SDSG
4.2

CBE IN SPC:

$$\frac{\partial f}{\partial t}+\dot{r}\frac{\partial f}{\partial r}+\dot{\theta}\frac{\partial f}{\partial \theta}+\dot{\phi}\frac{\partial f}{\partial \phi}+\dot{v}_r\frac{\partial f}{\partial v_r}+\dot{v}_\theta\frac{\partial f}{\partial v_\theta}+\dot{v}_\phi\frac{\partial f}{\partial v_\phi}=0$$

NO θ & ϕ DEPENDENCE $\Rightarrow \frac{\partial f}{\partial \theta}=\frac{\partial f}{\partial \phi}=0$

CONSIDER TERMS SEPARATELY:

$$\dot{r}=v_r$$

$$\dot{v}_r=\ddot{r}=a_r+r\dot{\theta}^2+r\sin^2\theta\,\dot{\phi}^2$$

$$=-\frac{\partial \Phi}{\partial r}+\frac{1}{r}\left(v_\theta^2+v_\phi^2\right)$$

$$\dot{v}_\theta=\frac{d}{dt}\left(r\dot{\theta}\right)=\dot{r}\dot{\theta}+r\ddot{\theta}$$

$$=a_\theta-\dot{r}\dot{\theta}+r\sin\theta\cos\theta\,\dot{\phi}^2$$

$$=-\underbrace{\frac{\partial \Phi}{\partial \theta}}_{\to 0}-\frac{1}{r}\left(\dot{r}\,r\dot{\theta}-\left(\sin\theta\,\dot{\phi}\,r\right)^2\cot\theta\right)$$

$$=-\frac{1}{r}\left(v_r v_\theta-v_\phi^2\cot\theta\right)$$

$$\dot{v}_\phi=\frac{d}{dt}\left(r\sin\theta\,\dot{\phi}\right)=\dot{r}(\sin\theta)\dot{\phi}+r(\cos\theta)\dot{\theta}\dot{\phi}+r(\sin\theta)\ddot{\phi}$$

$$=a_\phi-\dot{r}(\sin\theta)\dot{\phi}-r(\cos\theta)\dot{\theta}\dot{\phi}$$

$$=-\underbrace{\frac{\partial \Phi}{\partial \phi}}_{\to 0}-\frac{1}{r}\left(r\sin\theta\dot{\phi}\,\dot{r}+r\dot{\theta}\,r\sin\theta\dot{\phi}\cot\theta\right)$$

$$=-\frac{1}{r}v_\phi\left(v_r+v_\theta\cot\theta\right)$$

COMBINING TERMS TOGETHER:

$$\frac{\partial f}{\partial t}+v_r\frac{\partial f}{\partial r}+\frac{1}{r}\left(v_\theta^2+v_\phi^2\right)\frac{\partial f}{\partial v_r}-\frac{1}{r}\left(v_r v_\theta-v_\phi^2\cot\theta\right)\frac{\partial f}{\partial v_\theta}-\frac{1}{r}v_\phi\left(v_r+v_\theta\cot\theta\right)\cdot\frac{\partial f}{\partial v_\phi}-\frac{\partial \Phi}{\partial r}\frac{\partial f}{\partial v_r}=0,$$ AS REQUIRED.

SDSG 4.4(a)

CBE:

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\nabla f - \nabla\Phi\cdot\frac{\partial f}{\partial \underline{v}} = 0$$

$$f(x,v) = f(E) = f\left(\frac{1}{2}v^2 + \Phi(x)\right) = f\left(\frac{1}{2}\sum_i v_i^2 + \Phi(x)\right)$$

