2/

$$N = \int \begin{matrix}\text{ENERGY}\\ \text{DISTRIBUTION}\end{matrix} \times \begin{matrix}\text{DENSITY}\\ \text{OF STATES}\end{matrix} \; d \begin{matrix}\text{ALL POSSIBLE}\\ \text{ENERGIES}\end{matrix}$$

$$= \int_0^\infty dE \, \frac{g(E)}{z^{-1} e^{\beta E} - 1}$$

USE:

$g(E) = C \qquad x = \beta E \Rightarrow dE = \beta^{-1} dx$

REWRITE INTEGRAL:

$$N = \int_0^\infty \beta^{-1} dx \, \frac{C}{z^{-1} e^{x} - 1}$$

NOTE THAT THE INTEGRAL $\int \frac{C}{z^{-1} e^{\beta E} - 1} dE$ IS NONZERO EVERYWHERE SO WE'RE COUNTING GROUND STATES TOO, AS LONG AS $z < 1$.

$$N = k_B T \int_0^\infty \frac{C}{z^{-1} e^{\beta E} - 1} dE$$

DECREASE T : TO KEEP N FIXED, $\int_0^\infty \frac{C}{z^{-1} e^{x} - 1} dx$ MUST INCREASE.

WHEN $z = 1$, THIS INTEGRAL DIVERGES. AS z IS GETTING CLOSER TO 1, IT TENDS TO INFINITY. SO THERE IS NO CRITICAL T WHICH WOULD BE T_c.

$\Rightarrow$ NO BEC IS FORMING.

3.a

SCHRÖDINGER:

$$i\hbar \frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2}\psi + V\psi$$

, $V=0$ IN BOX, ∞ OUTSIDE

$$\Rightarrow \psi \propto e^{i\underline{k}\cdot\underline{x}}$$

BOUNDARY CONDITIONS REQUIRE: $k_i = \frac{2\pi n_i}{L}$ w/ $n_i \in \mathbb{Z}$

$$E_n \psi = -\frac{\hbar^2}{2m}\partial_x^2\psi = -\frac{\hbar^2}{2m}(i)^2 k^2\psi$$

$$\Rightarrow E_n = \frac{k^2\hbar^2}{2m} = \frac{4\pi^2\hbar^2}{2mL^2}\left(n_1^2+n_2^2+n_3^2\right)$$

$$\Rightarrow \underline{E_n \propto \frac{1}{mL^2}}$$

TOTAL NUMBER OF PARTICLES IN BOX $= \int$ ENERGY DISTRIBUTION $\times$ DENSITY OF STATES d ALL POSSIBLE ENERGIES

$$= \int \frac{1}{z^{-1}e^{\beta E}-1}\, g(E)\, dE$$

3.1

$$\sum_{\vec{n}} \approx \int_{\substack{\text{ALL}\\ n\text{ SPACE}}} d^3 n = \int_{\substack{\text{ALL } q\\ \text{SPACE}}} d^3\left(\frac{L q}{2\pi}\right) = \frac{L^3}{(2\pi)^3}\int_{\substack{\text{ALL}\\ q\text{ SPACE}}} d^3 q = \frac{4\pi L^3}{(2\pi)^3}\int_0^{\infty} q^2\, dq$$

$$= \frac{V}{2\pi^2}\int \frac{m}{\hbar^2 q}\, dE\, \frac{2mE}{\hbar^2}$$

USING: $E = \frac{\hbar^2 q^2}{2m} \Rightarrow q^2 = \frac{2mE}{\hbar^2}$

$$\Downarrow$$

$$dE = \frac{\hbar^2 q}{m}\, dq$$

$$= \frac{V}{2\pi^2}\int \frac{m}{\hbar^2\sqrt{\frac{2mE}{\hbar^2}}}\, \frac{2mE}{\hbar^2}\, dE = \frac{V}{2\pi^2}\int dE \sqrt{\frac{2mE}{\hbar^2}}\, \frac{m}{\hbar^2}\, dE$$

