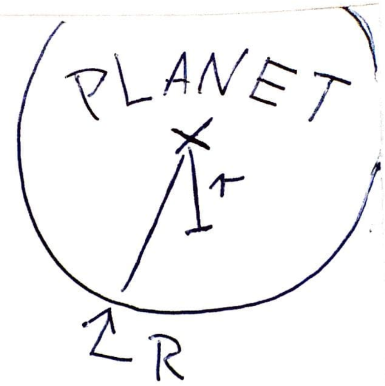
AFD II



1/ HYDROSTATIC EQUILIBRIUM: $\frac{1}{\rho}\nabla p = -\nabla\psi$

$$\nabla\psi(r) = -\frac{GM_{(ENCLOSED)}}{r^2}$$

$$= -G\frac{1}{r^2}\frac{4}{3}r^3\pi\rho = \frac{4}{3}r(-G\pi\rho)$$

$$\frac{1}{\rho}\nabla p = -\frac{4}{3}G\pi\rho r$$

$$\frac{1}{\rho}\frac{d}{dr}p = -\frac{4}{3}G\pi\rho r$$

$$p = -\frac{2}{3}G\pi\rho^2 r^2 + C$$

WE NEED:

$$p\big|_{r=R} = 0 \Rightarrow C = \frac{2}{3}G\pi\rho^2 R^2$$

$$\Rightarrow p = \frac{2}{3}G\pi\rho^2(R^2 - r^2)$$

$$p_{MAX} = p\big|_{r=0} = \frac{2}{3}G\pi\rho^2 R^2$$

THIS MUST NOT BE GREATER THAN p_0.

$$p_0 = \frac{2}{3}G\pi\rho^2 R^2$$

$$\rho = \sqrt{\frac{3p_0}{G}\frac{1}{2\pi}}\frac{1}{R} \Rightarrow R_{MAX} = \sqrt{\frac{3p_0}{G}\frac{1}{2\pi}}\frac{1}{\rho}$$

$$\cancel{M_{MAX} = \rho_{MAX}V = \frac{4}{3}R^3\pi\sqrt{\frac{3p_0}{G}\frac{1}{2\pi}}\frac{1}{R}}$$

$$M_{MAX} = \rho R_{MAX}^{3} \frac{4}{3}\pi$$

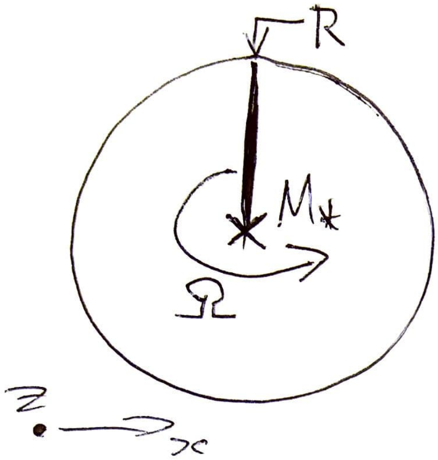
$$= \frac{4}{3}\pi \sqrt{\left(\frac{3P_0}{G}\right)^3 \left(\frac{1}{2\pi}\right)^3 \frac{1}{\rho^2}}$$

$$= \frac{2}{3\rho^2} \sqrt{\frac{1}{2\pi}\left(\frac{3P_0}{G}\right)^3}$$

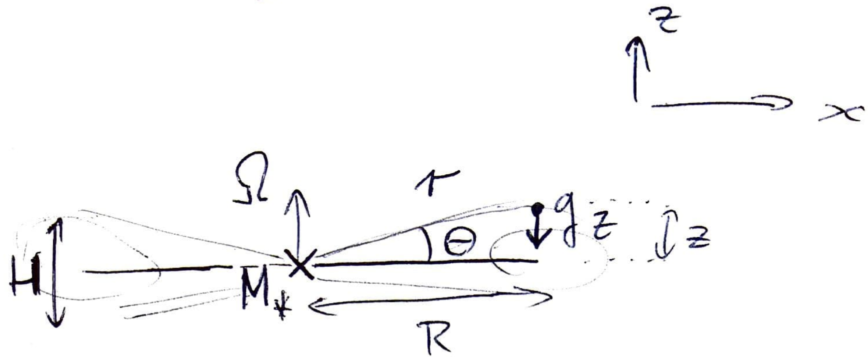
AS REQUIRED.