$\frac{\partial f}{\partial t} = 0$ (IE NO EXPLICIT t DEPENDENCE)

$$\underline{v}\cdot\nabla f = v_i \frac{\partial}{\partial x_i} f = v_i \frac{\partial f}{\partial E}\frac{\partial E}{\partial x_i} = v_i \frac{\partial f}{\partial E}\frac{\partial \Phi}{\partial x_i}$$

$$\nabla\Phi\cdot\frac{\partial f}{\partial \underline{v}} = \frac{\partial}{\partial x_i}(\Phi)\cdot\frac{\partial}{\partial v_i}(f) = \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial E}\frac{\partial E}{\partial v_i} = v_i\frac{\partial f}{\partial E}\frac{\partial \Phi}{\partial x_i}$$

$$\Rightarrow \underline{v}\cdot\nabla f = \nabla\Phi\cdot\frac{\partial f}{\partial \underline{v}} \Rightarrow \text{CBE LHS} = 0$$

$$f(E) = \frac{\rho_0}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}v^2 - \Phi(x)\right)$$

$$\rho = \int_{\forall v} f\,dv = \frac{\rho_0}{\sqrt{2\pi}}\exp(-\Phi(x))\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}v^2\right)dv$$

$$= \frac{\rho_0}{\sqrt{2\pi}}\exp(-\Phi(x))\sqrt{\pi} = \frac{\rho_0}{\sqrt{2}}\exp(-\Phi(x))$$

POISSON EQ: $\nabla^2\Phi = 4\pi G\rho \Rightarrow \Phi'' = 4\pi G\frac{\rho_0}{\sqrt{2}}\exp(-\Phi) = 4\pi G\rho_0 e^{-\Phi}$

2010, PAPER 3, QUESTION 6 | I

i/
COLLISIONLESS BOLTZMANN EQUATION:

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f - \nabla \Phi \frac{\partial f}{\partial v} = 0$$

FIRST MOMENT OF CBE:

$$\int_{\substack{\text{VELOCITY}\\\text{SPACE}}} \left(\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} \right) \cdot v_j \, d^3\underline{v} = 0$$

CONSIDER:

I) $$\frac{\partial}{\partial t}(f v_j) = v_j \frac{\partial f}{\partial t} + f \underbrace{\cancel{\frac{\partial v_j}{\partial t}}}_{=0} = v_j \frac{\partial f}{\partial t}$$

$$\Rightarrow \int \frac{\partial f}{\partial t} v_j \, d^3\underline{v} = \int \frac{\partial}{\partial t}(f v_j) \, d^3\underline{v} = \frac{\partial}{\partial t} \int f v_j \, d^3\underline{v}$$

II) $\frac{\partial \Phi}{\partial x_i}$ DOES NOT HAVE $\underline{v}$ DEPENDENCE

$$\text{I \& II} \Rightarrow \frac{\partial}{\partial t} \int f v_j \, d^3\underline{v} + \int v_i v_j \frac{\partial f}{\partial x_i} \, d^3\underline{v} - \frac{\partial \Phi}{\partial x_i} \int v_j \frac{\partial f}{\partial v_i} \, d^3\underline{v} = 0$$

PROCEEDING TERM BY TERM:

$$\frac{\partial}{\partial t} \int f v_j \, d^3\underline{v} = \frac{\partial}{\partial t}(\rho \langle v_j \rangle) = \rho \frac{\partial \langle v_j \rangle}{\partial t}$$

$$\int v_i v_j \frac{\partial f}{\partial x_i} \, d^3\underline{v} \underset{\uparrow}{=} \frac{\partial}{\partial x_i} \int v_i v_j f \, d^3\underline{v} = \frac{\partial}{\partial x_i}\left(\rho \langle v_i v_j \rangle\right)$$

$v_i v_j$ DOES
NOT HAVE
x_i DEP.

2010
P3Q6 (II)

$$\int v_j \frac{\partial f}{\partial v_i} d^3\underline{v} = \Big[\underbrace{f v_j}_{\to 0}\Big]_{-\infty}^{\infty} - \int \frac{\partial v_j}{\partial v_i} f \, d^3\underline{v} = -\delta_{ij}\int f \, d^3\underline{v}$$

AS WE WANT PHYSICAL BCS TO PREVAIL

COLLECT TERMS:

$$\rho \frac{\partial \langle v_j\rangle}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho \langle v_i v_j\rangle\right) + \frac{\partial \Phi}{\partial x_j}\overbrace{\int f \, d^3\underline{v}}^{\rho} = 0 \qquad (A)$$

RECALL (OR LOOK UP IN FORMULAE BOOKLET):

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho\langle v_i\rangle)}{\partial x_i} = 0 \qquad (B)$$

$$\cancel{\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_i}\langle v_i\rangle + \frac{\partial\langle v_i\rangle}{\partial x_i}\rho = 0 \qquad (B)}$$

