A CHARACTERIZATION OF CO-COMMUTATIVE HOPF ALGEBRAS

1. Introduction

A recent result of Aliniaeifard and Theim [1] states that two non-commutative, co-commutative graded connected Hopf algebras are isomorphic if and only if their graded dimensions are equal. We found this result to be quite surprising because it answered a basic question that we set out to ask about the Hopf algebra of symmetric functions in non-commuting variables, NCSym.

Problem 1.1. Identify all of the Hopf subalgebras of NCSym of graded dimension Catalan.

One does not have to search very hard to find at least one Hopf subalgebra of dimension Catalan (see Example **??**). A consequnce of the Aliniaeifard-Theim result is that it implies that any other Hopf subalgebra of NCSym of graded dimension Catalan must be isomorphic.

Changing perspectives, one can then ask the following related problem:

Problem 1.2. Identify conditions on the graded dimensions that imply the existence of a free graded connected co-commutative Hopf algebra.

We state the answer to this problem as a calculation that one can perform on the sequence of graded dimensions. The answer (stated in Theorem **??**) is that one should calculate a related sequence representing the number of free generators of the Hopf algebra. If the sequence is non-negative, then we can construct a Hopf algebra with these graded dimensions.

This provides a complete characterization of Hopf algebras of this type. Every sequence of generators determines completely the Hopf algebra. This motivates a second problem.

Problem 1.3. Given two Hopf algebras that have the same graded dimension, find a procedure for determining an (all) isomorphism(s) between the Hopf algebras.

2. Notation

A *composition* of n is a sequence $\beta = (\beta_1, \beta_2, \ldots, \beta_\ell)$ of positive integers such that $\beta_1 + \beta_2 + \cdots + \beta_\ell = n$. We refer the the integers β_i as the parts of β and write $\ell(\beta)$ for the length of β, which is the number of parts. We will use the notation $\beta \vDash n$ to indicate that β is a composition of n.

We say that λ is a *partition* of n, if $\lambda \vDash n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)}$. We will indicate that λ is a partition of n with the notation $\lambda \vdash n$. We will also use the notation $m_d(\lambda)$ to be the number of times that d appears as a part in λ.

Let $\mathbb{C}^{\mathbb{N}}$ denote the space of infinite sequences $(c_n)_{n\geq 1} = (c_1, c_2, \ldots)$ of complex numbers, i.e. $c_n \in \mathbb{C}$ for all $n \geq 1$. Given two sequences $\vec{c} = (c_1, c_2, \ldots)$ and $\vec{d} = (d_1, d_2, \ldots)$, we write

$$\vec{c} \leq \vec{d} \qquad \text{if and only if} \qquad c_n \leq d_n \text{ for all } n \geq 1.$$

Let $\vec{0} \in \mathbb{C}^{\mathbb{N}}$ denote the zero sequence, so that $\vec{c} \geq \vec{0}$ is and only if $\vec{c}$ consists of entirely nonnegative entries. For a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \vDash n$, define

$$c_\alpha = c_{\alpha_1} c_{\alpha_2} \cdots c_{\alpha_\ell}.$$

Note that $c_\alpha = c_\beta$ whenever $\beta \vDash n$ is a composition with the same parts as α in a possibly different order, i.e. $\beta_i = \alpha_{\sigma(i)}$ for some permutation σ of the integers $\{1, 2, \ldots, \ell(\alpha)\}$.

Given a sequence $\vec{a}$ consisting of nonnegative integers, let

$$X(\vec{a}) = \biguplus_{n \geq 1} \{x_i^{(n)} \mid 1 \leq i \leq a_n\}.$$

and let $\mathfrak{L}(\vec{a})$ denote the free graded Lie algebra on the graded set $X(\vec{a})$.

For a graded Lie algebra L, $\mathcal{U}(L)$ will denote the universal enveloping algebra of L. We can define $\mathcal{U}(L)$ as a quotient of the tensor algebra $T(L)/I$ where I is the ideal generated by $x \otimes y - y \otimes x - [x, y]$ for $x, y \in L$. Because I is generated by homogeneous elements, $\mathcal{U}(L)$ inherits a grading from L. The Poincaré-Birkhoff-Witt Theorem implies that for a given ordered basis $\{x_i\}$ of L, a basis for $\mathcal{U}(L)$ is given by $\{x_{i_1} x_{i_2} \cdots x_{i_k} : i_1 \leq i_2 \leq \cdots \leq i_k\}$.

For a graded Hopf algebra H, $\mathcal{P}(H)$ will denote graded Lie algebra of primitives, so that

$$\mathcal{P}(H) = \{x \in H \mid \Delta(x) = 1 \otimes x + x \otimes 1\}.$$

Add reference/definition for universal enveloping algebra and primitives. Add statement of Milnor–Moore and PBW theorem. — Lucas

For a graded vector space $V = \bigoplus_{n \geq 0} V_n$ with $V_n = \mathbb{C}$, let

$$\overrightarrow{\dim}(V) = (\dim(V_n))_{n \geq 1}.$$

Similarly, for a graded set $X = \biguplus_{n \geq 1} X^{(n)}$, let

$$\overrightarrow{\mathrm{card}}(X) = \left(|X^{(n)}|\right)_{n \geq 0}$$

3. Combinatorics of sequences of numbers

Our main results make use of three interrelated sequences that we will denote by $\vec{h}$, $\vec{a}$, and $\vec{p}$. In this section we take a purely enumerative perspective to these sequences, assuming only that they satisfy the relation given in Equation (3.1) and may have values from a field of characteristic 0.

However, in later sections these sequences come from a combinatorial Hopf algebra H and have an interpretation when they are non-negative integers. This interpretation will provide useful motivation here:

- $\vec{h} = (h_1, h_2, \ldots)$ will be the graded dimensions of H,
- $\vec{a} = (a_1, a_2, \ldots)$ will be the graded numbers of free generators of the algebra, and
- $\vec{p} = (p_1, p_2, \ldots)$ will be the graded dimension of the Lie algebra of primitives $\mathcal{P}(H)$.

