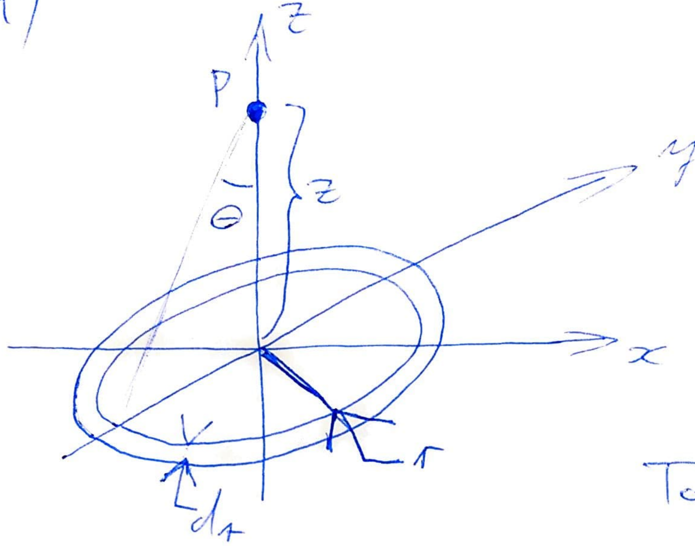
2011 P4Q6 (I)

(i)



force on point P from ring with radius r, thickness dr:

$$dF = G\frac{2\pi r\Sigma_0 dr}{r^2+z^2}\cdot\cos\theta$$

$$= 2\pi G\Sigma_0 \frac{r}{r^2+z^2}\frac{z}{\sqrt{r^2+z^2}}dr$$

Total force:

$$F = \int dF = -2\pi G\Sigma_0 \int_0^\infty \frac{rz}{(r^2+z^2)^{\frac{3}{2}}}\,dr$$

$$= -2\pi G\Sigma_0$$ (WHERE ⊖ SIGN SIGNIFIES THAT FORCE IS TOWARDS LAYER)

$$F = -\nabla\Phi \Rightarrow \Phi = 2\pi G\Sigma_0|z|$$

ABSOLUTE VALUE SIGN IS THERE SO THAT FORCE IS STILL IN LAYER'S DIRECTION WHEN $z<0$ TOO.

(INTEGRATION STEP:

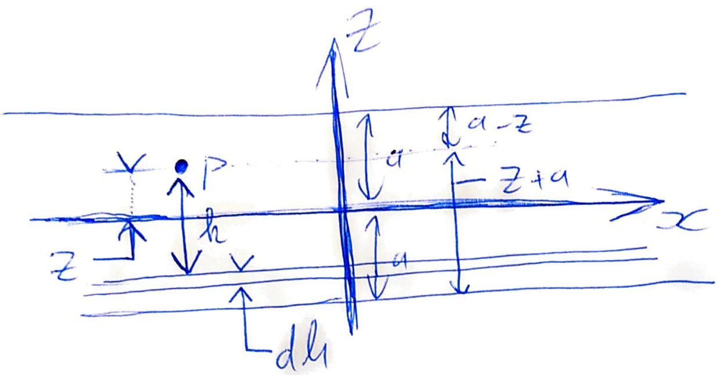
$$\int_0^\infty \frac{rz}{(r^2+z^2)^{\frac{3}{2}}}dr = \int_0^\infty \frac{\cancel{r}z}{u^{\frac{3}{2}}}\frac{du}{2\cancel{r}} = \frac{1}{2}\int_0^\infty u^{-\frac{3}{2}}z\,du = \frac{1}{2}(-2)u^{-\frac{1}{2}}\Big|_0^\infty z$$

$$= -\frac{1}{\sqrt{r^2+z^2}}\Big|_0^\infty z = -\left(0-\frac{z}{|z|}\right) = \underline{\text{SIGN}(z)}\ )$$

$u = r^2+z^2$

$\frac{du}{dr} = 2r$

2011
P4Q6 (II)



POTENTIAL AT P FROM LAYER THICKNESS dh, DISTANCE h:

$2\pi G \Sigma_0 |h|$

$$\Phi = \int d\Phi = \int_{z+a}^{a-z} 2\pi G \Sigma_0 |h| \, dh = 2\pi G \Sigma_0 \left(\int_0^{z+a} h\,dh + \int_0^{a-z} h\,dh \right)$$

$$\cancel{= 2\pi G \Sigma_0 \frac{1}{2} h^2 \Big|_{z+a}^{a-z}}$$

$$= 2\pi G \Sigma_0 \frac{1}{2} \left(h^2 \Big|_0^{z+a} + h \Big|_0^{a-z} \right)$$

$$= 2\pi G \Sigma_0 \frac{1}{2} \left(z^2 + a^2 + \cancel{2az} + a^2 + z^2 - \cancel{2az} \right)$$

$$= \underline{2\pi G \Sigma_0 \left(z^2 + a^2 \right)}$$

2011 P4Q6 (III)

(ii) INTRODUCE RELATIVE POTENTIAL & RELATIVE ENERGY

$$\Psi = -\Phi + \Phi_0$$

$$\mathcal{E} = -E + \Phi_0$$

IE. FOR $|z| < a$:

$$\Psi = -2\pi G \rho_0 (z^2 + a^2) + \Phi_0$$

$$\mathcal{E} = -\left(2\pi G \rho_0 (z^2 + a^2) + \tfrac{1}{2} v^2\right) + \Phi_0$$

