11.1

FR II: $\left(\frac{\dot{R}}{R}\right)^2 + \frac{K}{R^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}$

WE ALSO KNOW THAT:

$\rho = aT^4 \frac{g_i}{2}$ BOSONS (IE PHOTONS)

$\rho = \frac{7}{8} aT^4 \frac{g_i}{2}$ FERMIONS (IE NEUTRINOS)

USE:

ENERGY CONSERVATION:

$$\frac{d(\rho R^3)}{dR} = -3PR^2 \quad \text{WITH} \quad P = w\rho$$

$$\frac{d(\rho R^3)}{dR} = -3w\rho R^2$$

$$R^3 \frac{d\rho}{dR} + \rho\, 3R^2 = -3w\rho R^2$$

$$\frac{d\rho}{dR} = -3(1+w)\rho \frac{1}{R}$$

$$\cancel{\ln R = \ln(-3(1+w)\rho)}$$

$$\frac{d\rho}{\rho} = -3(1+w)\frac{dR}{R}$$

$$\ln\rho = -3(1+w)\ln R + C$$

$$\rho \propto R^{-3(1+w)}$$

IE I CAN REWRITE OUR EARLIER EXPRESSION TO BE:

$$R^{-3(1+w)} \propto a T^4 \frac{g_i}{2} \qquad \text{BOSONS}$$

$$R^{-3(1+w)} \propto \frac{7}{8} a T^4 \frac{g_i}{2} \qquad \text{FERMIONS}$$

RADIATION-DOMINATED UNIVERSE: $w = \frac{1}{3}$

$$R^{-4} \propto a T^4 \frac{g_i}{2} \qquad \text{BOSONS}$$

$$R^{-4} \propto \frac{7}{8} a T^4 \frac{g_i}{2} \qquad \text{FERMIONS}$$

$$R^{-4} \propto T^4 \Rightarrow R \propto T^{-1} \Rightarrow \frac{dR}{dt} \propto \frac{d}{dt}\frac{1}{T} = -T^{-2}\frac{dT}{dt}$$

$$\Rightarrow \left(\frac{\dot R}{R}\right)^2 = \left(\frac{-T^{-2}\dot T}{T^{-1}}\right)^2 = \left(\frac{\dot T}{T}\right)^2$$

REWRITE FR II:

$$\left(\frac{\dot T}{T}\right)^2 + \frac{K}{R^2} = \frac{8\pi G}{3}\left\{\begin{matrix}1\\ \frac{7}{8}\end{matrix}\right\} a T^4 \frac{g_i}{2} + \frac{\Lambda}{3}$$

FOR PHOTONS: $g_i = 2$

SET: $K=0$, $\Lambda = 0$ (WHY CAN I?)

$$\left(\frac{\dot T}{T}\right)^2 = \frac{8\pi G}{3} a T^4$$

AS WANTED (WITHIN c FACTORS)

INCLUDING 3 NEUTRINO SPECIES, WITH THE SAME DISTRIBUTION (IE SAME T) AS PHOTONS:
(WITH K=0=1)

$$\cancel{\left(\frac{\dot{T}}{T}\right)^2 = \frac{8\pi G}{3}\left(aT^4 + 3\cdot\frac{7}{8}\right.}$$

FOR EACH SPECIES, $g_i = 2$

$$\left(\frac{\dot{T}}{T}\right)^2 = \frac{8\pi G}{3}\left(1 + \frac{7}{8}\cdot 3\right)aT^4$$

$1 \rightarrow$ PHOTONS

$\frac{7}{8} \rightarrow$ 2) PRE FACTOR

$3 \rightarrow 3\cdot\frac{g_i\big|_2}{2}$

REWRITE:

$$\left(\frac{\dot{T}}{T}\right)^2 = \frac{29}{3}\pi G\, aT^4$$

CALCULATE AGE OF UNIVERSE AT MATTER-RADIATION EQUALITY.

FOR MATTER: $w=0$

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{K}{R^2} = \frac{8\pi G}{3}\left(\rho_{MATTER} + \rho_{RADIATION}\right) + \frac{\Lambda}{3}$$

TO PROCEED IN A SIMILAR WAY AS BEFORE, DON'T I NEED A RELATION BETWEEN MATTER DENSITY & T?

11.5

$c=1$

BASED ON 4.13:

$$n = \frac{4\pi}{h^3} g_i \int p^2 \exp\left(-\frac{E-\mu}{kT}\right) dp$$

$$E = \sqrt{p^2+m^2} = m\sqrt{1+\frac{p^2}{m^2}} \cong m\left(1+\frac{1}{2}\frac{p^2}{m^2}\right) = m + \frac{p^2}{2m}$$

RELATIONSHIP BETWEEN CHEMICAL POTENTIALS:

CONSERVATION RULE:

$$\mu_p + \mu_n = \mu_D$$

$$n_D = \frac{4\pi}{h^3} g_D \int p^2 \exp\left(-\frac{E-\mu_D}{kT}\right) dp$$

$$= \frac{4\pi}{h^3} g_D \int p^2 \exp\left(-\frac{m_D + \frac{p_D^2}{2m_D} - \mu_D}{kT}\right) dp$$

$$= \frac{4\pi}{h^3} g_D \exp\left(-\frac{m_D - \mu_D}{kT}\right) \int p_D^2 \exp\left(-\frac{p_D^2}{2m_D kT}\right) dp$$

SIMILARLY, FOR p OR n:

$$n_{p/n} = \frac{4\pi}{h^3} g_{p/n} \exp\left(-\frac{m_{p/n} - \mu_{p/n}}{kT}\right) \int p_{p/n}^2 \exp\left(-\frac{p_{p/n}^2}{2m_{p/n} kT}\right) dp$$