To define our sequences, we make use of the fact—recorded in Proposition 3.3 below—that any formal power series in $\mathbb{C}[[t]]$ with constant term 1 can be expressed in three equivalent ways,

$$1 + \sum_{k \geq 1} h_k t^k = \frac{1}{1 - \sum_{m \geq 1} a_m t^m} = \prod_{d \geq 1} \frac{1}{(1 - t^d)^{p_d}} \ . \tag{3.1}$$

which determines a triple of sequences: $(\vec{h}, \vec{a}, \vec{p})$ with $\vec{h} = (h_1, h_2, \ldots)$, $\vec{a} = (a_1, a_2, \ldots)$, and $\vec{p} = (p_1, p_2, \ldots)$.

Example 3.1. Take $f(t) = 1 + t + 2t^2 + 5t^3 + 14t^4 + \cdots \in \mathbb{C}[[t]]$, so that h_n is equal to the n^{th} Catalan number $\frac{1}{n+1}\binom{2n}{n}$. It is well known that $f(t)$ satisfies the functional equation

$$f(t) = \frac{1}{1 - t - t^2 - 2t^3 - 5t^4 - 14t^5 - \cdots} = \frac{1}{1 - tf(t)}$$

so the $\vec{a}$ sequence is equal to $\vec{a} = (1, 1, 2, 5, 14, 42, \ldots)$ (that is, a_n is the $n - 1^{st}$ Catalan number). We can use (3.1) to calculate the first few values of $\vec{p} = (1, 1, 3, 8, 25, \ldots)$ and it is known [**?**] that p_n is the number of binary Lyndon words of length $2n$ with the same number of 0s as 1s.

Example 3.2. Take $f(t) = 1 + 2t + 3t^2 + \cdots \in \mathbb{C}[[t]]$, so that $\vec{h} = (1, 2, 3, \ldots)$. Then $f(t)$ is a geometric series,

$$f(t) = \frac{1}{1 - 2t^2 + t^2} = \frac{1}{(1-t)^2}$$

so the remaining sequences are $\vec{a} = (2, -1, 0, \ldots)$ and $\vec{p} = (2, 0, 0, \ldots)$.

This is a boring example, but I am putting it in as a placeholder.
— Lucas

We will make extensive use of explicit formulas relating the sequences $\vec{h}$, $\vec{p}$, and $\vec{a}$. While these are well-known, we include a proof for the sake of completeness.

Proposition 3.3. *Any one sequence $\vec{h}$, $\vec{a}$, or $\vec{p} \in \mathbb{C}^{\mathbb{N}}$ belongs to a unique triple $(\vec{h}, \vec{a}, \vec{p})$ of sequences that satisfy Equation (3.1), given by:*

(i) $h_n = \sum_{\beta \vDash n} a_\beta = \sum_{\lambda \vdash n} \prod_{d \geq 1} \binom{p_d + m_d(\lambda) - 1}{p_d - 1}$,

(ii) $a_n = \sum_{\beta \vDash n} (-1)^{\ell(\beta)-1} h_\beta = \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)-1} \prod_{d \geq 1} \binom{p_d}{m_d(\lambda)}$, *and*

(iii) $p_n = \sum_{d|n} \sum_{\beta \vDash d} \frac{d \cdot \mu(n/d)}{n \cdot \ell(\beta)} a_\beta = \sum_{d|n} \sum_{\beta \vDash d} \frac{d \cdot \mu(n/d)(-1)^{\ell(\beta)-1}}{n \cdot \ell(\beta)} h_\beta$.

Proof. In order to see (i), take the geometric series expansion of the second and third expressions in Equation (3.1) to obtain

$$1 + \sum_{k \geq 1} h_k t^k = \sum_{k \geq 0} \left(\sum_{m \geq 1} a_m t^m \right)^k = \prod_{d \geq 1} \left(\sum_{k \geq 0} \binom{p_d + k - 1}{p_d - 1} t^{kd} \right) . \tag{3.2}$$

Expanding each product of sums and isolating the coefficient of t^n in each expression gives the desired equation.

Now we prove (ii). Take the reciprocal of Equation (3.1) and isolate the sum over m to obtain

$$1-\frac{1}{1+\sum_{k\geq 1}h_kt^k}=\sum_{m\geq 1}a_mt^m=1-\prod_{d\geq 1}(1-t^d)^{p_d}.$$

Expanding the left- and rightmost expressions yields

$$\sum_{r\geq 1}(-1)^{r-1}\left(\sum_{k\geq 1}h_kt^k\right)^r=\sum_{m\geq 1}a_mt^m=1-\prod_{d\geq 1}\sum_{k\geq 0}(-1)^k\binom{p_d}{k}t^{dk}.$$

Isolating the coefficient of t^n, we obtain equation (ii).

Finally, we deduce (iii). Beginning with Equation (3.1) as above, take the logarithm of each term and Taylor expand about 1 to obtain

$$\sum_{r\geq 1}\frac{(-1)^{r-1}(\sum_{k\geq 1}h_kt^k)^r}{r}=\sum_{r\geq 1}\frac{(\sum_{m\geq 1}a_mt^m)^r}{r}=\sum_{d\geq 1}\sum_{j\geq 1}p_d\frac{t^{jd}}{j}.$$

Now we isolate the coefficient of t^n in each of the expressions:

$$\sum_{\beta\vDash n}\frac{(-1)^{\ell(\beta)-1}h_\beta}{\ell(\beta)}=\sum_{\beta\vDash n}\frac{a_\beta}{\ell(\beta)}=\frac{1}{n}\sum_{d|n}dp_d.$$

Lastly, multiply the equation by n and apply Möbius inversion to solve for np_n. □

Proposition 3.3 can be stated as biconditional theorem in the sense that any one equality in the proposition implies, by reversing the calculations, that the corresponding generating function relations in Equation (3.1) will hold.