(A) $-\langle v_j\rangle$ (B):

$$\rho \frac{\partial \langle v_j\rangle}{\partial t} - \cancel{\frac{\partial \rho}{\partial t}\langle v_j\rangle} - \langle v_j\rangle \frac{\partial}{\partial x_i}\left(\rho\langle v_i\rangle\right) +$$

$$+ \frac{\partial}{\partial x_i}\left(\rho\langle v_i v_j\rangle\right) + \rho\frac{\partial \phi}{\partial x_j} = 0$$

$$\rho \frac{\partial \langle v_j\rangle}{\partial t} - \langle v_j\rangle \frac{\partial}{\partial x_i}\left(\rho\langle v_i\rangle\right) + \frac{\partial}{\partial x_i}\left(\rho\langle v_i v_j\rangle\right) = -\rho\frac{\partial \Phi}{\partial x_j}$$

CONSIDER:

$$\sigma^2_{ij} = \langle (v_i - \langle v_i\rangle)(v_j - \langle v_j\rangle)\rangle =$$

2010
P3Q6 (iii)

$$= \langle v_i v_j - v_i \langle v_j \rangle - \langle v_i \rangle v_j + \langle v_i \rangle \langle v_j \rangle \rangle$$

$$= \langle v_i v_j \rangle - \cancel{\langle v_i \rangle \langle v_j \rangle} - \langle v_i \rangle \langle v_j \rangle + \cancel{\langle v_i \rangle \langle v_j \rangle}$$

$$= \langle v_i v_j \rangle - \langle v_i \rangle \langle v_j \rangle = \sigma_{ij}^2$$

$$\Rightarrow \langle v_i v_j \rangle = \sigma_{ij}^2 + \langle v_i \rangle \langle v_j \rangle$$

REWRITE EQUATION:

$$\rho \frac{\partial \langle v_j \rangle}{\partial t} - \langle v_j \rangle \frac{\partial}{\partial x_i}\left(\rho \langle v_i \rangle\right) + \frac{\partial}{\partial x_i}\left(\rho \sigma_{ij}^2\right) + \frac{\partial}{\partial x_i}\left(\rho \langle v_i \rangle \langle v_j \rangle\right) = -\rho \frac{\partial \Phi}{\partial x_j}$$

$$\Rightarrow \underbrace{\rho \frac{\partial \langle v_j \rangle}{\partial t} - \langle v_j \rangle \frac{\partial}{\partial x_i}\left(\rho \langle v_i \rangle\right) + \frac{\partial}{\partial x_i}\left(\rho \langle v_i \rangle \langle v_j \rangle\right)} = -\rho \frac{\partial \Phi}{\partial x_j} - \frac{\partial}{\partial x_i}\left(\rho \sigma_{ij}^2\right)$$

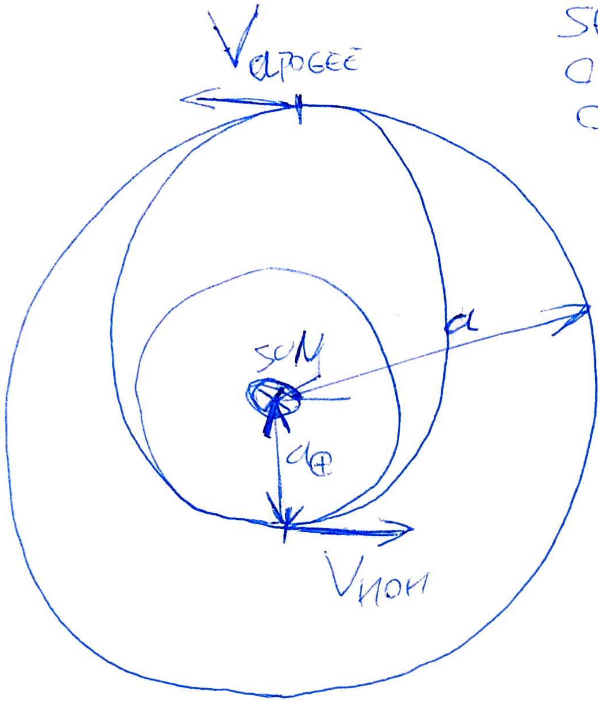
$$\cancel{-\langle v_j \rangle \frac{\partial}{\partial x_i}\left(\rho \langle v_i \rangle\right)} + \cancel{\frac{\partial}{\partial x_i}\left(\rho \langle v_i \rangle\right)\langle v_j \rangle} + \frac{\partial}{\partial x_i}\left(\rho \langle v_j \rangle\right)\langle v_i \rangle$$

