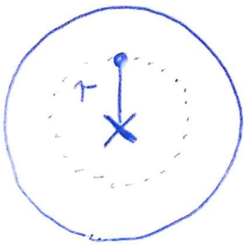
SDSG2

2.1.

• POTENTIAL FROM PART OF SPHERE $> r$

NEWTON'S FIRST THM: NO FORCE DUE TO EXTERNAL SPHERICALLY SYMMETRIC MASS DISTRIBUTION



$\Rightarrow \Phi =$ CONSTANT CONTRIB. FROM THIS BIT.

• POTENTIAL FROM INSIDE SPHERE:

NEWTON SECOND THM: TREAT INSIDE AS POINT MASS.

$$f = -\nabla\Phi \Rightarrow \Phi = -\int_{\infty}^{r} f\,dr$$

$$= -\int_{\infty}^{r} \frac{m_{<r'}G}{r'^2}\,dr'$$

$$= -\int_{\infty}^{r} \frac{4}{3} r'^3 \pi S \frac{1}{r'^2} G\,dr' = \int_{\infty}^{r} \frac{4}{3}\pi S G r'\,dr' = \frac{4}{3}\pi S G \left[\frac{r'^2}{2}\right]_{\infty}^{r}$$

$$= \frac{2}{3}\pi S G r^2 = \underline{\frac{1}{2}\Omega^2 r^2 + C}$$

WHERE $\Omega^2 = \frac{4}{3}\pi S G$

THE NOTATION Ω SUGGESTS THAT $\sqrt{\frac{4}{3}\pi S G}$ SHOULD HAVE DIMENSIONS $\sec^{-1}$

CHECK: $\sqrt{\frac{kg}{m^3} N \frac{m^2}{kg^2}} = \sqrt{\frac{N}{m\,kg}} = \sqrt{\frac{kg \frac{m}{s^2}}{m\,kg}} = \frac{1}{\sec}$. GOOD.

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2.2 BASED ON LECTURE HANDOUT 3, PAGE 6:

THE PART WHICH WORKS

UNSTABLE IF: $\frac{d}{dR}(R^3 f) > 0$

RECALL: FOR CIRCULAR ORBIT: $-R\dot{\phi}^2 = f$

$$\frac{d}{dR}\left(R^3(-R\dot{\phi}^2)\right) > 0$$

$$\frac{d}{dR}\left(R^4\dot{\phi}^2\right) < 0$$

RENAME $R \to r$
USE: $h = r^2\dot{\phi}$

$$\frac{d}{dr}\left(h^2(r)\right) < 0$$

AS REQUIRED.

ARISING QUESTIONS

- WHY TAKING SECOND DERIVATIVE OF $U_{EFFECTIVE}$ DOESN'T WORK? IE

$$U_{EFF}(r) = U(r) + \frac{h^2}{2r^2}$$

$$\frac{dU_{EFF}}{dr} = \frac{dU(r)}{dr} + \frac{h}{r^2}h' - h^2 r^{-3}$$

CONSIDER:

$$\frac{dU(r)}{dr} = -f(r) = r\dot{\phi}^2 - \underbrace{\cancel{\ddot{r}}}_{\to 0 \text{ FOR CIRCULAR}} = r\dot{\phi}^2$$

$$\frac{dU_{EFF}}{dr} = r\dot{\phi}^2 + \underbrace{\cancel{\frac{hh'}{r^2}}}_{\to 0 \text{ BEC } h'=0} - h^2 r^{-3}$$

ORBIT IS WHERE $\frac{dU_{EFF}}{dr} = 0$, IE $r\dot{\phi}^2 - h^2 r^{-3} = 0$

$$\dot{\phi}^2 = h^2 r^{-4}$$

$$h^2 = \dot{\phi}^2 r^4 \quad \text{(AS EXPECTED)}$$

$$\frac{dU_{EFF}}{dr} = r^{-3}h^2 - h^2 r^{-3} = 0$$

OH I SEE. SHOULDN'T THROW OUT THE $\frac{hh'}{r^2}$ TERM BEFORE TAKING SECOND DERIVATIVE. IS THE METHOD CORRECT THOUGH IN PRINCIP?

