

• FIRST, LETS FND $F(x)$.

Q1

$$e^{-F(x)} = e^{-x} x^n$$

$$= e^{-x} e^{n \ln x}$$

$$= e^{-x} e^{n \ln x}$$

$$= e^{n \ln x - x} \Rightarrow F(x) = -n \ln x + x$$

• WHERE IS MINIMUM OF $F(x)$?

$$\frac{dF(x)}{dx} = -n \frac{1}{x} + 1 = 0 \Rightarrow \underbrace{x_0 = n}_{F(x) \text{ minimum}}$$

~~$F(x)$~~

$$F''(x) = n \frac{1}{x^2}$$

$$F(x) \approx F(x_0) + F''(x_0) \frac{(x-x_0)^2}{2}$$

$$\approx \underbrace{-n \ln n + n}_{\text{min}} + \frac{n}{n^2} \frac{(x-n)^2}{2}$$

$$\approx n(1 - \ln n) + \frac{1}{n} \frac{(x-n)^2}{2}$$

FOR $n \gg 1$, $\ln n \gg 1$, so we have:

$$x - n \ln n + \frac{1}{n} \frac{(x-n)^2}{2}$$

PUT THIS BACK TO OUR INTEGRAL:

$$n! = \int_0^\infty \exp\left(n \ln n - \frac{1}{n} \frac{(x-n)^2}{2}\right) dx$$

$$= \int_0^{\infty} n^n \cdot \exp\left(-\frac{1}{n} \frac{(x-n)^2}{2}\right) dx$$

AND NOW I DON'T KNOW WHAT.

CONJECTURE WE OBTAIN: $n! \sim \sqrt{2\pi n} \cdot n^n e^{-n}$

$$\ln n! \sim \ln \sqrt{2\pi} + \ln \sqrt{n} + \ln(n^n e^{-n})$$

$$\sim \frac{1}{2} \ln n + (\ln n^n) + \ln e^{-n}$$

$$\sim \frac{1}{2} \ln n - n + n \ln n$$

$$\sim \frac{1}{2} \ln n - n + n \ln n$$

$$\sim n \ln n - n$$

AS REQUIRED.

Q2 N PARTICLES

$$E \propto \sum_{i=1}^N (p_i^2 + q_i^2 + r_i^2)$$

WE WANT NUMBER OF
POSSIBLE p_i, q_i, r_i
WHICH MAKES

$$\sum_{i=1}^N (p_i^2 + q_i^2 + r_i^2) < E$$

CONSIDER:



1D CASE:

$$x^2 < E$$

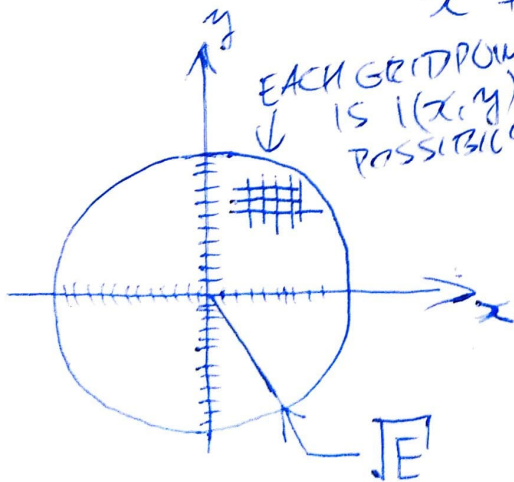
x INTEGER & $E \gg 1$

NUMBER OF POSSIBILITIES FOR x
IS PROPORTIONAL TO $E^{\frac{1}{2}}$

2D CASE:

$$x^2 + y^2 < E$$

EACH GRIDPOINT
IS (x, y)
POSSIBILITY



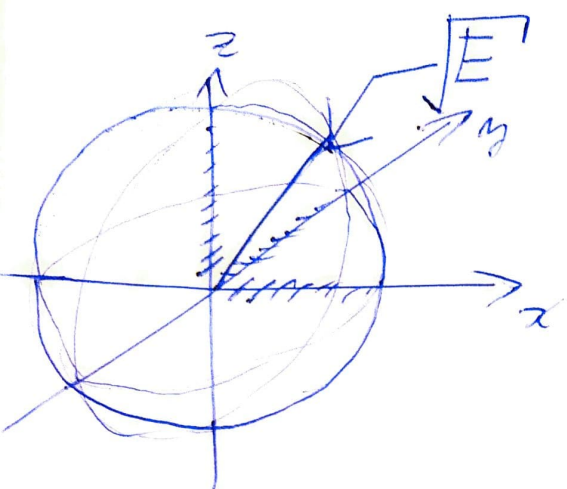
AREA OF CIRCLE \propto RADIUS²

$$\propto \sqrt{E}^2$$

$$\propto E$$

\Rightarrow NUMBER OF POSSIBILITIES $\propto E$

3D CASE:



$$x^2 + y^2 + z^2 < E$$

~~AREA~~

VOLUME OF SPHERE \propto

NUMBER OF POSSIBILITIES
FOR x, y, z

VOLUME OF SPHERE \propto

$$\text{RADIUS}^3 \propto \sqrt{E}^3 \propto E^{\frac{3}{2}}$$

OBSERVATION: NUMBER OF POSSIBILITIES SCALES AS

$$E^{\frac{\text{NUMBER OF DIMENSIONS}}{2}}$$

WE HAVE:

$$\sum_{i=1}^N (p_i^2 + q_i^2 + r_i^2) < E$$

IE A ~~3D~~ 3N DIMENSIONAL CASE.

SO WE HOPE, BASED ON THE ABOVE:

$$G(E) \propto E^{\frac{3N}{2}} \Rightarrow \alpha = \frac{3}{2}$$

Q3 (i)

$$E = \text{const.} \Rightarrow \Omega(E) = \text{const.}$$

$$\frac{d\Omega(E_1)}{dE_1} \propto \frac{d}{dE_1} \left[\exp(S_1(E_1)) + \exp(S_2(E - E_1)) \right]$$

$$\propto e^{S_1(E_1)} e^{S_2(E - E_1)} \left(\frac{dS_1(E_1)}{dE_1} + \frac{dS_2(E - E_1)}{dE_1} \right) = 0$$

$$\Rightarrow \frac{dS_1(E_1)}{dE_1} + \frac{dS_2(E - E_1)}{dE_1} = 0$$

$$\frac{d}{dE_1} \ln \Omega_1(E_1) + \frac{d}{dE_1} \ln \Omega_2(E - E_1) = 0$$

$$\frac{d}{dE_1} \ln \left(C_1 E_1^{\alpha_1 N_1} \right) + \frac{d}{dE_1} \ln \left(C_2 (E - E_1)^{\alpha_2 N_2} \right) = 0$$

$$\frac{1}{C_1 E_1^{\alpha_1 N_1}} \cdot \alpha_1 N_1 E_1^{\alpha_1 N_1 - 1} + \frac{1}{C_2 (E - E_1)^{\alpha_2 N_2}} \cdot C_2^{\alpha_2 N_2} (E - E_1)^{\alpha_2 N_2 - 1} \cdot \frac{d}{dE_1} (E - E_1) = 0$$

$$\alpha_1 N_1 E_1^{-1} - \alpha_2 N_2 (E - E_1)^{-1} = 0$$

$$\alpha_1 N_1 (E - E_1) - \alpha_2 N_2 E_1 = 0$$

$$-(\alpha_1 N_1 + \alpha_2 N_2) E_1 + \alpha_1 N_1 E = 0$$

$$\Rightarrow E_1^* = \alpha_1 N_1 E / (\alpha_1 N_1 + \alpha_2 N_2)$$

Q3(ii)

THIS BIT IS ZERO,
FROM PREVIOUS PAGE'S
RESULTS

$$\begin{aligned}
 & S_1(E_1) + S_2(E - E_1) = \\
 & = S_1(E_1^*) + S_2(E_1^* - E_1) + \frac{d}{dE_1} \left(S_1(E_1) + S_2(E - E_1) \right) \bigg|_{E_1^*} (E_1^* - E_1) \\
 & \quad + \frac{1}{2} \frac{d^2}{dE_1^2} \left(S_1(E_1) + S_2(E - E_1) \right) \bigg|_{E_1^*} (E_1^* - E_1)^2
 \end{aligned}$$

$$= S_1(E_1^*) + S_2(E_1^* - E_1) + \frac{1}{2} \frac{d^2}{dE_1^2} \left(S_1(E_1) + S_2(E - E_1) \right) \bigg|_{E_1^*} (E_1^* - E_1)^2$$

