1. (i)

IN FLUID'S IRF: $u^\mu = (c, \vec{0})$

$u_\nu \nabla_\mu u^\nu =$ BY DEF. OF ∇

$= u_\nu \left(\frac{\partial u^\nu}{\partial x^\mu} + \Gamma^\nu_{\mu\alpha} u^\alpha \right)$ KEEPING ONLY NON-ZERO TERMS:

$= u_0 \left(\frac{\partial u^0}{\partial x^0} + \Gamma^0_{00} u^0 \right)$

$= c \left(\underbrace{\frac{\partial c}{\partial t}}_{\to 0} + \Gamma^0_{00} c \right)$

USING FORMULA BOOKLET:

$\Gamma^0_{00} = - \frac{1}{2g_{00}} \frac{\partial g_{00}}{\partial x^0} = \frac{1}{2g_{00}} \frac{\partial g_{00}}{\partial x^0}$

$\Rightarrow \Gamma^0_{00} = 0$

$\Rightarrow c \left(\frac{\partial c}{\partial t} + \Gamma^0_{00} c \right) = 0$

$u_\nu \nabla_\mu u^\nu = 0$ IN IRF $\Rightarrow u_\nu \nabla_\mu u^\nu = 0$

IN ALL FRAMES.

CONSERVATION OF E-M TENSOR:

$$\nabla_\mu T^{\mu\nu} = 0$$

IE:

$$0 = \nabla_\mu \left[(\rho + p/c^2) u^\mu u^\nu - p g^{\mu\nu} \right]$$

$$\cancel{= (\rho + p/c^2) \nabla_\mu (u^\mu u^\nu) - p \nabla_\mu g^{\mu\nu}}$$

~~BY CHAIN RULE~~

BY PRODUCT RULE:

$$
\begin{aligned}
0 = \; & \nabla_\mu (\rho + p/c^2) \cdot u^\mu u^\nu \\
& + (\rho + p/c^2)(\nabla_\mu u^\mu) u^\nu + \\
& + (\rho + p/c^2)(\nabla_\mu u^\nu) u^\mu \\
& - (\nabla_\mu p) \cdot g^{\mu\nu} \\
& - \underbrace{(\nabla_\mu g^{\mu\nu})}_{=0} p
\end{aligned}
$$

(EQUATION I)

CONTRACT WITH u_ν:

TERM BY TERM:

$$\nabla_\mu(\rho + p/c^2)\, u^\mu \underbrace{u^\nu u_\nu}_{c^2} =$$

$$= c^2 u^\mu \nabla_\mu(\rho + p/c^2)$$

$$(\nabla_\mu u^\mu)\, u^\nu u_\nu = c^2 \nabla_\mu u^\mu$$

$$(\nabla_\mu u^\nu)\, u^\mu u_\nu = 0$$ BY OUR EARLIER RESULT.

$$g^{\mu\nu} u_\nu = u^\mu$$

COLLECTING THESE TOGETHER:

$$0 = c^2 u^\mu \nabla_\mu(\rho + \cancel{p}/c^2) +$$

$$+ (\rho + p/c^2)\, c^2 \nabla_\mu u^\mu +$$

$$- \cancel{(\nabla_\mu p)}\, u^\mu$$

$$0 = c^2 u^\mu \nabla_\mu S +$$
$$+ c^2 S \nabla_\mu u^\mu +$$
$$+ p \nabla_\mu u^\mu \qquad \text{(EQUATION II)}$$

$$\Rightarrow \nabla_\mu (S u^\mu) + \frac{p}{c^2} \nabla_\mu u^\mu = 0$$

$$\text{EQUATION I} - \frac{u^\nu}{c^2} \cdot \text{EQUATION II} =$$
$$= \nabla_\mu (S + p/c^2) u^\mu u^\nu +$$
$$+ (S + p/c^2)(\nabla_\mu u^\mu) u^\nu +$$
$$+ (S + p/c^2)(\nabla_\mu u^\nu) u^\mu$$
$$- (\nabla_\mu p) g^{\mu\nu}$$
$$- u^\nu u^\mu \nabla_\mu S$$
$$- u^\nu S \nabla_\mu u^\mu$$
$$- u^\nu \frac{p}{c^2} \nabla_\mu u^\mu$$