CHOOSE Φ_0 S.T. $f > 0 \;\forall \mathcal{E} > 0$

$f = 0 \;\forall \mathcal{E} \leq 0$

We don't want stars outside the layer

$\Rightarrow f(|z| \geq a) = 0$

~~f is continuous~~ ~~$\Rightarrow f(|z|=a) = 0$~~ At $z=a$, $v=0$, so stars don't wonder off from layer.

$$\mathcal{E}\Big|_{z=a} = -\left(2\pi G \rho_0 (a^2 + a^2) + \tfrac{1}{2} 0^2\right) + \Phi_0$$

$$= -4\pi G \rho_0 a^2 + \Phi_0 = 0$$

$$\Rightarrow \Phi_0 = +4\pi G \rho_0 a^2$$

$$\Rightarrow \mathcal{E} = -2\pi G \rho_0 (z^2 + a^2) - \tfrac{1}{2} v^2 + 4\pi G \rho_0 a^2$$

$$= -2\pi G \rho_0 (z^2 - a^2) - \tfrac{1}{2} v^2$$

$$= \tfrac{1}{2} \underbrace{(4\pi G \rho_0)}_{\omega^2} (a^2 - z^2) - \tfrac{1}{2} v^2$$

AS REQUIRED.

2011
P4Q6 $\boxed{(\text{IV})}$

$$\Psi = -2\pi G \rho_0 (z^2+a^2) + 4\pi G \rho_0 a^2$$

$$= \frac{1}{2}\omega^2 (a^2 - z^2)$$

SO WE HAVE: $\mathcal{E} = \Psi - \frac{1}{2}v^2 \Longrightarrow d\mathcal{E} = -v\,dv$

$$\rho(z) = \rho_0 = \int_{-\infty}^{\infty} f\,dv = 2\int_{0}^{\infty} f\,dv$$

BY SYMMETRY

$$= 2\int_0^{V_{MAX}} f\,dv$$

V_{MAX}: WHERE $f \geq 0$, IE $\mathcal{E} \geq 0$,

v IS MAX WHEN $\mathcal{E} = \Psi - \frac{1}{2}v^2 = 0 \Rightarrow v = \sqrt{2(\Psi - \mathcal{E})}$

$$\Rightarrow v_{MAX} = \sqrt{2\Psi}$$

$$\rho(z) = 2\int_0^{\sqrt{2\Psi}} f\,dv = 2\int_{\mathcal{E}|_{v=0}}^{\mathcal{E}|_{V_{MAX}}} f\,\frac{-1}{v}\,d\mathcal{E} = 2\int_{\Psi}^{0} \frac{-f\,d\mathcal{E}}{\sqrt{2(\Psi-\mathcal{E})}}$$

$$= \sqrt{2}\int_0^{\Psi} \frac{f\,d\mathcal{E}}{\sqrt{\Psi - \mathcal{E}}} = \sqrt{2}\int_0^{\frac{1}{2}\omega^2(a^2-z^2)} \frac{f(\mathcal{E})\,d\mathcal{E}}{\sqrt{\frac{1}{2}\omega^2(a^2-z^2) - \mathcal{E}}}$$

2011 P4Q6 (V)

$$\mathcal{E}\Big|_{z=0} = \frac{1}{2}\omega^2 a^2 - \frac{1}{2}v^2$$

$$\cancel{f\left(\mathcal{E}\big|_{z=0}\right)} = \cancel{\sqrt{2}\int_0^{\frac{1}{2}\omega^2 a^2} \frac{f(\mathcal{E})\,d\mathcal{E}}{\sqrt{\frac{1}{2}\omega^2 a^2 - \mathcal{E}}}}$$

FOR $z = 0$:

$$\int_0^{\frac{1}{2}\omega^2 a^2} \frac{f(\mathcal{E})\,d\mathcal{E}}{\sqrt{\frac{1}{2}\omega^2 a^2 - \mathcal{E}}} = \frac{S_0}{\sqrt{2}}$$

USING HINT:

$$f(\mathcal{E}) = \frac{1}{\pi}\frac{d}{d\mathcal{E}}\int_0^{\mathcal{E}} \frac{\frac{S_0}{\sqrt{2}}}{\sqrt{\mathcal{E} - \frac{1}{2}\omega^2 a^2}}\, d\left(\frac{1}{2}\omega^2 a^2\right)$$

$$= \frac{S_0}{\sqrt{2}\pi}\frac{d}{d\mathcal{E}}\int_0^{\mathcal{E}} \frac{1}{\sqrt{\mathcal{E} - x}}\,dx = \frac{S_0}{\sqrt{2}\pi}\frac{d}{d\mathcal{E}}\left[(-2)\sqrt{\mathcal{E} - x}\,\Big|_0^{\mathcal{E}}\right]$$

$$= -\frac{\sqrt{2}S_0}{\pi}\frac{d}{d\mathcal{E}}\left(-\sqrt{\mathcal{E}}\right) = \frac{\sqrt{2}S_0}{\pi}\frac{1}{2}\mathcal{E}^{-\frac{1}{2}} = \frac{S_0}{\sqrt{2}\pi}\mathcal{E}^{-\frac{1}{2}}$$

$$= \frac{S_0}{\sqrt{2}\pi}\left(\Psi - \frac{1}{2}v^2\right)^{-\frac{1}{2}} = \frac{S_0}{\sqrt{2}\pi}\left(\frac{1}{2}\omega^2 a^2 - \frac{1}{2}v^2\right)^{-\frac{1}{2}} = f(z=0, v)$$