3/ VIEW FROM

TOP



SIDE



$$g_z = -\frac{GM}{r^2}\sin\theta = -\frac{GM}{r^2}\frac{z}{r} \sim$$

$$\sim -\frac{GM}{R^3} z$$

HYDROSTAT. EQ.:

$$\frac{1}{\rho}\nabla p = -\nabla\psi$$

LETS MOVE TO ROTATING FRAME SO GAS IS STATIONARY AT $z=0$, AT $z \neq 0$ THE ONLY ACCELERATION IS g_z.

ISOTHERMAL GAS: $p = \frac{R_*}{\mu}\rho T \Rightarrow p = A\rho$ WITH $A = \frac{R_*}{\mu}T$

$$\frac{1}{\rho}\frac{\partial}{\partial z}(A\rho) = -\frac{GM_*}{R^3}z \qquad \Big/ \text{INTEGRATE}$$

$$A \ln\rho = -\frac{GM_*}{2R^3}z^2 + C$$

$$\rho = \rho_0 \exp\left(-\frac{GM_* z^2}{2AR^3}\right) \quad \text{WHERE } \rho_0 = \rho\big|_{z=0}$$

NOTING THAT:

$$\frac{GM_*}{R^2} = \Omega^2 R$$

REWRITE:

$$S = S_0 \exp\left(-\frac{\Omega^2 z^2}{2A}\right)$$

SUB FOR A:

$$S = S_0 \exp\left(-\frac{\Omega^2 z^2 \mu}{2R_* T}\right)$$

THIS IS INDEED A GAUSSIAN.

e-FOLDING LENGTH:

$$\cancel{\frac{2R_* 1}{\Omega^2}} \quad \cancel{\frac{\Omega^2 \mu}{2R_* T}} \quad \sqrt{\frac{2R_* T}{\Omega^2 \mu}}$$

RING IS THIN IF E-FOLDING $\ll$ R.

$$\sqrt{\frac{2R_* T}{\Omega^2 \mu}} \ll R$$

$$T \ll \frac{R^2 \Omega^2 \mu}{2R_*}$$

IF $R = 1.5 \cdot 10^{11}$ m, $\Omega = \frac{2\pi}{365 \cdot 24 \cdot 60^2}$, $\mu = 2$ (FOR H, WHICH IS PROBABLY MOST ABUNDANT)

$R_* = 8400$,

THEN:

$$T \ll 10^5 \text{ K}$$

SEEMS A BIT HIGH.

3/ POISSON: $\nabla^2 \psi = 4\pi G \rho$

IN SPC:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) = 4\pi G \rho$$

IN OUR CASE:

$$\frac{\partial \psi}{\partial r} \propto \frac{\partial}{\partial r}\frac{1}{\sqrt{r^2+b^2}} \propto -\frac{1}{2}\left(r^2+b^2\right)^{-\frac{3}{2}} 2r \propto \left(r^2+b^2\right)^{-\frac{3}{2}} r$$

SUBSTITUTE IN:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left(r^2+b^2\right)^{-\frac{3}{2}} r\right) \propto 4\pi G \rho$$

$$\frac{\partial}{\partial r}\left(r^3\left(r^2+b^2\right)^{-\frac{3}{2}}\right) \propto \rho r^2$$

$$3r^2\left(r^2+b^2\right)^{-\frac{3}{2}} + r^3\left(-\frac{3}{2}\right)\left(r^2+b^2\right)^{-\frac{5}{2}} 2r \propto \rho r^2$$

$$\underbrace{3\cancel{r^2}\left(r^2+b^2\right)^{-\frac{3}{2}}}_{\propto \psi^3 \cdot c} - \underbrace{3r^{\cancel{4}\,2}\left(r^2+b^2\right)^{-\frac{5}{2}}}_{\propto r^2 \psi^5 \cdot c} \propto \rho \cancel{r^2}$$

$$\psi^3\left(1 - r^2\psi^2\right) \propto \rho$$

$$\Rightarrow \underline{\rho \propto r^2 \psi^5 + O\left(\psi^3\right)}$$

IT IS SUSPICIOUS THAT I'VE DONE STH WRONG.