$\downarrow$ BASED ON HOPE & SOME NOT CONCLUSIVE ALGEBRA.

$$= \rho u^\mu \nabla_\mu u^\nu +$$

$$+ \frac{p}{c^2} u^\mu \nabla_\mu u^\nu$$

$$- g^{\mu\nu} \nabla_\mu p$$

$$+ u^\mu u^\nu \frac{1}{c^2} \nabla_\mu p$$

$$\Rightarrow (\rho + p/c^2) u^\mu \nabla_\mu u^\nu =$$

$$= (g^{\mu\nu} - u^\mu u^\nu / c^2) \nabla_\mu p$$

"IF THE FLUID IS DUST, SHOW THAT"... : IDK.

(ii)

$$\nabla_\mu(S u^\mu) + \frac{P}{c^2}\nabla_\mu u^\mu = 0$$

$$\nabla_\mu(S u^\mu) = \frac{\partial(S u^\mu)}{\partial x^\mu} + \Gamma^\mu_{\mu\alpha}(S u^\alpha)$$

$$= \frac{\partial(S u^\mu)}{\partial x^\mu} + \Gamma^\mu_{\alpha\mu}(S u^\alpha)$$

USE: $u^\mu = (c, \vec{0})$

$$\nabla_\mu(S u^\mu) = \frac{\partial(Sc)}{\partial(ct)} + \Gamma^\mu_{0\mu}(Sc)$$

USE $\Gamma^\mu_{0\mu}$ GIVEN & $\Gamma^0_{00} = 0$

$$= \dot{S} + 3c^2\frac{\dot{R}}{R}S$$

~~SIMILARLY:~~

$$\cancel{\nabla_\mu u^\mu = \underbrace{\frac{\partial c}{\partial t}}_{\to 0} + \Gamma^\mu_{0\mu} u^\mu = 0}$$

$$\cancel{\Longrightarrow c\dot{S} = -3c^2\frac{\dot{R}}{R}S}$$

SIMILARLY:

$$\nabla_\nu u^r = \frac{\partial u^r}{\partial x^\nu} + \Gamma^r_{\nu m} u^m$$

$$= \underbrace{\cancel{\frac{\partial (c, \vec{0})}{\partial x^\nu}}}_{\to 0} + \Gamma^r_{\nu 0}\, c$$

$$\nabla_\mu u^\mu = \Gamma^\mu_{\mu 0} c = \Gamma^\mu_{0\mu} c$$

$$= 3c^2 \frac{\dot{R}}{R}$$

COLLECT TERMS:

$$\dot{\rho} + 3c^2 \frac{\dot{R}}{R} \rho + 3\cancel{c^2} \frac{\dot{R}}{R} \frac{p}{\cancel{c^2}} = 0$$

$$\dot{\rho} + 3\frac{\dot{R}}{R}(\rho c^2 + p) = 0$$

$$\frac{d\rho}{c\,dt} + 3\frac{dR}{c\,dt}\frac{1}{R}(\rho c^2 + p) = 0$$

$$\frac{d\rho}{dR} + 3\frac{1}{R}(\rho c^2 + p) = 0$$

$$R^3 \frac{d\rho}{dR} + 3R^2 \rho c^2 = -3R^2 p$$

~~SHOVE A FACTOR OF C IN:~~

$$R^3 \frac{dS}{dR} + 3R^2 S = -3pR^2/c^2$$

$$\frac{d(SR^3)}{dR} = -3pR^2/c^2$$

PHYSICAL SIGNIFICANCE OF THIS EQ.?

$$R_{\mu\nu} = -8\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right) + \Lambda g_{\mu\nu}$$

