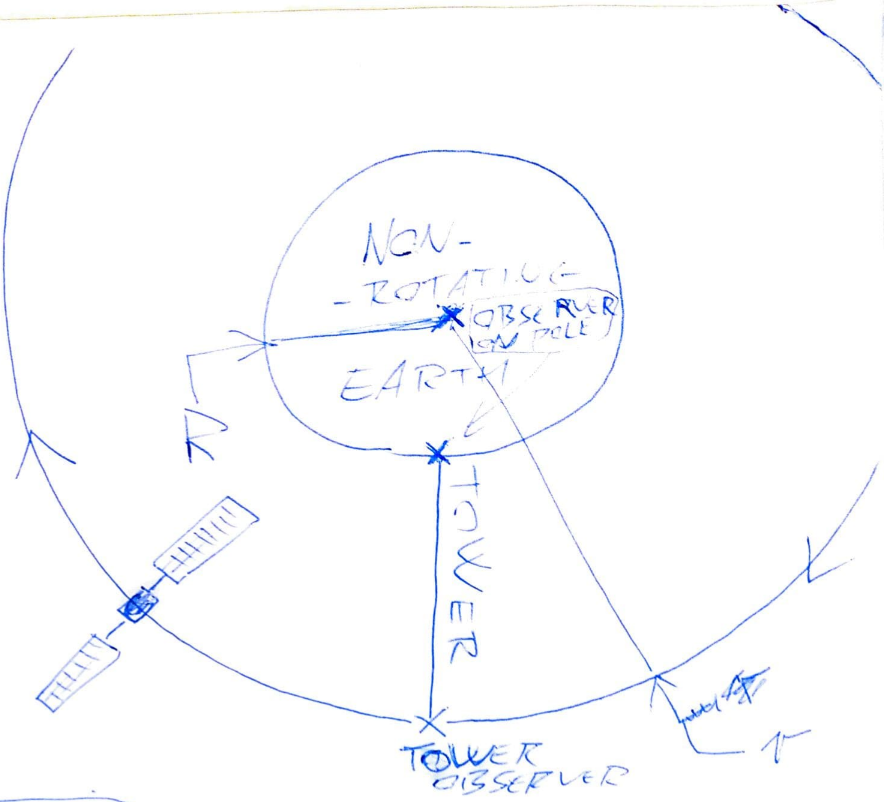
(THERE ARE TWO Γs FOR $\dot{\theta}\dot{\varphi}$: $\Gamma\dot{\theta}\dot{\varphi}$ & $\Gamma\dot{\varphi}\dot{\theta}$)

$$\Gamma^{\varphi}_{\varphi r} = \Gamma^{\varphi}_{r\varphi} = \frac{\cancel{2}\,1}{r} \;!$$

$$\Gamma^{\varphi}_{\theta\varphi} = \Gamma^{\varphi}_{\varphi\theta} = \cancel{2} \cot\theta$$

ALL OTHER Γ^{φ}_{XY} IS ZERO.

REL. IV / SHEET 2 / IV



$$g_{\mu\nu}u^\mu u^\nu = \left(\frac{ds}{d\tau}\right)^2 = c^2$$

$$\Downarrow$$

$$g_{\mu\nu}u^\mu u^\nu d\tau^2 = c^2 d\tau^2$$

$$g_{\mu\nu}u^\mu u^\nu d\tau^2 =$$

$$\cancel{= g_{00}u^0u^0 d\tau^2 + 2u\,dx}$$

$$= \underbrace{g_{00}}u^0u^0 d\tau^2 + g_{11}u^1u^1 d\tau^2 + g_{22}u^2u^2 d\tau^2 + g_{33}u^3u^3 d\tau^2 = c^2 d\tau^2$$

SINCE METRIC IS DIAGONAL.

$$u^\mu = \frac{dt}{d\tau}\left(c, \frac{dr}{dt}, \frac{d\theta}{dt}, \frac{d\phi}{dt}\right)$$ WHERE $d\tau$: SATELLITE PROPER TIME
dt: SATELLITE COORDINATE TIME

