2.

$$\underbrace{c_s = \sqrt{\frac{R_* T}{\mu}}} \qquad r_s = \underbrace{\frac{GM}{2c_s^2}}$$

I DERIVE THIS IN THE NEXT PROBLEM.

$$r_s = \frac{GM}{2} \frac{\mu}{R_* T}$$

$$r_s = \frac{G \cdot 2 \cdot 10^{30}}{2} \cdot \frac{1}{10^3 R \cdot 2 \cdot 10^6} = \underline{\underline{4 \cdot 10^9 \text{ m}}}$$

THIS COMES FROM:

$$c_s^2 = \left.\frac{dp}{dS}\right|_T \quad \& \quad p = \frac{R_*}{\mu} S T$$

$$c_s^2 = \frac{RT}{\mu}$$

3

BERNOULLI FOR FLOW

$$\text{EVERYWHERE} = \text{AT SONIC RADIUS}$$

$$\frac{1}{2}u^2 + \underbrace{c_s^2 \ln \rho}_{} - \frac{GM}{r} = \frac{1}{2}c_s^2 + c_s^2 \ln \rho_s - \frac{GM}{r_s}$$

THIS BIT COMES FROM:

$$\int \frac{dp}{\rho} = \int \frac{R_* T}{\mu}\frac{d\rho}{\rho} = R_*T/\mu \; \ln \rho = c_s^2 \ln \rho$$

USE: $r_s = \dfrac{GM}{2c_s^2}$

REWRITE:

$$\frac{GM}{r_s} = \frac{GM}{GM}2c_s^2 = 2c_s^2$$

SUBSTITUTE:

$$\frac{1}{2}u^2 + c_s^2 \ln \rho - \frac{GM}{r} = c_s^2\left(\ln \rho_s - \frac{3}{2}\right)$$

$$\frac{1}{2}u^2 = c_s^2\left(\ln \rho_s - \ln \rho - \frac{3}{2}\right) - \frac{GM}{r}$$

$$u^2 = 2c_s^2\left[\ln\left(\frac{\rho_s}{\rho}\right) - \frac{3}{2}\right] - \frac{2GM}{r}$$

AS $r \rightarrow \infty$, $u \rightarrow 0$ (STATIONARY FLUID BY ASSUMPTION)

$$\Rightarrow \; 2c_s^2\left[\ln\left(\frac{\rho_s}{\rho_\infty}\right) - \frac{3}{2}\right] = 0$$

$$\Downarrow$$

$$\ln\frac{\rho_s}{\rho_\infty} = \frac{3}{2}$$

$$\Downarrow$$

$$\frac{\rho_s}{\rho_\infty} = e^{\frac{3}{2}}$$

$$\rho_\infty = \rho_s e^{-3/2}$$

$$\rho_0 = \rho_s e^{-\frac{3}{2}}$$

USE:

$$\dot{M} = 4\pi r_s^2 \rho_s c_s$$

SUBSTITUTE: $r_s = \frac{GM}{2c_s^2}$ $\quad \rho_s = \rho_0 e^{3/2}$

$$\dot{M} = 4\pi\left(\frac{GM}{2c_s^2}\right)^2 \rho_0 e^{3/2} c_s = \frac{\pi G^2 M^2 e^{3/2} \rho_0}{c_s^3}$$

ACCRETION RATE ↑

MOMENTUM EQUATION:

$$\rho \frac{\partial \underline{u}}{\partial t} + \rho(\underline{u}\cdot\nabla)\underline{u} = -\nabla p + \rho \underline{g}$$

STEADY ACCRETION CASE:

$$\rho u \frac{d}{dr} u = -\frac{dp}{dr} - \frac{GM}{r^2}$$

$$u^2 \frac{d\ln u}{dr} = -\frac{1}{\rho}\frac{dp}{dr} - \frac{GM}{r^2}$$

USE:

$$c_s = \sqrt{\frac{dp}{d\rho}} \Rightarrow c_s^2 \frac{d\ln\rho}{dr} = \frac{dp}{\cancel{d\rho}}\frac{1}{\rho}\frac{d\rho}{dr}$$

REWRITE:

$$u^2 \frac{d\ln u}{dr} = -c_s^2 \underbrace{\frac{d\ln\rho}{dr}}_{\text{WE'RE GOING TO SUBTITUTE FOR THIS BIT.}} - \frac{GM}{r^2}$$