$$\Rightarrow \boxed{\rho \frac{\partial \langle v_j \rangle}{\partial t} + \rho \langle v_i \rangle \frac{\partial}{\partial x_i} \langle v_j \rangle = -\rho \frac{\partial \Phi}{\partial x_j} - \frac{\partial}{\partial x_i}\left(\rho \sigma_{ij}^2\right)}$$

AS REQUIRED.

2010 P3Q6 IV

KEPLER III: $T^2 \propto a^3$



SEMI MAJOR AXIS OF TRANSFER ORBIT: $\frac{a_\oplus + a}{2}$

$$\frac{T_{\text{FULL TRANSFER ORB.}}^2}{T_{\text{EARTH}}^2} = \frac{\left(\frac{a_\oplus + a}{2}\right)^3}{a_\oplus^3}$$

T_{EARTH} – 1 YR

$$T_{FTO} = \left[\frac{1}{2}\left(1 + \frac{a}{a_\oplus}\right)\right]^{\frac{3}{2}} \text{ YR}$$

$$T = \frac{T_{FTO}}{2} = \frac{1}{2}\left(\frac{1}{2}\right)^{\frac{3}{2}}\left(1 + \frac{a}{a_\oplus}\right)^{\frac{3}{2}} = \frac{1}{2\sqrt{8}}\left(1 + \frac{a}{a_\oplus}\right)^{\frac{3}{2}} \text{YR}$$

(WE'RE ONLY GOING UP, NOT UP & DOWN)

$$= \frac{1}{4\sqrt{2}}\left(1 + \frac{a}{a_\oplus}\right)^{\frac{3}{2}} \text{YR}$$

AS REQUIRED.

ENERGY CONSERVATION:

$$\frac{1}{2}V_{HOH}^2 - \frac{GM}{a_\oplus} = \frac{1}{2}V_{APOGEE}^2 - \frac{GM}{a}$$

ANGULAR MOM CONSERVATION:

$$V_{HOH}\, a_\oplus = V_{APOGEE}\, a$$

$$\frac{1}{2}V_{HOH}^2 - \frac{GM}{a_\oplus} = \frac{1}{2}V_{HOH}^2\left(\frac{a_\oplus}{a}\right)^2 - \frac{GM}{a}$$

2010
P3Q6 V

$$\frac{1}{2}V_{Hoh}^2\left(1-\left(\frac{a_\oplus}{a}\right)^2\right)=GM\left(\frac{1}{a_\oplus}-\frac{1}{a}\right)$$

$$V_{Hoh}^2=GM\cdot 2\,\frac{\frac{a-a_\oplus}{aa_\oplus}}{1-\left(\frac{a_\oplus}{a}\right)^2}$$

$$=GM\cdot 2\,\frac{(a-a_\oplus)\left(\frac{a}{a_\oplus}\right)}{a^2-a_\oplus^2}=\frac{\frac{a}{a_\oplus}}{a+a_\oplus}2GM$$

EARTH CASE:

$$\frac{GM}{a_\oplus^2}=\frac{V_\oplus^2}{a_\oplus}\Rightarrow V_\oplus^2=\frac{GM}{a_\oplus}$$

$$V_{Hoh}^2=\frac{a}{a+a_\oplus}2\cdot\frac{GM}{a_\oplus}$$

$$=\frac{a}{a+a_\oplus}\cdot 2\cdot V_\oplus^2$$

$$\Rightarrow V_{Hoh}=\sqrt{2}\left(\frac{a}{a+a_\oplus}\right)^{\frac{1}{2}}V_\oplus$$

IE

$$V_{ADD}=V_{Hoh}-V_\oplus=V_\oplus\left[\sqrt{2}\left(\frac{a}{a+a_\oplus}\right)^{\frac{1}{2}}-1\right]$$

~~QED~~ AS REQUIRED.