$\boxed{\text{CONSIDER MOVEMENT OF SATELLITE}}$

$$c^2 d\tau^2 = \left(1 - \frac{2\mu}{r}\right)c^2 dt^2 + -r^2\left(\frac{d\theta}{dt}\right)^2 dt^2$$

dr & $d\phi$ ARE ZERO ON CIRCULAR ORBIT.

$$\frac{d\theta}{dt} = \sqrt{\frac{GM}{r^3}} = \sqrt{\frac{\mu c^2}{r^3}}$$

$$c^2 d\tau^2 = \left(1 - \frac{2\mu}{r}\right)c^2 dt^2 - r^2\frac{\mu c^2}{r^3}dt^2$$

$$= \left(1 - \frac{3\mu}{r}\right)c^2 dt^2$$

$$\Rightarrow \frac{d\tau}{dt} = \left(1 - \frac{3\mu}{r}\right)^{\frac{1}{2}}$$

REL SHEET IV | 2 | V |

CONSIDER POLE & TOWER OBSERVERS

UNDERSTANDING IS WEAK HERE. CAN I THINK OF dt AS PROPER TIME INCREMENT OF SOMEONE IN CENTRE OF EARTH, DISTANCE FROM OBSERVER?

$$c^2 d\tau^2 = c^2 dt^2 g_{00} = c^2 dt^2 \left(1 - \frac{2M}{R}\right)$$

$$\cancel{dt = d\tau}\left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} = \frac{dt}{d\tau}$$

CONCLUDE:

$$\frac{\text{PERIOD IN SATELLITE PROPER TIME}}{\text{PERIOD IN POLE OBSERVER PROPER TIME}} = \frac{\Delta \tau_c}{\Delta \tau_{c_0}} = \frac{\Delta \tau_{c}}{\Delta t}\frac{\delta t}{\delta \tau_{c_0}}$$

COORDINATE TIME

$$= \left(1 - \frac{3M}{r}\right)^{\frac{1}{2}}\left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}}$$

CONCEPTUAL WORRY: r IN SCHWARTZCHILD LINE ELEMENT IS NOT THE SAME AS "NORMAL" RADIUS, IS IT?

WORKING TO FIRST ORDER:

$$\frac{\Delta \tau_c}{\Delta \tau_{c_0}} \simeq \left(1 - \frac{3M}{2r}\right)\left(1 + \frac{M}{R}\right) \simeq 1 - \frac{3M}{2r} + \frac{M}{R}$$

$$= 1 - \frac{3GM}{2rc^2} + \frac{GM}{Rc^2}$$

3

$$ds^2 = c^2\left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2$$

$$u^\mu_{ALICE} = A\delta^\mu_0$$

ALICE ISN'T MASSLESS SO:

$$\frac{ds^2}{d\tau^2} = \frac{c^2 d\tau^2}{d\tau^2} = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

$$= g_{\mu\nu}u^\mu u^\nu$$

$$= g_{00}u^0 u^0 = c^2\left(1-\frac{2M}{r}\right)A^2$$

$$\Rightarrow A = \left(1-\frac{2M}{r}\right)^{-\frac{1}{2}}$$

FOR MASSLESS PARTICLES, IE FOR THE PHOTON ALICE MEASURES:

$$p^\mu = c\frac{dx^\mu}{d\lambda}$$

ALICE MEASURES PHOTON E TO BE:

$$E = g_{\mu\nu}u^\mu p^\nu = g_{00}A\, c\frac{dx^0}{d\lambda}$$

$$= c^2\left(1-\frac{2M}{r}\right)\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}}c\dot{x}^0$$

$$= c^2\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}c\dot{t}$$

CONSIDER:

$$\left(\frac{ds}{d\lambda}\right)^2 = L = c^2\left(1-\frac{2M}{r}\right)\dot{t}^2 - \left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\,\dot{\phi}^2$$

