Q1.1

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}$$

WE WANT: GET RID OF R

PLAN: CONTRACT W/ g, EXPRESS R, SUBSTITUTE BACK

$/ \cdot g^{\mu\nu}$

$$R_{\mu\nu} g^{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\mu\nu} R + \Lambda g_{\mu\nu} g^{\mu\nu} =$$
$$= -8\pi G T_{\mu\nu} g^{\mu\nu}$$

$$R_\mu{}^\mu - \frac{1}{2} \underbrace{g_\mu{}^\mu}_{4} R + \Lambda \underbrace{g_\mu{}^\mu}_{4} = -8\pi G T_\mu{}^\mu$$

$$R - 2R + \Lambda \cdot 4 = -8\pi G T_\mu{}^\mu$$

$$R - 4\Lambda = 8\pi G T_\mu{}^\mu$$

$$R = 8\pi G T_\sigma{}^\sigma + 4\Lambda$$

SUBSTITUTE BACK:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(8\pi G T_\sigma{}^\sigma + 4\Lambda) + \Lambda g_{\mu\nu} =$$
$$= -8\pi G T_{\mu\nu}$$

REARRANGE:

$$R_{\mu\nu} - \Lambda g_{\mu\nu} = -8\pi G\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T_\sigma{}^\sigma\right)$$

AS REQUIRED.

Q1.2

FRZ: $$\left(\frac{\dot{R}}{R}\right)^2 + \frac{K}{R^2} = \frac{8\pi G}{3}\rho \cancel{\left(+\frac{\Lambda}{3}\right)} \to 0$$

NOTING THAT:

$$\dot{R} = \frac{dR}{dt} = \frac{dR}{R\frac{1}{R}dt} = \frac{1}{R}\frac{dR}{dt} = H$$

REWRITE FRZ:

$$\frac{H^2}{R^2} + \frac{K}{R^2} = \frac{8\pi G}{3}\rho$$

$$H^2 + K = \frac{8\pi G}{3}\rho R^2 \qquad /:H^2 \ /-1$$

$$\frac{K}{H^2} = \underbrace{\frac{8\pi G}{3}\rho\frac{R^2}{H^2}}_{=\frac{\rho}{\rho_{crit}} = \Omega} - 1$$

$$\frac{K}{H^2} = \Omega - 1$$

WE'RE OFF BY A c^2 BUT THAT PROBABLY HAS TO DO WITH HOW WE DEFINE K.

Q1.2

FR1:

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho+3P\right) + \cancelto{0}{\frac{\Lambda}{3}}$$

NOTING THAT:

$$\frac{d^2R}{dt^2} = \frac{d}{dt}\frac{d}{dt}R = \frac{d}{Rd\tau}\left(\frac{d}{Rdt}R\right) = \frac{1}{R}\frac{d}{d\tau}\mathcal{H}$$

REWRITE FR1:

$$\frac{1}{R^2}\frac{d\mathcal{H}}{d\tau} = -\frac{4\pi G}{3}\left(\rho+3P\right) \qquad /\cdot 2R^2$$

$$2\frac{d\mathcal{H}}{d\tau} = -\frac{8\pi G R^2}{3}\left(\rho+3w\rho c^2\right)$$

$$= -\frac{8\pi G R^2}{3}\rho\left(1+3wc^2\right)$$

$$= -\Omega\mathcal{H}^2\left(1+3wc^2\right)$$

USING:

$$\frac{kc^2}{\mathcal{H}^2} = \Omega - 1$$

REWRITE:

$$= -\left(\frac{kc^2}{\mathcal{H}^2}+1\right)\mathcal{H}^2\left(1+3wc^2\right)$$

$$\Rightarrow 2\frac{d\mathcal{H}}{d\tau} = -\left(\mathcal{H}^2+kc^2\right)\left(1+3wc^2\right)$$

APART FROM C^2 FACTORS, WE'RE GOOD

$\boxed{\text{Q 1.2}}$

USE THE RESULTS I WAS SUPPOSED TO GET:

$$\frac{\kappa c^2}{H^2} = \Omega - 1 \qquad 2\frac{dH}{dt} = -(3w+1)(H^2+\kappa c^2)$$

$$\frac{d\Omega}{dt} = \frac{d}{dt}\left(\frac{\kappa c^2}{H^2} + 1\right) = \kappa c^2 \frac{d}{dt}\frac{1}{H^2} = \kappa c^2 \cdot -2\frac{1}{H^3}\cdot\frac{dH}{dt}$$

$$= \kappa c^2 \cdot \cancel{-2}\frac{1}{H^3}\cancel{\frac{1}{2}}\cdot -(3w+1)(H^2+\kappa c^2)$$

$$= \underbrace{\kappa c^2 \frac{1}{H}}_{H(\Omega-1)}(3w+1)\underbrace{\left(1 + \frac{\kappa c^2}{H^2}\right)}_{\Omega}$$

$$= \underline{\underline{(3w+1)H\Omega(\Omega-1)}}$$ AS REQUIRED.