- ~~ISN'T $(h^2(r))'$ NEVER SA~~
  IS $\frac{d}{dr}(h^2) < 0$ EVER SATISFIED? $\frac{d}{dr}(h^2) = 2hh'$. h IS CONSTANT SO $h' = 0 \Rightarrow \frac{d}{dr}(h^2) = 0$ IF ANGULAR MOMENTUM IS CONSERVED IE ALWAYS: SO HOW ARE THERE UNSTABLE ORBITS?

SDSG2

2.3

RECALL ORBIT EQ.:

$$\frac{d^2u}{d\varphi^2} + u = -\frac{f_r}{h^2 u^2} \qquad \text{with } u = \frac{1}{r}$$

$$h = r^2\dot\varphi$$

$$u = \frac{1}{r} = a^{-1}\exp(-b\varphi)$$

$$\frac{1}{u^2h^2} = \frac{1}{u^2 u^{-4}\dot\varphi^2} = u^2\dot\varphi^{-2}$$

$$\frac{b^2}{a}\exp(-b\varphi) + \frac{1}{a}\exp(-b\varphi) = \nabla\Phi\, u^2\frac{1}{\dot\varphi^2}$$

$$\frac{1}{a}(b^2+1)\exp(-b\varphi) = \nabla\Phi\,\frac{u^2}{\dot\varphi^2}$$

$$= \nabla\Phi\,\frac{1}{\dot\varphi^2}\,a^{-2}\exp(-2b\varphi)$$

$$\underbrace{\dot\varphi^2}\, a(b^2+1)\exp(b\varphi) = \nabla\Phi$$

$$\frac{h^2}{r^4}\,\overbrace{a(b^2+1)\exp(b\varphi)}^{r} = \nabla\Phi$$

$$h^2(b^2+1)r^{-3} = \nabla\Phi$$

$$\Rightarrow \underline{\underline{\Phi \propto \frac{1}{r^2}}}$$

CONSTANT OF PROPORTIONALITY CAN BE OBTAINED IF WE KNOW h OF THE ORBIT AS WELL, CORRECT?

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2.4

LECTURE NOTES I, EQUATION 1.56 : $u^2 = 2\dfrac{E-\Phi}{h^2}$

$$\Rightarrow \frac{d^2\Phi}{du^2} = \frac{d^2}{du^2}\left(E - \frac{1}{2}u^2h^2\right) = -h^2$$

$$\rho(\underline{r}) > 0 \ \forall \underline{r} \Rightarrow \Phi < 0 \ \forall u$$

$$\left(\text{USING: } \Phi(\underline{r}) = -\int \frac{G\rho(\underline{r}')d^3\underline{r}'}{|\underline{r}-\underline{r}'|}\right)$$

WANT TO PROVE: THERE ARE 0 OR 2 ROOTS.
IE DISPROVE THAT THERE CAN BE 1 ROOT.

1 ROOT IF: $E - \Phi = 0$

$\dfrac{d^2\Phi}{du^2} = -h^2 \Rightarrow \dfrac{d\Phi}{du}$ IS DECREASING AS u INCREASES

$\Rightarrow \Phi$ IS CONCAVE. (AND THEN WHAT?)

$E = \underbrace{KE}_{\text{KINETIC ENERGY}} + \Phi$

$KE > 0$
(IE PARTICLE IN CENTRAL FORCE FIELD DOES NOT STAY STATIONARY)

$\Rightarrow E > \Phi \Rightarrow E - \Phi > 0$

$\Downarrow$

$E - \Phi \neq 0$

$\Downarrow$

CANNOT BE 1 ROOT

$E - \Phi > 0 \Rightarrow u^2 = \dfrac{2}{h^2}(E-\Phi)$ <u>MUST</u> HAVE 2 ROOTS.