Definition 3.4. The *sequence transfer maps* relate sequences $(\vec{a},\vec{h},\vec{p})$ satisfying Equation (3.1):

$$\begin{array}{rcl}\phi_{p,a}:\mathbb{C}^{\mathbb{N}} & \to & \mathbb{C}^{\mathbb{N}}\\ \vec{p} & \mapsto & \vec{a}\end{array},\qquad \begin{array}{rcl}\phi_{h,a}:\mathbb{C}^{\mathbb{N}} & \to & \mathbb{C}^{\mathbb{N}}\\ \vec{h} & \mapsto & \vec{a}\end{array},\qquad\text{and}\qquad \begin{array}{rcl}\phi_{p,h}:\mathbb{C}^{\mathbb{N}} & \to & \mathbb{C}^{\mathbb{N}}\\ \vec{p} & \mapsto & \vec{h}\end{array}$$

as well as their inverses

$$\phi_{a,p}=\phi_{p,a}^{-1},\qquad \phi_{a,h}=\phi_{h,a}^{-1},\qquad\text{and}\qquad \phi_{h,p}=\phi_{p,h}^{-1}.$$

If the sequence $\vec{a}$ consists of entirely non-negative integers, then $\vec{h}=\phi_{a,h}(\vec{a})$ and $\vec{p}=\phi_{a,p}(\vec{a})$ will also have combinatorial interpretation in terms of words. Recall that a word in a set X is a finite sequence $w=w_1w_2\dots w_\ell$ of "letters" $w_i\in X$. If $X=\bigcup_{n\geq 1}X^{(n)}$ is a graded set, then define the *degree* of a word to be

$$\mathsf{deg}(w_1w_2\dots w_\ell)=\sum_{i=1}^{\ell}\mathsf{deg}(w_i)\qquad\text{where }\mathsf{deg}(x)=n\text{ for all }x\in X^{(n)}.$$

For a fixed order on X, we order the words on X lexicographically. The *rotation* of a word $w = w_1 w_2 \dots w_\ell$ is the word

$$\mathsf{cyc}(w) = w_2 \dots w_\ell w_1.$$

This defines an operation of order ℓ on words of with ℓ letters. A word is *Lyndon* if it is strictly smaller than each of $\mathsf{cyc}(w), \mathsf{cyc}^2(w), \dots, \mathsf{cyc}^{\ell-1}(w)$. For instance, if $X = \{x < y\}$, then $xyxyy$ is a Lyndon word, but neither $xyxy$ nor xyx are: $xyxy = \mathsf{cyc}^2(xyxy)$, while $xyx > xxy = \mathsf{cyc}^2(xyx)$.

The proposition below is well known, but the presentation of the formulae in Proposition 3.3 does not match a reference that we could find in the literature. Hence we include the details of the following result for completeness.

Proposition 3.5. *Let $\vec{a} \in \mathbb{C}^{\mathbb{N}}$ contain of only nonnegative integers, so that there exists a graded set $X = \bigcup_{n\geq 1} X^{(n)}$ with $\overrightarrow{\mathrm{card}}(X) = \vec{a}$. If $\vec{h} = \phi_{a,h}(\vec{a})$ and $\vec{p} = \phi_{a,p}(\vec{p})$, then for each $n \geq 1$, we have:*

(1) h_n is equal to the number of words of degree n in the alphabet X for all $n \geq 1$, and

(2) p_n is equal to the number of Lyndon words of degree n in the alphabet X of degree n for all $n \geq 1$.

Proof. To see (1), note that any degree n word $w = w_1 w_2 \cdots w_\ell$ in X defines a unique composition $(\mathsf{deg}(w_1), \mathsf{deg}(w_2), \dots, \mathsf{deg}(w_\ell))$, and for a particular composition $\beta \vDash n$, every $w \in X^{(\beta_1)} \times X^{(\beta_2)} \times \cdots \times X^{(\beta_\ell)}$ has this property. Thus, the number of degree n words in X agrees with the formula for h_n given in Proposition 3.3 (i).

For the second point the reader may refer to [6, Theorem 4.9, Theorem 5.1] for a more complete exposition.

The proof will make use of the set

$$I_n = \{(i, w) \mid \text{degree } n \text{ words } w \text{ in } X \text{ and } 1 \leq i \leq \mathsf{deg}(w_1)\},$$

which has size $|I_n| = \sum_{\beta \vDash n} \beta_1 a_\beta$. Note that since $a_\beta = a_{\mathsf{cyc}^k(\beta)}$, then we have

$$|I_n| = \sum_{\beta \vDash n} \frac{1}{\ell(\beta)} \sum_{k=1}^{\ell(\beta)} \beta_1 a_{\mathsf{cyc}^k(\beta)} = \sum_{\beta \vDash n} \frac{1}{\ell(\beta)} \sum_{k=1}^{\ell(\beta)} \mathsf{cyc}^{-k}(\beta)_1 a_\beta = \sum_{\beta \vDash n} \frac{n a_\beta}{\ell(\beta)} .$$

We will show that

$$\sum_{\beta \vDash n} \frac{n a_\beta}{\ell(\beta)} = |I_n| = \sum_{d \mid n} d\, |\{\text{degree } d \text{ Lyndon words in } X\}|, \tag{3.3}$$

from which Möbius inversion shows that the number of degree n Lyndon words is equal to the formula for p_n in terms of $\vec{a}$ given in Proposition 3.3 (iii).

To start, we define a "faux-cycling" operation on I_n by

$$\mathsf{fcyc}(i, w) = \begin{cases} (i+1, w) & \text{if } i < \mathsf{deg}(w_1) \\ (1, \mathsf{cyc}(w)) & \text{otherwise.} \end{cases}$$

For any degree n word $w = w_1 w_2 \cdots w_\ell$ and $1 \le a \le \ell$,

$$(0, \mathsf{cyc}^a(w)) = \mathsf{fcyc}^{\mathsf{deg}(w_1)+\cdots+\mathsf{deg}(w_{a-1})}(0, w).$$

Thus, fcyc is periodic of order n. Moreover, if k is the minimal positive integer for which $\mathsf{cyc}^k(w) = w$, then the fcyc orbit of $(0, w)$—or (i, w) for any $1 \le i \le \mathsf{deg}(w_1)$—has size

$$d = \mathsf{deg}(w_1 w_2 \cdots w_k).$$

Finally, the cyclic shifts of $w' = w_1 w_2 \cdots w_k$ are distinct, so taking the unique minimal one, we obtain a Lyndon word of degree d. Since w is the (ℓ/k)-fold concatenation of w' with itself, the d-elements of the fcyc-orbit of $(0, w)$ are uniquely determined by w'. $\square$