$w = 0$ CASE:

$$\frac{d\Omega}{dt} = H\Omega(\Omega-1) \quad \& \quad \frac{dH}{dt} = -\frac{1}{2}(H^2+\kappa c^2)$$

$$\Omega = \int_0^{\Omega_F} d\Omega = \left[\left(\frac{1}{3}\Omega^3 - \frac{1}{2}\Omega^2\right)H\right]_0^{\Omega_F} - \int_0^{\Omega_F}\left(\frac{1}{3}\Omega^3 - \frac{1}{2}\Omega^2\right)\cdot -\frac{1}{2}(H^2+\kappa c$$

~~I DON'T SEE HOW TO SKETCH THIS WITHOUT KNOWING~~

Q1.3

FRI: $$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3P) + \cancel{\frac{\Lambda}{3}}^{\,0}$$

MATTER DOMINATED UNIVERSE $\Rightarrow P = 0$

$\cancel{\rho \propto R^{3}}$

$$2 \cdot q_0 = 2 \cdot \frac{1}{H_0^2}\left(-\frac{1}{R}\frac{d^2R}{dt^2}\right)$$

FRI FOR MATTER DOMINATED UNIVERSE

$$= 2 \cdot \frac{1}{H_0^2} \frac{4\pi G}{3}(\rho)$$

$$= \frac{8\pi G \rho}{3H_0^2} = \frac{\rho}{\rho_{CRIT}} \frac{1}{R^2} = \frac{\Omega_0}{R^2}$$

THIS IS RATHER UNFORTUNATE.
AFTER TALKING TO A FRIEND I REALIZE THAT I AM CONFUSING CURLY $\mathcal{H}$ WITH HUBBLE H.

THIS IS TRUE: $$\frac{8\pi G R^2}{3\mathcal{H}_0^2} = \rho_{0\,CRIT}$$

THIS IS NOT: $$\frac{8\pi G R^2}{8H_0^2} = \rho_{0\,CRIT}$$

$$H = \frac{dR}{dt}\frac{1}{R} = \frac{1}{R} \cdot \frac{d}{R\,dt} R = \frac{1}{R^2}\frac{dR}{d\tau} = \frac{1}{R}\mathcal{H}$$

$$\Rightarrow \frac{8\pi G}{3H^2} = \rho_{CRIT}$$

SO, ARMED WITH THIS NEW RELATION, TRY AGAIN:

$$\frac{8\pi G \rho}{3H_0^2} = \frac{8\pi G \rho R^2}{3H_0^2} = \frac{\rho}{\rho_{CRIT}} = \underline{\underline{\Omega_0}}$$

$\Longrightarrow$ $2q_0 = \Omega_0$ AS REQUIRED.

FR II: $$\left(\frac{\dot R}{R}\right)^2 + \frac{K}{R^2} = \underbrace{\frac{8\pi G}{3}\rho}_{-2\frac{\ddot R}{R} \text{ FROM FR I}} + \cancel{\frac{\Lambda}{3}}^{\,0}$$

$$\left(\frac{\dot R}{R}\right)^2 = -2\frac{\ddot R}{R} - \frac{K}{R^2}$$

$$= \cancel{2q_0 H_0}$$

$$\left(\frac{\dot R}{R}\right)^2 = +2q_0 H_0^2 - \frac{K}{R^2}$$

$$\boxed{q_0 = \frac{1}{H_0^2}\left(-\frac{1}{R}\frac{d^2R}{dt^2}\right) \quad \Downarrow \quad \frac{\ddot R}{R} = -q_0 H_0^2}$$

$/ : R_0^2 \cdot R^2$

$$\left(\frac{\dot R}{R_0}\right)^2 = 2q_0 H_0^2 \frac{R^2}{R_0^2} - \frac{K}{R_0^2}$$