= ALL POSSIBLE STATES

$= \int$ DENSITY OF STATES dENERGIES (ALL)

$$= \int g(E)\, dE$$

$$\Rightarrow g(E) = \frac{V}{2\pi^2}\sqrt{\frac{2mE}{\hbar^2}}\,\frac{m}{\hbar^2} = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} E^{\frac{1}{2}}$$

RETURN TO EQUATION ON PREVIOUS PAGE.

$$N = \int_0^{\infty} \frac{1}{z^{-1} e^{\beta E} - 1}\, \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} E^{\frac{1}{2}}\, dE$$

3.11

SUB: $x = \beta E$

$$= \int \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \frac{\overbrace{x^{\frac{1}{2}}\beta^{-\frac{1}{2}}}^{\sim E^{\frac{1}{2}}}}{z^{-1}e^{x}-1} \underbrace{dx\,\beta^{-1}}_{dE}$$

$$= \frac{V}{4\pi^2}\left(\frac{2m k_B T}{\hbar^2}\right)^{\frac{3}{2}} \int_0^\infty dx \frac{x^{\frac{1}{2}}}{z^{-1}e^{x}-1}$$

TO GET T_c, SET $z=1$:

(ACTUALLY, WE DON'T CARE AS LONG AS INTEGRAL GIVES CONSTANT MULTIPLICATIVE FACTOR)

$$N \propto V (m T_c)^{\frac{3}{2}}$$

$$\Rightarrow \left(\frac{N}{V}\right)^{\frac{2}{3}} \frac{1}{m} \propto T_c$$

$$T_c \propto N^{2/3} / m V^{\frac{2}{3}}$$

RECALL: $V = L^3$

$V^{\frac{2}{3}} = L^2$

$$\underline{T_c \propto N^{2/3} / m L^2}$$

TONG NOTES 3.31:

FRACTION OF PARTICLES IN GROUND STATE $= \dfrac{n_0}{N} = 1 - \left(\dfrac{T}{T_c}\right)^{\frac{3}{2}}$

3.III

$$\Rightarrow n_0 = N - N\left(\frac{T}{T_c}\right)^{\frac{3}{2}}$$

NOT GROUND STATE:

$$N - n_0 = N - \left(N - N\left(\frac{T}{T_c}\right)^{\frac{3}{2}}\right) = N\left(\frac{T}{T_c}\right)^{\frac{3}{2}}$$

WHEN $T < T_c$:

$$N - n_0 = N\left(\frac{T}{T_c}\right)^{\frac{3}{2}} < O(N)$$

LETS DERIVE THE QUOTED EXPRESSION.

$$\frac{n_0}{N} = \frac{\frac{1}{z^{-1}-1}}{\frac{V}{\lambda^3} g_{3/2}(z) + \frac{z}{1-z}} = \frac{1}{\frac{\frac{V}{\lambda^3} g_{3/2}(z) + 1}{\frac{z}{1-z}}}$$

$$= \frac{1}{\frac{V}{\lambda^3} g_{3/2}(z) \frac{1-z}{z} + 1} = \frac{\frac{z}{1-z}}{\frac{V}{\lambda^3} g_{3/2}(z) + \frac{z}{1-z}}$$

$$= \frac{1}{N} \frac{z}{1-z} = \frac{n_0}{N}$$

AH, THIS IS WHERE I STARTED FROM.