Corollary 3.6. *If $\vec{a} \in \mathbb{C}^{\mathbb{N}}$ consists only of nonnegative integers, then $\vec{h} \ge \vec{p} \ge \vec{a} \ge \vec{0}$.*

Proof. Let $X = \biguplus_{n \ge 1} X^{(n)}$ be a graded set with $|X^{(n)}| = a_n$. Considered as a word with one letter, each element $x \in X$ is a Lyndon word, so by Proposition 3.5 the inequalities above correspond to the set inclusions

$$\{\text{degree } n \text{ words in } X\} \supseteq \{\text{degree } n \text{ Lyndon words in } X\} \supseteq X^{(n)}. \qquad \square$$

4. Preliminary statements about FGCCHAs

Add details to this section about universal enveloping algebra, CMM, PBW theorems. Include citations to Reutenauer's text. Pay attention to what we need in "classification by sequences" section.
— Lucas

We now give the algebraic context for the results of Section 3. As in the introduction, we use the abbreviation FGCCHA to mean Free Graded Connected Cocommutative Hopf Algebra. By definition, every FGCCHA H comes with a graded set of generators,

$$X = \biguplus_{n \ge 1} X^{(n)}$$

for which $X^{(n)} \subseteq H_n$ and X freely generates H as an algebra.

The structure of any connected cocommutative Hopf algebra H can be described entirely by its primitive elements. For a Lie algebra L, there is a cocommutative Hopf algebra structure on $\mathcal{U}(L)$, with the coproduct given by extending $\Delta(x) = 1 \otimes x + x \otimes 1$ for $x \in L$. The Cartier-Milnor-Moore theorem [4, see Theorem 5.18] states there is a Hopf algebra isomorphism $\mathcal{U}(\mathcal{P}(H)) \cong H$ induced by the inclusion $\mathcal{P}(H) \to H$. Furthermore, by [6, Theorem 1.4] we have $\mathcal{P}(\mathcal{U}(L)) = L$.

In the context that H is a FGCCHA generated by a graded set $X(\vec{a})$, we know H is isomorphic as an algebra to the free algebra on $X(\vec{a})$. It is also a well known result that $\mathcal{U}(\mathfrak{L}(\vec{a}))$ is isomorphic to the free algebra on $X(\vec{a})$ [6,], hence there is an isomorphism of algebras $H \cong \mathcal{U}(\mathfrak{L}(\vec{a}))$. This is not necessarily an isomorphism of Hopf algebras, however we will later see such an isomorphism does exist.

Proposition 4.1. *Let H be a FGCCHA with generating set X. The triple of sequences*

$$\left(\overrightarrow{\dim}(H), \overrightarrow{\mathrm{card}}(X), \overrightarrow{\dim}\left(\mathcal{P}(H)\right)\right)$$

satisfies Equation (3.1).

Proof. Let $\vec{h} = \overrightarrow{\dim}(H)$, $\vec{a} = \overrightarrow{\mathrm{card}}(X)$, and $\vec{p} = \overrightarrow{\dim}(\mathcal{P}(H))$. We will show that $\vec{h} = \phi_{a,h}(\vec{a})$ and $\vec{p} = \phi_{h,p}(\vec{p})$, from which the claim follows.

For $n \geq 1$, h_n is the number of degree n monomials in X, which is also the number of degree n words in X. Applying Proposition 3.5, $\vec{h} = \phi_{a,h}(\vec{a})$.

On the other hand, the Milnor–Moore theorem states that H is isomorphic to the universal enveloping algebra of $\mathcal{P}(H)$. By the Poincare–Birkhoff–Witt theorem, for any fixed homogeneous basis $Y = \biguplus_{n\geq 0} Y^{(n)}$ of $\mathcal{P}(H)$, h_n counts the multisets of Y whose elements have degree summing (with repetition) to n. Since $\vec{p} = \overrightarrow{\mathrm{card}} Y$, h_n is therefore the coefficient of t^n in

$$\prod_{d\geq 1} \frac{1}{(1-t^d)^{p_d}},$$

so it follows that $\vec{p} = \phi_{h,p}(\vec{p})$. ☐

Could also use Lyndon factorization if we added it to the previous Proposition — Lucas

Example 4.2. We consider two well-known FGCCHAs from the perspective of Proposition 4.1.

(1) NSym
(2) CQSym

re-written up to here — Lucas

5. Characterization of FGCCHAs by a sequences

section intro: 1. outline section, 2. Aliniaeifard–Thiem give us iso if hilbert series are equal. 3. we appraoch this from the sequences perspective. 4. state when Hopf algebra exists based on H-sequence. 5. example 6. state how to construct this iso using OPG's 6. example(?) 7. prove statements in subsections. — Lucas

In this section we will classify all FGCCHAs by their graded dimensions. Then, we determine for which sequences there exists a FGCCHA H with $\overrightarrow{\dim}(H)$ and $\overrightarrow{\dim}(\mathcal{P}(H))$.

In [1,], it is proved that two FGCCHAs H and K are isomorphic if and only if $\overrightarrow{\dim}(H) = \overrightarrow{\dim}(K)$. As in Proposition 4.1, $\overrightarrow{\dim}(H)$ uniquely determines the triple

$$(\overrightarrow{\dim}(H), \overrightarrow{\mathrm{card}}(X), \overrightarrow{\dim}(\mathcal{P}(H)))$$

and we can make the following conclusions.

Proposition 5.1. *Let H be a FGCCHA generated by a graded set $X(\vec{a})$. Then,*

(1) $H \cong \mathcal{U}(\mathfrak{L}(\vec{a}))$ as Hopf algebras.
(2) $\mathcal{P}(H) \cong \mathfrak{L}(\vec{a})$.

Proof. (1) Because $\overrightarrow{\dim}(H) = \overrightarrow{\dim}(\mathcal{U}(\mathfrak{L}(\vec{a})))$.
(2) Given (1), we have $\mathcal{P}(H) \cong \mathcal{P}(\mathcal{U}(\mathfrak{L}(\vec{a}))) = \mathfrak{L}(\vec{a})$. ☐

Prop 6.1 and 6.2 can be moved here — Eric

Example 5.2. *A Cocommutative Hopf algebra on permutations.*

Let $\vec{a}$ be the sequence determined by

$$\sum_{n\geq 1} a_n t^n = 1 - \frac{1}{\sum_{m\geq 0} m! t^m}.$$

Then $\vec{a} = (1, 1, 3, 13, 71, \ldots)$ corresponds to [2, A003319] which counts the number of indecomposable permutations of $[n]$. By **??** there exists a FGCCHA of graded dimension $n!$, which we will construct explicitly.