CONTINUE WITH: $\rho \propto \psi^5$

$$\frac{1}{\rho}\nabla p = -\nabla \psi$$

$$\frac{dp}{dr} = -\rho \frac{d\psi}{dr} \Rightarrow \frac{dp}{d\psi} = -\rho$$

$$\Rightarrow \frac{dp}{d\psi} \propto \psi^5 \Rightarrow p \propto \psi^6 \Rightarrow p \propto \rho^{\frac{6}{5}} = \rho^{1+\frac{1}{5}}$$

$\Rightarrow$ POLYTROPIC EoS w/ $n=5$.

REST: PROBABLY UNDOABLE
WITHOUT GETTING
FIRST PART RIGHT.

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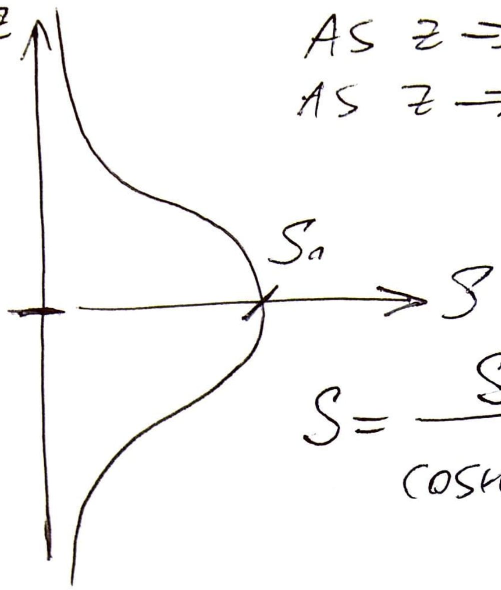


AS $z \to 0$, $\rho \to \rho_0$

AS $z \to \infty$, $\rho \to 0$

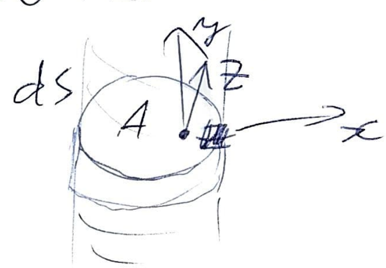
$$\rho = \frac{\rho_0}{\cosh^2\left(\sqrt{\frac{2\pi G \rho_0}{A}}\, z\right)} \quad \text{w/ } A = \frac{R_* T}{\mu}$$

(FROM NOTES)

POISSON: $\nabla^2 \psi = 4\pi G \rho$

FROM UNIT AREA OF SLAB LOOKING FROM ABOVE:

$$\int \nabla \cdot \nabla \psi \, dV = 4\pi G \int \rho \, dV$$



$$\int \nabla \psi \cdot ds = 4\pi G \int \rho \, dV$$

WHEN AT z, ACCELERATION:

$\ddot{z}$ = ACCELERATION FROM MASS ABOVE z − ACCELERATION FROM MASS BELOW z

GRAV. FIELD LINES ARE PARALLEL TO z, SO:

$$\int \nabla \psi \cdot ds = \nabla \psi \, A = -g A$$

ACCELERATION FROM MASS ABOVE z $= -\frac{1}{A} 4\pi G \int_z^\infty \rho \, dV$

$$= -\frac{1}{A} 4\pi G \int_z^\infty \rho A \, dz'$$

$$= -4\pi G \int_z^\infty \rho \, dz'$$

$$= -4\pi G \int_z^{\infty} \frac{\rho_0}{\cosh^2\left(\sqrt{\frac{2\pi G \rho_0}{A}}\, z'\right)}\, dz'$$

$$= -4\pi G \rho_0 \tanh\left(\sqrt{\frac{2\pi G \rho_0}{A}}\, z'\right) \sqrt{\frac{A}{2\pi G \rho_0}}\,\Bigg|_z^{\infty}$$

$$= -\sqrt{8\pi G A \rho_0}\left[1 - \tanh\left(\sqrt{\frac{2\pi G \rho_0}{A}}\, z\right)\right]$$

ACCELERATION FROM MASS BELOW z $= -4\pi G \int_{-\infty}^{z} \rho\, dz$

$$= -4\pi G \int_{-\infty}^{z} \frac{\rho_0 \, dz'}{\cosh^2\left(\sqrt{\frac{2\pi G \rho_0}{A}}\, z'\right)}$$

$$= -\sqrt{8\pi G A \rho_0} \tanh\left(\sqrt{\frac{2\pi G \rho_0}{A}}\, z'\right)\Bigg|_{-\infty}^{z}$$

$$= -\sqrt{8\pi G A \rho_0}\left[\tanh\left(\sqrt{\frac{2\pi G \rho_0}{A}}\, z\right) + 1\right]$$