STEADY FLOW:

$$\frac{d}{dr}\ln\dot{M} = 0 \quad \text{(IE NO PILING UP GAS)}$$

$$\frac{d}{dr}\ln(4\pi r^2 \rho u) = 0 \Rightarrow \frac{d}{dr}\ln\rho = -\frac{d}{dr}\ln u - \frac{d}{dr}\ln r^2$$

$$= -\frac{d}{dr}\ln u - \frac{2}{r}$$

SUBSTITUTE:

$$\cancel{u^2 \frac{d\ln u}{dr} =}$$

$$u^2 \frac{d}{dr}\ln u = c_s^2\left(\frac{d}{dr}\ln u + \frac{2}{r}\right) - \frac{GM}{r^2}$$

$$(u^2 - c_s^2)\frac{d}{dr}\ln u = \frac{2c_s^2}{r}\left(1 - \frac{GM}{2c_s^2 r}\right)$$

Where $u = c_s \Rightarrow 1 - \frac{GM}{2c_s^2 r} \Rightarrow \underline{\underline{r_s = \frac{GM}{2c_s^2}}}$

WITH SPECIFIC VALUES:

$$r_s = \frac{G\cdot 2\cdot 10^{30}}{2c_s^2} = \frac{10^{30}G}{c_s^2} = \frac{10^{30}G}{(1000)^2} = \underline{6.7\cdot 10^{13}\,m}$$

$: (7\cdot 10^8)$

$\rightarrow \approx \underline{10^5}$ SOLAR RADIUS

ACCRETION RATE IN kg/sec:

$$\dot M = \frac{\pi G^2 (2\cdot 10^{30})^2 e^{3/2}\, 10^9 \cdot \overbrace{2}^{1 H_2 \text{ HAS } 2H \text{ ATOMS}} \cdot 1.67\cdot 10^{-27}}{(1000)^3} = \underline{8.4\cdot 10^{14}\ \frac{kg}{sec}}$$

HOW LONG WILL IT TAKE TO DOUBLE...

$$\dot M = \frac{\pi G^2 M^2 e^{3/2} \rho_0}{c_s^3}$$

LET $\pi G^2 e^{3/2} \rho_0 / c_s^3 = A$

$$\frac{dM}{dt} = A M^2$$

$$\frac{dM}{M^2} = A\,dt$$

$$-M^{-1} = At + C$$

WE WANT: $-M^{-1}\big|_{t=0} = M_*^{-1} \Rightarrow C = -M_*^{-1}$

$$-M^{-1} = At - M_*^{-1}$$

$$M = \frac{1}{-At + \frac{1}{M_*}}$$

SUB IN A:

$$M = \frac{1}{-\frac{\pi G^2 e^{3/2} S_0}{c_s^3}\,t + \frac{1}{M_*}}$$

THERE IS A PROBLEM:
I AM GETTING $M \to \infty$ IN
A VERY MUCH FINITE TIME.

IGNORING THIS CONCERN: M WILL DOUBLE
WHEN DENOMINATOR
HALVENS.

$$\frac{1}{M_*} - \frac{\pi G^2 e^{3/2} S_0}{c_s^3}\,t = \frac{1}{2}\,\frac{1}{M_*}$$

$$\frac{1}{2}\frac{1}{M_*} = \frac{\pi G^2 e^{3/2} S_0}{c_s^3} t$$

$$t = \frac{1}{2M_*}\frac{c_s^3}{\pi G^2 e^{3/2} S_0}$$

$$= \frac{1}{2(2\cdot 10^{30})}\frac{(1000)^3}{\pi G^2 e^{3/2}\cdot 10^9\cdot 2\cdot 1.67\cdot 10^{-27}}$$

$$\cancel{= 6.67\cdot 10} \quad \cancel{1.67\cdot 10^{-5}\ \text{sec}}$$

$$\cong 2.4\cdot 10^{15}\ \text{SEC} \cong \underline{75\ \text{MILLION YRS}}.$$

QUICK GOOGLING "HOW LO~~N~~NG STAR ACCRETION LASTS"

$\Rightarrow$ THIS IS AN OVERESTIMATE.

"HOW DOES THIS TIME DEPEND ON THAT INITIAL MASS?"

$t \propto \frac{1}{M_*}$ AS SEEN ON THE ABOVE FORMULAE.

