AFD L3

main topic: gravitation

recap:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0$$ Eulerian

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \underline{u} = 0$$ Lagrangian

cont. eq.

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p + \rho \underline{g}$$ Lagrangian

momentum eq.

$$\rho \frac{\partial \underline{u}}{\partial t} + \rho (\underline{u} \cdot \nabla) \underline{u} = -\nabla p + \rho \underline{g}$$ Eulerian

$$\partial_t (\rho \underline{u}) = -\nabla \cdot (\rho \underline{u} \times \underline{u} + p \underline{\underline{I}}) + \rho \underline{g}$$ conservative form

grav. potential Ψ: $\underline{g} = -\nabla \Psi$

conservative force.

i.e work done is indep. of path

$$-\int_{\underline{r}}^{\infty} \underline{g} \cdot \underline{dl} = \int_{\underline{r}}^{\infty} \nabla \Psi \cdot \underline{dl} = \Psi(\infty) - \Psi(\underline{r})$$

Newton's law for point mass

$$\Psi = -\frac{GM}{r}$$ mass at origin

for system of masses:

$$\Psi = -\sum_i \frac{GM_i}{|\underline{r} - \underline{r}_i|} \Rightarrow \underline{g} = -\nabla \Psi = -\sum_i \frac{GM_i (\underline{r} - \underline{r}_i)}{|\underline{r} - \underline{r}_i|^3}$$

in continuum limit:

$$\underline{g} = -G \int \rho(\underline{r}') \frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3} dV'$$

take divergence:

$$\nabla_{\underline{r}} \cdot \underline{g} = -G \int \rho(\underline{r}') \underbrace{\nabla_{\underline{r}} \cdot \left[\frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3}\right]}_{4\pi\delta(\underline{r}-\underline{r}')} dV'$$

$$= -4\pi G \int \rho(\underline{r}')\delta(\underline{r}-\underline{r}') dV'$$

ie $$= -4\pi G \rho(\underline{r})$$

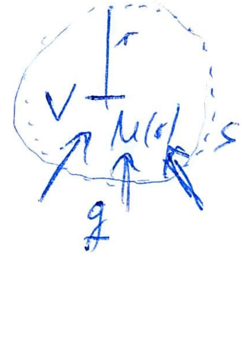
$$\underline{\nabla \cdot \underline{g} = -\nabla^2 \psi = -4\pi G \rho}$$ POISSON EQUATION

integral form: $\int_V \nabla \cdot \underline{g} \, dV = -4\pi G \int_V \rho \, dV$

$$\Rightarrow \int_S \underline{g} \cdot d\underline{S} = -4\pi G M$$

Example I

$$\int \underline{g} \cdot d\underline{S} = -4\pi G \underbrace{M(r)}_{\text{enclosed mass}}$$



$$-4\pi r^2 \underline{g} = -4\pi G M(r)$$

$$\Rightarrow |\underline{g}| = \frac{GM(r)}{r^2}$$

$$\therefore \underline{g} = -\frac{GM(r)}{r^2}\hat{\underline{r}}$$

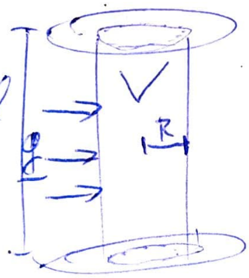
consequence: inside of ~~shell (than)~~ spherical shell:
there is no force.
it you squeeze it, it doesn't work.

application: velocity curves of stellar galaxies.
i.e "rotation curves"

main argument for DM.

note: observed galaxy is not spherical.
DM is thought to have much more spherical distrib. than observed matter.

Example II

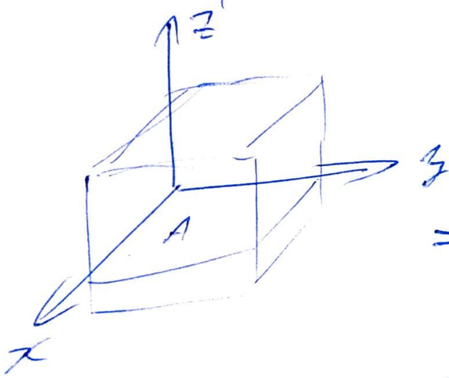


$$\int \underline{g} \cdot d\underline{S} = -4\pi G \int_V \rho \, dV$$

$$-2\pi R l |\underline{g}| = -4\pi G l \underbrace{M(r)}_{\text{enclosed mass per unit length}}$$

$$\underline{g} = -\frac{2GM(R)}{R} \hat{\underline{R}}$$

Example III
Planar geometry w/ reflection symmetry in $z=0$



$$\int_S \underline{g} \, d\underline{S} = -4\pi G \int_V \rho \, dV$$

$$\Rightarrow -2|\underline{g}| A = -4\pi G A \int_{-z}^{z} \rho(z) \, dz$$

$$\Rightarrow \underline{g} = -4\pi G \hat{\underline{z}} \int_0^z \rho(z) \, dz$$

Potential of spherically symmetric system

spherical sys:

$$\underline{g} = -|\underline{g}|\,\hat{\underline{r}} \qquad |\underline{g}| = \frac{G}{r^2}\underbrace{\int_0^r 4\pi\rho(r')r'^2\,dr'}_{M(r)} = \frac{d\Psi}{dr}$$