TOTAL ACCELERATION IN $(+)z$ DIR = ACCELERATION FROM MASS ABOVE z − ACCELERATION FROM MASS BELOW z

$$= 2\sqrt{8\pi G A \rho_0} \tanh\left(\sqrt{\frac{2\pi G \rho_0}{A}}\, z\right)$$

VERTICAL VELOCITY $= \int \ddot{z}\, dt = 2\sqrt{8\pi G A \rho_0} \ln\left(\cosh\left(\sqrt{\frac{2\pi G \rho_0}{A}}\, z\right)\right) \cdot \sqrt{\frac{A}{2\pi G \rho_0}} + C$

$$= 4A \ln\left[\cosh\left(\sqrt{\frac{2\pi G \rho_0}{A}}\, z\right)\right] + C$$

STAR IS AT REST AT z_0, IE $\dot{z}=0$, SO:

$$C = -4A \ln\left[\cosh\left(\sqrt{\frac{2\pi G\rho_0}{A}}\, z_0\right)\right]$$

LET $a = \sqrt{\frac{2\pi G \rho_0}{A}}$

WE HAVE:

$$\dot{z} = 4A \ln \cosh(az) - 4A \ln \cosh(a z_0)$$

$$= 4 \frac{R_* T}{\mu} \ln \frac{\cosh(az)}{\cosh(a z_0)}$$

WHICH IS NOT WHAT WE WANT,
BUT ~~VERY CLOSE~~ SEEMS TO BE CLOSE to IT.

CHECK DIMENSIONS:

$$[R_*] = J/K$$

$$\left[\frac{R_* T}{\mu}\right] = \frac{J/K \cdot K}{kg} = \frac{kg\,\frac{m^2}{s^2}}{kg} = \frac{m^2}{s^2} \Rightarrow$$ THE ABOVE RESULT MUST BE WRONG.

5/ [EVEN IF THIS IS CORRECT (I HAVE DOUBTS)
I AM UNSURE WHAT I AM DOING]

ADIABATIC CASE $\Rightarrow \gamma = 1 + 1/n$

$\Rightarrow \text{đ}Q = 0$

FIRST LAW:

$$\text{đ}Q = d\mathcal{E} + \underbrace{p\,dV}_{-dW}$$

BECOMES:

$$d\mathcal{E} = -p\,dV$$

LAGRANGIAN VIEW:

$$\frac{D\mathcal{E}}{Dt} = \frac{DW}{Dt} + \frac{\text{đ}Q}{dt}$$

(OHH, WHY ISN'T THIS LAST TERM DQ/Dt, RATHER THAN $\text{đ}Q/dt$?)

$$\frac{DW}{Dt} = \frac{D}{Dt}\left(-pV\right)\Big|_p = -p\frac{D}{Dt}V = -p\frac{D}{Dt}\frac{1}{\rho} = \frac{p}{\rho^2}\frac{D\rho}{Dt}$$

$$\text{đ}Q = 0 \Rightarrow \text{đ}Q/dt = 0$$

$$\Rightarrow \frac{D\mathcal{E}}{Dt} = \frac{DW}{Dt} \Rightarrow \mathcal{E} = W + C$$

IF I STRETCH WHAT OUR NOTATION CAN BEAR:

$$D\mathcal{E} = DW$$

$$= \frac{DW}{Dt}Dt = \frac{p}{\rho^2}D\rho \Rightarrow \mathcal{E} = \int_{\text{ALL } \mathcal{E} \text{ LEVELS}} D\mathcal{E} = \int_0^{\rho} \frac{p}{\rho'^2}D\rho'$$

$$= \int_0^{s} \frac{r}{s'^2}\, ds' \qquad (?)$$

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$$P = K\rho^{1+1/n}$$
$$P = \frac{R_*}{\mu}\rho T$$
$$\Rightarrow T_c = \frac{\mu K}{R_*}\rho_c^{1/n}$$

$$T \text{ const} \Rightarrow K \propto \rho_c^{-1/n}$$

$$M = \int_0^{r_{MAX}} 4\pi r^2 \rho \, dr \qquad dr = \sqrt{\frac{K(1+n)}{4\pi G \rho_c^{1-1/n}}}\, d\xi$$

