STAT PHYS SHEET I

Q1

• FIRST, LETS FIND $F(x)$.

$$e^{-F(x)} = e^{-x} x^{n}$$
$$= e^{-x} e^{\ln x^{n}}$$
$$= e^{-x} e^{n \ln x}$$
$$= e^{n \ln x - x} \Rightarrow F(x) = -n \ln x + x$$

• WHERE IS MINIMUM OF $F(x)$?

$$\frac{dF(x)}{dx} = -n \frac{1}{x} + 1 = 0 \Rightarrow \underbrace{x_0}_{F(x)\text{ MINIMUM}} = n$$

$$F''(x) = n \frac{1}{x^2}$$

$$F(x) \approx \underbrace{F(x_0)}_{\downarrow} + F''(x_0) \frac{(x - x_0)^2}{2}$$

$$\approx \overbrace{-n \ln n + n} + \frac{n}{n^2} \frac{(x-n)^2}{2}$$

$$\approx n(1 - \ln n) + \frac{1}{n} \frac{(x-n)^2}{2}$$

FOR $n \gg 1$, $\ln n \gg 1$, so we have:

$$x - n \ln n + \frac{1}{n} \frac{(x-n)^2}{2}$$

PUT THIS BACK TO OUR INTEGRAL:

$$n! = \int_0^{\infty} \exp\left(n \ln n - \frac{1}{n} \frac{(x-n)^2}{2}\right) dx$$

$$= \int_0^\infty n^n \cdot \exp\left(-\frac{1}{n}\frac{(x-n)^2}{2}\right) dx$$

AND NOW I DON'T KNOW WHAT.

SOMEHOW WE OBTAIN: $n! \sim \sqrt{2\pi n}\, n^n e^{-n}$

$$\ln n! \sim \ln\sqrt{2\pi} + \ln\sqrt{n} + \ln\left(n^n e^{-n}\right)$$

$$\sim \frac{1}{2}\ln n + \left(\ln n^n\right) + \ln e^{-n}$$

$$\sim \frac{1}{2}\ln n - n + \ln e^{n \ln n}$$

$$\sim \frac{1}{2}\ln n - n + n\ln n$$

$$\sim n\ln n - n$$

AS REQUIRED.

Q2 N PARTICLES

$$E \propto \sum_{i=1}^{N} (p_i^2 + q_i^2 + r_i^2)$$

WE WANT NUMBER OF POSSIBLE p_i, q_i, r_i WHICH MAKES

$$\sum_{i=1}^{N} (p_i^2 + q_i^2 + r_i^2) < E$$

CONSIDER:

1D CASE:

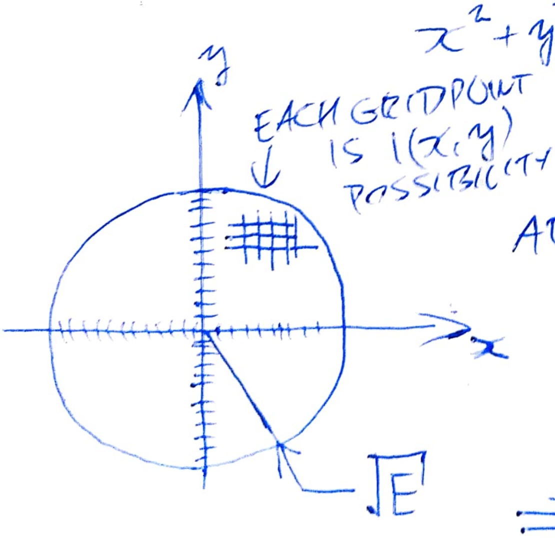
$$x^2 < E$$

x INTEGER & $E \gg 1$

NUMBER OF POSSIBILITIES FOR x IS PROPORTIONAL TO $E^{\frac{1}{2}}$

2D CASE:

$$x^2 + y^2 < E$$



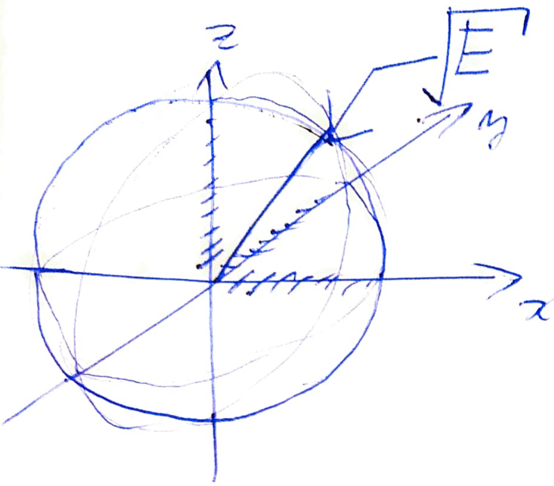
AREA OF CIRCLE $\propto$ RADIUS^2

$\propto \sqrt{E}^2$

$\propto E$

$\Rightarrow$ NUMBER OF POSSIBILITIES $\propto E$

3D CASE:



