PQM3

Q1 (I) START FROM:

$|4,4\rangle = |3,3\rangle|1,1\rangle$

(A) TO GET $|4,3\rangle$, KEEP $\underline{J}^2$ THE SAME BUT LOWER J_z:

$$J_-|4,4\rangle = |4,3\rangle$$

$$= (J_-\otimes I + I\otimes J_-)|3,3\rangle|1,1\rangle$$

$$= (J_-|3,3\rangle)|1,1\rangle + |3,3\rangle(J_-|1,1\rangle)$$

$$= \sqrt{6}\,|3,2\rangle|1,1\rangle + \sqrt{2}\,|3,3\rangle|1,0\rangle$$

USING: $J_-|j,m\rangle = \sqrt{j(j+1)-m(m-1)}\,\hbar\,|j,m-1\rangle$

NORMALIZE:

$$|4,3\rangle = \sqrt{\frac{6}{8}}\,|3,2\rangle|1,1\rangle + \sqrt{\frac{2}{8}}\,|3,3\rangle|1,0\rangle$$

$$= \sqrt{\frac{3}{4}}\,|3,2\rangle|1,1\rangle + \sqrt{\frac{1}{4}}\,|3,3\rangle|1,0\rangle$$

$|3,3\rangle$: USE: $|j-1,j-1\rangle =$

$$= \sqrt{\frac{j_2}{j}}\,|j_1,j_1-1\rangle|j_2,j_2\rangle - \sqrt{\frac{j_1}{j}}\,|j_1,j_1\rangle|j_2,j_2-1\rangle$$

PQM3 | Q1 | (II)

$j = 4$; $j_1 = 3$ $j_2 = 1$

$$|3,3\rangle = \sqrt{\tfrac{1}{4}}\,|3,2\rangle|1,1\rangle - \sqrt{\tfrac{3}{4}}\,|3,3\rangle|1,0\rangle$$

LOWER $|3,3\rangle$ TO GET $|3,2\rangle$:

$$J_-|3,3\rangle = (J_-\otimes I + I\otimes J_-)\left(\sqrt{\tfrac{1}{4}}\,|3,2\rangle|1,1\rangle - \sqrt{\tfrac{3}{4}}\,|3,3\rangle|1,0\rangle\right)$$

PROCEED
TERM BY TERM:

- $J_-\otimes I\,|3,2\rangle|1,1\rangle =$

  $= \sqrt{3(3+1)-2(2-1)}\,|3,1\rangle|1,1\rangle = \underline{\sqrt{10}\,|3,1\rangle|1,1\rangle}$

- $J_-\otimes I\,|3,3\rangle|1,0\rangle =$

  $= \sqrt{3(3+1)-3(3-1)}\,|3,2\rangle|1,0\rangle = \underline{\sqrt{6}\,|3,2\rangle|1,0\rangle}$

- $I\otimes J_-\,|3,2\rangle|1,1\rangle =$

  $= \sqrt{1(1+1)-1(1-1)}\,|3,2\rangle|1,1\rangle = \underline{\sqrt{2}\,|3,2\rangle|1,0\rangle}$

- $I\otimes J_-\,|3,3\rangle|1,0\rangle =$

  $= \sqrt{1(1+1)-0(0-1)}\,|3,3\rangle|1,-1\rangle = \underline{\sqrt{2}\,|3,3\rangle|1,-1\rangle}$

PQM3 | Q1 | (III)

COMBINE TERMS:

$$|3,2\rangle = \sqrt{\frac{1}{4}}\left(\sqrt{10}\,|3,1\rangle|1,1\rangle + \sqrt{2}\,|3,2\rangle|1,0\rangle\right) +$$

$$-\sqrt{\frac{3}{4}}\left(\sqrt{6}\,|3,2\rangle|1,0\rangle + \sqrt{2}\,|3,3\rangle|1,-1\rangle\right)$$

SIMPLIFY:

$$= \sqrt{\frac{5}{2}}\,|3,1\rangle|1,1\rangle - \sqrt{2}\,|3,2\rangle|1,0\rangle - \sqrt{\frac{3}{2}}\,|3,3\rangle|1,-1\rangle$$