$$r^2 = \xi^2 \frac{K(1+n)}{4\pi G\rho_c^{1-1/n}}$$

$$= 4\pi \int_{\xi|_{r=0}}^{\xi|_{r_{MAX}}} \left(\frac{K(1+n)}{4\pi G \rho_c^{1-1/n}}\right)^{\frac{3}{2}} \rho_c \xi^2 \theta^3 d\xi$$

$$\Rightarrow M \propto \rho_c \left(\rho_c^{-1+\frac{1}{n}}\right)^{\frac{3}{2}} K^{\frac{3}{2}} \propto \rho_c^{\frac{1}{2}\left(\frac{3}{n}-1\right)} K^{\frac{3}{2}}$$

SUB IN FOR K:

$$\Rightarrow M \propto \rho_c^{\frac{1}{2}\left(\frac{3}{n}-1\right)} \left(\rho_c^{-\frac{1}{n}}\right)^{\frac{3}{2}} \propto \rho_c^{-1/2}$$

WE NOTE SOMEWHAT SURPRISEDLY THAT n DEPENDENCE HAS BEEN LOST.

$$\xi \propto \rho_c^{\frac{1}{2}(1-1/n)} K^{-\frac{1}{2}} r \propto \rho_c^{\frac{1}{2}(1-\frac{1}{n})} \left(\rho_c^{-\frac{1}{n}}\right)^{-\frac{1}{2}} r \propto \rho_c^{\frac{1}{2}} r$$

$$\xi|_{r_{MAX}} = \text{CONSTANT} \Rightarrow r \propto \rho_c^{-\frac{1}{2}}$$

NOTING THAT: $M \propto \rho_c^{-\frac{1}{2}}$ & $r_{MAX} = R \propto \rho_c^{-\frac{1}{2}} \Rightarrow \underline{M \propto R}$

7/

FLUID EQUATIONS:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0 \qquad \text{I (CONTINUITY)}$$

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\underline{u} = -\frac{1}{\rho} \nabla p \qquad \text{II (MOMENTUM)}$$

CONSIDER SMALL PERTURBATIONS:

$$\rho = \rho_0 + \Delta\rho \text{ WHERE } \Delta\rho = \Delta\rho_0 e^{i(kx-\omega t)} \qquad A$$

$$u = u_0 + \Delta u \text{ WHERE } \Delta u = \Delta u_0 e^{i(kx-\omega t)} \qquad B$$

[I HAVE CONCEPTUAL DIFFICULTIES WITH WHAT Δu IS EXACTLY IN THE LAGRANGIAN VIEW; I THINK I CAN VISUALIZE EULERIAN Δu]

SUBSTITUTE A&B TO I:

$$\frac{\partial}{\partial t}(\rho_0 + \Delta\rho) + \frac{\partial}{\partial x}\left([\rho_0 + \Delta\rho][u_0 + \Delta u]\right) =$$

$$= \frac{\partial \Delta\rho}{\partial t} + \frac{\partial}{\partial x}(\rho_0 \Delta u) + \frac{\partial}{\partial x}(u_0 \Delta\rho) + \frac{\partial}{\partial x}(\Delta\rho \Delta u)$$

$$= \frac{\partial \Delta\rho}{\partial t} + \rho_0 \frac{\partial}{\partial x}\Delta u + u_0 \frac{\partial \Delta\rho}{\partial x} + \text{HIGHER ORDER TERM}$$

NOTING THAT WE CAN SET $u_0 = 0$ BECAUSE LET'S HAVE THE WAVE IN A FLUID WHICH IS PREVIOUSLY ~~IN EQUILIBRIUM~~ AT REST.

OR u_0 IS 0 BECAUSE I AM TAKING LAGRANGIAN VIEW? SLIGHTLY CONFUSED.

$$\frac{\partial \Delta S}{\partial t} + S_0 \frac{\partial}{\partial x} \Delta u = 0 \qquad (1)$$

SUBSTITUTE TO MOMENTUM EQ.:

$$\frac{\partial}{\partial t}(u_0 + \Delta u) + (u_0 + \Delta u)\frac{\partial}{\partial x}(u_0 + \Delta u)$$

$$= \frac{\partial \Delta u}{\partial t} + \underbrace{u_0}_{\to 0} \frac{\partial}{\partial x} \Delta u \quad + \text{HIGHER ORDER TERM} =$$

$$= -\frac{1}{S_0 + \Delta S} \frac{\partial}{\partial x} \underbrace{(p_0 + \Delta p)}_{p}$$