2010
P3QE VI

~~APHELION DIST = a (1+e)~~

APHELION DIST = SEMI-MAJOR AXIS · (1+e)

$$a = \frac{a + a_\oplus}{2}(1+e)$$

$$\frac{2a}{a + a_\oplus} - 1 = e$$

$$e = \frac{2 \cdot 1.524}{1.524 + 1} - 1 \cong \underline{\underline{0.21}}$$

$$T = \frac{1}{4\sqrt{2}}\left(1 + \frac{1.524}{1}\right)^{\frac{3}{2}} \cong \underline{\underline{0.71 \text{ YR}}}$$

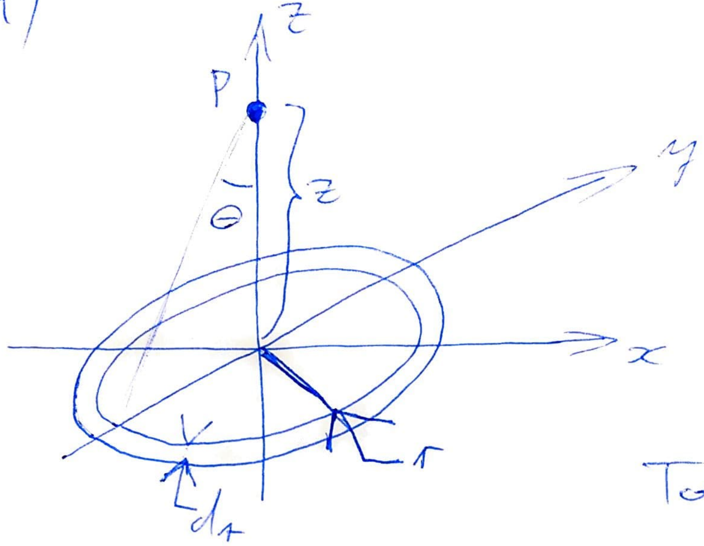
$$V_{ADD} = V_\oplus\left(\sqrt{2}\left(\frac{\cancel{a}\,1.524}{\cancel{a}\,1 + 1.524}\right)^{\frac{1}{2}} - 1\right) \cong 0.10 V_\oplus$$

$$\cancel{= V_\oplus \cdot 2} \qquad V_\oplus = \frac{150 \cdot 10^6 \cdot 10^3 \cdot 2\pi}{365 \cdot 24 \cdot 60 \cdot 60} = 4.8 \cdot 10^3 \cdot 2\pi \ \frac{m}{s}$$

$$\Rightarrow V_{ADD} \cong 4.8 \cdot 10^2 \ \cancel{10^3} \cdot 2\pi \cong \underline{\underline{3}} \ \frac{km}{s}$$

2011 P4Q6 (I)

(i)



force on point P from ring with radius r, thickness dr:

$$dF = G\frac{2\pi r\,\Sigma_0\,dr}{r^2+z^2}\cdot\cos\theta$$

$$= 2\pi G\Sigma_0\,\frac{r}{r^2+z^2}\,\frac{z}{\sqrt{r^2+z^2}}\,dr$$

Total force:

$$F = \int dF = -2\pi G\Sigma_0\int_0^\infty \frac{rz}{(r^2+z^2)^{\frac{3}{2}}}\,dr$$

$$= -2\pi G\Sigma_0$$ (WHERE ⊖ SIGN SIGNIFIES THAT FORCE IS TOWARDS LAYER)

$$F = -\nabla\Phi \Rightarrow \underline{\underline{\Phi = 2\pi G\Sigma_0|z|}}$$

ABSOLUTE VALUE SIGN IS THERE SO THAT FORCE IS STILL IN LAYER'S DIRECTION WHEN $z<0$ TOO.