NORMALIZE:

$$\frac{5}{2} + 2 + \frac{3}{2} = 6 \rightarrow \text{DIVIDE COEFFS BY } \sqrt{6}.$$

$$\underline{|3,2\rangle = \sqrt{\frac{5}{12}}\,|3,1\rangle|1,1\rangle - \sqrt{\frac{1}{3}}\,|3,2\rangle|1,0\rangle - \sqrt{\frac{1}{4}}\,|3,3\rangle|1,-1\rangle}$$

~~FOR $|2,2\rangle$, USE AGAIN:~~

$$\cancel{|j-1,j-1\rangle = \sqrt{\frac{j_2}{j}}\,|j_1,j_1-1\rangle|j_2,j_2\rangle +}$$

$$\cancel{-\sqrt{\frac{j_1}{j}}\,|j_1,j_1\rangle|j_2,j_2-1\rangle}$$

$\cancel{j=3,\ j_1=2,\ j_2=1}$

~~SO WE END UP WITH:~~

PQM3 | Q1 | (IV)

$|2,2\rangle$

WE HAVE:

$$J_z J_z |j-2, j-2\rangle = j-2 |j-2, j-2\rangle$$

$$\cancel{= j_1 + j_2 |j-2, j-2\rangle}$$

$$= (j_1 + j_2 - 2) |j-2, j-2\rangle$$

WE ALSO HAVE:

$$J_- J_- |j, j\rangle = (J_- \otimes I + I \otimes J_-)(J_- \otimes I + I \otimes J_-) |j_1, j_1\rangle |j_2, j_2\rangle$$

$$\cancel{= (J_- \otimes I + I \otimes J_-) \left[\sqrt{2j_1} |j_1, j_1 - 1\rangle + \sqrt{2j_2} |j_2, j_2 - 1\rangle\right]}$$

$$\cancel{= (J_- \otimes I + I \otimes J_-) [\sqrt{2j}}$$

$$= (J_- \otimes I + I \otimes J_-) \left[\sqrt{2j_1} |j_1, j_1 - 1\rangle |j_2, j_2\rangle + \right.$$

$$\left. + \sqrt{2j_2} |j_1, j_1\rangle |j_2, j_2 - 1\rangle \right]$$

$$= \sqrt{2j_1} \sqrt{2j_1 - 2} |j_1, j_1 - 2\rangle |j_2, j_2\rangle +$$

$$+ \sqrt{2j_1} \sqrt{2j_2} |j_1, j_1 - 1\rangle |j_2, j_2 - 1\rangle +$$

$$+ \sqrt{2j_2} \sqrt{2j_1} |j_1, j_1 - 1\rangle |j_2, j_2 - 1\rangle$$

$$+ \sqrt{2j_2} \sqrt{2j_2 - 2} |j_1, j_1\rangle |j_2, j_2 - 2\rangle$$

PQM3 | Q1 | (V)

$$= \sqrt{2 j_1 (j_1 - 2)} \, |j_1, j_1 - 2\rangle |j_2, j_2\rangle$$

$$+ 2\sqrt{2} \sqrt{j_1 j_2} \, |j_1, j_1 - 1\rangle |j_2, j_2 - 1\rangle$$

$$+ \sqrt{2 j_2 (j_2 - 2)} \, |j_1, j_1\rangle |j_2, j_2 - 2\rangle$$

THIS EQUALS TO:

$$J_- J_- |j, j\rangle = J_- \sqrt{2j} \, |j, j-1\rangle$$

$$= \sqrt{2 j (j-2)} \, |j, j-2\rangle$$

SET UP EQUATIONS TO FIND CONSTANTS:

$$|j-2, j-2\rangle =$$

$$= A \, |j_1, j_1 - 2\rangle |j_2, j_2\rangle +$$

$$+ B \, |j_1, j_1 - 1\rangle |j_2, j_2 - 1\rangle +$$

$$+ C \, |j_1, j_1\rangle |j_2, j_2 - 2\rangle$$

NORMALIZATION:

$$|A|^2 + |B|^2 + |C|^2 = 1$$

ORTHOGONALITY:

$$\langle j, j-2 | j-2, j-2\rangle = 0$$

PQM3 | Q1 | VI

FROM TOP OF PREVIOUS PAGE, WE HAVE:

$$|j, j-2\rangle =$$

$$= \sqrt{\frac{j_1(j_1-2)}{j(j-2)}} |j_1, j_1-2\rangle |j_2, j_2\rangle +$$

$$+ \frac{2\sqrt{j_1 j_2}}{\sqrt{j(j-2)}} |j_1, j_1-1\rangle |j_2, j_2-1\rangle +$$

$$+ \sqrt{\frac{j_2(j_2-2)}{j(j-2)}} |j_1, j_1\rangle |j_2, j_2-2\rangle$$

USING ORTHOGONALITY (BOTTOM OF PREV. PAGE):

$$A \sqrt{\frac{j_1(j_1-2)}{j(j-2)}} + B \cdot 2 \cdot \sqrt{\frac{j_1 \cdot j_2}{j(j-2)}} + C \sqrt{\frac{j_2(j_2-2)}{j(j-2)}} = 0$$

$$/ \cdot \sqrt{j(j-2)}$$

$$A \sqrt{j_1(j_1-2)} + 2B \sqrt{j_1 j_2} + C \sqrt{j_2(j_2-2)} = 0$$

GUESS A SOLUTION:

$$A = \sqrt{j_2(j_2-2)} \qquad B = 0 \qquad C = -\sqrt{j_1(j_1-2)}$$

(WHAT IF THERE ARE OTHER SOLS THOUGH? THERE PROBABLY AREN'T)

PQM3 | Q1 | VII

NORMALIZATION:

$$j_2(j_2-2)+j_1(j_1-2)=1$$

$$j_2^2-2j_2+j_1^2-2j_1=1$$

LET'S NOT BOTHER WITH THIS NOW, DEAL WITH IT LATER.

WE HAVE:

$$|j-2,j-2\rangle=\sqrt{j_2(j_2-2)}\,|j_1,j_1-2\rangle|j_2,j_2\rangle+$$

$$-\sqrt{j_1(j_1-2)}\,|j_1,j_1\rangle|j_2,j_2-2\rangle$$

TO GET $|2,2\rangle$, WE HAVE: $j=4$, $j_1=3$, $j_2=1$

$$|2,2\rangle=\sqrt{1(1-2)}\,|j_1,j_1-2\rangle|j_2,j_2\rangle+$$

$$-\sqrt{3(3-2)}\,|j_1,j_1\rangle|j_2,j_2-2\rangle$$

$$=i\,|j_1,j_1-2\rangle|j_2,j_2\rangle$$

$$-\sqrt{3}\,|j_1,j_1\rangle|j_2,j_2-2\rangle$$

$$=i\,|3,1\rangle|1,1\rangle+\sqrt{3}\,|3,3\rangle|1,-1\rangle$$

THIS IS PROBABLY WRONG $\Rightarrow$ MOVE ON.

PQM3
Q2 (i)

i/ $\underline{\sigma = 2:}$

$$|2,2\rangle = |1,1\rangle|1,1\rangle$$

$\underline{\sigma = 1}$

$$|2,1\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle|1,1\rangle + |1,1\rangle|1,0\rangle\right)$$

OR

$$|1,1\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle|1,1\rangle - |1,1\rangle|1,0\rangle\right)$$

$\underline{\sigma = 0:}$

$$\cancel{|2,0\rangle = \frac{1}{\sqrt{6}}|1,-1\rangle + \sqrt{\frac{2}{3}}|0,0\rangle + \frac{1}{\sqrt{6}}}$$

$$|2,0\rangle = \frac{1}{\sqrt{6}}|1,1\rangle|1,-1\rangle + \sqrt{\frac{2}{3}}|1,0\rangle|1,0\rangle + \frac{1}{\sqrt{6}}|1,-1\rangle|1,1\rangle$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}|1,1\rangle|1,-1\rangle - \frac{1}{\sqrt{2}}|1,-1\rangle|1,1\rangle$$

$$|0,0\rangle = \frac{1}{\sqrt{3}}\left(|1,1\rangle|1,-1\rangle - |1,0\rangle|1,0\rangle + |1,-1\rangle|1,1\rangle\right)$$

TRIED TO USE WIKIPEDIA HERE ↑

UNSURE IF I HAVE A DEEP ENOUGH UNDERSTANDING OR NOT (PROBABLY NOT)