Let (placeholder) be the graded vector space spanned by permutations of $[n]$ in degree n. We will denote by $\pi = \pi_1\pi_2\cdots\pi_n$ the permutation of $[n]$ mapping $m \mapsto \pi_m$. For permutations $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ of $[n]$ and $\tau = \tau_1\tau_2\cdots\tau_k$ of $[k]$, we define their concatenation product to be the permutation of $[n+k]$,

$$\sigma|\tau = \sigma_1\sigma_2\cdots\sigma_n(\tau_1+n)(\tau_2+n)\cdots(\tau_k+n)$$

which gives () the structure of a graded algebra.

The *indecomposable permutations* are the permutations π which cannot be written as the concatenation product of two non-trivial permutations, hence they are free generators for (). We now define the coproduct by declaring

$$\Delta(\iota) = 1 \otimes \iota + \iota \otimes 1$$

for all indecomposable permutations ι, and extending algebraically. Under this coproduct () is endowed with the structure of a FGCCHA, with graded dimension $n!$.

6. Classification by Sequences

Everything in this section should be moved to the previous one, I think. Leave the section here for numbering reasons until next meeting though. — Lucas

In this section we will characterize free graded connected cocommutative Hopf algebras in terms of sequences.

Proposition 6.1. *Given a sequence of nonnegative integers $\vec{h} = (h_0, h_1, h_2, \ldots)$, there exists a FGCCHA H with $\overrightarrow{\dim}(H) = \vec{h}$ if and only if $\phi_{h,a}(\vec{h}) \geq 0$. Moreover, if this is the case, then $H \cong \mathcal{U}(\mathfrak{L}(\vec{a}))$.*

Proof. Suppose that we have a free, graded, connected, cocommutative Hopf algebra H with $\dim(H_i) = h_i$. Then by Proposition 3.3, H is freely generated by a graded set $X = \biguplus_{i\geq 0} X_i$ with $|X_i| = a_n$ for each $n \geq 0$, and therefore $a_n \geq 0$.

On the other hand, assuming $a_n \geq 0$ for all $n \geq 0$, as a non-negative graded set, a_n generates a free graded connected cocommutative Hopf algebra H. By Proposition 3.3 we have that $\vec{a}$ also corresponds to $\vec{h} = (h_0, h_1, h_2, \ldots)$, where $h_n \geq 0$ for all $n \geq 0$ and $\dim H_n = h_n$. Furthermore, $H \cong \mathcal{U}(\mathfrak{L}(\vec{a}))$ by Corollary **??**. ☐

It would be nice to have an example of a sequence $\vec{h}$ that is not obvious that it doesn't have a Hopf algebra. For instance, $\vec{h} = (1, 2, 2, 2, 2, \ldots)$ is there a Hopf algebra? If yes, find me a different example. If no, apply this theorem to explain why. — Mike

Proposition 6.2. *Given a sequence of nonnegative integers $\vec{p} = (p_0, p_1, p_2, \ldots)$, there exists a FGCCHA H with $\overrightarrow{\dim}(\mathcal{P}(H)) = \vec{p}$ if and only if $\phi_{p,a} \geq 0$.*

Proof. Let $\vec{p} = (p_0, p_1, p_2, \ldots)$ be a nonnegative integer sequence and let H be a free graded connected cocommutative Hopf algebra such that $\dim \mathcal{P}(H)_n = p_n$. By Proposition 3.3 we have that

$$h_n = \sum_{\lambda \vdash n} \prod_{d \geq 1} \binom{p_d + m_d(\lambda) - 1}{p_d - 1}$$

so that $\vec{p}$ corresponds to a sequence $\vec{h} = (h_0, h_1, h_2, \ldots)$ which is clearly nonnegative. Hence $a_n \geq 0$ for all $n \geq 0$ by Proposition 6.1.

Supposing $a_n \geq 0$ for all $n \geq 0$, then by Proposition 6.1 we have that $\vec{a}$ generates a free graded connected cocommutative Hopf algebra H, and we let p_n count the number of primitive elements of H at degree n. Hence $\overrightarrow{\dim}(\mathcal{P}(H)_n) = \vec{p} \geq 0$.

□

6.1. **Constructing isomorphisms.**

Definition 6.3. A graded set $X = \biguplus_{n\geq 1} X_n$ will be called *homogeneously ordered* if each X_n is ordered. We will write the ordered homogeneous componenents as

$$\vec{X}_n = (x_n^1, x_n^2, \dots, x_n^{a_n}).$$

For a free graded, connected, cocommutative Hopf algebra H, let

$$\mathrm{OPG}(H) = \{\text{Graded sets of homogeneously ordered primitive generators of } H\}.$$

Proposition 6.4. *Let H be a free graded connected cocommutative Hopf algebra, $X \subset H$ a set of homogeneously ordered primitive generators with $|X_n| = a_n$. For any $Y \in \mathrm{opg}(H)$ there exists a unique sequence $((A_n, \vec{B}_n))_{n=1}^{\infty}$ with $A_n \in \mathrm{GL}_{a_n}(\mathbb{C})$ and $\vec{B}_n = (b_n^1, b_n^2, \dots, b_n^{a_n})$ with $b_n^i \in [\mathcal{P}(H), \mathcal{P}(H)]_n$, such that for each ordered graded component $\vec{Y}_n = (y_n^1, y_n^2, \dots, y_n^{a_n})$ we have*

$$\vec{Y}_n = A_n \vec{X}_n + \vec{B}_n.$$

Is it possible to find an example of this construction that is small enough to me manageable but big enough to be interesting here? Maybe with 3 generators?
— Lucas