(INTEGRATION STEP:

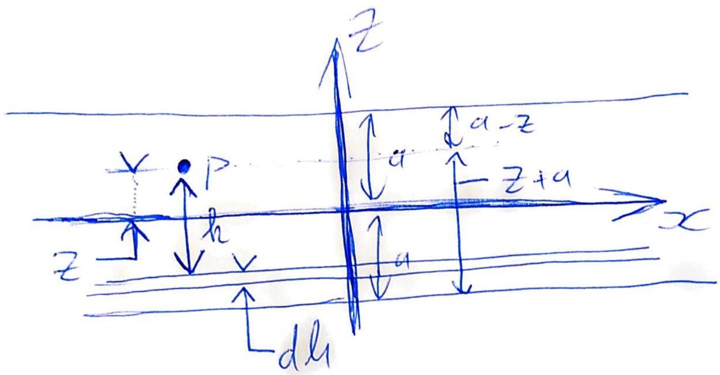
$$\int_0^\infty \frac{rz}{(r^2+z^2)^{\frac{3}{2}}}\,dr = \int_0^\infty \frac{\cancel{r}z}{u^{\frac{3}{2}}}\,\frac{du}{2\cancel{r}} = \frac{1}{2}\int_0^\infty u^{-\frac{3}{2}}z\,du = \frac{1}{2}(-2)u^{-\frac{1}{2}}\Big|_0^\infty z$$

$$= -\frac{1}{\sqrt{r^2+z^2}}\Big|_0^\infty z = -\left(0-\frac{z}{|z|}\right) = \underline{\mathrm{SIGN}(z)}\;)$$

$u = r^2+z^2$

$\frac{du}{dr} = 2r$

2011
P4Q6 (II)



POTENTIAL AT P FROM LAYER THICKNESS dh, DISTANCE h:

$$2\pi G\Sigma_0 |h|$$

$$\Phi = \int d\Phi = \int_{z+a}^{a-z} 2\pi G\Sigma_0 |h| dh = 2\pi G\Sigma_0 \left(\int_0^{z+a} h\,dh + \int_0^{a-z} h\,dh \right)$$

$$\cancel{= 2\pi G\Sigma_0 \frac{1}{2} h^2 \Big|_{z+a}^{a-z}}$$

$$= 2\pi G\Sigma_0 \frac{1}{2}\left(h^2 \Big|_0^{z+a} + h \Big|_0^{a-z} \right)$$

$$= 2\pi G\Sigma_0 \frac{1}{2}\left(z^2 + a^2 + \cancel{2az} + a^2 + z^2 - \cancel{2az} \right)$$

$$= 2\pi G\Sigma_0 \left(z^2 + a^2 \right)$$

2011 P4Q6 (III)

(ii) INTRODUCE RELATIVE POTENTIAL & RELATIVE ENERGY

$$\Psi = -\Phi + \Phi_0$$

$$\mathcal{E} = -E + \Phi_0$$

IE. FOR $|z| < a$:

$$\Psi = -2\pi G \rho_0 (z^2 + a^2) + \Phi_0$$

$$\mathcal{E} = -\left(2\pi G \rho_0 (z^2 + a^2) + \tfrac{1}{2} v^2\right) + \Phi_0$$

CHOOSE Φ_0 S.T. $f > 0 \ \forall \mathcal{E} > 0$

$f = 0 \ \forall \mathcal{E} \leq 0$

We don't want stars outside the layer

$\Rightarrow f(|z| \geq a) = 0$

~~$\Rightarrow f(|z| = a) = 0$~~

~~f is continuous~~

At $z = a$, $v = 0$, so stars don't wonder off from layer.

$$\mathcal{E}\Big|_{z=a} = -\left(2\pi G \rho_0 (a^2 + a^2) + \tfrac{1}{2} 0^2\right) + \Phi_0$$

$$= -4\pi G \rho_0 a^2 + \Phi_0 = 0$$

$$\Rightarrow \Phi_0 = +4\pi G \rho_0 a^2$$

$$\Rightarrow \mathcal{E} = -2\pi G \rho_0 (z^2 + a^2) - \tfrac{1}{2} v^2 + 4\pi G \rho_0 a^2$$

$$= -2\pi G \rho_0 (z^2 - a^2) - \tfrac{1}{2} v^2$$

$$= \tfrac{1}{2} \underbrace{(4\pi G \rho_0)}_{\omega^2} (a^2 - z^2) - \tfrac{1}{2} v^2$$

AS REQUIRED.