$x^2+y^2+z^2 < E$

~~AREA~~ VOLUME OF SPHERE $\propto$ NUMBER OF POSSIBILITIES FOR x, y, z

VOLUME OF SPHERE $\propto$ RADIUS^3 $\propto \sqrt{E}^3 \propto E^{\frac{3}{2}}$

OBSERVATION: NUMBER OF POSSIBILITIES SCALES AS:

$$E^{\frac{\text{NUMBER OF DIMENSIONS}}{2}}$$

WE HAVE:

$$\sum_{i=1}^{N} \left(n_i^2 + q_i^2 + r_i^2\right) < E$$

IE A ~~3D~~ 3N DIMENSIONAL CASE.

SO WE HOPE, BASED ON THE ABOVE:

$$G(E) \propto E^{\frac{3N}{2}} \Rightarrow \underline{\alpha = \frac{3}{2}}$$

Q3 (i)

$$E = \text{const.} \Rightarrow \Omega(E) = \text{const.}$$

$$\frac{d\Omega(E_1)}{dE_1} \propto \frac{d}{dE_1}\left[\exp\left(S_1(E_1)\right) + \exp\left(S_2(E - E_1)\right)\right]$$

$$\propto e^{S_1(E_1)} e^{S_2(E-E_1)} \left(\frac{dS_1(E_1)}{dE_1} + \frac{dS_2(E-E_1)}{dE_1}\right) = 0$$

$$\Rightarrow \frac{dS_1(E_1)}{dE_1} + \frac{dS_2(E-E_1)}{dE_1} = 0$$

$$\frac{d}{dE_1} \ln \Omega_1(E_1) + \frac{d}{dE_1} \ln \Omega_2(E-E_1) = 0$$

$$\frac{d}{dE_1} \ln\left(C_1 E_1^{\alpha_1 N_1}\right) + \frac{d}{dE_1} \ln\left(C_2 (E-E_1)^{\alpha_2 N_2}\right) = 0$$

$$\frac{1}{\cancel{C_1} E_1^{\alpha_1 N_1}} \cancel{C_1} \alpha_1 N_1 E_1^{\alpha_1 N_1 - 1} + \frac{1}{\cancel{C_2}(E-E_1)^{\alpha_2 N_2}} C_2 \alpha_2 N_2 (E-E_1)^{\alpha_2 N_2 - 1} \cdot \frac{d}{dE_1}(E-E_1) = 0$$

$$\alpha_1 N_1 E_1^{-1} - \alpha_2 N_2 (E-E_1)^{-1} = 0$$

$$\alpha_1 N_1 (E-E_1) - \alpha_2 N_2 E_1 = 0$$

$$-(\alpha_1 N_1 + \alpha_2 N_2) E_1 + \alpha_1 N_1 E = 0$$

$$\Rightarrow \boxed{E_1^* = \alpha_1 N_1 E / (\alpha_1 N_1 + \alpha_2 N_2)}$$

Q3(ii)

$$S_1(E_1)+S_2(E-E_1)=$$

$$= S_1(E_1^*) + S_2(E_1^* - E_1) + \overbrace{\frac{d}{dE_1}\left(S_1(E_1)+S_2(E-E_1)\right)\Big|_{E_1^*}}^{\text{THIS BIT IS ZERO, FROM PREVIOUS PAGE'S RESULTS}}(E_1^*-E_1^*)$$

$$+\frac{1}{2}\frac{d^2}{dE_1^2}\left(S_1(E_1)+S_2(E-E_1)\right)\Big|_{E_1^*}(E_1-E_1^*)^2$$

$$= S_1(E_1^*) + S_2(E_1^* - E_1) + \frac{1}{2}\frac{d^2}{dE_1^2}\left(S_1(E_1)+S_2(E-E_1)\right)\Big|_{E_1^*}(E_1-E_1^*)^2$$