Proof. For a fixed $X \in \mathrm{OPG}(H)$, setting $V = \mathrm{span}_{\mathbb{C}}(X)$ we have

$$\mathcal{P}(H) = \mathrm{span}_{\mathbb{C}}(X) \oplus \mathrm{span}_{\mathbb{C}}\{[[[x_{i_1}, x_{i_2}], \dots,]x_{i_k}] : x_{i_j} \in X, k > 1\} = V \oplus [\mathcal{P}(H), \mathcal{P}(H)]$$

because $\mathcal{P}(H) \cong \mathfrak{L}(X)$. Similarly, for an arbitrary $Y \in \mathrm{OPG}(H)$ we have

$$\mathcal{P}(H) = W \oplus [\mathcal{P}(H), \mathcal{P}(H)] \text{ where } W = \mathrm{span}_{\mathbb{C}}(Y).$$

Letting $\overline{p}$ denote the coset of p mod $[\mathcal{P}(H), \mathcal{P}(H)]$, we see that $\{\overline{x} : x \in X\}$ and $\{\overline{y} : y \in Y\}$ are both bases for $\dfrac{\mathcal{P}(H)}{[\mathcal{P}(H), \mathcal{P}(H)]}$. Fixing the component of degree n, we have for any $y_n^i \in Y_n$,

$$\begin{aligned} \overline{y_n^i} &= a_n^{i1}\overline{x_n^1} + a_n^{i2}\overline{x_n^2} + \dots + a_n^{ia_n}\overline{x_n^{a_n}} \\ &= \overline{a_n^{i1}x_n^1 + a_n^{i2}x_n^2 + \dots + a_n^{ia_n}x_n^{a_n}} \end{aligned}$$

for some coefficients a_n^{ij}. Therefore,

$$y_n^i = a_n^{i1}x_n^1 + a_n^{i2}x_n^2 + \dots + a_n^{ia_n}x_n^{a_n} + b_n^i$$

for some $b_n^i \in [\mathcal{P}(H), \mathcal{P}(H)]$, and for $A_n = [a_n^{ij}] \in \mathrm{GL}_{a_n}(\mathbb{C})$ we have $\vec{Y}_n = A_n \vec{X}_n + \vec{B}_n$ as desired. □

Conversely, given $X \in \mathrm{OPG}(H)$ and $((A_n, \vec{B}_n))_{n=1}^{\infty}$ as above, $\vec{Z}_n = A_n\vec{X}_n + \vec{B}_n$ descends to a basis for $\left(\dfrac{\mathcal{P}(H)}{[\mathcal{P}(H), \mathcal{P}(H)]}\right)_n$, hence $Z = \biguplus_{n\geq 1} \vec{Z}_n \in \mathrm{OPG}(H)$. Therefore the above conditions completely characterize $\mathrm{OPG}(H)$.

Proposition 6.5. *Suppose that H and K are free graded connected cocommutative Hopf algebras with $\dim H_n = \dim K_n$ for all $n \geq 0$. Then $H \cong K$. Moreover, for any fixed $X \in \mathrm{OPG}(H)$, there is a bijection*

$$\begin{array}{rcl} \{\textit{isomorphisms } H \to K\} & \leftrightarrow & \mathrm{OPG}(K) \\ \phi & \mapsto & \phi(X) \end{array}$$

Proof. Let $\phi : H \to K$ be an isomorphism, let $X = \biguplus_{n\geq 1} \vec{X}_n \in \mathrm{OPG}(H)$ be fixed. Then $\phi(X)$ generates K, since for any $y \in K$ we have for some $z \in H$,

$$y = \phi(z) = \phi(\sum_I x_{i_1}x_{i_2}\cdots x_{i_k}) = \sum_I \phi(x_{i_1})\phi(x_{i_2})\cdots\phi(x_{i_k}).$$

For any nontrovial polynomial p in $\phi(X)$ such that

$$p(\phi(x_{i_1}), \phi(x_{i_2}), \ldots, \phi(x_{i_n})) = 0 = \phi(p(x_{i_1}, x_{i_2}, \ldots, x_{i_n})),$$

we must have $p(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) = 0$, a contradiction, hence $\phi(X)$ is algebraically independent. Because ϕ is a Hopf isomorphism the $\phi(X)$ are primitive, and with the ordering inherited from X we have $\phi(X) \in \mathrm{OPG}(K)$. We know $\phi(X)$ is uniquely determined by ϕ as a homogeneously ordered set, since an isomorphism is completely determined by it's action on generators.

Now given any $Y = \biguplus_{n\geq 1} \vec{Y}_n \in \mathrm{OPG}(K)$, we will construct an isomorphism $H \to K$ whose image on X is Y and respects ordering. Define the bijection $f : X \to Y$ such that $f(x_n^i) = y_n^i$ for all $n \geq 1$. Extending algebraically, we have the desired isomorphism $\tilde{f} : H \to K$. □

Lemma 6.6. *Let H be a graded connected cocommutative Hopf algebra, freely generated by a graded set $X = \biguplus_{n\geq 1} X_n$. Then the graded set*

$$\mathbf{e}(X) = \biguplus_{n\geq 1} \{\mathbf{e}(x) \mid x \in X_n\}$$

comprises a complete set of primitive generators of H.

Proof. We will first show $\mathbf{e}(X)$ is algebraically independent in H. Fix a total order in each X_n, and extend to a total order on X, such that if $x \in X_i, y \in X_j$ and $i > j$, then $x < y$. For any $x \in X$, we have

$$\mathbf{e}(x) = \mathrm{id}(x) - \frac{1}{2}\mu \circ \tilde{\Delta}(x) + \cdots = x + p((x_i)),$$

where $p((x_i))$ is some polynomial in generators x_i of lesser degree than x. Then the minimal term in $\mathbf{e}(x)$ is x, and any polynomial $q(\mathbf{e}(x_1), ..., \mathbf{e}(x_n))$ has the same minimal term as

$q(x_1, ..., x_n)$. Hence if f is a non-trivial polynomial and $f(\mathbf{e}(x_1), \ldots, \mathbf{e}(x_n)) = 0$, then $f(x_1, \ldots, x_n) = 0$, a contradiction. Therefore $\mathbf{e}(X)$ is an algebraically independent set.

