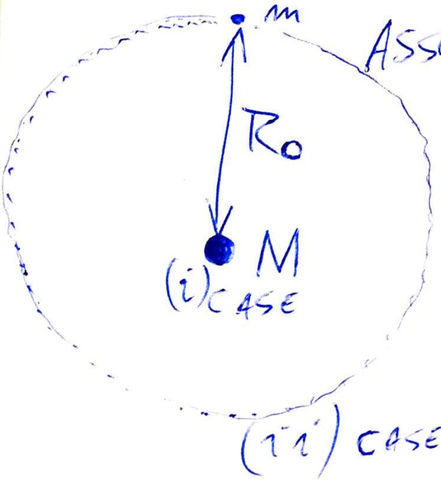
8.



ASSUME $m \ll M$

(a)(i) MINUS SIGN TO INDICATE IT IS FALLING INWARDS.

$$a_0 = -\frac{GM}{R_0^2}$$

(a)(ii)

$$a_0 = -\frac{GM}{R_0^2}$$

BECAUSE BY GAUSS'S LAW, THIS IS JUST CASE (i)

(b)(i) $$\ddot{R} = -\frac{GM}{R^2} \qquad / \cdot \dot{R}$$

$$\dot{R}\ddot{R} = -\frac{GM}{R^2}\dot{R} \qquad / \text{REARRANGE}$$

$$\Rightarrow \frac{d}{dt}\left[\frac{1}{2}\dot{R}^2 \cancel{+} - \frac{GM}{R}\right] = 0 \qquad / \text{REMOVE } \frac{d}{dt}$$

$$\frac{1}{2}\dot{R}^2 - \frac{GM}{R} = C \qquad / \text{REARRANGE}$$

$$\frac{dR}{dt} = \pm\sqrt{\frac{2GM}{R} + C}$$

~~Wait, what? Why do I have positive $\dot{R}$, when I want to fall inwards?~~

We want to fall inwards, so we choose the $\ominus$ sign.

$$\frac{dR}{dt} = -\sqrt{\frac{2GM}{R} + C}$$

Initially we're at rest, so:

$$\left.\frac{dR}{dt}\right|_{t=0} = 0 \quad \text{i.e.} \quad \left.\frac{dR}{dt}\right|_{R=R_0} = 0$$

Rewrite constant in our equation to reflect this:

$$\frac{dR}{dt} = -\sqrt{2GM}\left(\sqrt{\frac{1}{R} - \frac{1}{R_0}}\right)$$

$$t_{FALL} = \int_{t|_{R_0}}^{t|_{R=0}} dt = \int_{R_0}^{0} \frac{1}{-\sqrt{2GM}} \frac{1}{\sqrt{\frac{1}{R} - \frac{1}{R_0}}} dR$$

SUBSTITUTE: $R = R_0 \sin^2\theta$

LIMIT GOES AS: $\theta: \frac{\pi}{2} \to 0$

$$= -\frac{1}{\sqrt{2GM}} \int_{\theta=\frac{\pi}{2}}^{\theta=0} \left(\frac{1}{R_0 \sin^2\theta} - \frac{1}{R_0}\right)^{-\frac{1}{2}} \overbrace{R_0\, 2\sin\theta\cos\theta\, d\theta}$$

$$= -\frac{1}{\sqrt{2GM}} \sqrt{R_0}\, R_0\, 2 \int_{\theta=\frac{\pi}{2}}^{0} \underbrace{\left(\frac{1}{\sin^2\theta} - 1\right)}_{\cot^2\theta}^{-\frac{1}{2}} \sin\theta\cos\theta\, d\theta$$

$$= -\sqrt{\frac{2R_0^3}{GM}} \int_{\theta=\frac{\pi}{2}}^{0} \tan\theta \sin\theta \cos\theta\, d\theta$$

$$= -\sqrt{\frac{2R_0^3}{GM}} \int_{\theta=\frac{\pi}{2}}^{0} \sin^2\theta \, d\theta = -\sqrt{\frac{2R_0^3}{GM}} \left(-\frac{\pi}{4}\right)$$

$$= \frac{\pi}{2}\sqrt{\frac{R_0^3}{2GM}}$$

(b)(ii)

$$\ddot{R} = -\frac{G}{R^2} \underbrace{M_{ENCLOSED}(R)}_{\text{MASS OF VOLUME WITHIN RADIUS } R}$$

$$M_{ENCLOSED}(R) = \cancel{\frac{4}{3}R^3\pi} \left(\frac{R}{R_0}\right)^3 M$$

REWRITE EQUATION ABOVE:

$$\cancel{\ddot{R} = -\frac{G}{R^2}\frac{4}{3}R^3\pi = -\frac{4G\pi}{3}R}$$

$$\cancel{\Rightarrow \ddot{R} + \frac{4\pi G}{3}} \qquad \ddot{R} = -\frac{G}{R^2}\left(\frac{R}{R_0}\right)^3 M = -\frac{GR}{R_0^3} M$$

$$\Rightarrow \ddot{R} + \frac{GM}{R_0^3} R = 0$$

THIS IS JUST SHM EQ.

$$\text{FALL TIME} = \text{QUARTER OF PERIOD} = \frac{1}{4} \cdot \sqrt{\frac{R_0^3}{GM}}\, 2\pi = \frac{\pi}{2}\sqrt{\frac{R_0^3}{GM}}$$

(SAW THESE IN TOPICS LECTURES)

"A CLUSTER CONSISTS INITIALLY..." PART

$$t = \frac{\pi}{2}\sqrt{\frac{R_0^3}{GM}}$$

MASS OF A GLOBULAR CLUSTER $\sim 10^6 M_\odot$

SIZE $\sim$ few parsecs

$$t = \frac{\pi}{2}\sqrt{\frac{(3 \cdot 10^{16})^3}{G \cdot 10^6 \cdot 2 \cdot 10^{30}}} \sim 10^{12}\ \text{sec}$$

$\hookrightarrow \sim 2 \cdot 10^4$ YRS

This is convincingly high enough to be believable.