PQM 3
Q2 (II)

J=2 STATES:

$$|2,2\rangle = |1,1\rangle|1,1\rangle$$

$$|2,1\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle|1,1\rangle + |1,1\rangle|1,0\rangle\right)$$

$$|2,0\rangle = \frac{1}{\sqrt{6}}|1,1\rangle|1,-1\rangle + \sqrt{\frac{2}{3}}|1,0\rangle|1,0\rangle + \frac{1}{\sqrt{6}}|1,-1\rangle|1,1\rangle$$

I HAVE 2 SUBSYSTEMS ONLY, SO $|2,0\rangle$ IS OUT OF THE GAME BECAUSE I NEED 3 FOR THAT. (DO I?)

THIS SEEMS WRONG NOW

$$|2,2\rangle = |1,1\rangle|1,1\rangle$$

SYMMETRIC TO SWAP

$$|2,1\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle|1,1\rangle + |1,1\rangle|1,0\rangle\right)$$

SYMMETRIC TO SWAP.

~~J=1 STATE,~~ WHICH FORMS FROM A COMBINATION OF TWO SUBSYSTEMS:

$$|1,1\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle|1,1\rangle - |1,1\rangle|1,0\rangle\right)$$

THIS IS ANTISYMMETRIC TO SWAPPING THESE TWO:

[illegible]

PQM 3
Q2 (iii)

J=1 STATES

$$|1,1\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle|1,1\rangle - |1,1\rangle|1,0\rangle\right)$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|1,1\rangle|1,-1\rangle - |1,-1\rangle|1,1\rangle\right)$$

ANTISYMMETRIC THESE ARE, BECAUSE:

$$(\sigma_1 \leftrightarrow \sigma_2)|1,1\rangle =$$

$$= \frac{1}{\sqrt{2}}\left(|1,1\rangle|1,0\rangle - |1,0\rangle|1,1\rangle\right) = -|1,1\rangle$$

$$(\sigma_1 \leftrightarrow \sigma_2)|1,0\rangle =$$

$$= \frac{1}{\sqrt{2}}\left(|1,-1\rangle|1,1\rangle - |1,1\rangle|1,-1\rangle\right) = -|1,0\rangle$$

~~###~~

PQM 3
Q2 (IV)

"Show that $l+s$ must be even"

$\boxed{\text{If } s=1,}$ possible SPIN states of the combined system: $|1,1\rangle$ & $|1,0\rangle$

Both of these are antisymmetric, as we ~~the~~ have shown previously.

Spin 1 particles are bosons
$\Rightarrow$ combination of them is also boson
(dodgy reasoning)

$\Rightarrow$ combined system must be symmetric if we ~~(change)~~ exchange all quantum numbers

Spin state is antisymmetric $\Rightarrow$ spatial state is antisymmetric too.

~~Antisy~~ Antisymmetric spatial state $\Rightarrow l = \text{ODD}$
(recalling: $Y_l^m(-\hat{x}) = (-1)^l Y_l^m(\hat{x})$)

$s + l = 1 + \text{ODD NUMBER} = \underline{\text{EVEN}}$

PQM3
Q2 (v)

$\underline{s = \text{EVEN}}$

POSSIBLE SPIN STATES:

$|2,2\rangle \quad |2,1\rangle \quad |2,0\rangle \quad |0,0\rangle$

These are all symmetric to $\sigma_1 \leftrightarrow \sigma_2$ swap.

(For example:

$$(\sigma_1 \leftrightarrow \sigma_2)|0,0\rangle = \frac{1}{\sqrt{3}}\Big(|1,-1\rangle|1,1\rangle - |1,0\rangle|1,0\rangle + |1,1\rangle|1,-1\rangle\Big)$$
$$= |0,0\rangle\Big)$$

Spin wavefunction is symmetric.
Overall, we have to ~~have~~ have symmetry if we exchange all quantum numbers, since we are dealing with a boson.

$\Rightarrow$ Spatial wavefunction has to be symmetric $\Rightarrow l = \text{EVEN}$

$s + l = \text{EVEN} + \text{EVEN} = \underline{\underline{\text{EVEN}}}$

GOOD.

