2010, PAPER 3, QUESTION 6 | I

i/ COLLISIONLESS BOLTZMANN EQUATION:

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f - \nabla \Phi \frac{\partial f}{\partial v} = 0$$

FIRST MOMENT OF CBE:

$$\int_{\substack{\text{VELOCITY}\\\text{SPACE}}} \left(\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} \right) \cdot v_j \, d^3\underline{v} = 0$$

CONSIDER:

I) $$\frac{\partial}{\partial t}(f v_j) = v_j \frac{\partial f}{\partial t} + f \underbrace{\cancel{\frac{\partial v_j}{\partial t}}}_{=0} = v_j \frac{\partial f}{\partial t}$$

$$\Rightarrow \int \frac{\partial f}{\partial t} v_j \, d^3\underline{v} = \int \frac{\partial}{\partial t}(f v_j) \, d^3\underline{v} = \frac{\partial}{\partial t} \int f v_j \, d^3\underline{v}$$

II) $\frac{\partial \Phi}{\partial x_i}$ DOES NOT HAVE $\underline{v}$ DEPENDENCE

$$\text{I \& II} \Rightarrow \frac{\partial}{\partial t} \int f v_j \, d^3\underline{v} + \int v_i v_j \frac{\partial f}{\partial x_i} d^3\underline{v} - \frac{\partial \Phi}{\partial x_i} \int v_j \frac{\partial f}{\partial v_i} d^3\underline{v} = 0$$

PROCEEDING TERM BY TERM:

$$\frac{\partial}{\partial t} \int f v_j \, d^3\underline{v} = \frac{\partial}{\partial t}\left(S \langle v_j \rangle \right) = S \frac{\partial \langle v_j \rangle}{\partial t}$$

$$\int v_i v_j \frac{\partial f}{\partial x_i} d^3\underline{v} \underset{\uparrow}{=} \frac{\partial}{\partial x_i} \int v_i v_j f \, d^3\underline{v} = \frac{\partial}{\partial x_i}\left(S \langle v_i v_j \rangle \right)$$

$v_i v_j$ DOES NOT HAVE x_i DEP.

$\boxed{\text{2010 P3Q6 (II)}}$

$$\int v_j \frac{\partial f}{\partial v_i} d^3\underline{v} = \left[\cancel{f v_j}\right]_{-\infty}^{\infty} - \int \frac{\partial v_j}{\partial v_i} f \, d^3\underline{v} = -\delta_{ij}\int f \, d^3\underline{v}$$

$\rightarrow 0$
AS WE WANT
PHYSICAL BCS
TO PREVAIL

COLLECT TERMS:

$$\rho \frac{\partial\langle v_j\rangle}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho\langle v_i v_j\rangle\right) + \frac{\partial\Phi}{\partial x_j}\overbrace{\int f d^3\underline{v}}^{\rho} = 0 \qquad (A)$$

RECALL (OR LOOK UP IN FORMULAE BOOKLET):

$$\cancel{\frac{\partial\rho}{\partial t}} \qquad \frac{\partial\rho}{\partial t} + \frac{\partial(\rho\langle v_i\rangle)}{\partial x_i} = 0 \qquad (B)$$

$$\cancel{\frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x_i}\langle v_i\rangle + \frac{\partial\langle v_i\rangle}{\partial x_i}\rho = 0 \qquad (B)}$$

(A) $-\langle v_j\rangle$ (B):

$$\rho\frac{\partial\langle v_j\rangle}{\partial t} - \cancel{\frac{\partial\rho}{\partial t}\langle v_j\rangle} - \langle v_j\rangle\frac{\partial}{\partial x_i}\left(\rho\langle v_i\rangle\right) +$$

$$+ \frac{\partial}{\partial x_i}\left(\rho\langle v_i v_j\rangle\right) + \rho\frac{\partial\phi}{\partial x_j} = 0$$

$$\rho\frac{\partial\langle v_j\rangle}{\partial t} - \langle v_j\rangle\frac{\partial}{\partial x_i}\left(\rho\langle v_i\rangle\right) + \frac{\partial}{\partial x_i}\left(\rho\langle v_i v_j\rangle\right) = -\rho\frac{\partial\Phi}{\partial x_j}$$

CONSIDER:

$$\sigma^2_{ij} = \left\langle (v_i - \langle v_i\rangle)(v_j - \langle v_j\rangle)\right\rangle =$$