THIS IS NOT WHAT WE WERE SUPPOSED TO BE PROVING.

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2.5 (I) $\nabla^2 \Phi = 4\pi G \rho$

$$\text{TOTAL MASS} = \int_{\text{ALL SPACE}} \rho \, dV = \int_{\text{ALL SPACE}} \frac{1}{4\pi G} \nabla^2 \Phi \, dV$$

$$= \int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{1}{4\pi G} \nabla^2 \Phi \; r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= \frac{1}{G} \int_0^{\infty} \nabla^2 \Phi \, r^2 \, dr = \frac{1}{G} \int_0^{\infty} \frac{1}{\cancel{r^2}} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) \cancel{r^2} \, dr$$

$$= \int_0^{\infty} \frac{\partial}{\partial r} \left[r^2 \frac{\partial}{\partial r} \left(\frac{-M}{b + \sqrt{b^2 + r^2}} \right) \right] dr$$

$$= M \int_0^{\infty} -\frac{\partial}{\partial r} \left[r^2 \frac{\partial}{\partial r} \left(\frac{1}{b + \sqrt{b^2 + r^2}} \right) \right] dr$$

- TRUST OURSELVES THAT WE'RE GOOD IN SETTING UP INTEGRALS

OR

- BRUTE FORCE ALGEBRA

OR

- PUT IT INTO COMPUTER (THIS WAS MY CHOICE)

ARRIVE AT

$= M$

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2.5(II)

$$\rho(0) = \frac{1}{4\pi G} \nabla^2 \Phi \Big|_{r=0}$$

$$= \frac{1}{4\pi \cancel{G}} \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\left(\frac{-\cancel{G}M}{b+\sqrt{b^2+r^2}} \right)\right)\Bigg|_{r=0}$$

$$= \frac{3M}{16\pi b^3}$$

[DONE BY COMPUTER, RESULT CAN BE VERIFIED ON DESMOS: DESMOS.COM/CALCULATOR/ECNFBTM4MC]

$$\rho(r \gg b) = \frac{1}{4\pi G} \nabla^2 \Phi \Big|_{r \gg b}$$

~~USING: $b+\sqrt{b^2+r^2} \approx r$ WHEN $r \gg b$~~

$$\cancel{\approx \frac{1}{4\pi} \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\left(\frac{-M}{r} \right)\right)\Big|_{r \gg b}}$$

$$= \frac{1}{4\pi} \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\left(\frac{-M}{b+\sqrt{b^2+r^2}} \right)\right)$$

$$= \frac{bM}{2\pi r^4}$$

[VERIFIED BY DESMOS] (LINK ABOVE ↑)

SDSG2

2.7.

(I)

HANDOUT 3 SLIDE 22: $\kappa^2 = \left.\frac{\partial^2 \Phi}{\partial R^2}\right|_{(R_c,0)} + \frac{3L_z^2}{R_c^4}$

EQUATION 3.11 $\Rightarrow \frac{L_z^2}{R_c^4} = \dot{\phi}^2 = \Omega^2$

$\cancel{\Rightarrow \kappa^2 = 3\Omega_0^2} \Rightarrow \kappa^2 = 3\Omega_0^2 + \left.\frac{\partial^2 \Phi}{\partial R^2}\right|_{(R_c,0)}$

HANDOUT 3 SLIDE 22: $\nu^2 = \left.\frac{\partial^2 \Phi}{\partial z^2}\right|_{(R_c,0)}$

$\cancel{(\kappa^2+\nu^2) = \Omega^2 \left.\frac{\partial^2 \Phi}{\partial R^2}\right|_{(R_c,0)}}$

$$\kappa^2 + \nu^2 = \left.\frac{\partial^2 \Phi}{\partial R^2}\right|_{(R_c,0)} + \left.\frac{\partial^2 \Phi}{\partial z^2}\right|_{(R_c,0)} + \frac{3L_z^2}{R_c^4}$$

$$= \partial_R^2 \Phi\Big|_{(R_c,0)} + \partial_z^2 \Phi\Big|_{(R_c,0)} + 3\Omega_0^2$$