EXPAND:

$$\frac{1}{S_0 + \Delta S} = \frac{1}{S_0} \frac{1}{1 + \frac{\Delta S}{S_0}} = S_0^{-1}\left(1 + \frac{\Delta S}{S_0}\right)^{-1} \cong S_0^{-1}\left(1 - \frac{\Delta S}{S_0}\right) \approx S_0^{-1}$$

WE GET:

$$\frac{\partial \Delta u}{\partial t} = -\frac{1}{S_0} \frac{\partial}{\partial x} \Delta p \qquad (2)$$

$\frac{\partial}{\partial t}$ OF (1):

$$\frac{\partial^2 \Delta S}{\partial t^2} + S_0 \frac{\partial}{\partial t} \frac{\partial}{\partial x} \Delta u = 0$$

$$\frac{\partial^2 \Delta S}{\partial t^2} = -S_0 \frac{\partial}{\partial x} \frac{\partial}{\partial t} \Delta u$$

USING (2):

$$\frac{\partial^2 \Delta S}{\partial t^2} = \cancel{-S_0}\left(\cancel{-\frac{1}{S_0}}\right) \frac{\partial^2}{\partial x^2} \Delta p$$

IF WE HAVE:

$$\Delta p = \left.\frac{dp}{d\rho}\right|_{\rho=\rho_0} \Delta\rho$$

CAN REWRITE:

$$\frac{\partial^2 \Delta\rho}{\partial t^2} = \left.\frac{dp}{d\rho}\right|_{\rho=\rho_0} \frac{\partial^2}{\partial x^2}\Delta\rho$$

THIS HAS SOLUTIONS IN THE FORM:

$$\Delta\rho = \Delta\rho_0 e^{i(kx-\omega t)}$$

LET: $\Delta u = \Delta u_0 e^{i(kx-\omega t)}$

(HMMM, I HAVEN'T DERIVED WHY EXACTLY THIS FORM IS JUSTIFIED FOR Δu)

SUBSTITUTE THIS TO ①, ~~ALONG WITH~~ $\cancel{u = u_0 + \Delta u}$

GET:

$$-i\omega\Delta\rho + \rho_0 ik\Delta u = 0$$

$$\Rightarrow \Delta u = \frac{\omega}{k}\frac{\Delta\rho}{\rho_0} = c_s\frac{\Delta\rho}{\rho_0}$$

$$\text{if } \frac{\Delta\rho}{\rho} \ll 1 \Rightarrow c_s \gg \Delta u$$

HOW DO I KNOW THAT $\Delta\rho$ & Δu HAS THE SAME FORM? MAYBE FROM THE NOTION THAT CHANGES IN DENSITY MUST BE DIRECTLY LINKED TO FLUID MOTION? CAN DO THE MATH BUT PHYSICAL UNDERSTANDING IS WEAK.

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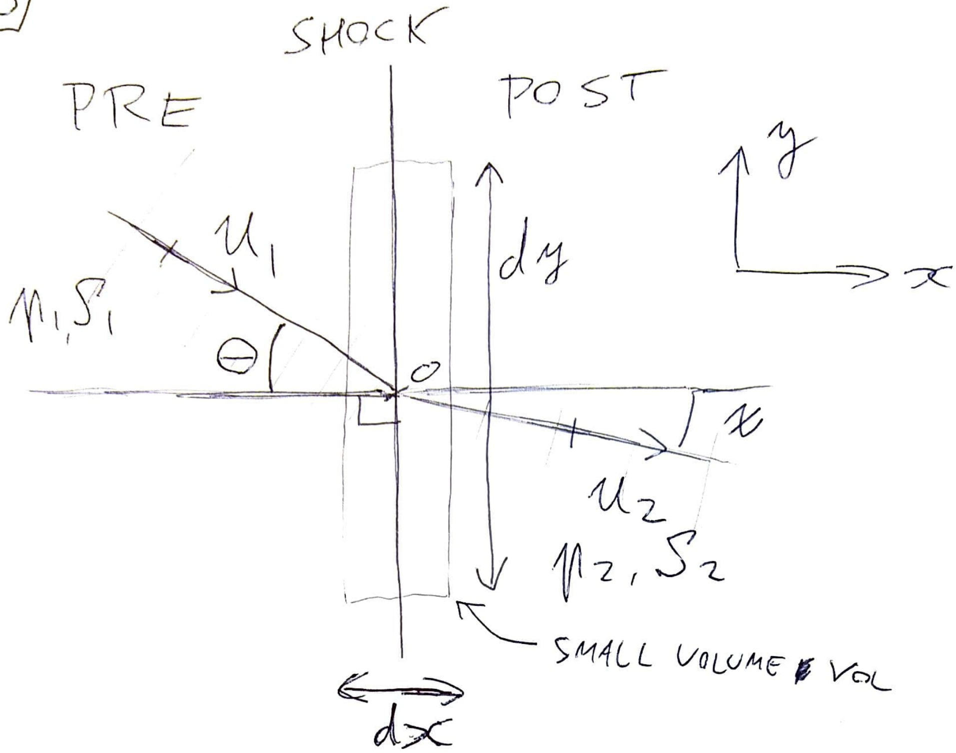