2011 P4Q6 (IV)

$$\psi = -2\pi G \rho_0 (z^2 + a^2) + 4\pi G \rho_0 a^2$$

$$= \frac{1}{2}\omega^2 (a^2 - z^2)$$

SO WE HAVE: $\varepsilon = \psi - \frac{1}{2} v^2 \Longrightarrow d\varepsilon = -v\,dv$

$$\rho(z) = \rho_0 = \int_{-\infty}^{\infty} f\,dv = 2\int_{0}^{\infty} f\,dv$$

BY SYMMETRY

$$= 2\int_0^{V_{MAX}} f\,dv$$

V_{MAX}: WHERE $f \geq 0$, IE $\varepsilon \geq 0$,

v IS MAX WHEN $\varepsilon = \psi - \frac{1}{2}v^2 = 0 \Rightarrow v = \sqrt{2(\psi - \varepsilon)}$

$$\Rightarrow v_{MAX} = \sqrt{2\psi}$$

$$\rho(z) = 2\int_0^{\sqrt{2\psi}} f\,dv = 2\int_{\varepsilon|_{V=0}}^{\varepsilon|_{V_{MAX}}} f \frac{-1}{v}\,d\varepsilon = 2\int_{\psi}^{0} \frac{-f\,d\varepsilon}{\sqrt{2(\psi-\varepsilon)}}$$

$$= \sqrt{2}\int_0^{\psi} \frac{f\,d\varepsilon}{\sqrt{\psi - \varepsilon}} = \sqrt{2}\int_0^{\frac{1}{2}\omega^2(a^2-z^2)} \frac{f(\varepsilon)\,d\varepsilon}{\sqrt{\frac{1}{2}\omega^2(a^2-z^2) - \varepsilon}}$$

$\boxed{\text{2011 P4Q6 (V)}}$

$$\mathcal{E}\Big|_{z=0} = \frac{1}{2}\omega^2 a^2 - \frac{1}{2}v^2$$

$$\cancel{f\left(\mathcal{E}\big|_{z=0}\right) = \sqrt{2}\int_0^{\frac{1}{2}\omega^2 a^2} \frac{f(\mathcal{E})\,d\mathcal{E}}{\sqrt{\frac{1}{2}\omega^2 a^2 - \mathcal{E}}}}$$

FOR $z=0$:

$$\int_0^{\frac{1}{2}\omega^2 a^2} \frac{f(\mathcal{E})\,d\mathcal{E}}{\sqrt{\frac{1}{2}\omega^2 a^2 - \mathcal{E}}} = \frac{S_0}{\sqrt{2}}$$

USING HINT:

$$f(\mathcal{E}) = \frac{1}{\pi}\frac{d}{d\mathcal{E}}\int_0^{\mathcal{E}} \frac{\frac{S_0}{\sqrt{2}}}{\sqrt{\mathcal{E} - \frac{1}{2}\omega^2 a^2}}\, d\left(\frac{1}{2}\omega^2 a^2\right)$$

$$= \frac{S_0}{\sqrt{2}\pi}\frac{d}{d\mathcal{E}}\int_0^{\mathcal{E}} \frac{1}{\sqrt{\mathcal{E}-x}}\,dx = \frac{S_0}{\sqrt{2}\pi}\frac{d}{d\mathcal{E}}\left[(-2)\sqrt{\mathcal{E}-x}\,\Big|_0^{\mathcal{E}}\right]$$

$$= -\frac{\sqrt{2}S_0}{\pi}\frac{d}{d\mathcal{E}}\left(-\sqrt{\mathcal{E}}\right) = \frac{\sqrt{2}S_0}{\pi}\,\frac{1}{2}\mathcal{E}^{-\frac{1}{2}} = \frac{S_0}{\sqrt{2}\pi}\,\mathcal{E}^{-\frac{1}{2}}$$

$$= \frac{S_0}{\sqrt{2}\pi}\left(\psi - \frac{1}{2}v^2\right)^{-\frac{1}{2}} = \frac{S_0}{\sqrt{2}\pi}\left(\frac{1}{2}\omega^2 a^2 - \frac{1}{2}v^2\right)^{-\frac{1}{2}} = f(z=0, v)$$