4.a

$$N = \int dE\, G(E) P(E)$$

$$= \int dE\, C E^{\alpha-1} \frac{1}{z^{-1} e^{\beta E} - 1}$$

SUBSTITUTE: ~~βE~~

$x = \beta E \Rightarrow$ $dE = \beta^{-1} dx$

$E^{\alpha-1} = x^{\alpha-1} \beta^{-(\alpha-1)}$

$$N = \int \beta^{-1} dx\, C\, x^{\alpha-1} \beta^{-(\alpha-1)} \frac{1}{z^{-1} e^{x} - 1}$$

$$= \beta^{-1-(\alpha-1)} C \int \frac{x^{\alpha-1} dx}{z^{-1} e^{x} - 1}$$

$$\Rightarrow N \propto \beta^{-\alpha} \propto (k_B T)^{\alpha}$$

$$N = (k_B T)^{\alpha} C \int \frac{x^{\alpha-1} dx}{z^{-1} e^{x} - 1}$$

N IS CONSTANT: AS T GOES DOWN, $\int \ldots dx$ MUST GO UP.

$\Rightarrow$ z MUST APPROACH 1

FOR CRITICAL T, z = 1

$$N = (k_B T_c)^{\alpha} C \int_0^{\infty} \frac{x^{\alpha-1} dx}{e^{x} - 1}$$

4.1

NOTE:

$$g_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty dx \, \frac{x^{n-1}}{z^{-1} e^{x} - 1}$$

OUR INTEGRAL IS:

$$\int_0^\infty \frac{x^{\alpha-1}}{e^{x}-1} \, dx = g_\alpha(1) \cdot \Gamma(\alpha)$$

$$N = (k_B T_c)^\alpha C \, g_\alpha(1) \Gamma(\alpha)$$

$$N^{\frac{1}{\alpha}} = k_B T_c \left(C g_\alpha(1) \Gamma(\alpha) \right)^{\frac{1}{\alpha}}$$

$$T_c = \frac{1}{k_B} \left(\frac{C g_\alpha(1) \Gamma(\alpha)}{N} \right)^{\alpha}$$

"DETERMINE T_c FOR BOSONS IN A THREE-DIMENSIONAL TRAP"

$\boxed{5}$

2 STATES, EACH OCCUPIED OR NOT $\Rightarrow 2 \cdot 2 = 4$

$$GPF = \sum_{n=0,1} \left(\sum_{m=0,1} e^{-\beta n(E_n - \mu)} e^{-\beta m(E_m - \mu)} \right)$$

$$E_n = 0 \qquad E_m = \epsilon$$

$$= \sum_{n=0,1} \left(\sum_{m=0,1} e^{-\beta n(-\mu)} e^{-\beta m(\epsilon - \mu)} \right)$$

$$= \sum_{n=0,1} \left(e^{-\beta n(-\mu)} \right) \left(\underset{m=0}{\underset{\downarrow}{1}} + \underset{m=1}{\underset{\downarrow}{e^{-\beta(\epsilon - \mu)}}} \right)$$

$$= \left(1 + e^{\beta\mu}\right)\left(1 + e^{-\beta\epsilon} e^{\beta\mu}\right)$$

$$= (1 + z)(1 + e^{-\beta\epsilon} z)$$

NICE FACTORS HERE

$$= \underline{1 + z + z e^{-\beta\epsilon} + z^2 e^{-\beta\epsilon}}$$

$$Z_0 = \sum_{m=0,1} e^{-\beta n(0 - \mu)} = 1 + z$$

$$Z_1 = \sum_{m=0,1} e^{-\beta m(\epsilon - \mu)} = 1 + e^{-\beta\epsilon} z$$

MEAN OCCUPATION NUMBER $= \frac{1}{\beta}\frac{\partial}{\partial\mu}\log Z_\epsilon$

$$= \frac{1}{\beta}\frac{\partial}{\partial\mu}\log\left(1+e^{-\beta\epsilon}e^{\beta\mu}\right)$$

$$= \frac{1}{\beta}\frac{1}{1+e^{-\beta\epsilon}e^{\beta\mu}}e^{-\beta\epsilon}\beta e^{\beta\mu}$$

$$= \frac{e^{-\beta(\epsilon-\mu)}}{1+e^{-\beta(\epsilon-\mu)}}$$

$$= \frac{1}{e^{\beta(\epsilon-\mu)}+1}$$

FDD: $n_r = \frac{1}{e^{\beta(E_r-\mu)}+1}$, SAME AS ABOVE WITH $E_r = \epsilon$

FERMION INTERACTIONS:

COULD INCLUDE MULTIPLICATIVE TERMS IN THE GPF DEPENDING ON WHICH LEVELS INTERACT.