$$\propto \ln S_1(E_1^*) + \ln S_2(E_1^* - E_1) +$$

$$\left(\frac{1}{2} \frac{d^2}{dE_1^2} \left(S_1(E_1) + S_2(E - E_1) \right) \right) \bigg|_{E_1^*} (E_1^* - E_1)^2$$

$$+ \frac{1}{2} \frac{d^2}{dE_1^2} \left(\ln S_1(E_1) + \ln S_2(E - E_1) \right) \bigg|_{E_1^*} (E_1^* - E_1)^2$$

$$\propto \ln S_1(E_1^*) + \ln S_2(E_1^* - E_1) + \frac{1}{2} \frac{d^2}{dE_1^2} \left(\ln S_1(E_1) \right) \bigg|_{E_1^*} (E_1^* - E_1)^2 +$$

$$+ \frac{1}{2} \frac{d^2}{dE_1^2} \left(\ln S_2(E - E_1) \right) \bigg|_{E_1^*} (E - E_1^*)^2$$

EVALUATE SECOND DERIVATIVE TERMS SEPARATELY:

CONSIDER:

$$\frac{d^2}{dE_1^2} \left(h \Omega_1(E_1) \right) = \frac{d}{dE_1} \left(\frac{1}{\Omega_1(E_1)} \frac{d\Omega_1(E_1)}{dE_1} \right)$$

$$= -\frac{1}{\Omega_1(E_1)^2} \frac{d\Omega_1(E_1)}{dE_1} + \frac{1}{\Omega_1(E_1)} \frac{d^2 \Omega_1(E_1)}{dE_1^2}$$

~~Also we have~~

$$= -\frac{1}{c_1^2 E_1^{2\alpha_1 N_1}} \frac{d}{dE_1} \left(c_1 E_1^{\alpha_1 N_1} \right) + \frac{1}{c_1 E_1^{\alpha_1 N_1}} \frac{d^2}{dE_1^2} \left(c_1 E_1^{\alpha_1 N_1} \right)$$

$$= -\frac{1}{c_1^2 E_1^{2\alpha_1 N_1}} c_1 \alpha_1 N_1 E_1^{\alpha_1 N_1 - 1} +$$

$$+ \frac{1}{c_1 E_1^{\alpha_1 N_1}} c_1 \alpha_1 N_1 (\alpha_1 N_1 - 1) E_1^{\alpha_1 N_1 - 2}$$

$$= -\frac{1}{c_1} \alpha_1 N_1 E_1^{-\alpha_1 N_1 - 1} + \alpha_1 N_1 (\alpha_1 N_1 - 1) (E_1)^{-2}$$

WE ALSO HAVE:

$$\frac{d^2}{dE_1^2} \ln \Sigma_2(E-E_1) = \frac{d}{dE_1} \left(\frac{1}{\Sigma_2(E-E_1)} \frac{d\Sigma_2(E-E_1)}{dE_1} \right)$$

$$= \left(-\frac{1}{\Sigma_2(E-E_1)^2} \frac{d\Sigma_2(E-E_1)}{dE_1} + \frac{1}{\Sigma_2(E-E_1)} \frac{d^2\Sigma_2(E-E_1)}{dE_1^2} \right)$$

$$\cdot \frac{d}{dE_1} (E-E_1)$$

$$= + \frac{1}{C_2^2(E-E_1)^{2\alpha_2 N_2}} \frac{d}{dE_1} \left(C_2(E-E_1)^{\alpha_2 N_2} \right)$$

$$- \frac{1}{C_2(E-E_1)^{\alpha_2 N_2}} \frac{d^2}{dE_1^2} \left(C_2(E-E_1)^{\alpha_2 N_2} \right)$$

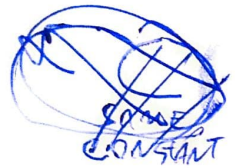
$$= \frac{1}{C_2^2(E-E_1)^{2\alpha_2 N_2}} C_2 \alpha_2 N_2 (E-E_1)^{\alpha_2 N_2 - 1} \cdot (-1)$$

$$- \frac{1}{C_2(E-E_1)^{\alpha_2 N_2}} C_2 \alpha_2 N_2 (\alpha_2 N_2 - 1) (E-E_1)^{\alpha_2 N_2 - 2}$$

$$= \frac{1}{C_2} \alpha_2 N_2 (E - E_1)^{-\alpha_2 N_2 - 1} + \alpha_2 N_2 (\alpha_2 N_2 - 1) (E - E_1)^{-2}$$