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P3Q6 (iii)

$$= \langle v_i v_j - v_i \langle v_j \rangle - \langle v_i \rangle v_j + \langle v_i \rangle \langle v_j \rangle \rangle$$

$$= \langle v_i v_j \rangle - \langle v_i \rangle \langle v_j \rangle \cancel{- \langle v_i \rangle \langle v_j \rangle + \langle v_i \rangle \langle v_j \rangle}$$

$$= \langle v_i v_j \rangle - \langle v_i \rangle \langle v_j \rangle = \sigma_{ij}^2$$

$$\Rightarrow \langle v_i v_j \rangle = \sigma_{ij}^2 + \langle v_i \rangle \langle v_j \rangle$$

REWRITE EQUATION:

$$\rho \frac{\partial \langle v_j \rangle}{\partial t} - \langle v_j \rangle \frac{\partial}{\partial x_i}\left(\rho \langle v_i \rangle\right) + \frac{\partial}{\partial x_i}\left(\rho \sigma_{ij}^2\right) + \frac{\partial}{\partial x_i}\left(\rho \langle v_i \rangle \langle v_j \rangle\right) = -\rho \frac{\partial \Phi}{\partial x_j}$$

$$\Rightarrow \rho \frac{\partial \langle v_j \rangle}{\partial t} \underbrace{- \langle v_j \rangle \frac{\partial}{\partial x_i}\left(\rho \langle v_i \rangle\right) + \frac{\partial}{\partial x_i}\left(\rho \langle v_i \rangle \langle v_j \rangle\right)} = -\rho \frac{\partial \Phi}{\partial x_j} - \frac{\partial}{\partial x_i}\left(\rho \sigma_{ij}^2\right)$$

$$\cancel{- \langle v_j \rangle \frac{\partial}{\partial x_i}\left(\rho \langle v_i \rangle\right)} + \cancel{\frac{\partial}{\partial x_i}\left(\rho \langle v_i \rangle\right) \langle v_j \rangle} + \frac{\partial}{\partial x_i}\left(\rho \langle v_j \rangle\right) \langle v_i \rangle$$

$$\Rightarrow \boxed{\rho \frac{\partial \langle v_j \rangle}{\partial t} + \rho \langle v_i \rangle \frac{\partial}{\partial x_i} \langle v_j \rangle = -\rho \frac{\partial \Phi}{\partial x_j} - \frac{\partial}{\partial x_i}\left(\rho \sigma_{ij}^2\right)}$$

AS REQUIRED.

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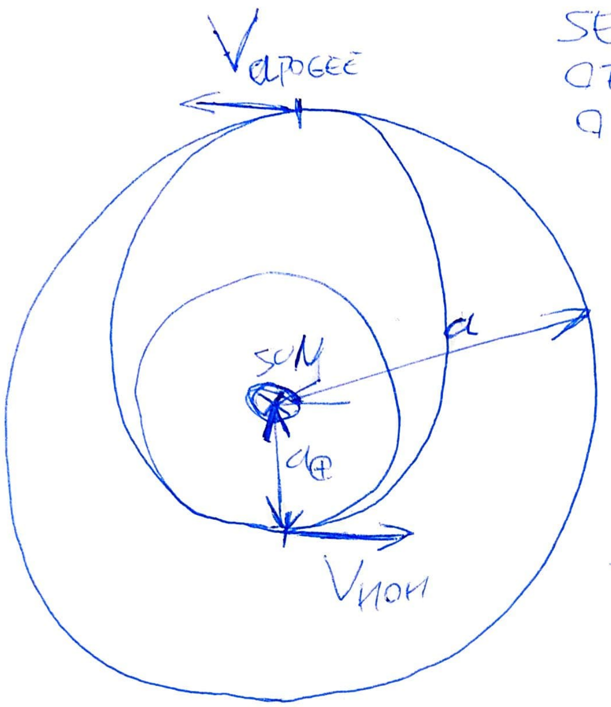
P3Q6 IV

KEPLER III: $T^2 \propto a^3$



SEMI MAJOR AXIS OF TRANSFER ORBIT: $\frac{a_\oplus + a}{2}$

$$\frac{T_{\text{FULL TRANSFER ORB.}}^2}{T_{\text{EARTH}}^2} = \frac{\left(\frac{a_\oplus + a}{2}\right)^3}{a_\oplus^3}$$