WRITE K AS: $K = \left[\frac{8\pi G}{3}\rho_0 - H_0^2\right] R_0^2$

Q 1.3

$$H_0^2 \cdot \left(1 - 2q_0 + 2q_0 \frac{R_0}{R}\right)$$

DEF OF q_0

$$= H_0^2 + 2\frac{1}{R_0}\frac{d^2R}{dt^2}\Big|_{t_0} - 2\frac{1}{R_0}\frac{d^2R}{dt^2}\Big|_{t_0}\frac{R_0}{R}$$

/ REARRANGE

$$= H_0^2 + 2\left(\frac{1}{R_0} - \frac{1}{R}\right)\frac{d^2R}{dt^2}\Big|_{t_0}$$

USE FRI

$$= H_0^2 + 2\left(\frac{1}{R_0} - \frac{1}{R}\right)\left(-\frac{4\pi G}{3}\rho_0 R_0\right)$$

/ REARRANGE

$$= H_0^2 - \frac{8\pi G}{3}\rho_0 + \frac{8\pi G}{3}\rho_0\frac{R_0}{R}$$

/ USE EXPRESSION FOR K (LEC I. SLIDE 23)

$$= -\frac{K}{R_0^2} + \frac{8\pi G}{3}\rho_0\frac{R_0}{R}$$

/ REWRITE

$$= \frac{1}{R_0^2}\left(-K + 8\pi G/3 \cdot \rho_0\frac{R_0^3}{R}\right)$$

/ LEC I. SLIDE 21

$$= \frac{\dot{R}^2}{R_0^2}$$

AS WANTED.

Q1.4

$$t_0 = \frac{1}{H_0}\int_0^{R_0} \frac{dR}{R\,E(z)} \qquad E(z) = \sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda}$$

/ SUB. IN

$$= \frac{1}{H_0}\int_0^{R_0} \frac{dR}{R\sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda}}$$

/ REARRANGE, AIMING FOR A FORM SUGGESTED

$$= \frac{1}{H_0}\int_0^{R_0} \frac{dR}{R\,\Omega_\Lambda^{1/2}\sqrt{\frac{\Omega_M}{\Omega_\Lambda}(1+z)^3 + 1}}$$

LET: $\frac{\Omega_M}{\Omega_\Lambda}(1+z)^3 = \tan^2\theta$

USE: $\tan^2\theta + 1 = \sec^2\theta$

$$= \frac{1}{H_0}\int_0^{R_0} \frac{\cos\theta\, dR}{\Omega_\Lambda^{1/2} R}$$

USE: $R = \frac{R_0}{1+z} = \frac{R_0}{\left(\frac{\Omega_\Lambda}{\Omega_M}\right)^{\frac{1}{3}} \tan^{\frac{2}{3}}\theta}$

$$= \frac{1}{H_0}\int_0^{R_0} \frac{\cos\theta \left(\frac{\Omega_\Lambda}{\Omega_M}\right)^{\frac{1}{3}} \tan^{\frac{2}{3}}\theta \,\frac{1}{R_0}\, dR}{\Omega_\Lambda^{\frac{1}{2}}}$$

GET dR:

$$\frac{dR}{d\Theta} = \frac{R_0}{\left(\frac{\Omega_\Lambda}{\Omega_M}\right)^{\frac{1}{3}}} \frac{d}{d\Theta}\left(\tan^{-\frac{2}{3}}\Theta\right)$$

$$= \frac{R_0}{\left(\frac{\Omega_\Lambda}{\Omega_M}\right)^{\frac{1}{3}}} \cdot -\frac{2}{3}\left(\tan\Theta\right)^{-\frac{5}{3}} \cdot \sec^2\Theta$$

SUBSTITUTE THIS INTO THE INTEGRAL:

$$= \frac{1}{H_0}\int_{\Theta|_{R=0}}^{\Theta|_{R_0}} \frac{\cos\Theta \cancel{\left(\frac{\Omega_\Lambda}{\Omega_M}\right)^{\frac{1}{3}}} \tan^{\frac{2}{3}}\Theta \cancel{\frac{1}{R_0}} \cancel{\frac{R_0}{\left(\frac{\Omega_\Lambda}{\Omega_M}\right)^{\frac{1}{3}}}} \cdot -\frac{2}{3}\tan^{-\frac{5}{3}}\Theta \sec^2\Theta d\Theta}{\Omega_\Lambda^{\frac{1}{2}}}$$

$$= \frac{1}{H_0}\int_{\Theta|_{R=0}}^{\Theta|_{R_0}} \frac{\cos\Theta \cot\Theta \sec^2\Theta \cdot \left(-\frac{2}{3}\right)}{\Omega_\Lambda^{\frac{1}{2}}} d\Theta$$

$$= \frac{1}{H_0}\int_{\Theta|_{R=0}}^{\Theta|_{R_0}} -\frac{2}{3}\frac{1}{\Omega_\Lambda^{\frac{1}{2}}}\frac{1}{\sin\Theta} d\Theta$$

$$= \frac{1}{H_0} \cdot -\frac{2}{3} \cdot \frac{1}{\Omega_\Lambda^{\frac{1}{2}}} \cdot \left[-\ln\frac{\cos\Theta + 1}{\sin\Theta}\right]_{\Theta|_{R=0}}^{\Theta|_{R_0}}$$