PUT THESE TERMS BACK TO OUR EXPANSION:

$$S_1(E_1) + S_2(E - E_1) \propto$$



$$\propto \ln S_1(E_1^*) + \ln S_2(E_1^* - E_1) +$$

$$+ \frac{1}{2} \left(-\frac{1}{C_1} \alpha_1 N_1 E_1^{-\alpha_1 N_1 - 1} + \alpha_1 N_1 (\alpha_1 N_1 - 1) E_1^{-2} \right) (E_1 - E_1^*)^2 +$$

$$+ \frac{1}{2} \left(\frac{1}{C_2} \alpha_2 N_2 (E - E_1)^{-\alpha_2 N_2 - 1} + \alpha_2 N_2 (\alpha_2 N_2 - 1) (E - E_1)^{-2} \right) (E_1 - E_1^*)^2$$

AND NOW I DON'T KNOW WHAT.

Q4

(i)

ENTROPY IS MAXIMIZED IF

$$\frac{\partial S}{\partial E} = 0 \quad \& \quad \frac{\partial^2 S}{\partial E^2} < 0$$

ENTROPY CHANGE WHEN SEPARATED SYSTEMS CONNECTED:

$$\begin{aligned} \delta S &= \frac{\partial S_1(E_1)}{\partial E} \delta E_1 + \frac{\partial S_2(E_2)}{\partial E} \delta E_2 \\ &= \left(\frac{\partial S_1(E_1)}{\partial E} - \frac{\partial S_2(E_2)}{\partial E} \right) \delta E_1 \quad \begin{array}{l} \text{USING} \\ \delta E_2 = -\delta E_1 \end{array} \\ &= \left(\frac{1}{T_1} - \frac{1}{T_2} \right) \delta E_1 \quad \begin{array}{l} \text{USING} \\ \text{TEMP.} \\ \text{DEFINITION} \end{array} \end{aligned}$$

$$\text{IF } T_1 = T_2 \Rightarrow \frac{\delta S}{\delta E_1} = 0$$

CONSIDER:

$$\frac{\partial^2 S}{\partial E^2} = \frac{\partial}{\partial E} \left(\frac{\partial S}{\partial E} \right) = \frac{\partial T}{\partial E} \frac{\partial}{\partial T} \left(\frac{\partial S}{\partial E} \right)$$

$$= \frac{1}{C_V} \frac{\partial}{\partial T} \frac{1}{T} = -\frac{1}{C_V T^2} < 0$$

$\Rightarrow C_V$ MUST BE POSITIVE

Q4

$$Z = \sum_n e^{-\beta E_n}$$

$$\frac{\partial Z}{\partial \beta} = \sum_n (-E_n) e^{-\beta E_n}$$

(ii)

$$\langle E \rangle = \sum_n \underbrace{\frac{1}{Z} e^{-\beta E_n}}_{p(n)} E_n = \frac{1}{Z} (-1) \frac{\partial Z}{\partial \beta} = - \frac{\partial}{\partial \beta} \ln Z$$

$$\langle E^2 \rangle = \sum_n \frac{1}{Z} e^{-\beta E_n} E_n^2 = \frac{1}{Z} \frac{\partial^2}{\partial \beta^2} Z$$

~~$$= \frac{1}{Z} \frac{\partial}{\partial \beta} \left(\beta \frac{1}{\beta} \frac{\partial Z}{\partial \beta} \right)$$~~

~~$$= \frac{1}{Z} \frac{1}{\beta} \frac{\partial Z}{\partial \beta} + \frac{\beta}{Z} \frac{\partial}{\partial \beta} \left(\frac{1}{\beta} \frac{\partial Z}{\partial \beta} \right)$$~~

~~$$= \frac{1}{Z} \frac{1}{\beta} \frac{\partial Z}{\partial \beta} + \frac{\beta}{Z} \left(-\frac{1}{\beta^2} \right) \frac{\partial Z}{\partial \beta} + \frac{\beta}{Z} \frac{1}{\beta} \frac{\partial^2 Z}{\partial \beta^2}$$~~

