AFD SHEET I

5.

(a) $\underline{b} \times \underline{\nabla} \times \underline{b} =$

ASSUMING THIS MEANS:

$= b \times (\underline{\nabla} \times \underline{b}) = \epsilon_{ijk} b_j (\underline{\nabla} \times \underline{b})_k$

$= \epsilon_{ijk} b_j \epsilon_{klm} \partial_l b_m$

$= \epsilon_{kij} \epsilon_{klm} b_j \partial_l b_m$

$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) b_j \partial_l b_m$

$= b_m \partial_i b_m - b_l \partial_l b_i$

$\cancel{= \partial_i (b_m b_m) - b_m \partial_i b_m - b_l \partial_l b_i}$

~~WHERE I HAVE USED~~

NOTE THAT:

$\partial_i (b_m b_m) = b_m \partial_i b_m + b_m \partial_i b_m$

$= 2 b_m \partial_i b_m$

$\Rightarrow \partial_i \left(\frac{1}{2} b_m b_m\right) = b_m \partial_i b_m$

USE THIS RESULT, REPLACE FIRST TERM:

$= \partial_i \left(\frac{1}{2} b_m b_m\right) - b_l \partial_l b_i$

$= \nabla\left(\frac{1}{2} \underline{b} \cdot \underline{b}\right) - \underline{b} \cdot \nabla \underline{b}$

AFD
SHEET II

NOW LET'S ASSUME ANOTHER ORDER:

$\underline{b} \times \underline{\nabla} \times \underline{b} =$

$= (\underline{b} \times \underline{\nabla}) \times \underline{b} = \cancel{\epsilon_{ijk} b_j \partial_k}$

$= \epsilon_{ijk} (\underline{b} \times \underline{\nabla})_j b_k$

$= \epsilon_{ijk} \epsilon_{jlm} b_l \partial_m b_k$

$\cancel{= \epsilon_{kij}}$

$= \epsilon_{jki} \epsilon_{jlm} b_l \partial_m b_k$

~~RELABEL: j → k, k → i, i → j,~~

$= (\delta_{kl}\delta_{im} - \delta_{il}\delta_{km}) b_l \partial_m b_k$

$= b_k \partial_i b_k - b_i \partial_k b_k$

$= \underline{\nabla}(\frac{1}{2} \underline{b} \cdot \underline{b}) - \cancel{\underline{b} \times} \underline{\nabla} \cdot \underline{b}\, \underline{b}$

AS FOUND PREVIOUSLY,
SO WE'RE GOOD.

(b)

$$\nabla\times(\nabla a)=$$

$$=\epsilon_{ijk}\,\partial_j(\nabla a)_k$$

$$=\epsilon_{ijk}\,\partial_j\,\partial_k\,a$$

THIS PART IS SYMMETRIC UNDER $j \leftrightarrow k$ SWAP

THIS PART IS ANTISYMMETRIC UNDER $j \leftrightarrow k$ SWAP

$$\Longrightarrow \underline{\nabla\times(\nabla a)=0}$$

(c) $\nabla\times(a\underline{b})=$

$$=\epsilon_{ijk}\,\partial_j(a\underline{b})_k=\cancel{a\,\epsilon_{ijk}\,\partial_j\,b_k}$$

$$=\epsilon_{ijk}\,\partial_j(a\,b_k)$$

$$=\epsilon_{ijk}(a\,\partial_j b_k+b_k\,\partial_j a)$$

$$=a\,\epsilon_{ijk}\,\partial_j b_k-\cancel{b_k\,\epsilon_{kij}\,\partial_j a}\,\epsilon_{ikj}\,b_k\,\partial_j a$$

$$\underline{=a\,\nabla\times\underline{b}-\underline{b}\times\underline{\nabla}a}$$

"USING THE ABOVE IDENTITIES"... PART

EULERIAN MOMENTUM EQUATION:

$$\rho \frac{\partial \underline{u}}{\partial t} + \rho(\underline{u}\cdot\nabla)\underline{u} = -\nabla p - \rho\nabla\Phi$$

CURL OF THIS:

$$\nabla\times\rho\frac{\partial \underline{u}}{\partial t} + \nabla\times\rho(\underline{u}\cdot\nabla)\underline{u} =$$
$$= \nabla\times-\nabla p - \nabla\times\rho\nabla\Phi$$

RHS = 0 USING RESULT FROM b.

WE HAVE:

$$\nabla\times\cancel{\rho}\frac{\partial \underline{u}}{\partial t} + \nabla\times\cancel{\rho}(\underline{u}\cdot\nabla)\underline{u} = 0$$

$$\nabla\times\frac{\partial \underline{u}}{\partial t} + \nabla\times(\underline{u}\cdot\nabla)\underline{u} = 0$$

$$\frac{\partial}{\partial t}\left(\nabla\times\underline{u}\right) + \nabla\times(\underline{u}\cdot\nabla)\underline{u} = 0$$

~~= 0 BECAUSE THE PROBLEM ASKS~~
~~SAYS SO.~~

$$\cancel{\nabla\times(\underline{u}\cdot\nabla)\underline{u} = 0}$$

USING RESULTS FROM (a), REWRITE $(\underline{u}\cdot\nabla)\underline{u}$ AS:

$$\underline{u}\cdot\nabla\underline{u} = \nabla\left(\frac{1}{2}\underline{u}\cdot\underline{u}\right) - \underline{u}\times\nabla\times\underline{u}$$

REWRITE $\nabla\times(\underline{u}\cdot\nabla\underline{u})$ AS:

$$\nabla\times(\underline{u}\cdot\nabla\underline{u}) = \underbrace{\nabla\times\nabla\left(\tfrac{1}{2}\underline{u}\cdot\underline{u}\right)}_{=0 \text{ BY RESULT (e)}} - \nabla\times(\underline{u}\times\nabla\times\underline{u})$$

$$= -\nabla\times(\underline{u}\times\nabla\times\underline{u})$$

WE HAVE

$$\frac{\partial}{\partial t}(\nabla\times\underline{u}) + \nabla\times(\underline{u}\times\nabla\times\underline{u}) = 0$$

AT $t = t_0$, WE HAVE:

$$\frac{\partial}{\partial t}(\nabla\times\underline{u}) = 0$$

$\Rightarrow$ $\nabla\times\underline{u}$ MUST REMAIN CONSTANT, IE 0.