$$\propto \ln\Omega_1(E_1^*) + \ln\Omega_2(E_1^*-E_1) + \cancel{\frac{1}{2}\frac{d^2}{dE_1^2}\left(S_1(E_1)+S_2(E-E_1)\right)\Big|_{E_1^*}}$$

$$+\frac{1}{2}\frac{d^2}{dE_1^2}\left(\ln\Omega_1(E_1)+\ln\Omega_2(E-E_1)\right)\Big|_{E_1^*}(E_1-E_1^*)^2$$

$$\propto \ln\Omega_1(E_1^*)+\ln\Omega_2(E_1^*-E_1)+\frac{1}{2}\frac{d^2}{dE_1^2}\left(\ln\Omega_1(E_1)\right)\Big|_{E_1^*}(E_1-E_1^*)^2+$$

$$+\frac{1}{2}\frac{d^2}{dE_1^2}\left(\ln\Omega_2(E-E_1)\right)\Big|_{E_1^*}(E-E_1^*)^2$$

EVALUATE SECOND DERIVATIVE TERMS SEPARATELY:

CONSIDER:

$$\frac{d^2}{dE_1^2}\left(\ln \Omega_1(E_1)\right) = \frac{d}{dE_1}\left(\frac{1}{\Omega_1(E_1)}\,\frac{d\Omega_1(E_1)}{dE_1}\right)$$

$$= -\frac{1}{\Omega_1(E_1)^2}\,\frac{d\Omega_1(E_1)}{dE_1} + \frac{1}{\Omega_1(E_1)}\,\frac{d^2 \cancel{\ln}\,\Omega_1(E_1)}{dE_1^2}$$

~~ALSO, WE HAVE~~

$$= -\frac{1}{C_1^2 E_1^{2\alpha_1 N_1}}\,\frac{d}{dE_1}\left(C_1 E_1^{\alpha_1 N_1}\right) + \frac{1}{C_1 E_1^{\alpha_1 N_1}}\,\frac{d^2}{dE_1^2}\left(C_1 E_1^{\alpha_1 N_1}\right)$$

$$= -\frac{1}{C_1^2 E_1^{2\alpha_1 N_1}}\, C_1\, \alpha_1 N_1\, E_1^{\alpha_1 N_1 - 1} +$$

$$+ \frac{1}{C_1 E_1^{\alpha_1 N_1}}\, C_1\, \alpha_1 N_1 (\alpha_1 N_1 - 1)\, E_1^{\alpha_1 N_1 - 2}$$

$$= -\frac{1}{C_1}\alpha_1 N_1\, E_1^{-\alpha_1 N_1 - 1} + \alpha_1 N_1(\alpha_1 N_1 - 1)\left(E_1\right)^{-2}$$

WE ALSO HAVE:

$$\frac{d^2}{dE_1^2} \ln \Omega_2(E-E_1) = \frac{d}{dE_1}\left(\frac{1}{\Omega_2(E-E_1)} \frac{d\Omega_2(E-E_1)}{dE_1}\right)$$

$$= \left(-\frac{1}{\Omega_2(E-E_1)^2} \frac{d\Omega_2(E-E_1)}{dE_1} + \frac{1}{\Omega_2(E-E_1)} \frac{d^2\Omega_2(E-E_1)}{dE_1^2}\right)$$

$$\cdot \frac{d}{dE_1}(E-E_1)$$

$$= + \frac{1}{c_2^2(E-E_1)^{2\alpha_2 N_2}} \frac{d}{dE_1}\left(c_2(E-E_1)^{\alpha_2 N_2}\right)$$

$$- \frac{1}{c_2(E-E_1)^{\alpha_2 N_2}} \frac{d^2}{dE_1^2}\left(c_2(E-E_1)^{\alpha_2 N_2}\right)$$

$$= \frac{1}{c_2^2(E-E_1)^{2\alpha_2 N_2}} c_2 \alpha_2 N_2 (E-E_1)^{\alpha_2 N_2 - 1} \cdot (-1)$$

$$- \frac{1}{c_2(E-E_1)^{\alpha_2 N_2}} c_2 \alpha_2 N_2 (\alpha_2 N_2 - 1) (E-E_1)^{\alpha_2 N_2 - 2}$$

$$= \frac{1}{C_2}\alpha_2 N_2 (E-E_1)^{-\alpha_2 N_2 - 1} + \alpha_2 N_2(\alpha_2 N_2 - 1)(E-E_1)^{-2}$$