00 COMPONENT:

$$R_{00} = -8\left(T_{00} - \frac{1}{2} T g_{00}\right) + \Lambda g_{00}$$

$$g_{00} = c^2$$

$$T_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} T^{\alpha\beta}$$

$$T_{00} = g_{0\alpha} g_{0\beta} T^{\alpha\beta}$$

$$= g_{00} g_{00} T^{00}$$

$$= c^2\left[(S + p/c^2)c^2 - p\underbrace{g^{00}}_{=1}\right]$$

IF $u^\mu = (c, \vec{0})$, THEN $g^{00} = 1$,
IF $u^\mu = (1, \vec{0})$ THEN $g^{00} = \frac{1}{c^2}$

$$= \rho c^4 + p c^2 - p c^2 = \rho c^4$$

$$T = g_{\mu\nu} T^{\mu\nu}$$

$$= g_{\mu\nu} (\rho + p/c^2) u^\mu u^\nu - g_{\mu\nu}\, p\, g^{\mu\nu}$$

$$= \rho c^2 + p - p \delta^\mu_\mu$$

$$= \rho c^2 + p - 4p$$

$$= \rho c^2 - 3p$$

SUBSTITUTE BACK:

$$R_{00} = -\kappa \left(\rho c^4 - \tfrac{1}{2} (\rho c^2 - 3p) c^2 \right) + \Lambda c^2$$

$$= -\kappa \left(+ \tfrac{1}{2} \rho c^4 + \tfrac{1}{2} 3 p c^2 \right) + \Lambda c^2$$

$$= -\frac{\kappa c^4}{2} \left(\rho + 3p \frac{1}{c^2} \right) + \Lambda c^2$$

$$= -4\pi G \left(\rho + 3p \frac{1}{c^2} \right) + \Lambda c^2 = \frac{3\ddot{R}}{R}$$

IT IS POSSIBLE THAT I'VE BEEN LOUSY WITH C FACTORS.

$$\rightarrow \frac{\ddot{R}}{R} = -\overbrace{\frac{4\pi G}{3}}^{\frac{\kappa c^4}{6}}\left(\rho + 3p\frac{1}{c^2}\right) + \frac{\Lambda}{3}c^2$$

BACK TO EFE, R22 CASE:

$$\cancel{R_{22} = -\kappa\left(T_{22} - \frac{1}{2}T g_{22}\right)}$$

$$R_{22} = -\kappa\left(T_{22} - \frac{1}{2}T g_{22}\right) + \Lambda g_{22}$$

$$T_{22} = g_{2\alpha} g_{2\beta} T^{\alpha\beta}$$

$$= g_{22} g_{22} T^{22}$$

$$= \left(-R^2 r^2\right)^2 T^{22}$$

$$= R^4 r^4 \left(-p g^{22}\right)$$

$$= R^4 r^4 (-p)\frac{1}{(-R^2 r^2)} = R^2 r^2 p$$

EARLIER WE FOUND:

$$T = \rho c^2 - 3p$$

SUBSTITUTE IN:

$$R_{22} = -\kappa\left(R^2 r^2 p - \frac{1}{2}(\rho c^2 - 3p) R^2 r^2\right) +$$
$$+ \Lambda r^2 R^2$$

$$= -\left(R\ddot{R} + 2\dot{R}^2 + 2c^2 k\right) r^2 / c^2$$

$$\Downarrow$$

$$-\kappa R^2 p + \frac{1}{2}\kappa(\rho c^2 - 3p) R^2 + \Lambda R^2 =$$
$$= \left(-R\ddot{R} - 2\dot{R}^2 - 2c^2 k\right)/c^2$$

$$/ : R^2$$

$$c^2\left(-\kappa p + \frac{1}{2}\kappa(\rho c^2 - 3p) + \Lambda\right) = -\frac{\ddot{R}}{R} - 2\frac{\dot{R}^2}{R^2} - 2c^2 k \frac{1}{R^2}$$

$$c^2\left(-\frac{5}{2}\kappa p + \frac{1}{2}\kappa \rho c^2 + \Lambda\right) = -\frac{\ddot{R}}{R} - 2\frac{\dot{R}^2}{R^2} - 2c^2 k \frac{1}{R^2}$$

