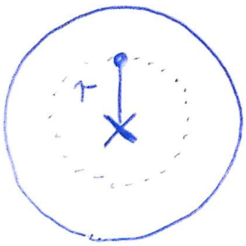
SDSG2

2.1.

• POTENTIAL FROM PART OF SPHERE > r

NEWTON'S FIRST THM: NO FORCE DUE TO EXTERNAL SPHERICALLY SYMMETRIC MASS DISTRIBUTION



$\Rightarrow \Phi$ = CONSTANT CONTRIB. FROM THIS BIT.

• POTENTIAL FROM INSIDE SPHERE:

NEWTON SECOND THM: TREAT INSIDE AS POINT MASS.

$$f = -\nabla \Phi \Rightarrow \Phi = -\int_{\infty}^{r} f \, dr$$

$$= -\int_{\infty}^{r} \frac{m_{<r'} G}{r'^2} dr'$$

$$= -\int_{\infty}^{r} \frac{4}{3} r'^3 \pi S \frac{1}{r'^2} G \, dr' = \int_{\infty}^{r} \frac{4}{3} \pi S G r' dr' = \frac{4}{3}\pi S G \left[\frac{r'^2}{2}\right]_{\infty}^{r}$$

$$= \frac{2}{3} \pi S G r^2 = \frac{1}{2} \Omega^2 r^2 + C$$

where $\Omega^2 = \frac{4}{3}\pi S G$

THE NOTATION Ω SUGGESTS THAT $\sqrt{\frac{4}{3}\pi S G}$ SHOULD HAVE DIMENSIONS sec^{-1}

CHECK: $\sqrt{\frac{kg}{m^3}} \sqrt{N \frac{m^2}{kg^2}} = \sqrt{\frac{N}{m \, kg}} = \sqrt{\frac{kg \frac{m}{s^2}}{m \, kg}} = \frac{1}{\text{sec}}$ GOOD.

SDSG2

2.2 BASED ON LECTURE HANDOUT 3, PAGE 6:

THE PART WHICH WORKS

UNSTABLE IF: $\frac{d}{dR}(R^3 f) > 0$

RECALL: FOR CIRCULAR ORBIT: $-R\dot{\phi}^2 = f$

$$\frac{d}{dR}\left(R^3(-R\dot{\phi}^2)\right) > 0$$

$$\frac{d}{dR}\left(R^4\dot{\phi}^2\right) < 0$$

RENAME $R \Rightarrow r$

USE: $h = r^2\dot{\phi}$

$$\frac{d}{dr}\left(h^2(r)\right) < 0$$

AS REQUIRED.

ARISING QUESTIONS

- WHY TAKING SECOND DERIVATIVE OF $U_{EFFECTIVE}$ DOESN'T WORK? IE

$$U_{EFF}(r) = U(r) + \frac{h^2}{2r^2}$$

$$\frac{dU_{EFF}}{dr} = \frac{dU(r)}{dr} + \frac{h}{r^2}h' - h^2 r^{-3}$$

CONSIDER:

$$\frac{dU(r)}{dr} = -f(r) = r\dot{\phi}^2 - \cancel{\ddot{r}} = r\dot{\phi}^2$$

$\ddot{r} \to 0$ FOR CIRCULAR

$$\frac{dU_{EFF}}{dr} = r\dot{\phi}^2 + \cancel{\frac{hh'}{r^2}} - h^2 r^{-3}$$

$\to 0$ BEC $h' = 0$

ORBIT IS WHERE $\frac{dU_{EFF}}{dr} = 0$, IE $r\dot{\phi}^2 - h^2 r^{-3} = 0$

$$\dot{\phi}^2 = h^2 r^{-4}$$

$$h^2 = \dot{\phi}^2 r^4 \quad \text{(AS EXPECTED)}$$

$$\frac{dU_{EFF}}{dr} = \cancel{\dot{\phi}^2}\, r^{-3}h^2 - h^2 r^{-3} = 0$$

OH I SEE. SHOULDN'T THROW OUT THE $\frac{hh'}{r^2}$ TERM BEFORE TAKING SECOND DERIVATIVE.

IS THE METHOD CORRECT THOUGH IN PRINCIP?

- ~~ISN'T $(h^2(r))'$ NEVER SA~~

IS $\frac{d}{dr}(h^2) < 0$ EVER SATISFIED? $\frac{d}{dr}(h^2) = 2hh'$. h IS CONSTANT SO $h' = 0 \Rightarrow \frac{d}{dr}(h^2) = 0$ IF ANGULAR MOMENTUM IS CONSERVED IE ALWAYS: SO HOW ARE THERE UNSTABLE ORBITS?