T_{EARTH} – 1 YR

$$T_{FTO} = \left[\frac{1}{2}\left(1 + \frac{a}{a_\oplus}\right)\right]^{\frac{3}{2}} \text{ YR}$$

$$T = \frac{T_{FTO}}{2} = \frac{1}{2}\left(\frac{1}{2}\right)^{\frac{3}{2}}\left(1 + \frac{a}{a_\oplus}\right)^{\frac{3}{2}} = \frac{1}{2\sqrt{8}}\left(1 + \frac{a}{a_\oplus}\right)^{\frac{3}{2}} \text{YR}$$

(WE'RE ONLY GOING UP, NOT UP & DOWN)

$$= \frac{1}{4\sqrt{2}}\left(1 + \frac{a}{a_\oplus}\right)^{\frac{3}{2}} \text{YR}$$

AS REQUIRED.

ENERGY CONSERVATION:

$$\frac{1}{2}V_{HOH}^2 - \frac{GM}{a_\oplus} = \frac{1}{2}V_{APOGEE}^2 - \frac{GM}{a}$$

ANGULAR MOM CONSERVATION:

$$V_{HOH}\, a_\oplus = V_{APOGEE}\, a$$

$$\frac{1}{2}V_{HOH}^2 - \frac{GM}{a_\oplus} = \frac{1}{2}V_{HOH}^2\left(\frac{a_\oplus}{a}\right)^2 - \frac{GM}{a}$$

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P3Q6 V

$$\frac{1}{2} V_{Hoh}^2 \left(1 - \left(\frac{a_\oplus}{a}\right)^2\right) = GM\left(\frac{1}{a_\oplus} - \frac{1}{a}\right)$$

$$V_{Hoh}^2 = GM \cdot 2 \frac{\frac{a - a_\oplus}{a a_\oplus}}{1 - \left(\frac{a_\oplus}{a}\right)^2}$$

$$= GM \cdot 2 \frac{(a - a_\oplus)\left(\frac{a}{a_\oplus}\right)}{a^2 - a_\oplus^2} = \frac{\frac{a}{a_\oplus}}{a + a_\oplus} 2GM$$

EARTH CASE:

$$\frac{GM}{a_\oplus^2} = \frac{V_\oplus^2}{a_\oplus} \Rightarrow V_\oplus^2 = \frac{GM}{a_\oplus}$$

$$V_{Hoh}^2 = \frac{a}{a + a_\oplus} 2 \cdot \frac{GM}{a_\oplus}$$

$$= \frac{a}{a + a_\oplus} \cdot 2 \cdot V_\oplus^2$$

$$\Rightarrow V_{Hoh} = \sqrt{2}\left(\frac{a}{a + a_\oplus}\right)^{\frac{1}{2}} V_\oplus$$

IE

$$V_{ADD} = V_{Hoh} - V_\oplus = V_\oplus\left[\sqrt{2}\left(\frac{a}{a + a_\oplus}\right)^{\frac{1}{2}} - 1\right]$$

~~REQ.D~~

AS REQUIRED.

2010
P3QE [VI]

~~APHELION DIST = a (1+e)~~

APHELION DIST = SEMI-MAJOR AXIS $\cdot$ (1+e)

$$a = \frac{a + a_\oplus}{2}(1+e)$$

$$\frac{2a}{a+a_\oplus} - 1 = e$$

$$e = \frac{2 \cdot 1.524}{1.524+1} - 1 \cong \underline{\underline{0.21}}$$

$$T = \frac{1}{4\sqrt{2}}\left(1 + \frac{1.524}{1}\right)^{\frac{3}{2}} \cong \underline{\underline{0.71\ \text{YR}}}$$

$$V_{ADD} = V_\oplus\left(\sqrt{2}\left(\frac{\cancel{a}\,1.524}{\cancel{a}\,1+1.524}\right)^{\frac{1}{2}} - 1\right) \cong 0.10 V_\oplus$$

$$\cancel{= V_\oplus \cdot\ ?} \qquad V_\oplus = \frac{150 \cdot 10^6 \cdot 10^3 \cdot 2\pi}{365 \cdot 24 \cdot 60 \cdot 60} = 4.8 \cdot 10^3 \cdot 2\pi \ \frac{m}{s}$$

$$\Rightarrow V_{ADD} \cong 4.8 \cdot 10^2\ \cancel{10^3} \cdot 2\pi \cong \underline{\underline{3}}\ \frac{km}{s}$$