LET: $X_i = \int p_i^2 \exp\left(-\frac{p_i^2}{2m_i kT}\right) dp$

THEN WE HAVE:

$$\frac{n_D}{n_n n_p} = \frac{h^3}{4\pi} \frac{g_D}{g_n g_p} \exp\left(-\frac{m_D - \mu_D - m_p - m_n + \mu_p + \mu_n}{kT}\right) \cdot \frac{X_D}{X_n X_p}$$

NOTE:

=0 BY CONSERVATION RULE

$$-(m_D - \mu_D - m_p - m_n + \mu_p + \mu_n) = +B_D$$

REWRITE, USING: $g_D = g_n = g_p = 2$

$$\frac{n_D}{n_n n_p} = \frac{h^3}{4\pi} \frac{1}{2} \exp\left(+\frac{B_D}{kT}\right) \frac{X_D}{X_n X_p}$$

EVALUATE THE X_i PARTS:

$$X_i = \int p_i^2 \exp\left(-\frac{p_i^2}{2m_i kT}\right) dp_i =$$

LET: $q_i = \frac{p_i}{\sqrt{2m_i kT}}$

$$X_i = \int q_i^2 (2m_i kT) \exp\left(-q_i^2\right) \sqrt{2m_i kT}\, dq_i$$

$$= (2 m_i kT)^{\frac{3}{2}} \int q_i^2 \exp(-q_i^2) dq_i$$

REWRITE PART OF OUR EXPRESSION FOR $\frac{n_D}{n_n n_p}$:

$$\frac{h^3}{4\pi} \frac{1}{2} \frac{X_D}{X_n X_p} = \frac{h^3}{4\pi} \frac{1}{2} \left(\frac{\cancel{2} M_D \cancel{kT}}{\cancel{2} M_n \cancel{kT} 2 m_p kT} \right)^{\frac{3}{2}} \cdot \text{INTEGRAL TERMS}$$

$$= \frac{h^3}{8\pi} \left(\frac{M_D}{2 M_n m_p kT} \right)^{\frac{3}{2}} \cdot \text{INTEGRAL TERMS}$$

USE
$M_D \approx 2 m_p$ & $m_n \approx m_p$

$$\simeq \frac{h^3}{8\pi} \left(\frac{\cancel{2 m_p}}{\cancel{2 M_p} m_p kT} \right)^{\frac{3}{2}} \cdot \text{INTEGRAL TERMS}$$

$$= \frac{1}{8\pi} \left(\frac{m_p kT}{h^2} \right)^{-\frac{3}{2}} \cdot \text{INTEGRAL TERMS.}$$

WE NOW HAVE:

$$\frac{n_D}{n_n n_p} = \frac{1}{8\pi} \left(\frac{m_p kT}{h^2} \right)^{-\frac{3}{2}} \exp\left(\frac{B_D}{kT} \right) \cdot \frac{\int q_D^2 e^{-q_D^2} dq_D}{\int q_p^2 e^{-q_p^2} dq_p \int q_n^2 e^{-q_n^2} dq_n}$$

• WE HOPE:

$$\frac{1}{8\pi} \frac{\int q_D^2 e^{-q_D^2} dq_D}{\int q_p^2 e^{-q_p^2} dq_p \int q_n^2 e^{-q_n^2} dq_n} = \pi^{-\frac{3}{2}}$$

• WE KNOW:

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \sqrt{\frac{\pi}{4}} \quad \& \quad \int (ax)^2 e^{-(ax)^2} dx = \sqrt{\frac{\pi}{4a^2}}$$

• SUSPECT:

$$\int q_p^2 e^{-q_n^2} dq_p \approx \int q_n^2 e^{-q_n^2} dq_n$$

• REARRANGE THE HOPED RELATION

$$\frac{\int q_D^2 e^{-q_D^2} dq_D}{\left(\int q_p^2 e^{-q_p^2} dq_p\right)^2} \approx \frac{8}{\sqrt{\pi}}$$

• RECALL FROM EARLIER:

$$q_D \propto \frac{p_D}{\sqrt{m_D}} \quad \& \quad q_p \propto \frac{p_p}{\sqrt{m_p}}$$

$$\Downarrow$$

$$q_D \propto \cancel{\frac{p_D}{2m_p}} \quad \frac{p_D}{\sqrt{2m_p}}$$

I WANTED A RELATIONSHIP BTWN p_D & p_p BUT WE DON'T CARE ACTUALLY.

$$\int q_i^2 e^{-q_i^2} dq_i = \sqrt{\frac{\pi}{4}}$$

$$\frac{\int q_D^2 e^{-q_D^2} dq_D}{\left(\int q_p^2 e^{-q_p^2} dq_p\right)^2} = \frac{\sqrt{\frac{\pi}{4}}}{\sqrt{\frac{\pi}{4}}^2} = \frac{4}{\sqrt{\pi}} \neq \frac{8}{\sqrt{\pi}}$$

WHAT WE HOPED IS NOT QUITE TRUE
BUT ALMOST.

ARRIVE TO:

$$\frac{n_D}{n_n n_p} = \frac{1}{8\pi}\left(\frac{m_p kT}{h^2}\right)^{-\frac{3}{2}} \exp\left(\frac{B_D}{kT}\right) \cdot \frac{4}{\sqrt{\pi}}$$

$$= \frac{1}{2}\left(\frac{\pi m_p kT}{h^2}\right)^{-\frac{3}{2}} \exp\left(\frac{B_D}{kT}\right)$$