REVERSE-ENGINEERING THE RESULT:

$$\begin{aligned} \frac{\partial^2}{\partial \beta^2} \ln Z + \left(\frac{\partial}{\partial \beta} \ln Z \right)^2 &= \\ &= \frac{\partial}{\partial \beta} \left(\frac{1}{Z} \frac{\partial Z}{\partial \beta} \right) + \left(\frac{1}{Z} \frac{\partial Z}{\partial \beta} \right)^2 \\ &= \left(\frac{\partial}{\partial \beta} \frac{1}{Z} \right) \frac{\partial Z}{\partial \beta} + \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2} + \left(\frac{1}{Z} \frac{\partial Z}{\partial \beta} \right)^2 \\ &= -\frac{1}{Z^2} \left(\frac{\partial Z}{\partial \beta} \right)^2 + \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2} + \left(\frac{1}{Z} \frac{\partial Z}{\partial \beta} \right)^2 = \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2} \end{aligned}$$

IE WHAT WE HAVE.

HOW CAN I GO THIS KNOWING WAY WITHOUT WHAT I'M ASKED TO GET?

I CAN GO THIS DIR.

$$\Rightarrow \langle E^2 \rangle = \frac{\partial^2}{\partial \beta^2} \ln Z + \left(\frac{\partial}{\partial \beta} \ln Z \right)^2$$

$$(\Delta E)^2 = \langle E^2 \rangle - \langle E \rangle^2$$

$$= \frac{\partial^2}{\partial \beta^2} \ln Z + \left(\frac{\partial}{\partial \beta} \ln Z \right)^2 - \left(-\frac{\partial}{\partial \beta} \ln Z \right)^2$$

$$= \underline{\underline{\frac{\partial^2}{\partial \beta^2} \ln Z}}$$

$$C_V = \frac{\partial \langle E \rangle}{\partial T} = \frac{\partial}{\partial T} \left(-\frac{\partial}{\partial \beta} \ln Z \right)$$

$$\text{USING: } \beta = \frac{1}{k_B T} \Rightarrow \frac{\partial}{\partial T} = \frac{\partial \beta}{\partial T} \frac{\partial}{\partial \beta} = \frac{\partial \frac{1}{k_B T}}{\partial T} \frac{\partial}{\partial \beta}$$



$$= -\frac{1}{k_B} T^{-2} \frac{\partial}{\partial \beta}$$

REWRITE EXPRESSION FOR C_V :

$$C_V = -\frac{1}{k_B T^2} \frac{\partial}{\partial \beta} \left(-\frac{\partial}{\partial \beta} \ln Z \right) = \frac{1}{k_B T^2} \frac{\partial^2}{\partial \beta^2} \ln Z$$

$$= \frac{1}{k_B T^2} (\Delta E)^2$$

$$\Rightarrow \underline{\underline{(\Delta E)^2 = C_V k_B T^2 \propto C_V}} \quad \text{AS REQUIRED.}$$

Q6

(a)(i)

"HOW MANY STATES" ...

$$N \uparrow C_N \text{ i.e.}$$

$$\frac{\cancel{N!}}{N! (N-N!)!}$$

"EXPRESS THE ENERGY OF SUCH A STATE" ...

$$E = N \uparrow \epsilon$$

where ϵ is the difference in energy of spin up & down states

$$S(E) = k_B \ln \Omega(E) = k_B \ln \frac{\cancel{N!}}{N! (N-N!)!}$$

FOR LARGE ~~N~~
 $N! : N! \sim N \ln N - N$

~~$$S(E) \sim k_B (\ln N! - \ln N!)$$~~

$$\begin{aligned} S(E) &\sim k_B (N \ln N - N - N \ln N + N \\ &\quad - (N - N) \ln (N - N) + (N - N)) \\ &\sim k_B (N \ln N - N \ln N - (N - N) \ln (N - N)) \\ &\sim k_B (N - N) \end{aligned}$$

$$\sim \mathcal{E}_B \left(-N \uparrow \ln N \uparrow + N \uparrow \ln N + (N - N \uparrow) \ln N - (N - N \uparrow) \ln (N - N \uparrow) \right)$$

$$\sim \mathcal{E}_B \left(-(N - N \uparrow) \ln \frac{N - N \uparrow}{N} + N \uparrow \ln \frac{N}{N \uparrow} \right)$$

$$\sim -\mathcal{E}_B \left((N - N \uparrow) \ln \frac{N - N \uparrow}{N} + N \uparrow \ln \frac{N \uparrow}{N} \right)$$