CONTINUITY EQUATION FOR VOL:

$$\cancel{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}} \qquad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0$$

$$\cancel{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial x}}$$

$$\cancel{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho \cos\theta\, u_1) + \frac{\partial}{\partial y}(\rho \sin\theta\, u_1)}$$

~~INTEGRATE UP:~~

$$\cancel{\int_{-\frac{dx}{2}}^{\frac{dx}{2}} \dot{\rho}\, dx + \rho u_x \Big|_{x=-\frac{dx}{2}} + \rho u_y \Big|_{x}}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) = 0$$

INTEGRATE UP:

$$\underbrace{\iint \dot{S}\,dx\,dy}_{0 \text{ IN STEADY STATE (NO PILING UP)}} + \iint \frac{\partial}{\partial x}(S u_x)\,dx\,dy + \iint \frac{\partial}{\partial y}(S u_y)\,dx\,dy = 0$$

$$\iint \frac{\partial}{\partial x}(S u_x)\,dx\,dy = \int_{-\frac{dy}{2}}^{\frac{dy}{2}} \Big[S u_x\Big]_{x=-\frac{dx}{2}}^{x=\frac{dx}{2}} dy$$

$$= \Big[S u_x\Big]_{-\frac{dx}{2}}^{\frac{dx}{2}} dy$$

$$\iint \frac{\partial}{\partial y}(S u_y)\,dx\,dy = S u_y \Big|_{-\frac{dy}{2}}^{\frac{dy}{2}} \cdot dx$$

REWRITE:

$$\Big[S u_x\Big]_{-\frac{dx}{2}}^{\frac{dx}{2}} dy + \Big[S u_y\Big]_{-\frac{dy}{2}}^{\frac{dy}{2}} dx = 0$$

I THINK I HAVE IMPLICITLY ASSUMED THAT $S = S(x)$ & NOT $S(x, y)$.

WHERE DO I EVALUATE THIS S THEN?

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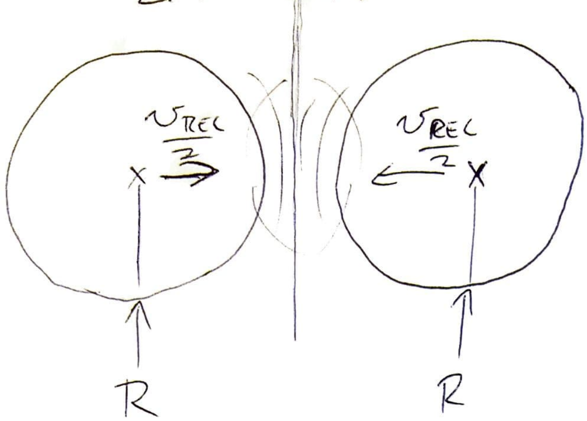
ZMF:



$$t_{COLL} = \frac{2R}{\frac{v_{REL}}{2}} = \frac{2 \cdot 3 \cdot 10^{16}}{4000} = \cancel{1.5}\,3 \cdot 10^{13}\ \text{SEC} \approx \cancel{500}\ \cancel{\sim 0.5\ \text{MYR}}$$

~ 1 MYR

$$Q = 10^{-4}\ \frac{J}{s\,kg} \sim 3000\ \frac{J}{YR\ kg}$$

COMPARE THESE TWO: ISOTHERMAL