EULER-LAGRANGE EQ:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{t}} = \frac{\partial L}{\partial t} \Rightarrow \frac{\partial L}{\partial \dot{t}} = \text{CONST} \Rightarrow \left(1-\frac{2M}{r}\right)\dot{t} = \mathcal{E}$$

REWRITE ALICE'S ENERGY MEASUREMENT:

$$E = c^2\left(1-\frac{2\mu}{r}\right)^{\frac{1}{2}} c\dot{t}$$

$$= c^2\left(1-\frac{2\mu}{r}\right)^{-\frac{1}{2}} c\,\mathcal{E} = h\nu$$

(WE MIGHT BE HAVING TOO MANY C-S)

LET'S HAVE AN OBSERVER WHO IS WITH BOB WHEN HE EMITS A PHOTON, BUT IS ALWAYS STATIONARY.

LETS CALL HIM CHARIE.

RATIO OF FREQUENCY MEASUREMENTS BY ALICE & CHARLIE:

$$\frac{\nu_{ALICE}}{\nu_{CHARLIE}} = \frac{E_{ALICE}}{E_{CHARLIE}} = \sqrt{\frac{1-\frac{2\mu}{r}\Big|_{CHARLIE}}{1-\frac{2\mu}{r}\Big|_{ALICE}}}$$

$$1+z = \frac{\nu_{CHARLIE}}{\nu_{ALICE}}$$

$$z = \frac{\nu_{CHARLIE}}{\nu_{ALICE}} - 1 = \sqrt{\frac{1-2\mu/R}{1-2\mu/r_e}} - 1$$

IF $r_e \to \frac{2GM}{c^2}$ IE $r_e \to 2\mu \Rightarrow \frac{2\mu}{r_e} \to 1 \Rightarrow 1-\frac{2\mu}{r_e} \to 0$

$$\Rightarrow \sqrt{\frac{1-2\mu/R}{1-2\mu/r_e}} \to \infty \Rightarrow z \to \infty$$

AM I RIGHT IN THINKING THAT $1+z$ IS RATIO OF PHOTONS MEASURED BY STATIONARY OBSERVERS AT THE PLACE OF EMISSION & RECEPTION? IE WHAT I HAVE CALCULATED $1+z$ IS NOT THE FREQUENCY RATIO MEASURED BY ALICE & BOB.

REL SHEET IV | 4

$$b = \frac{h}{ck} = \frac{r^2 \dot{\phi}}{c\left(1 - \frac{2M}{r}\right)\dot{t}}$$

WE WANT

$$b = r\left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}}$$

WE THEREFORE HOPE

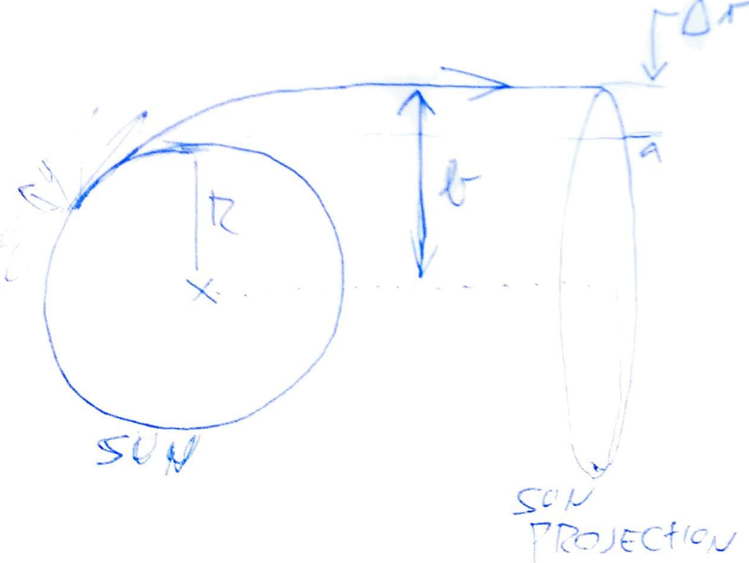
$$\frac{r\dot{\phi}}{c\dot{t}} = \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}}$$