SUB IN FOR $\frac{\ddot{R}}{R}$:

$$c^2\left(-\frac{5}{2}\kappa p + \frac{1}{2}\kappa \rho c^2 + \Lambda\right) = -\frac{\kappa c^4}{6}\left(\rho + 3p\frac{1}{c^2}\right) - \frac{\Lambda}{3}c^2$$
$$- 2\frac{\dot{R}^2}{R^2} - 2c^2 k \frac{1}{R^2}$$

$$-\frac{5}{2}\kappa p c^2 + \frac{1}{2}\kappa \rho c^4 + \Lambda c^2 = -\frac{\kappa c^4}{6}\rho - \frac{1}{2}\kappa p c^2 - \frac{\Lambda}{3}c^2$$
$$- 2\frac{\dot{R}^2}{R^2} - 2c^2 k^2 \frac{1}{R^2}$$

$$-2\kappa\eta c^2+\frac{2}{3}\kappa S c^4+\frac{4}{3}\Lambda c^2=-2\frac{\dot{R}^2}{R^2}-2c^2 k\frac{1}{R^2}$$

/ : -2

$$\left(\frac{\dot{R}}{R}\right)^2+\frac{k c^2}{R^2}=-\frac{1}{3}\kappa c^4 S+\kappa\eta c^2-\frac{2}{3}\Lambda c^2$$

SOMETHING WENT WRONG
WITH ALGEBRA.

"CONSIDER A MASSIVE PARTICLE..."

II.

Symmetry: otherwise it introduces a translational term, which is not viscosity.

$v_\varphi(r_1) = \Omega_1 r_1$

$v_\varphi(r_2) = \Omega_2 r_2$

$\frac{\partial P}{\partial \varphi} = 0$

cylindrical symmetry $\Rightarrow \frac{\partial v_\varphi}{\partial \varphi} = 0$ $\qquad \frac{\partial^2 v_\varphi}{\partial \varphi^2} = 0$

symmetry $\Rightarrow \frac{\partial v_\varphi}{\partial z} = 0$ $\qquad \frac{\partial^2 v_\varphi}{\partial^2 v_z^2} = 0$

steady state: $\partial v_\varphi / \partial t = 0$

NS FOR v_φ USING ABOVE ~~SYMMETRY CONSIDERATIONS~~ CONSIDERATIONS:

$$v_r \frac{\partial v_\varphi}{\partial r} + \cancel{\frac{v_\varphi}{r}} \frac{v_\varphi v_r}{r} =$$

$$= \nu \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial v_\varphi}{\partial r}\right) - \nu \frac{v_\varphi}{r^2}$$

$$\cancel{v_r \frac{\partial v_\varphi}{\partial r}} + \cancel{\frac{v_\varphi v_r}{r}} = \nu \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial v_\varphi}{\partial r}\right) - \nu \frac{v_\varphi}{r^2}$$

USE: $v_r = 0$ TOO

$$\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial v_\varphi}{\partial r}\right) = \frac{v_\varphi}{r^2}$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial v_\phi}{\partial r}\right) = \frac{v_\phi}{r}$$

LET: $v_\phi = \sum_{-\infty}^{\infty} a_n x^n$

$$\frac{\partial v_\phi}{\partial r} = \sum_{-\infty}^{\infty} a_n n x^{n-1}$$

$$r\frac{\partial v_\phi}{\partial r} = \sum_{-\infty}^{\infty} a_n n x^n$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial v_\phi}{\partial r}\right) = \sum_{-\infty}^{\infty} a_n n^2 x^{n-1}$$

$$\frac{v_\phi}{r} = \sum_{-\infty}^{\infty} a_n x^{n-1}$$

$$\Rightarrow a_n n^2 = a_n \Rightarrow n^2 = 1 \Rightarrow n = \pm 1$$

$$\Rightarrow v_\phi = a_{-1} x^{-1} + a_1 x^1 \quad \text{IE}$$

$$v_\phi = Ar + B/r$$

$$v_\phi(r_1) = \Omega_1 r_1 = A r_1 + B/r_1$$

$$v_\phi(r_2) = \Omega_2 r_2 = A r_2 + B/r_2$$

$\mathrm{I} \Rightarrow$

$$A = \frac{\Omega_1 r_1 - B/r_1}{r_1} = \Omega_1 - \frac{B}{r_1^2}$$

$\mathrm{II} \Rightarrow$

$$\left(\Omega_2 r_2 - A r_2\right) r_2 = B$$

SOLVE FOR A, B.