$\boxed{6}$

$$Z = \prod_{\substack{ALL \\ STATES}} z_r$$

$$= \prod_{\substack{ALL \\ STATES}} \left(1 + e^{-\beta(E_r - \mu)}\right)$$

$$S = \frac{\partial}{\partial T}\left(k_B T \ln Z\right)$$

$$= \frac{\partial}{\partial T}\left(k_B T \ln\left[\prod\left(1 + e^{-\beta(E_r-\mu)}\right)\right]\right)$$

$$= \frac{\partial}{\partial T}\left[k_B T \sum \ln\left(1 + e^{-\beta(E_r-\mu)}\right)\right]$$

$$= k_B \sum \ln\left(1 + e^{-\beta(E_r-\mu)}\right) + k_B T \sum \frac{\partial}{\partial T} \ln\left(1 + e^{-\beta(E_r-\mu)}\right)$$

$$= k_B \sum \ln\left(1 + e^{-\beta(E_r-\mu)}\right) + \sum k_B T \frac{1}{1 + e^{-\beta(E_r-\mu)}} \cdot \frac{\partial}{\partial T}\left(e^{-\beta(E_r-\mu)}\right)$$

$$= k_B \sum \ln\left(1 + e^{-\beta(E_r-\mu)}\right) + \sum k_B T \frac{1}{1 + e^{-\beta(E_r-\mu)}} e^{\beta\mu}(-E_r) e^{-\beta E_r} \frac{\partial \beta}{\partial T}$$

NOTE: $\frac{\partial \beta}{\partial T} = \frac{\partial}{\partial T}\left(\frac{1}{k_B T}\right) = -\frac{1}{k_B}\frac{1}{T^2}$

$$= k_B \sum \ln\left(1 + e^{-\beta(E_r-\mu)}\right) + \sum k_B T \frac{1}{1 + e^{-\beta(E_r-\mu)}} e^{\beta\mu}(-E_r)$$

$$\cdot e^{-\beta E_r} \cdot -\frac{1}{k_B}\frac{1}{T^2}$$

$$= k_B \sum \ln\left(1 + e^{-\beta(E_r - \mu)}\right) + \frac{1}{T} \sum \frac{e^{-\beta(E_r - \mu)}}{1 + e^{-\beta(E_r - \mu)}} E_r$$

$$= k_B \sum \ln\left(1 + e^{-\beta(E_r - \mu)}\right) + \frac{1}{T} \sum \frac{1}{e^{\beta(E_r - \mu)} + 1} E_r$$

AND NOW IDK WHAT.

7

MEAN NUMBER OF EXCITED ELECTRONS $= \int_{E>0}$ DENSITY OF STATES X PROBABILITY OF A PARTICULAR ENERGY LEVEL BEING FILLED UP dE

$\downarrow$

FERMI-DIRAC

$$= \int_0^\infty g(E) \frac{1}{e^{(E-\mu)/k_BT}+1} dE$$

THE LOWER LIMIT IS 0 BUT THE E=0 LEVEL DOES NOT CONTRIBUTE BECAUSE ~~THAT~~ ~~...~~ THEY ARE KILLED BY ~~...~~ $g(E)\big|_{E=0} = 0$ TERM.

$$\langle N \rangle = \int_0^\infty A\sqrt{E}\, dE \frac{1}{e^{(E-\mu)/k_BT}+1}$$

$$= \int_0^\infty A\sqrt{E} \frac{1}{e^{-(\Delta+\mu)/k_BT}+1} dE$$

$$= \int_0^\infty A\sqrt{E} \frac{e^{(\Delta+\mu)/k_BT}}{e^{(\Delta+\mu)/k_BT}+1} dE$$

$$= \int_0^\infty A\sqrt{E} \frac{1}{e^{(\Delta+\mu)/k_BT}+1} dE\, e^{(\Delta+\mu)/k_BT} = N e^{(\Delta+\mu)/k_BT}$$

