6

Poisson: $\nabla^2 \psi = 4\pi G \rho$

Hydrostat eq.: $\frac{1}{\rho} \nabla p = -\nabla \psi$

Is our fluid isothermal? If yes, I can follow the notes. If not, I don't know what.

Isothermal $\Rightarrow p = \frac{R_*}{\mu} \rho T \Rightarrow p = A\rho$

$$A = \frac{R_* T}{\mu}$$


$\nabla = \frac{\partial}{\partial z}$, $p = p(z)$, $\psi = \psi(z)$

• FIRST, USE HYDROSTATIC EQUILIBRIUM EQUATION TO GET $\psi(\rho)$ & $\rho(\psi)$

$$\frac{1}{\rho} \nabla p = A \frac{1}{\rho} \frac{\partial}{\partial z} \rho = -\frac{\partial}{\partial z} \psi$$

$$A \frac{d}{dz} \ln \rho = -\frac{\partial}{\partial z} \psi$$

$$A \ln \rho = -\psi + C$$

FROM THIS, WE HAVE: $\Rightarrow \psi = -A \ln \rho + C =$

$$= -A \ln\left(\frac{\rho}{\rho_0}\right) + \psi_0$$

INCORPORATE SOME OF THE ADDITIVE CONSTANT TO THE ln, KEEP SOME OUTSIDE TOO

WE ALSO HAVE:

$$\rho = C \cdot \exp\left(-\frac{\psi}{A}\right)$$

REWRITE TO MORE CONVINIENT FORM

$$\rho = \rho_0 e^{-(\psi-\psi_0)/A}$$

• TURN TO POISSON TO CONNECT $\psi$ TO $z$

$$\frac{d^2\psi}{dz^2} = 4\pi G \rho_0 e^{-(\psi-\psi_0)/A}$$

CHANGE VARIABLES: $x = -(\psi-\psi_0)/A$, $Z_1 = z\sqrt{2\pi G \rho_0/A}$

NOTE THAT: $\cancel{\frac{d}{dz} = \frac{dZ}{dz}\frac{d}{d}}$

$$\frac{d}{dz} = \frac{dZ_1}{dz}\frac{d}{dZ_1} = \sqrt{2\pi G \rho_0/A}\;\frac{d}{dZ_1}$$

$$\Rightarrow \frac{d^2}{dz^2} = \frac{2\pi G \rho_0}{A}\frac{d}{dZ_1}$$

REWRITE POISSON: $\cancel{\frac{d^2\psi}{dZ_1^2} = 2Ae^{x}}$

$$\frac{d^2\psi}{dZ_1^2} = 2Ae^{x}$$

NOTING THAT: $-xA + \psi_0 = \psi$ & SETTING $\psi_0 = 0$, WE GET:

$$\frac{d^2 x}{d z_1^2} = -2e^{x}$$

MULTIPLY BY $\frac{dx}{dz_1}$. NOTE THAT THIS IS THE SAME TRICK AS QUESTION 8(b)(i)'s MULTIPLICATION BY $\dot{R}$ (RIGHT IN THE BEGINNING THERE).

$$\frac{dx}{dz_1}\frac{d^2 x}{dz_1^2} = -2\frac{dx}{dz_1}e^{x}$$

$$\frac{1}{2}\frac{d}{dz_1}\left[\left(\frac{dx}{dz_1}\right)^2\right] = -2\frac{d}{dz}e^{x}$$

$$\left(\frac{dx}{dz_1}\right)^2 = c_1 - 4e^{x}$$

We don't want a peak in the potential at $z=0$, so: $\left.\frac{dx}{dz_1}\right|_{x=0} = 0 \Longrightarrow c_1 = 4$

$$\frac{dx}{dz_1} = \pm 2\sqrt{1-e^{x}}$$

$$\int\frac{dx}{\sqrt{1-e^{x}}} = \pm 2\int dz_1$$

CHANGE VAR: $e^x = \sin^2\theta$

$$\Rightarrow e^x\,dx = 2\sin\theta\cos\theta\,d\theta$$

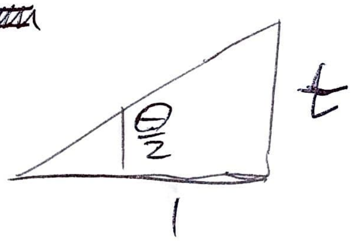
$$dx = \frac{2\sin\theta\cos\theta}{e^x}\,d\theta = \frac{2\cos\theta}{\sin\theta}\,d\theta$$

REWRITE INTEGRAL:

$$\int \frac{dx}{\sqrt{1-e^x}} = \int \frac{2\cos\theta\,d\theta}{\sin\theta\sqrt{1-\sin^2\theta}}$$

$$= \int \frac{2\,d\theta}{\sin\theta}$$

SET: $t = \tan\frac{\theta}{2} \Rightarrow dt = \sec^2\frac{\theta}{2}\,\frac{1}{2}\,d\theta = \frac{1}{2}\,\frac{1\,d\theta}{\frac{1}{1+t^2}} = \frac{1}{2}\,\frac{1+t^2}{1}\,d\theta$




$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= \frac{2t}{1+t^2}$$

CONTINUE WITH INTEGRAL:

$$\int \frac{dx}{\sqrt{1-e^x}} = \cancel{\int 2\,\frac{\frac{1}{2}(1+t^2)}{t^2}\,d\theta}$$

$$\cancel{= \int \frac{2\,\frac{1}{2}(1+t^2)\,d\theta}{\left(\frac{2t}{1+t^2}\right)^{-1}}} =$$

$$= \int \frac{2\overbrace{\frac{2\,dt}{1+t^2}}^{d\theta}}{\underbrace{\frac{2t}{1+t^2}}_{\sin\theta}} = \int \frac{2}{t}\,dt$$

$$= 2\ln t + C_2$$

USING RESULTS FROM 2 PAGES BEFORE (BOTTOM OF THAT PAGE)

$$2\ln t = \pm 2 z_1 + C_2$$

$$x\Big|_{z=0} = 0 \Rightarrow \theta = \frac{\pi}{2} \Rightarrow t = 1 \Rightarrow C_2 = 0$$