To show $\mathbf{e}(X)$ generates H, observe that in each degree n, the monomials in X of degree n form a basis for H_n. The degree n monomials in $\mathbf{e}(X)$ are linearly independent, because the $\mathbf{e}(X)$ are algebraically independent. Because every monomial $\mathbf{e}(x_1)\mathbf{e}(x_2)\cdots\mathbf{e}(x_k)$ corresponds uniquely to the monomial $x_1x_2\cdots x_k$, the degree n monomials in $\mathbf{e}(X)$ form a basis of H_n, hence $\mathbf{e}(X)$ freely generates H. □

Remark 6.7. Given two Hopf algebras H and K satisfying the criterion of Prop. 7.2, we may present a method to compute an explicit isomorphism. Given graded sets X and Y generating H and K respectively, we apply Lemma 6.6 to obtain primitive generators $\mathbf{e}(X)$ and $\mathbf{e}(Y)$. Choosing bijections $f_n : \mathbf{e}(X)_n \to \mathbf{e}(Y)_n$ for all $n \geq 1$ we obtain a bijection of primitive generators $f : \mathbf{e}(X) \to \mathbf{e}(Y)$ which completely determines an isomorphism $\tilde{f} : H \to K$.

Example 6.8. The Hopf algebra **CQSym** introduced in [5] is a free graded connected cocommutative Hopf algebra, with a basis $\mathbf{P}^\pi$ indexed by non-crossing set partitions π, and is freely generated by $\{\mathbf{P}^\pi : \pi \text{ is atomic and non-crossing}\}$ [5, Proposition 5.2].

There is a sub-Hopf algebra $\mathbf{\Pi}^{\mathrm{NC}}$ of NCSym spanned by the power-sum basis elements $\mathbf{p}_\pi$ indexed also by non-crossing set partitions, and freely generated by $\{\mathbf{p}_\pi : \pi \text{ is atomic and non-crossing}\}$ which can be seen from the product and coproduct formula,

$$\mathbf{p}_\pi \cdot \mathbf{p}_\sigma = \mathbf{p}_{\pi|\sigma}$$

$$\Delta(\mathbf{p}_\pi) = \sum_{S \subseteq [\ell(\pi)]} \mathbf{p}_{\pi_S} \otimes \mathbf{p}_{\pi_{S^c}},$$

where for $\pi = \{B_1, B_2, \ldots, B_{\ell(\pi)}\}$ and $S = \{s_1, s_2, \ldots, s_k\} \subseteq [\ell(\pi)]$, we define

$$\pi_S = \mathrm{st}\{B_{s_1}, B_{s_2}, \ldots, B_{s_k}\}.$$

For all $n \geq 0$, we have $\dim(\mathbf{CQSym}_n) = \dim(\mathbf{\Pi}^{\mathrm{NC}}_n) = c_n$ where c_n is the n-th Catalan number. By Theorem **??** we have $\mathbf{CQSym} \cong \mathbf{\Pi}^{\mathrm{NC}}$, and remarkably due to the identity

$$\sum_{n\geq 0} c_n t^n = \frac{1}{1 - \sum_{n\geq 1} c_{n-1} t^n},$$

we see that the number of generators in degree n for each algebra is c_{n-1}, or the number of atomic non-crossing partitions of $[n]$.

To realize an explicit isomorphism as outlined above, $\mathbf{e}(\mathbf{P}^\pi)$ and $\mathbf{e}(\mathbf{p}_\pi)$ running over all atomic non-crossing set partitions π constitute free primitive generators for their respective algebras, and the map

$$f : \mathbf{e}(\mathbf{P}^\pi) \mapsto \mathbf{e}(\mathbf{p}_\pi)$$

induces a Hopf algebra isomorphism $\tilde{f} : \mathbf{CQSym} \to \mathbf{\Pi}^{\mathrm{NC}}$

7. Classifying SubHopf algebras by sequence

Mike will read + make suggestions — Lucas

Let H be a free graded connected cocommutative Hopf algebra. In this section we construct all the isomorphism classes of graded Hopf algebra which occur as sub-Hopf algebras of H. To begin, recall the function $\phi_{p,a}$ on $\mathbb{C}^{\mathbb{N}}$ defined by

$$\phi_{p,a}(\vec{p}) = \Big(\sum_{\lambda\vdash n}(-1)^{\ell(\lambda)-1}\prod_{d\geq 0}\binom{p_d}{m_d(\lambda)}\Big)_{n=1}^{\infty}.$$

Theorem 1. *Let H be a free graded connected cocommutative Hopf algebra and let $\vec{p} = \overrightarrow{\dim}(\mathcal{P}(H)_n)$. Then there is a bijection*

$$\left\{\begin{array}{c}\text{Isomorphism classes of}\\ \text{sub-Hopf algebras } G\subseteq H\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{Sequences } \vec{q}=(q_1,q_2,\dots) \text{ such}\\ \text{that } \vec{p}\geq\vec{q}\geq\vec{0} \text{ and } \phi_{p,a}(\vec{q})\geq\vec{0}\end{array}\right\}$$

$$\begin{array}{rcl} G & \mapsto & \overrightarrow{\dim}\big(\mathcal{P}(G)\big)\\ \mathcal{U}(\mathfrak{L}(\phi_{p,a}(\vec{q}))) & \leftarrow\!\shortmid & \vec{q}\end{array}$$

The proof of Theorem 1 will construct a sub-Hopf algebra in H isomorphic to each $\mathcal{U}(\mathfrak{L}(\phi_{p,a}(\vec{q})))$. Before completing the proof we will give several motivating examples and intermediate results.