THIS IS SUPPOSED TO BE EQUAL TO:

$= 2\Omega_0^2 \Rightarrow$ WE WANT TO PROVE:

$$\partial_R^2 \Phi\Big|_{(R_c,0)} + \partial_z^2 \Phi\Big|_{(R_c,0)} = -\Omega_0^2$$

$\cancel{\Phi(R,z) = \Phi(R,-z) \Rightarrow \partial_z^2 \Phi\big|_{(R_c,0)} = 0}$

~~WE HAVE:~~

$\cancel{\partial_R^2 \Phi\big|_{(R_c,0)} = -\Omega_0^2}$

$\cancel{\partial_R \Phi\big|_{(R_c,0)} = -\Omega_0^2 R + C}$

SDSG2

2.7(II)

HANDOUT 2, EQUATION 2.3:

$$\nabla^2 \Phi(\underline{x}) = 4\pi G \rho(\underline{x}) = 0$$

PROBLEM TELLS US THAT $\rho(\underline{x}) = 0$

IN CPC:

$$\nabla^2 \Phi = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \Phi}{\partial R}\right) + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\Rightarrow -\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \Phi}{\partial R}\right) = \frac{\partial^2 \Phi}{\partial z^2}$$

RECALL, FOR CIRCULAR ORBITS:

$$\ddot{R} - R\dot{\varphi}^2\Big|_{R=\text{CONST}} = -R_c \Omega_0^2 = -\frac{\partial \Phi}{\partial R}\Big|_{(R_c,0)}$$

SUBSTITUTE:

$$-\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \Phi}{\partial R}\right)\Bigg|_{R=R_c} = -\frac{1}{R}\frac{\partial}{\partial R}\left(R^2 \Omega_0^2\right)\Bigg|_{(R_c,0)}$$

$$= -\frac{1}{R}\left(2R\Omega_0^2\right)\Bigg|_{(R_c,0)} = -2\Omega_0^2$$

$$\Rightarrow \frac{\partial^2 \Phi}{\partial z^2}\Bigg|_{(R_c,0)} = -2\Omega_0^2\Big|_{(R_c,0)} = -2\Omega_0^2$$

WE WANTED TO PROVE:

$$\partial_R^2 \Phi\Big|_{(R_c,0)} + \partial_z^2 \Phi\Big|_{(R_c,0)} = -\Omega_0^2$$

TRANSFORMED IT INTO:

$$\partial_R^2 \Phi\Big|_{(R_c,0)} - 2\Omega_0^2 = -\Omega_0^2$$

SDSGZ
2.7 III

$$\Rightarrow \partial_R^2 \Phi \Big|_{(R_c,0)} = \Omega_0^2$$

THIS IS WHAT WE WANT TO PROVE NOW.

$$/ \int dR$$

$$\partial_R \Phi \Big|_{(R_c,0)} = \Omega_0^2 R + C = -\underbrace{f}_{\text{FORCE ON PARTICLE.}}$$

AT $R=0$, $z=0$, $f=0$ (SYMMETRY) $\Rightarrow C=0$

$$\Rightarrow \partial_R \Phi \Big|_{(R_c,0)} = \Omega_0^2 R$$

& CONSIDERING THAT THE EQUATION:

$$\ddot{R} - R\dot{\phi}^2 \Big|_{R=R_c} = -\frac{\partial \Phi}{\partial R}\Big|_{R=R_c}$$

IN OUR CASE IS:

$$-R\dot{\phi}^2 \Big|_{\substack{R=R_c \\ z=0}} = -\partial_R \Phi \Big|_{\substack{R=R_c \\ z=0}}$$

$$\Downarrow$$

$$R\Omega_0^2 = \partial_R \Phi \Big|_{(R_c,0)}$$ AS DESIRED.

~~SO WHAT WE WANTED TO BE IS TRUE~~

$$\Rightarrow \kappa^2 + \nu^2 = 2\Omega_0^2$$