(IE NO POTENTIAL IN THE MIDDLE)

$$\Rightarrow t = e^{z_1}$$

WE HAD:

$$e^{x} = \sin^2\theta \Rightarrow \sin\theta = e^{x/2} = \frac{2t}{1+t^2} = \frac{2e^{z_1}}{1+e^{2z_1}}$$

$$= \frac{1}{\cosh z_1}$$

$$\Rightarrow x = 2\ln\left(\cosh z_1\right)^{-1} = +2\ln\cosh\left(\sqrt{\frac{2\pi G S_0}{A}}\, z\right)$$

WE SET $\Psi_0 = 0$ 3 PAGES BEFORE.

$$\Rightarrow \Psi = +2A\ln\cosh\left(\sqrt{\frac{2\pi G S_0}{A}}\, z\right)$$

USING WHAT WE'VE FOUND EARLIER, IE:

$$\rho = \rho_0 \exp\left[-\frac{\psi - \psi_0}{A}\right]$$

IF $\psi_0 = 0$:

$$= \rho_0 \exp\left(\ln \cosh^{-2}\left(\sqrt{\frac{2\pi G \rho_0}{A}}\, z\right)\right)$$

$$= \frac{\rho_0}{\cosh^{2}\left(\sqrt{\frac{2\pi G \rho_0}{A}}\, z\right)}$$

HEAVY RELIANCE ON NOTES.

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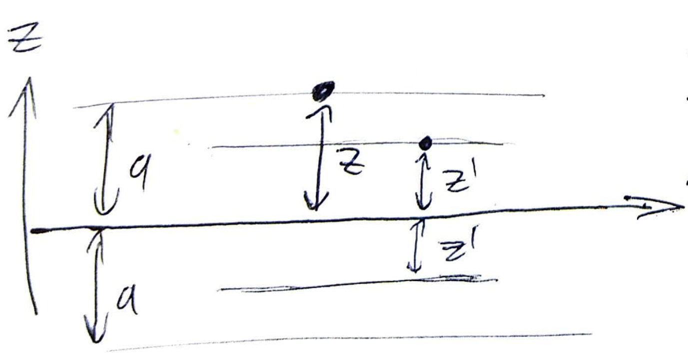
"IF A GALACTIC DISK" PART



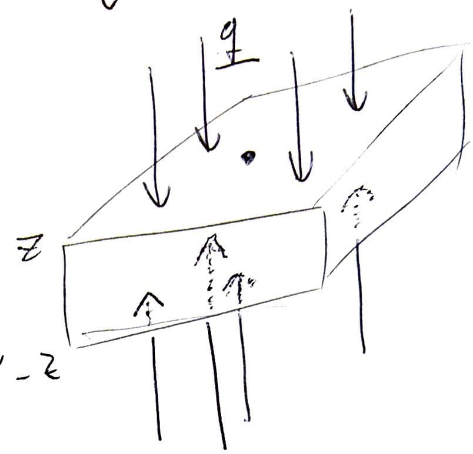

LETS APPROACH THIS QUESTION WITH A BIT OF ELECTROMAGNETISM MINDSET.
WE WANT TO KNOW FIELD SRENGTH AS A FUNCTION OF Z.

WHEN STAR IS $z'$ AWAY FROM MIDPLANE, THE ONLY MATTER MATTERS FOR US IS BETWEEN $-z'$ & $z'$!

GAUSS LAW:

$$\int \underline{g}\cdot d\underline{s} = -4\pi G\int_V \rho \, dV$$

CONSIDER A PART OF THE SLAB WITH AREA $A$:




$d\underline{s}$ POINTING OUTWARDS, $\underline{g}$ INWARDS

$$-|\underline{g}|\,2A = -4\pi G A\int_{-z}^{z}\rho(z)\,dz$$

UNIFORM DENSITY: $\rho(z)=\rho$

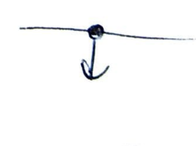
$$= -4\pi G A\, 2z\rho$$

$$|\underline{g}| = 4\pi G z\rho$$

$\underline{g}$ ALWAYS POINTS TOWARDS MID PLANE.

EQUATION OF MOTION:

$$\ddot{z} = \underbrace{-}_{\text{IF } z>0\text{, WE WANNA GO DOWNWARDS:}} 4\pi G z\rho$$



$$\ddot{z} + 4\pi G\rho z = 0$$

~~SHM~~ RECOGNIZE THAT THIS IS SHM, WITH $\omega^2 = 4\pi G\rho$

AT MIDPLANE, STAR HAS MAX SPEED.
MAX SPEED IN SHM: $\omega \cdot a$

$$v_{MAX} = \sqrt{4\pi G \rho}\, a$$

$$= \sqrt{4\pi G \cdot 10^{-18}}\; 10^{18} \approx 2.9 \cdot 10^{4} \frac{m}{s}$$

$$\approx \underline{10^{-4}\, c}$$

Believable.
(i.e. not absurdly low or absurdly high.)

$$\text{PERIOD} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{4\pi G \rho}} = 2.2 \cdot 10^{14}\, s$$

$\hookrightarrow \sim \underline{7 \cdot 10^{6}\, \text{YRS}}$

Seems realistic.