EVALUATE THE LIMITS:

$$R=0 \Rightarrow \lim_{R\to 0} z = \infty \Rightarrow \theta = \frac{\pi}{2} \qquad \ln \frac{\cos\theta + 1}{\sin\theta}\Bigg|_{\theta=\frac{\pi}{2}} = 0$$

WE WANT AGE AT z:

ie when $\tan\theta = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{\frac{1}{2}}(1+z)^{\frac{3}{2}}$

$$\cancel{t(z)} \quad t(z) = \frac{1}{H_0}\frac{2}{3}\frac{1}{\sqrt{\Omega_\Lambda}} \ln \frac{1+\cos\theta}{\sin\theta}\Bigg|_{\theta=\arctan\left(\sqrt{\frac{\Omega_m}{\Omega_\Lambda}}(1+z)^{\frac{3}{2}}\right)}$$

AS REQUIRED.

$$t(z)\Bigg|_{\substack{z=0 \\ H_0 = 67.5\,\frac{km}{Mpc\,s} = 3.23\cdot 10^{-18}\,\frac{1}{s} \\ \Omega_m = 0.31 \\ \Omega_\Lambda = 1-0.31}} = 1.4\cdot 10^{10}\ \text{YRS}$$

CONSISTENT WITH LITERATURE.

1.5

I MANAGED TO GET DONE ONLY THE VERY LAST BIT.

$$\frac{dS}{dR} = -3(1+\omega)\frac{S}{R}$$

$$\frac{dS}{S} = -3(1+\omega)\frac{dR}{R}$$

$$\ln S = -3(1+\omega)\ln R + C$$

$$= \ln R^{-3(1+\omega)} + \ln C$$

$$= \ln\left(C \cdot R^{-3(1+\omega)}\right)$$

$$\Rightarrow S \propto R^{-3(1+\omega)}$$

.6

ATTEMPTS FOR FIRST PART (DIDN'T WORK)

$$ds^2 = c^2 dt^2 - R^2(t)\frac{dr^2}{1-kr^2} \qquad \frac{ds}{d\tau} = c$$

↳ PROPER TIME OF PARTICLE

$$\cancel{\left(\frac{ds}{dt}\right)^2} = c^2 dt^2 - R^2(t)\,dx^2$$

AND I DON'T KNOW WHAT I DO WITH THESE.

HOWEVER, IN THE SECOND PART, CAN USE:

$$g_{\mu\nu} u^\mu u^\nu = \text{CONSTANT} = c^2$$

$$u^\mu = \gamma_v (c, v, 0, 0)$$

USING METRIC:

$$ds^2 = c^2 dt^2 - R^2(t)\,dx^2$$

$$g_{\mu\nu} u^\mu u^\nu = \cancel{\gamma_v^2 (c^2 \cdot c^2 - \gamma_v)}$$

$$= g_{00} u^0 u^0 + g_{11} u^1 u^1$$

$$= c^2 c \gamma_v c \gamma_v - R^2 \gamma_v v \gamma_v v$$

$$= c^4 \gamma_v^2 - R^2 \gamma_v^2 v^2$$

$$= \gamma_v^2 (c^4 - R^2 v^2) = \text{CONSTANT}$$

$$\gamma_{v_1}^2 \left(c^4 - R^2(t_1)\, v_1^2 \right) = \gamma_{v_2}^2 \left(c^4 - R^2(t_2)\, v_2^2 \right)$$

WE SHOULDN'T HAVE THE c^4 TERMS.

IF WE DIDN'T HAVE THEM, WE'D HAVE:

$$\gamma_{v_1} R(t_1)\, v_1 = \gamma_{v_2} R(t_2)\, v_2$$

$$\Downarrow$$

$$\frac{R(t_1)}{R(t_2)} = \frac{\gamma_{v_2} v_2}{\gamma_{v_1} v_1}$$

BUT WE DO HAVE THE C TERMS SO I'M PROBABLY WRONG.