PUT THESE TERMS BACK TO OUR EXPANSION:

$$S_1(E_1) + S_2(E-E_1) \propto$$

~~SOME CONSTANT~~

$$\propto \ln\Omega_1(E_1^*) + \ln\Omega_2(E_1^* - E_1) +$$

$$+ \frac{1}{2}\left(-\frac{1}{C_1}\alpha_1 N_1 E_1^{-\alpha_1 N_1 - 1} + \alpha_1 N_1(\alpha_1 N_1 - 1) E_1^{-2}\right)(E_1 - E_1^*)^2 +$$

$$+ \frac{1}{2}\left(\frac{1}{C_2}\alpha_2 N_2 (E-E_1)^{-\alpha_2 N_2 - 1} + \alpha_2 N_2(\alpha_2 N_2 - 1)(E-E_1)^{-2}\right)(E_1 - E_1^*)^2$$

AND NOW I DON'T KNOW WHAT.

Q4

(i)

ENTROPY IS MAXIMIZED IF

$$\frac{\partial S}{\partial E} = 0 \quad \& \quad \frac{\partial^2 S}{\partial E^2} < 0$$

ENTROPY CHANGE WHEN SEPARATED SYSTEMS CONNECTED:

$$\delta S = \frac{\partial S_1(E_1)}{\partial E}\delta E_1 + \frac{\partial S_2(E_2)}{\partial E}\delta E_2$$

USING $\delta E_2 = -\delta E_1$

$$= \left(\frac{\partial S_1(E_1)}{\partial E} - \frac{\partial S_2(E_2)}{\partial E}\right)\delta E_1$$

USING TEMP. DEFINITION

$$= \left(\frac{1}{T_1} - \frac{1}{T_2}\right)\delta E_1$$

IF $T_1 = T_2 \Rightarrow \frac{\delta S}{\delta E_1} = 0$

CONSIDER:

$$\frac{\partial^2 S}{\partial E^2} = \frac{\partial}{\partial E}\frac{\partial S}{\partial E} = \frac{\partial T}{\partial E}\frac{\partial}{\partial T}\left(\frac{\partial S}{\partial E}\right)$$

$$= \frac{1}{C_V}\frac{\partial}{\partial T}\frac{1}{T} = -\frac{1}{C_V T^2} < 0$$

$\Rightarrow C_V$ MUST BE POSITIVE

$\boxed{Q4}$

(ii)

$$Z = \sum_n e^{-\beta E_n} \qquad \frac{\partial Z}{\partial \beta} = \sum_n (-E_n) e^{-\beta E_n}$$

$$\langle E \rangle = \sum_n \underbrace{\frac{1}{Z} e^{-\beta E_n}}_{p(n)} E_n = \frac{1}{Z}(-1)\frac{\partial Z}{\partial \beta} = \underline{-\frac{\partial}{\partial \beta} \ln Z}$$

$$\langle E^2 \rangle = \sum_n \frac{1}{Z} e^{-\beta E_n} E_n^2 = \frac{1}{Z} \frac{\partial^2}{\partial \beta^2} Z$$

$$\cancel{= \frac{1}{Z} \frac{\partial}{\partial \beta}\left(\beta \frac{1}{\beta} \frac{\partial Z}{\partial \beta}\right)}$$

$$\cancel{= \frac{1}{Z} \frac{1}{\beta} \frac{\partial Z}{\partial \beta} + \frac{\beta}{Z} \frac{\partial}{\partial \beta}\left(\frac{1}{\beta} \frac{\partial Z}{\partial \beta}\right)}$$

$$\cancel{= \frac{1}{Z} \frac{1}{\beta} \frac{\partial Z}{\partial \beta} + \frac{\beta}{Z}\left(-\frac{1}{\beta^2}\right) \frac{\partial Z}{\partial \beta} + \frac{\beta}{Z} \frac{1}{\beta} \frac{\partial^2 Z}{\partial \beta^2}}$$

REVERSE-ENGINEERING THE RESULT:

$$\frac{\partial^2}{\partial \beta^2} \ln Z + \left(\frac{\partial}{\partial \beta} \ln Z\right)^2 =$$

$$= \frac{\partial}{\partial \beta}\left(\frac{1}{Z} \frac{\partial Z}{\partial \beta}\right) + \left(\frac{1}{Z} \frac{\partial Z}{\partial \beta}\right)^2$$

$$= \left(\frac{\partial}{\partial \beta} \frac{1}{Z}\right) \frac{\partial Z}{\partial \beta} + \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2} + \left(\frac{1}{Z} \frac{\partial Z}{\partial \beta}\right)^2$$

$$= -Z^{-2}\left(\frac{\partial Z}{\partial \beta}\right)^2 + \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2} + \left(\frac{1}{Z} \frac{\partial Z}{\partial \beta}\right)^2 = \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2}$$

I CAN GO THIS DIR.