4

$$r \propto L^a \rho_0^b t^c$$

$$[L] = \frac{J}{sec} = \frac{kg \frac{m}{sec^2} m}{sec} = kg\, m^2 \frac{1}{sec^3}$$

$$[\rho_0] = \frac{kg}{m^3}$$

$$[t] = sec$$

$$[r] = m$$

ORIGINAL PROPORTIONALITY TRANSLATES TO:

$$\left(kg\, m^2 sec^{-3}\right)^a \left(kg\, m^{-3}\right)^b \left(sec\right)^c = m$$

$$\Rightarrow a + b = 0$$

$$-3a + c = 0$$

$$2a - 3b = 1$$

$$\begin{pmatrix} 1 & 1 & 0 \\ -3 & 0 & 1 \\ 2 & -3 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow a = \frac{1}{5} \quad b = -\frac{1}{5} \quad c = \frac{3}{5}$$

$$\Rightarrow r \propto \left(\frac{L}{\rho_0}\right)^{\frac{1}{5}} t^{\frac{3}{5}}$$

WAIT, WHAT IF CONSTANT OF PROPORTIONALITY IS NOT DIMENSIONLESS?

COMBINE L, S_0 & t TO GIVE DIMENSIONALLY LENGTH QUANTITY:

$$\lambda = \left(\frac{L}{S_0}\right)^{\frac{1}{5}} t^{\frac{3}{5}}$$

DEFINE DIMENSIONLESS DISTANCE PARAMETER:

$$\xi = \frac{r}{\lambda} = r\left(\frac{S_0}{L}\right)^{\frac{1}{5}} t^{-\frac{3}{5}}$$

WE HOPE:

$$X = X_1(t)\tilde{X}(\xi)$$ WHERE X IS ANY QUANTITY.

$$\left(\begin{aligned} \frac{\partial X}{\partial r} &= X_1 \frac{d\tilde{X}}{d\xi} \frac{\partial \xi}{\partial r}\bigg|_t \\ \frac{\partial X}{\partial t} &= \tilde{X}(\xi)\frac{dX_1}{dt} + X_1 \frac{d\tilde{X}}{d\xi}\frac{\partial \xi}{\partial t}\bigg|_r \end{aligned}\right)$$

~~R (ONLY FUNCTION OF t~~

$$R\,(\text{FUNCTION OF } t \text{ ONLY}) = R_1(t)\tilde{R}(\xi)$$

R_1 HAS t DEPENDENCE,
$\tilde{R}(\xi)$ HAS ONLY ξ DEPENDENCE BUT
R MUST NOT DEPEND ON ξ $\Rightarrow$ $\tilde{R}(\xi)$ MUST BE CONSTANT

$$\Rightarrow R(t) = R_1(t)\xi_0$$

$$R(t) = \left(\frac{L}{S_0}\right)^{\frac{1}{5}} t^{\frac{3}{5}} \xi_0$$

SINCE THIS IS THE ONLY WAY TO HAVE LENGTH DIMENSIONALLY.

a, b, c ARE STILL $\frac{1}{5}, -\frac{1}{5}, \frac{3}{5}$ RESPECTIVELY.

MY FORMULA FOR $R(t)$ DOES NOT SHOW ANY SIGN OF STALLING WHICH ISN'T THAT GREAT.

$$A \propto R^2 \propto L^{\frac{2}{5}} \not\propto L$$

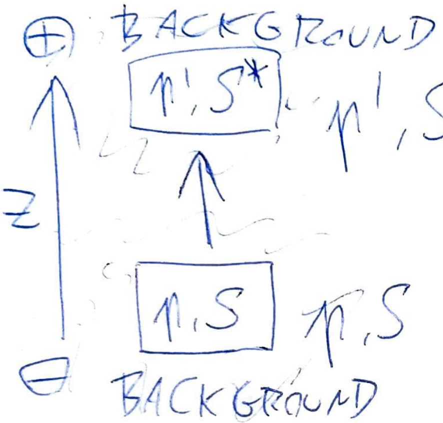
EXPANSION VELOCITY:

$$\frac{dR}{dt} = \left(\frac{L}{S_0}\right)^{\frac{1}{5}} \xi_0 \frac{3}{5} t^{-\frac{2}{5}} = \frac{3}{5}\frac{R}{t}$$

5

$$p = K\rho^{1+\frac{1}{n}} \quad \text{POLYTROPIC EOS}$$

TAKE A FLUID ELEMENT. LIFT IT A BIT, SLOW ENOUGH FOR PRESSURE TO EQUILIBRIZE. QUICK ENOUGH SO THERE IS NO HEAT EXCHANGE.