REARRANGE:

$$\left(1 - \frac{2M}{r}\right)c^2\dot{t}^2 - r^2\dot{\phi}^2 = 0$$

THIS IS TRUE AT CLOSEST APPROACH, IE WHEN $\dot{r} = 0$
(TOPIC 9 EQ 48)

~~THIS IS RATHER [illegible]~~
~~[illegible] IS DEEPER~~



$$\Delta r = b - R$$
$$= R\left(1 - \frac{2GM}{c^2 R}\right)^{-\frac{1}{2}} - R$$
$$\simeq 1500\ m$$

$\Longrightarrow$ DIAMETER IS BIGGER BY ~ 3 km

REL IV SHEET 5

THIS IS WRONG.

TOPIC XI EQ IS:

$$c(\tau-\tau_0) = \frac{2}{3}\left[\left(\frac{r_0^3}{2M}\right)^{\frac{1}{2}} - \left(\frac{r^3}{2M}\right)^{\frac{1}{2}}\right]$$

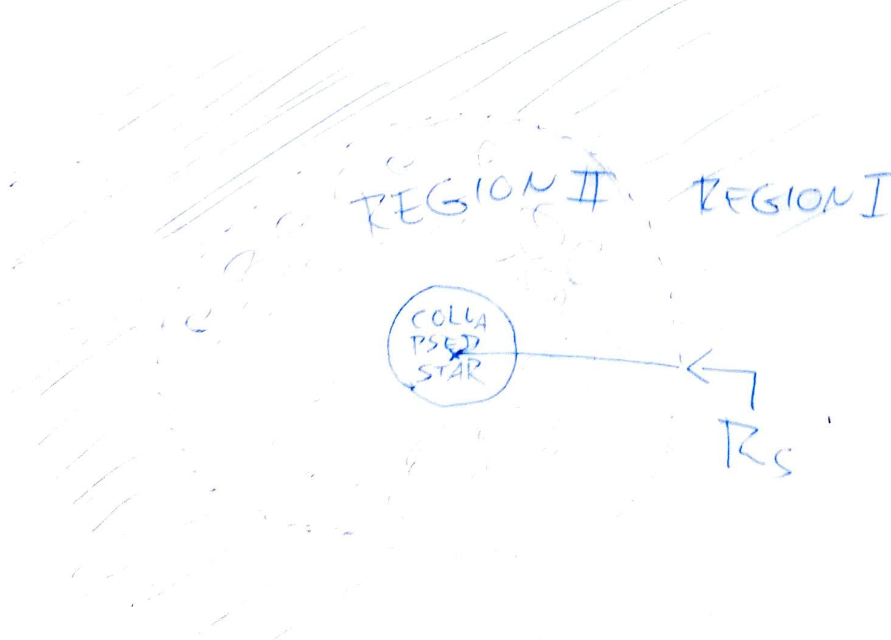
for $r(\tau_0=0)=2M$

$$c\tau\Big|_{r=0} = \frac{2}{3}\left[2M-0\right] = \frac{4}{3}M$$

(IE FALLING TO THE CENTRE)

$$\Rightarrow \underline{\tau = \frac{4}{3}\frac{M}{c}}$$

NOT WHAT'S EXPECTED. I AM MISUSING r PROBABLY: IT IS NOT THE CONVENTIONAL RADIUS IN THE FORMULA I AM QUOTING.



MY METRIC IS ONLY VALID FOR VACUUM REGIONS

WHY PARTICLES FALL IN TO THE SINGULARITY, WHERE THE METRIC ISN'T EVEN VALID? (BEC. IT IS NOT VACUUM)

REL. SHEET IV
Q 6 (I)

FRIEDMANN I EQ: $\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{1}{3}\Lambda c^2$

"EMPTY UNIVERSE WITH VANISHING COSMOLOGICAL CONSTANT"

$\Rightarrow \rho = \Lambda = 0$

WE END UP WITH:

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{kc^2}{a^2}$$

$$\dot{a} = \sqrt{-k}\,c$$

$$a = \sqrt{-k}\,ct$$

RW LINE ELEMENT: $ds^2 = c^2dt^2 - a^2(t)\underbrace{\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]}_{d\sigma^2}$