HOW CAN I GO THIS WAY WITHOUT KNOWING WHAT I'M SUPPOSED TO GET?

IE WHAT WE HAVE.

$$\Rightarrow \langle E^2 \rangle = \frac{\partial^2}{\partial \beta^2} \ln Z + \left(\frac{\partial}{\partial \beta} \ln Z\right)^2$$

$$(\Delta E)^2 = \langle E^2 \rangle - \langle E \rangle^2$$

$$= \frac{\partial^2}{\partial \beta^2} \ln Z + \left(\frac{\partial}{\partial \beta} \ln Z\right)^2 - \left(-\frac{\partial}{\partial \beta} \ln Z\right)^2$$

$$= \underline{\frac{\partial^2}{\partial \beta^2} \ln Z}$$

$$C_V = \frac{\partial \langle E \rangle}{\partial T} = \frac{\partial}{\partial T}\left(-\frac{\partial}{\partial \beta} \ln Z\right)$$

USING: $\beta = \frac{1}{k_B T} \Rightarrow \frac{\partial}{\partial T} = \frac{\partial \beta}{\partial T}\frac{\partial}{\partial \beta} = \frac{\partial \frac{1}{k_B T}}{\partial T}\frac{\partial}{\partial \beta}$

$$= -\frac{1}{k_B} T^{-2} \frac{\partial}{\partial \beta}$$

REWRITE EXPRESSION FOR C_V:

$$C_V = -\frac{1}{k_B T^2} \frac{\partial}{\partial \beta}\left(-\frac{\partial}{\partial \beta} \ln Z\right) = \frac{1}{k_B T^2} \frac{\partial^2}{\partial \beta^2} \ln Z$$

$$= \frac{1}{k_B T^2} (\Delta E)^2$$

$$\Rightarrow \underline{(\Delta E)^2 = C_V k_B T^2 \propto C_V}$$ AS REQUIRED.

Q6

(a)(i) "HOW MANY STATES"... $^{N\uparrow}C_N$ ie $\frac{N!}{N\uparrow!(N-N\uparrow)!}$

"EXPRESS THE ENERGY OF SUCH A STATE"...

$$E = N\uparrow \varepsilon$$

where ε is the difference in energy of spin up & down states

$$S(E) = k_B \ln \Omega(E) = k_B \ln \frac{N!}{N\uparrow!(N-N\uparrow)!}$$

$$\boxed{\text{FOR LARGE } n!: \quad n! \sim n \ln n - n}$$

$$\cancel{S(E) \sim k_B (\ln N\uparrow! - \ln}$$

$$S(E) \sim k_B \big(N \ln N \cancel{- N} - N\uparrow \ln N\uparrow \cancel{+ N\uparrow}$$
$$- (N-N\uparrow) \ln (N-N\uparrow) + \cancel{(N-N\uparrow)} \big)$$

$$\sim k_B \big(N \ln N - N\uparrow \ln N\uparrow - (N-N\uparrow) \ln (N-N\uparrow) \big)$$

$$\cancel{\sim k_B (N-N\uparrow)}$$

$$\sim k_B\left(-N\uparrow \ln N\uparrow + N\uparrow \ln N + (N - N\uparrow)\ln N - (N-N\uparrow)\ln(N-N\uparrow)\right)$$

$$\sim k_B\left(-(N-N\uparrow)\ln\frac{N-N\uparrow}{N} + N\uparrow \ln\frac{N}{N\uparrow}\right)$$

$$\sim -k_B\left((N-N\uparrow)\ln\frac{N-N\uparrow}{N} + N\uparrow \ln\frac{N\uparrow}{N}\right)$$

AS A FUNCTION OF E:

$$\sim -k_B N\left[\frac{N-N\uparrow}{N}\ln\frac{N-N\uparrow}{N} + \frac{N\uparrow}{N}\ln\frac{N\uparrow}{N}\right]$$

USE: $N\uparrow = \frac{E}{\epsilon}$

$$\sim -k_B N\left[\left(1-\frac{E}{N\epsilon}\right)\ln\left(1-\frac{E}{N\epsilon}\right) + \frac{E}{N\epsilon}\ln\left(\frac{E}{N\epsilon}\right)\right]$$