3.

$$\ddot{\delta}_m + 2\frac{\dot{R}}{R}\dot{\delta}_m - 4\pi G \bar{\rho}_m \delta_m = 0$$

EdS UNIVERSE: $R(t) \propto t^{2/3}$

$$\frac{\dot{R}}{R} = \frac{\frac{2}{3} t^{-\frac{1}{3}}}{t^{\frac{2}{3}}} = \frac{2}{3} t^{-1}$$

$$\ddot{\delta}_m + 2\,\frac{2}{3} t^{-1} \dot{\delta}_m - \underbrace{4\pi G \bar{\rho}_m}_{C} \delta_m = 0$$

$$\ddot{\delta}_m + \frac{4}{3} t^{-1} \dot{\delta}_m - C\delta_m = 0$$

$$\ddot{\delta}_m = \sum_{-\infty}^{\infty} a_n x^n$$

$$\cancel{n(n-1)x^{n-2} + \frac{4}{3} n x^{n-2} - C}$$

$$\sum_{-\infty}^{\infty} a_n n(n-1) x^{n-2} + \frac{4}{3} a_n n x^{n-2} - \sum_{\infty}^{\infty} C a_n x^n = 0$$

$$\Rightarrow a_n\left(n(n-1) + \frac{4}{3} n \right) - C a_{n-2} = 0$$

4 (i)

- RADIAL VELOCITY MEASUREMENTS
- ASTROMETRY
- DIRECT IMAGING.

Their BV color is different ⇒ they have vastly different timespan ⇒ if they formed together, we're unlikely to observe them.

Additional info: astrometry data.

Bigger dist, ie lower BV color, ie X.

4(ii)

$L = L_0 \quad m = m_0 \quad \kappa = \kappa_0 \dfrac{P^{\alpha-1}}{T^{\beta-1}}$

RADIATIVE DIFFUSION EQ.:

$$\frac{dT}{dr} = -\frac{3\kappa\rho L}{16\pi a c r^2 T^3}$$

ALSO: $\dfrac{dP}{dr} = -\dfrac{Gm\rho}{r^2} \Rightarrow \dfrac{dr}{dP} = -\dfrac{r^2}{Gm\rho}$

$$\frac{dT}{dr}\frac{dr}{dP} = \frac{dT}{dP}$$

$$\frac{dT}{dP} = -\frac{3\kappa\cancel{\rho}L}{16\pi a c\cancel{r^2}T^3} \cdot -\frac{\cancel{r^2}}{Gm\cancel{\rho}}$$

$$= \frac{3\kappa L}{16\pi a c T^3 G m}$$

$$= \frac{3L_0}{16\pi a c G m_0}\frac{\kappa}{T^3}$$

SUB IN FOR κ

$$= \frac{3L_0}{16\pi a c G m_0}\kappa_0\frac{P^{\alpha-1}}{T^{\beta-1}}$$

LET $C' = \frac{3 L_0 K_0}{16 \pi a c G M_0}$

$$\frac{dT}{dP} = C' \frac{P^{\alpha - 1}}{T^{\beta - 1}}$$

~~SUB IN PROVIDED P~~

$$T^{\beta - 1}\, dT = C' P^{\alpha - 1}\, dP$$

$$\cancel{\frac{1}{\beta - 1} T^{\beta}}$$

SOME CONSTANT

$$\frac{1}{\beta} T^{\beta} = C' \frac{1}{\alpha} P^{\alpha} + C''$$

$$\cancel{C' \frac{\alpha}{\beta} T^{\beta} - C''}$$

$$\frac{\alpha}{\beta} T^{\beta} = C' P^{\alpha} + C'' \alpha$$

$$\left[\frac{1}{C'} \frac{\alpha}{\beta} T^{\beta} - \frac{C'' \alpha}{C'} \right]^{\frac{1}{\alpha}} = P$$