PQM 3
Q2 ($\boxed{VI}$)

IN GENERAL, $j \in \{ l+s, l+s-1, \dots |l-s| \}$

~~IF $j=1$, & $s=1$ (THAT IS THE CASE BECAUSE WE'RE DEALING W/ SPIN-1 PARTICLES~~

IF $j=1$, & IF $s=0$, $\Rightarrow l=1$

BUT THIS IS NO GOOD, SINCE $s+l$ HAS TO BE EVEN.

IF $j=1$, & IF $\boxed{s=1}$, $\Rightarrow l=0$ OR 1 OR 2

$l=0,2$ IS NOT ACCEPTABLE FOR $s+l=$ EVEN CONDITION.

$\Rightarrow \boxed{l=1}$ ~~EVEN~~ ($1+1=$ EVEN)

IF $j=1$ & IF $\boxed{s=2}$, $\Rightarrow l=0,1,2,3$

$s+l=$ EVEN CONDITION RESTRICTS OUR CHOICES TO:

$\boxed{l=0,2}$

SUMMARY: POSSIBLE l & s VALUES:

l	s
1	1
0,2	2

FQM3
Q4 (I)

$$H = \frac{\underline{P}^2}{2m} + V_0 + \frac{1}{2} m \omega^2 \underline{X}^2$$

GIVEN THAT:

$$A_i = \sqrt{\frac{m\omega}{2\hbar}}\, x_i + i \frac{p_i}{\sqrt{2m\hbar\omega}}$$

WE HAVE:

$$A_i^+ = \sqrt{\frac{m\omega}{2\hbar}}\, x_i - i \frac{p_i}{\sqrt{2m\hbar\omega}}$$

$$A_i^+ A_i = \frac{m\omega}{2\hbar} x_i^2 + \frac{p_i^2}{2m\hbar\omega} + \frac{1}{2\hbar} i \overbrace{[x_i\, p]}^{i\hbar}$$

$$\hbar\omega A_i^+ A_i = \frac{1}{2} m\omega x_i^2 + \frac{1}{2m} p_i^2 \cancel{} - \frac{1}{2}\hbar\omega$$

$$\hbar\omega \sum A_i^+ A_i = \frac{1}{2} m\omega \underline{X}^2 + \frac{1}{2m} \cancel{}\, \underline{P}^2 - \frac{3}{2}\hbar\omega$$

$$\Rightarrow H = \hbar\omega\left(\sum A_i^+ A_i + \frac{3}{2} \cancel{\hbar\omega}\right) + V_0$$

~~$A_i^+ A_i$ CAN BE THOUGHT OF AS A COUNTING~~
~~OPERATOR: IT RETURNS NUMBER OF~~
~~PARTICLES OSCILLATING IN~~
~~i^{th} DIRECTION~~

~~THIS REASONING IS QUITE BAD THOUGH, THIS IS~~
~~WHAT I'M TRYING TO PROVE~~

PQM3
Q4(II)

$A_i^+ A_i$ COUNTS ANGULAR MOMENTUM QUANTA IN THE i TH DIRECTION (IS THAT CORRECT?)

$$H|\psi\rangle = \left[\hbar\omega\left(\sum A_i^+ A_i + \frac{3}{2}\right) + V_0\right]|\psi\rangle$$
$$= \left[\hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right) + V_0\right]|\psi\rangle$$
$$= \left[\hbar\omega\left(n + \frac{3}{2}\right) + V_0\right]|\psi\rangle$$

$$n_x + n_y + n_z = n$$

IE n_x, n_y, n_z EXPRESSES THE DISTRIBUTION OF TOTAL ANGULAR MOM. ALONG x, y, z AXIS.

~~CAUSE FOR CONCERN: THE~~

DEGENERACY ARISES FROM:
MULTIPLE WAYS OF DISTRIBUTING n TO 3 DIFFERENT GROUPS.

~~n THING TO DISTRIBUTE TO 3 GROUPS~~

PQM3
Q4(III)

DISTRIBUTE N OBJECTS & 2 SEPARATORS, ORDER WITHIN OBJECTS & WITHIN SEPARATORS DOESN'T MATTER.

× × × × ×|× × ×|× × ×

$$DEG = \frac{(N+2)!}{N!\,2!} = \frac{(N+1)(N+2)}{2}$$