LET'S CHOOSE $k=1$. $\Rightarrow a = \underline{i\,ct}$

$$d\sigma^2\Big|_{k=1} = \frac{dr^2}{1-r^2} + r^2 d\Omega^2$$

PARAMETRIZE BY: $r = \underline{\sinh \chi}$ $\quad \chi: 0 \to \infty$

$$dr = \cosh\chi\, d\chi$$

$$d\sigma^2\Big|_{k=1} = \frac{\cosh^2\chi (d\chi)^2}{1-\sinh^2\chi} + \sinh\chi\, d\Omega^2 = \underline{d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)}$$

ds^2 BECOMES:

$$ds^2 = c^2dt^2 + c^2t^2\left[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

AS WANTED

REL SHEET IV
Q6 (II)

GEOMETRY OF SPATIAL HYPERSURFACES?

$k = 1 \Rightarrow$ HYPERSURFACE IS A 2-SPHERE EMBEDDED IN 4D (IE: (t, χ, θ, ϕ)) SPACETIME.

- *SHOW THAT METRIC* IS MINKOWSKIAN.
- RECONCILE

I DON'T KNOW THESE.

REL SHEET IV
Q8

SOLUTION TO EINSTEIN'S EQUATIONS IF ISOTROPY & HOMOGENEITY ARE REQUIRED: FRIEDMANN EQUATIONS.

~~THESE, WITH $\Lambda = 0$~~:

$$\left(\frac{\dot a}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{1}{3}\Lambda c^2 \quad (\mathrm{I})$$

$$\left(\frac{\ddot a}{a}\right) = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{1}{3}\Lambda c^2 \quad (\mathrm{II})$$

$$\rho > 0, \; p \geq 0$$

$$(\mathrm{II})\Big|_{\Lambda=0} \Rightarrow \frac{\ddot a}{a} = -A^2, \text{ WHERE } A^2 = \frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) > 0$$

$$\Rightarrow \ddot a + A^2 a = 0$$

SHM EQUATION FOR a WITH FREQUENCY A.

$\Rightarrow$ NOT STATIC.

STATIC, PRESSURELESS SOL. IF $\Lambda > 0$; BUT THIS IS UNSTABLE.

WE WANT $\underset{\text{STATIC}}{\dot a = 0}$ & $\underset{\text{UNSTABLE}}{\ddot a < 0}$

$$(\mathrm{I}) \Rightarrow \left(\frac{\dot a}{a}\right) = \pm\left(\frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{1}{3}\Lambda c^2\right)^{\frac{1}{2}}$$

$\dot a = \pm$

$$\dot{a} = \pm a\sqrt{\frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{1}{3}\Lambda c^2}$$

FROM (II), IF $p=0$:

$$\frac{8\pi G}{3}\rho = -2\frac{\ddot{a}}{a} + \frac{2}{3}\Lambda c^2$$

SUB THIS IN:

$$\dot{a} = \pm\sqrt{-2\frac{\ddot{a}}{a} + \frac{2}{3}\Lambda c^2 - \frac{kc^2}{a^2} + \frac{1}{3}\Lambda c^2}$$

$$\pm\sqrt{-2\frac{\ddot{a}}{a} + \Lambda c^2 - \frac{kc^2}{a^2}}$$

~~IF $k=0$, THEN $\ddot{a}$ HAS TO BE NEGATIVE~~
~~FOR $\dot{a}$ TO BE PHYSICAL (IE NOT COMPLEX).~~

~~$\dot{a}$ CAN BE NEG~~

WE WANT: $\dot{a}=0 \Rightarrow -2\frac{\ddot{a}}{a} + \Lambda c^2 - \frac{kc^2}{a^2} = 0$

WITH SUITABLE CHOICE OF k THIS SEEMS POSSIBLE.

I CANNOT SHOW FROM THIS THAT $\ddot{a} < 0$,
SO I AM PROBABLY WRONG.