$$\left.\begin{array}{l} p' = K\rho^{*\,1+\frac{1}{n}} \\ p = K\rho^{1+\frac{1}{n}} \end{array}\right\} \Longrightarrow$$

$$\Longrightarrow \rho^* = \rho \left(\frac{p'}{p}\right)^{\frac{1}{1+\frac{1}{n}}}$$

EXPAND DENSITY CHANGE:

$$p' = p + \frac{dp}{dz}\delta z$$

SUB THIS IN:

$$\rho^* = \rho\left(\frac{p + \frac{dp}{dz}\delta z}{p}\right)^{\frac{1}{1+\frac{1}{n}}}$$

$$= \rho\left(1 + \frac{1}{p}\frac{dp}{dz}\delta z\right)^{\frac{1}{1+\frac{1}{n}}}$$

$$\approx \rho + \frac{\rho}{P}\frac{dP}{dz}dz \cdot \frac{1}{1+\frac{1}{n}} \quad \left(\text{APPROXIMATION WORKS BECAUSE } \frac{1}{P}\frac{dP}{dz}\delta z \ll 1\right)$$

USING: $(1+x)^n \approx 1 + n(1+x)^{n-1}x$

$$\approx 1 + nx \quad \text{IF } x \ll 1$$

BACKGROUND ATMOSPHERE:

$$\rho' = \rho + \frac{d\rho}{dz}\delta z$$

STABLE IF:

$$\rho^* > \rho' \quad \text{(SO OUR FLUID ELEMENT WILL SINK BACK)}$$

$$\cancel{\rho} + \frac{\rho}{P}\frac{dP}{dz}\delta z \frac{1}{1+\frac{1}{n}} > \cancel{\rho} + \frac{d\rho}{dz}\delta z$$

$$\frac{1}{1+\frac{1}{n}}\frac{\rho}{P}\frac{dP}{dz} > \frac{d\rho}{dz}$$

$$\frac{d}{dz}\ln P > \left(\frac{d}{dz}\ln \rho\right)\left(1+\frac{1}{n}\right)$$

$$> \frac{d}{dz}\ln \rho^{1+\frac{1}{n}}$$

$$\frac{d}{dz} \ln\left(n S^{-\left(1+\frac{1}{n}\right)}\right) > 0$$

$$\frac{d}{dz} \ln K > 0$$

$$\frac{dK}{dz} > 0$$

WHERE IS OUR n DEPENDENCE?

SEEMS LIKE ITS GONE $\Rightarrow$ I'M WRONG.

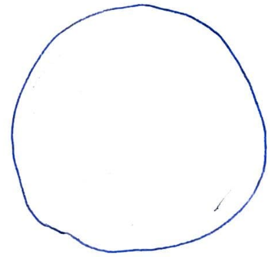
7.

$$M_J \sim \rho_0 \lambda_J^3$$

$$\sim \rho_0 \left(\frac{\pi c_s^2}{G \rho_0}\right)^3$$

$$\sim \frac{\pi^3 c_s^6}{G^3 \rho_0^2}$$

(I SOMEWHAT FAIL TO SEE HOW SUCH A UNIFORM SPHERE COULD EXIST, MAY BE I AM MISUNDERSTANDING THE QUESTION)



USING RESULT FROM SHEET I Q8c,
FREEFALL TIME ACROSS A UNIFORM SPHERE IS:

$$t_{FF} = \frac{\pi}{2}\sqrt{\frac{R^3}{2GM}}$$

SUBBING IN JEANS MASS:

$$t_{FF} = \frac{\pi}{2}\sqrt{\frac{R^3 G^{\cancel{3}2} \rho_0^2}{2 \cancel{G} \pi^3 c_s^6}}$$

$$= \frac{\pi}{2}\sqrt{\frac{R^3 G^2 \rho_0^2}{2\pi^3 c_s^6}}$$

SOUND WAVE CROSSING TIME:

$$t_{CROSS} = \frac{2R}{c_s}$$

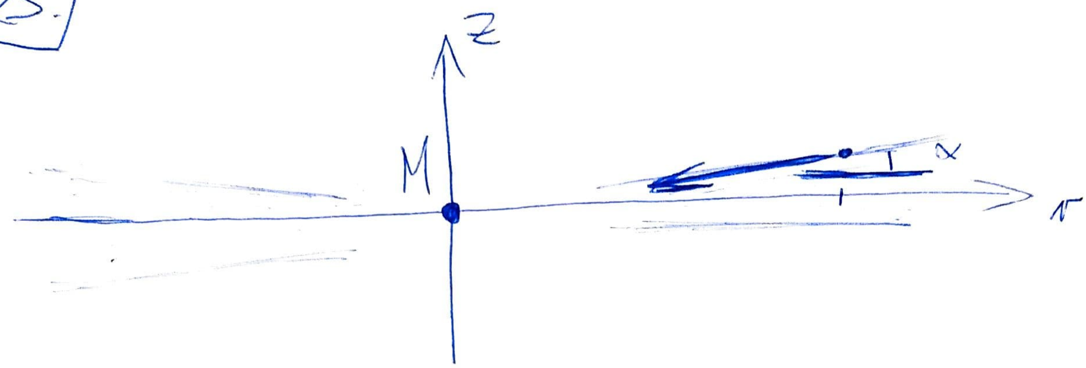
$$\frac{t_{CROSS}}{t_{FF}} = \frac{2R}{c_s} \frac{2}{\pi} \sqrt{\frac{2\pi^3 c_s^6}{R^3 G^2 S_0^2}}$$

$$= \frac{2}{\pi} \sqrt{\frac{8\pi^3 c_s^4}{R G^2 S_0^2}}$$

USE: $c_s^2 = \frac{dP}{dS}$

$$\frac{t_{CROSS}}{t_{FF}} = \frac{2}{\pi} \sqrt{\frac{8\pi^3}{R G^2 S_0^2}} \frac{dP}{dS}$$

8.



$$g_z = -\frac{GM}{r^2}\sin\alpha$$

$$\cong -\frac{GM}{r^2}\frac{z}{r}$$

$$\cong -\frac{GMz}{r^3}$$

HYDROSTATIC EQUILIB:

$$\boxed{\frac{1}{\rho}\frac{\partial p}{\partial z} = g_z = -\frac{GM}{r^3}z}$$

$$p = \frac{R}{\mu}\rho T \Rightarrow p = A\rho$$

$$A\frac{1}{\rho}\frac{\partial \rho}{\partial z} = -\frac{GM}{r^3}z \qquad \text{/INTEGRATE} \qquad (*)$$

$$\ln\rho = -\frac{GM}{2r^3}z^2 + C$$

$$\rho = \rho_0 \exp\left(-\frac{GMz^2}{2r^3}\right)\cdot C$$

TO SATISFY (*), REWRITE CONSTANT TERM TO:

$$\rho = \rho_0 \exp\left(-\frac{GMz^2}{2r^3A}\right)$$

$$P = K\rho^{1+\frac{1}{n}}$$

$$\frac{\partial P}{\partial z} = \frac{\partial}{\partial z}\left(K\rho^{1+\frac{1}{n}}\right)$$

$$= K\frac{\partial}{\partial z}\rho^{1+\frac{1}{n}}$$

$$= K\left(1+\frac{1}{n}\right)\rho^{\frac{1}{n}}\frac{d\rho}{dz}$$

SUBSTITUTE THIS IN TO HYDROSTATIC EQUILIB:

$$\frac{1}{\rho}K\left(1+\frac{1}{n}\right)\rho^{\frac{1}{n}}\frac{d\rho}{dz} = -\frac{GM}{r^3}z$$

$$K\left(1+\frac{1}{n}\right)\rho^{\frac{1}{n}-1}\frac{d\rho}{dz} = -\frac{GM}{r^3}z$$

USE: $\rho(z) \propto (z_m^2 - z^2)^2$

$$\frac{d\rho}{dz} = 2(z_m^2 - z^2)(-2z)$$

BACK TO EQUATION:

$$K\left(1+\frac{1}{n}\right)(z_m^2 - z^2)^{2\left(\frac{1}{n}-1\right)} 2(z_m^2 - z^2)(-2z) = -\frac{GM}{r^3}z$$

$$K\left(1+\frac{1}{n}\right)\left(z_m^2-z^2\right)^{\frac{2}{n}-1}(-4)z=-\frac{GM}{r^3}z$$

AT GIVEN r, WE HAVE:

$$\Rightarrow \left(z_m^2-z^2\right)^{\frac{2}{n}-1} = \text{CONSTANT}$$

$$\Rightarrow \frac{2}{n}-1=0 \Rightarrow \underline{n=2}$$