SDSG2

2.3

RECALL ORBIT EQ.:

$$\frac{d^2u}{d\varphi^2} + u = -\frac{f_r}{h^2u^2} \qquad \text{with } u = \frac{1}{r}$$

$$h = r^2\dot\varphi$$

$$u = \frac{1}{r} = a^{-1}\exp(-b\varphi)$$

$$\frac{1}{u^2h^2} = \frac{1}{u^2u^{-4}\dot\varphi^2} \qquad r = u^2\dot\varphi^{-2}$$

$$\frac{b^2}{a}\exp(-b\varphi) + \frac{1}{a}\exp(-b\varphi) = \nabla\Phi\, u^2\frac{1}{\dot\varphi^2}$$

$$\frac{1}{a}(b^2+1)\exp(-b\varphi) = \nabla\Phi\,\frac{u^2}{\dot\varphi^2}$$

$$= \nabla\Phi\,\frac{1}{\dot\varphi^2}\,a^{-2}\exp(-2b\varphi)$$

$$\dot\varphi^2 a(b^2+1)\exp(b\varphi) = \nabla\Phi$$

$$\frac{h^2}{r^4}\,a(b^2+1)\exp(b\varphi) = \nabla\Phi$$

$$h^2(b^2+1)r^{-3} = \nabla\Phi$$

$$\Rightarrow \Phi \propto \frac{1}{r^2}$$

CONSTANT OF PROPORTIONALITY CAN BE OBTAINED IF WE KNOW h OF THE ORBIT AS WELL, CORRECT?

SDSG2

2.4

LECTURE NOTES I, EQUATION 1.56 : $$u^2 = 2\frac{E - \Phi}{h^2}$$

$$\Rightarrow \frac{d^2\Phi}{du^2} = \frac{d^2}{du^2}\left(E - \frac{1}{2}u^2h^2\right) = -h^2$$

$$\rho(\underline{r}) > 0 \ \forall \underline{r} \Rightarrow \Phi < 0 \ \forall u$$

$$\left(\text{USING: } \Phi(\underline{r}) = -\int \frac{G\rho(\underline{r}')\,d^3\underline{r}'}{|\underline{r} - \underline{r}'|}\right)$$

WANT TO PROVE: THERE ARE 0 OR 2 ROOTS.
IE DISPROVE THAT THERE CAN BE 1 ROOT.

1 ROOT IF: $E - \Phi = 0$

$\frac{d^2\Phi}{du^2} = -h^2 \Rightarrow \frac{d\Phi}{du}$ IS DECREASING AS u INCREASES

$\Rightarrow \Phi$ IS CONCAVE. (AND THEN WHAT?)

$$\left.\begin{array}{l} E = \underbrace{KE}_{\text{KINETIC ENERGY}} + \Phi \\ KE > 0 \end{array}\right\} \Rightarrow E > \Phi \Rightarrow E - \Phi > 0$$

(IE PARTICLE IN CENTRAL FORCE FIELD DOES NOT STAY STATIONARY)

$\Downarrow$

$E - \Phi \neq 0$

$\Downarrow$

CANNOT BE 1 ROOT

$E - \Phi > 0 \Rightarrow u^2 = \frac{2}{h^2}(E - \Phi)$ MUST HAVE 2 ROOTS.

THIS IS NOT WHAT WE WERE SUPPOSED TO BE PROVING.

SDSG2

2.5 (I) $\nabla^2 \Phi = 4\pi G \rho$

$$\text{TOTAL MASS} = \int_{\text{ALL SPACE}} \rho \, dV = \int_{\text{ALL SPACE}} \frac{1}{4\pi G} \nabla^2 \Phi \, dV$$

$$= \int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{1}{4\pi G} \nabla^2 \Phi \; r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= \frac{1}{G} \int_0^{\infty} \nabla^2 \Phi \, r^2 \, dr = \frac{1}{G} \int_0^{\infty} \frac{1}{\cancel{r^2}} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) \cancel{r^2} \, dr$$

$$= \int_0^{\infty} \frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r}\left(\frac{-M}{b+\sqrt{b^2+r^2}}\right)\right] dr$$

$$= M \int_0^{\infty} -\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r}\left(\frac{1}{b+\sqrt{b^2+r^2}}\right)\right] dr$$

- TRUST OURSELVES THAT WE'RE GOOD IN SETTING UP INTEGRALS

OR

- BRUTE FORCE ALGEBRA

OR

- PUT IT INTO COMPUTER (THIS WAS MY CHOICE)

ARRIVE AT

$$= M$$

SDSG2

2.5(ii)

$$\rho(0) = \frac{1}{4\pi G}\nabla^2\Phi\Big|_{r=0}$$

$$= \frac{1}{4\pi \cancel{G}}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{-\cancel{G}M}{b+\sqrt{b^2+r^2}}\right)\right)\Bigg|_{r=0}$$

$$= \frac{3M}{16\pi b^3}$$

[DONE BY COMPUTER, RESULT CAN BE VERIFIED ON DESMOS: DESMOS.COM/CALCULATOR/ECNFBTM4MC]

$$\rho(r \gg b) = \frac{1}{4\pi G}\nabla^2\Phi\Big|_{r \gg b}$$

~~USING: $b+\sqrt{b^2+r^2} \approx r$ WHEN $r \gg b$~~

$$\cancel{\approx \frac{1}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{-M}{r}\right)\right)\Bigg|_{r \gg b}}$$

$$= \frac{1}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{-M}{b+\sqrt{b^2+r^2}}\right)\right)$$

$$= \frac{bM}{2\pi r^4}$$

[VERIFIED BY DESMOS] (LINK ABOVE ↑)