AFD L2

~~Continuum~~

Continuity & Momentum equations

Re:
- fluid: continuous media that flows
- collisional vs collisionless fluid
- Eulerian & Lagrangian frameworks

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + \underline{u} \cdot \nabla Q$$

Lagrangian Eulerian time derivative convective derivative

- concept of streamlines, particle paths & streaklines

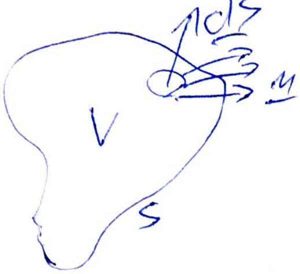
goal: set up PDEs describing properties as function of time
non-relativistic fluids

conservation of mass
continuity equation

conservation of momentum
pressure & stress tensor
momentum eq for fluid
concept of ram-pressure

conservation of mass

Eulerian volume, fluid flowing through it



$$\frac{\partial}{\partial t}\int_V \rho \, dV = -\int_S \rho \underline{u} \cdot d\underline{S}$$

$$\Rightarrow \int_V \frac{\partial \rho}{\partial t} dV = -\int_V \nabla \cdot (\rho \underline{u}) dV$$

$$\Rightarrow \int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) \right) dV = 0$$

arrive to:

Eulerian continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0$$

Lagrangian view   using Eulerian view

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \underline{u} \cdot \nabla \rho = -\nabla \cdot (\rho \underline{u}) + \underline{u} \cdot \nabla \rho$$

$$= -\rho \nabla \cdot u - \underline{u} \cdot \nabla \rho + \underline{u} \cdot \nabla \rho$$

$$= -\rho \nabla \cdot \underline{u}$$

arrive to:

Lagrangian continuity equation

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \underline{u} = 0$$

suppose incompressible fluid: $\frac{D\rho}{Dt} = 0$

$$\Leftrightarrow \nabla \cdot \underline{u} = 0$$

i.e. flow is not converging or diverging (so it is incompressible)

Conservation of Momentum

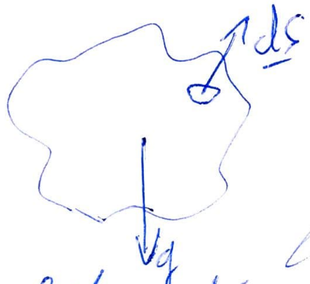
$d\underline{F} = p\, d\underline{S}$ continuing pressure as flux of momentum

$d\underline{F}$ & $d\underline{S}$ align.

more generally:

$dF_i = \sigma_{ij}\, dS_j$ if $\sigma_{ij} \not\propto I$, $d\underline{S}$ & $d\underline{F}$ do not align.

consider fluid element subject to forces & external force field.



total pressure force

$$\underline{F}\cdot\hat{\underline{n}} = -\int_S p\,\hat{\underline{n}}\cdot d\underline{S} = -\int_V \nabla\cdot(p\,\hat{\underline{n}})\,dV$$

$$= -\int_V \underline{n}\cdot\nabla p\, dV$$

$\hat{\underline{n}}$: arbitrary dir.

eq. of motion for fluid element:

$$\left(\frac{D}{Dt}\int_V \rho\,\underline{u}\,dV\right)\cdot\hat{\underline{n}} = -\int_V \hat{\underline{n}}\cdot\nabla p\, dV + \int_V \rho\,\underline{g}\cdot\hat{\underline{n}}\, dV$$

$$\Rightarrow \frac{D}{Dt}(\rho\,\underline{u}\,\delta V) = -\delta V\,\hat{\underline{n}}\cdot\nabla p + \delta V\rho\,\underline{g}\cdot\hat{\underline{n}}$$

Product rule is obeyed by lagrangian derivative:

$$\Rightarrow \underbrace{\hat{\underline{n}}\cdot\underline{u}\,\frac{D}{Dt}(\rho\,\delta V)}_{0\text{, by mass conservation}} + \rho\,\delta V\,\hat{\underline{n}}\cdot\frac{D\underline{u}}{Dt} =$$

$$\Rightarrow \delta V\,\hat{\underline{n}}\cdot\left(\rho\frac{D\underline{u}}{Dt} + \nabla p - \rho\,\underline{g}\right) = 0$$

$\forall\, \delta V, \hat{\underline{n}}$

$\Rightarrow \rho \frac{Du}{Dt} = -\nabla p + \rho g$ Lagrangian momentum equation

write Lagrangian deriv. explicitly:

$\rho \frac{\partial u}{\partial t} + \rho(\underline{u} \cdot \nabla)\underline{u} = -\nabla p + \rho g$ Eulerian momentum equation.

Notational convenience:

$\frac{\partial}{\partial t} \equiv \partial_t \qquad \frac{\partial}{\partial x_i} \equiv \partial_i$

$\frac{\partial}{\partial t}(\rho u_i) = \partial_t(\rho u_i)$

$= \rho \partial_t u_i + u_i \partial_t \rho$

$= \underbrace{-\rho u_j \partial_j u_i}_{-\rho \underline{u} \cdot \nabla u} \underbrace{- \partial_j p \delta_{ij}}_{\partial_i p \text{ ie } \nabla p} + \rho g_i - u_i \partial_j (\rho u_j)$

write ∂_j ..:

$\partial_t(\rho u_i) = -\partial_j(\underbrace{\rho u_i u_j}_{\text{bulk flow "Ram Pressure"}} + \underbrace{p \delta_{ij}}_{\text{random thermal motions}}) + \rho g_i$

stress tensor due to

bulk flow "Ram Pressure" | random thermal motions

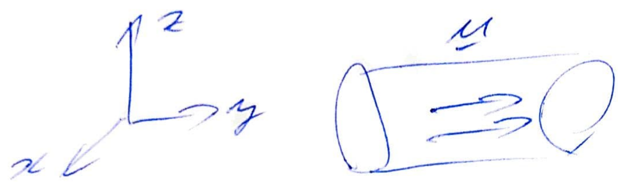
$= -\partial_j \sigma_{ij} + \rho g_i$ conservative form

$\partial_t(\rho \underline{u}) = -\nabla \cdot (\underbrace{\rho \underline{u} \otimes \underline{u} + p \underline{\underline{I}}}_{\text{flux of momentum density}}) + \rho g$ coordinate free language

flux of momentum density

Eulerian form: $\partial_t \rho = -\nabla \cdot (\rho \underline{u})$

Example: flow in a pipe



$$\sigma_{ij} = \begin{pmatrix} p & 0 & 0 \\ 0 & p + \rho u^2 & 0 \\ 0 & 0 & p \end{pmatrix}$$

ram pressure