RENAME CONSTANTS: $P = \left[B + \frac{\alpha}{\beta} A T^{\beta} \right]^{\frac{1}{\alpha}}$

$$d\ln P = \frac{1}{P}\,dP$$

$$d\ln T = \frac{1}{T}\,dT$$

$$\frac{d\ln P}{d\ln T} = \frac{T}{P}\frac{dP}{dT} = \frac{T}{P}\frac{d}{dT}\left(B + \frac{\alpha}{B}AT^{B}\right)^{\frac{1}{\alpha}}$$

$$= \frac{T}{P}\,\frac{1}{\alpha}\left(B + \frac{\alpha}{B}AT^{B}\right)^{\frac{1}{\alpha}-1}\,\cancel{\frac{\alpha}{B}}A\cancel{B}T^{B-1}$$

$$= \frac{AT^{B}}{\left(B + \frac{\alpha}{B}AT^{B}\right)^{\frac{1}{\alpha}}}\left(B + \frac{\alpha}{B}AT^{B}\right)^{\cancel{\frac{1}{\alpha}}-1}$$

$$= \frac{AT^{B}}{B + \frac{\alpha}{B}AT^{B}}$$

AS REQUIRED

$\boxed{7\ (ii)}$

$$\nabla^2 \Phi = -4\pi G \rho$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) = -4\pi G \rho$$

USE GIVEN SOLUTION:

$$\frac{\partial \Phi}{\partial r} = \frac{\partial}{\partial r}\left(\frac{A}{r+a}\right) = -\frac{A}{(r+a)^2}$$

SUBSTITUTE IN:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{-A}{(r+a)^2}\right) = -\frac{A}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{(a+r)^2}\right)$$

$$= -\frac{A}{r^2}\frac{\partial}{\partial r}\left((a+r)^{-2} r^2\right)$$

$$= -\frac{A}{r^2}\left[(-2)(a+r)^{-3} r^2 + (a+r)^{-2} 2r\right]$$

$$= \frac{2A}{r}\left[(a+r)^{-3} r - (a+r)^{-2}\right]$$

$$= \frac{2A}{r}\frac{1}{(a+r)^2}\left[(a+r)^{-1} r - 1\right]$$

$$= \frac{2A}{r} \frac{1}{(a+r)^2} \underbrace{\left[\frac{r}{a+r} - 1 \right]}_{= \frac{r-(a+r)}{a+r} = \frac{-a}{a+r}}$$

$$= \frac{2A}{r} \frac{1}{(a+r)^3} (-a) = - \frac{2Aa}{r} \frac{1}{(a+r)^3}$$

$$= -2A \frac{1}{\left(\frac{r}{a}\right)(a+r)^3} = - \frac{2A}{a^3} \frac{1}{\left(\frac{r}{a}\right)\left(1+\frac{r}{a}\right)^3}$$

$$= - \frac{2A}{a^3} \frac{\rho}{\rho_0} = -4\pi G \rho$$

$$\Rightarrow -\frac{2A}{a^3} \frac{1}{\rho_0} = -4\pi G$$

$$\underline{A = 2\pi G \rho_0 a^3}$$

Φ IS A SOLUTION to POISSON EQ &
SATISFIES BOUNDARY CONDITIONS
$\Rightarrow$ BY UNIQUENESS, IT IS THE ONLY SOLUTION.

"THE NUMBER DENSITY OF STARS..."

$$\beta \equiv 1 - \frac{\overline{v_\theta^2} + \overline{v_\phi^2}}{2\overline{v_r^2}}$$

REWRITE *:

$$\frac{d\left(\nu\overline{v_r^2}\right)}{dr} + \nu\left(\frac{d\Phi}{dr} + \frac{2\overline{v_r^2}}{r}\beta\right) = 0$$