WHICH IS WRONG

9

$$\langle E \rangle = \int \text{DENSITY OF STATES} \times \text{PROBABILITY OF BEING IN THAT E LEVEL} \times E \, dE$$

$$= \int dE \, \frac{E \, g(E)}{z^{-1} e^{\beta E} + 1}$$

$$pV = \frac{1}{\beta} \log Z$$

$$= \frac{1}{\beta} \log \prod_r \cancel{\frac{1}{1+e^{-\beta(E_r - \mu)}}} \left(1 + e^{-\beta(E_r - \mu)}\right)$$

$$= \frac{+1}{\beta} \sum_r \log\left(1 + e^{-\beta E_r} z\right)$$

$$= +\frac{1}{\beta} \int dE \, g(E) \log\left(1 + e^{-\beta E_r} z\right)$$

USE:

$$g(E) = 2\sqrt{E} \cdot c$$

$$pV = \frac{1}{\beta} \int dE \, 2\sqrt{E} \log\left(1 + e^{-\beta E_r} z\right)$$

$$= \frac{1}{\beta} \left[\begin{matrix} \text{BOUN-} \\ \text{DARY} \\ \text{TERMS} \end{matrix} \right] - \frac{1}{\beta} \int \frac{2}{3} E^{\frac{3}{2}} \, 2 \, \frac{1 \cdot c}{1 + z e^{-\beta E}} \cdot e^{-\beta E} (-\beta) z$$

$$= \frac{2}{3}\int E \underbrace{2\sqrt{E}\,c}_{g(E)} \frac{e^{-\beta E} z}{1+e^{-\beta E} z}\, dE$$

$$= \frac{2}{3}\int E\, g(E) \frac{1}{z^{-1}e^{\beta E}+1} = \frac{2}{3}E$$

$$g(E) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \sqrt{E}\, g_s$$

$$\frac{E}{V} = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} g_s \int_0^\infty dE \frac{E^{\frac{3}{2}}}{z^{-1}e^{\beta E}+1}$$

SUB:

$$x = \beta E \Rightarrow E^{\frac{3}{2}} = \beta^{-\frac{3}{2}} x^{\frac{3}{2}} \;\&\; dE = \beta^{-1} dx$$

$$= \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} g_s \frac{1}{\beta^{5/2}} \int_0^\infty dx \frac{x^{3/2}}{z^{-1}e^{x}+1}$$

WOULD NEED TO EXPAND THIS, BUT I DON'T GET HOW ITS DONE IN NOTES.

~~BASED ON HOPE NOT MATH~~

$$\cancel{= \frac{3z}{2\lambda^3\beta}\left(1 \mp \frac{\lambda^3 N}{4\sqrt{2}V} + \dots\right)}$$

$$= \frac{3z}{2\lambda^3\beta}\left(1 - \frac{z}{4\sqrt{2}} + \dots\right) g_1$$

BY SOME MAGIC

USE:

$$z = \frac{\lambda^3 N}{V}\left(1 + \frac{1}{2\sqrt{2}}\frac{\lambda^3 N}{V} + \dots\right)$$

(WHERE DID THIS COME FROM?)

$$E = \frac{3}{2}\frac{N}{\beta}\left(1 + \frac{1}{2\sqrt{2}}\frac{\lambda^3 N}{V} + \dots\right)\left(1 - \frac{1}{4\sqrt{2}}\frac{\lambda^3 N}{V} + \dots\right)$$

$\hookrightarrow \frac{3}{2} pV$

$$\Longrightarrow pV = N k_B T\left(1 + \frac{\lambda^3 N}{4\sqrt{2}V} + \dots\right)$$

I DON'T GET THE MATH HERE SO
I GIVE UP ON THIS.