— defining zero of potential at infinity

$$\Rightarrow \Psi = \int_\infty^{r_0} \frac{G}{r^2}\left\{\int_0^r 4\pi\rho(r')r'^2\,dr'\right\}dr$$

integration by parts

$$\Psi = -\left\{\frac{G}{r}\int_0^r 4\pi\rho(r')r'^2\,dr'\right\}\Bigg|_{r=\infty}^{r_0} + \int_\infty^{r_0}\frac{G}{r}4\pi\rho(r)r^2\,dr$$

$$\Rightarrow \Psi = -\frac{GM(r_0)}{r_0} + \int_\infty^{r_0} 4\pi G\rho(r)\,r\,dr$$

Gravitational potential energy

imagine dismantling of a system one particle at a time.

$$\Omega = -\frac{1}{2}\sum_{j\neq i}\sum_i \frac{GM_iM_j}{|\underline{r}_j-\underline{r}_i|} = \frac{1}{2}\sum_j M_j\Psi_j$$

to avoid double counting

particles don't feel their own field

continuum limit

$$\Omega = \frac{1}{2}\int \rho(\underline{r})\Psi(\underline{r})\,dV$$

for spherical system:

$$\Omega = \frac{1}{2}\int_0^\infty 4\pi\rho(r)r^2\Psi(r)\,dr \overset{\text{BY PARTS}}{=} \frac{1}{2}\underbrace{\left[M(r)\Psi(r)\right]_0^\infty}_{0} - \int_0^\infty M(r)\underbrace{\frac{d\Psi}{dr}}_{\frac{GM(r)}{r^2}}dr$$

$\Psi|_\infty = 0 \quad M|_0 = 0$

$$\Omega = -\frac{1}{2}\int_0^\infty \frac{GM^2(r)}{r^2}\,dr$$

integrate by parts again

$$\Omega = \frac{1}{2}\,GM^2(r)\,\frac{1}{r}\Big|_0^\infty - \frac{1}{2}\int_0^\infty -\frac{G}{r}\frac{d}{dr}(M^2)\,dr$$

$$\qquad\qquad \to 0$$

$$= -\frac{1}{2}\int_0^\infty \frac{1}{r}\,G2M\,\frac{dM}{dr}\,dr$$

$$\Rightarrow \Omega = -G\int_0^\infty \frac{M(r)}{r}\,dM$$

nice interpretation: peel away shells of spherical sys, take them to infinity

Virial Theorem

consider gravitating system of many particles, mass m_i pos $\underline{r}_i$

consider time deriv. of quantity $I_i = m_i r_i^2$

$$\frac{1}{2}\frac{d^2}{dt^2}(m_i \underline{r}_i^2) = m_i\frac{d}{dt}\left(\underline{r}_i \cdot \frac{d\underline{r}_i}{dt}\right)$$

$$= m_i\,\underline{r}_i \cdot \frac{d^2\underline{r}_i}{dt^2} + m_i\left(\frac{d\underline{r}_i}{dt}\right)^2$$

$$= \underline{r}_i \cdot \underline{F}_i + \underbrace{m_i\left(\frac{d\underline{r}_i}{dt}\right)^2}_{2\times KE}$$

Sum over particles:

$$\frac{1}{2}\frac{d^2 I}{dt^2} = \underbrace{\sum_i \underline{r}_i \cdot \underline{F}_i}_{V,\ \text{the "virial"}} + 2T$$

$\to$ gonna show now this is grav. pot.

assumption: system is isolated

$$\underline{F}_i = \sum_j \underline{F}_{ij} \qquad \underline{F}_{ij} = -\underline{F}_{ji}$$

$$V = \sum_i \sum_{j>i} \underline{F}_{ij} \cdot (\underline{r}_i - \underline{r}_j)$$

making sure
we don't
double count

Put: $\underline{F}_{ij} = -\frac{G m_i m_j}{r_{ij}^3} \underline{r}_{ij}$

V becomes:

$$V = -\sum_i \sum_{j>i} \frac{G m_i m_j}{r_{ij}}$$

this is just grav. potential energy

we derived:

$$\frac{1}{2} \frac{d^2 I}{dt^2} = 2T + \Omega$$

Now assume: system is in state of dynamical equi. LHS is zero

$$2T + \Omega = 0$$

- Implications of virial thm

$$T = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} M \langle v^2 \rangle$$

$$\Omega = -\frac{1}{2} \int_0^\infty \frac{G M^2(r)}{r^2} dr = -\int_0^\infty \frac{G M(r)}{r} dM = -\frac{G M^2}{\bar{r}}$$

from earlier

shell interpretation

– appropriate weighted average

$$2T = -\Omega$$

$$\Rightarrow M\langle v^2\rangle = \frac{GM^2}{\bar{r}}$$

$$\Rightarrow \langle v^2\rangle = \frac{GM}{\bar{r}}$$

remember that dynamical equilibrium needed for virial thm. to hold. in this case, particles are "virialized"

$$E_{TOTAL} = T + \Omega$$

$$\Rightarrow E_{TOTAL} = -T = -\frac{1}{2}M\langle v^2\rangle = -\frac{GM^2}{\bar{r}^2}$$

as v goes up, E_{TOTAL} goes more negative,

$\bar{r}$ goes up

gravothermal collapse of globular clusters

total E of glob cluster changes as high velocity stars leave

it gets smaller

speeds go up more

more stars leave

runaway process

may stop from binary formation in the middle.

gravitating systems have negative specific "heat" capacity