Example 7.1. We apply the theorem to our running example: let $H = \mathbb{C}\langle x,y\rangle$, so that $\vec{p} = \overrightarrow{\dim}(\mathcal{P}(H))$ is the sequence $\vec{p} = (2,1,2,3,6,9,18,30,\dots)$ [2, A001037] given by

$$p_n = \frac{1}{n}\sum_{d|n}\mu(n/d)2^d.$$

Then we can take

$$\vec{q} = \vec{p} - (1,0,0,\dots)$$

so that clearly $\vec{p}\geq\vec{q}\geq\vec{0}$. The sequence $\vec{q}$ is identified in [2, A059966], and direct computation then gives that

$$\phi_{p,a}(\vec{q}) = (1,1,1,\dots)\geq\vec{0},$$

so this sequence determines a subalgebra of $\mathcal{U}(\mathfrak{L}(1,1,1,\dots))$ that is freely generated by one primitive element in each degree. We have already identified we have identified $\mathcal{U}(\mathfrak{L}(1,1,1,\dots))\cong \mathsf{NSym}$, we conclude that H has a Hopf subalgebra isomorphic to NSym.

pretend that we have — Lucas

We now state and prove an intermediate result in the proof of Theorem 1; the proof of the theorem follows. Given a graded Lie algebra L, the (graded) derived subalgebra of L is

$$[L,L] = \bigoplus_{n\geq 1}[L,L]_n \qquad \text{where} \qquad [L,L]_n = \mathbb{C}\text{-span}\{\text{degree-}n \text{ commutators of } L\}.$$

Lemma 7.2. *Let $\vec{a} = (a_1,a_2,\dots)$ be a sequence of nonnegative integers. Then*

$$\overrightarrow{\dim}([\mathfrak{L}(\vec{a}),\mathfrak{L}(\vec{a})]) = \phi_{a,p}(\vec{a}) - \vec{a}.$$

Proof. Using the fact that $\phi_{a,p}(\vec{a}) = \overrightarrow{\dim}(\mathfrak{L}(\vec{a}))$, it is sufficient to show that

$$\vec{a} = \overrightarrow{\dim}\left(\mathfrak{L}(\vec{a})/[\mathfrak{L}(\vec{a}),\mathfrak{L}(\vec{a})]\right).$$

To begin, note that by Proposition 6.1, $\vec{a}$ enumerates a fixed choice of free homogeneous primitive generators for $\mathfrak{L}(\vec{a})$. The elements of this set are linearly independent modulo the derived sub-Lie algebra, so a_n is a lower bound on the dimension of the degree n part of the quotient. However, any element of homogeneous degree n in $\mathfrak{L}(\vec{a})$ must be, by definition, a linear combination of some element of $[\mathfrak{L}(\vec{a}),\mathfrak{L}(\vec{a})]$ and the degree n primitive generators, so the dimension of the quotient is at most a_n. □

this set = a fixed choice of free homogeneous generators — Mike

two assertions are being made in this paragraph that I think need to be justified: 1. the elements are lin ind mod $\mathfrak{L}(\vec{a})$ and 2. $r \in \mathfrak{L}(\vec{a})$ implies $r = s + t$ with $s \in [\mathfrak{L}(\vec{a}),\mathfrak{L}(\vec{a})]$ and t primitive — Mike

Proof of Theorem 1. First, suppose that G is a sub-Hopf algebra of H and let $\vec{g} = \overrightarrow{\dim}(G)$. Let $\vec{q} = \phi_{h,p}(\vec{b})$, so that $\vec{q} = \overrightarrow{\dim}(\mathcal{P}(G))$. By Proposition 6.1 the isomorphism class of G is uniquely determined by $\vec{q}$, so the given map is injective.

Now, suppose that $\vec{q}$ is a sequence satisfying the conditions that $\vec{p} \geq \vec{q} \geq \vec{0}$ and $\vec{b} = \phi_{p,a}(\vec{q}) \geq \vec{0}$. We will construct a sub-Hopf algebra G of H which is isomorphic to $\mathcal{U}(\mathfrak{L}(\vec{b}))$; by Proposition 3.3 this will show that the given map is surjective, completing the proof.

In order to do so, we construct a tower of sub-Lie algebras

$$0 = L^{(1)} \subseteq L^{(2)} \subseteq L^{(3)} \subseteq \cdots \subseteq \mathcal{P}(H)$$

such that

$$L^{(n)} \cong \mathfrak{L}(b_1,\ldots,b_{n-1},0,0,\ldots).$$

Taking $L^{(\infty)}$ to be the union of the $L^{(n)}$, we obtain G as $\mathcal{U}(L^{(\infty)}) \subseteq \mathcal{U}(\mathcal{P}(H)) = H$.

Our construction is inductive with base case $L^{(1)} = 0$. Begin by assuming that $L^{(n)}$ has been constructed. By Lemma 7.2 and our inductive hypothesis,

$$\overrightarrow{\dim}([L^{(n)},L^{(n)}]) = \phi_{a,p}(b_1,\ldots,b_{n-1},0,0,\ldots) - (b_1,\ldots,b_{n-1},0,0,\ldots).$$

Considering the nth term, we see that $\overrightarrow{\dim}(L^{(n)})_n = \overrightarrow{\dim}([L^{(n)},L^{(n)}])_n$, and we can further deduce—after a careful examination of the definition of $\phi_{a,p}$—that

$$\phi_{a,p}(b_1,\ldots,b_{n-1},0,0,\ldots)_n = \phi_{a,p}(\vec{b}) - b_n = q_n - b_n. \tag{7.1}$$

Therefore,

$$\overrightarrow{\dim}\left(\mathcal{P}/[L^{(n)},L^{(n)}]\right)_n = p_n - (q_n - b_n) = b_n + (p_n - q_n) \geq b_n.$$

The preceding inequality shows that we can choose b_n linearly independent elements $\{x_1^{(n)},\ldots,x_{b_n}^{(n)}\}$ of $\mathcal{P}(H)_n$ which remain linearly independent modulo $[L^{(n)},L^{(n)}]$ and define

$$L^{(n+1)} = \langle L^{(n)}, x_1^{(n)},\ldots,x_{b_n}^{(n)}\rangle.$$

Since $L^{(n+1)}$ is a sub-Lie algebra of a free Lie algebra, namely $\mathcal{P}(H)$, [3, Theorem 2.2] states that $L^{(n+1)}$ is also free. Moreover, as $L^{(n+1)}$ is generated by the generators of $L^{(n)}$ and homogeneous elements of degree n, we deduce that

$$L^{(n+1)} \cong \mathfrak{L}(b_1,\ldots,b_{n-1},b_n',0,\ldots) \qquad \text{for some } b_n' \leq b_n.$$

Moreover, using Lemma 7.2 and the definition of $L^{(n+1)}$,

$$\begin{aligned} b'_n &= \dim\left(L_n^{(n+1)}/[L^{(n+1)}, L^{(n+1)}]_n\right) \\ &= \dim\left(\mathbb{C}\text{-span}\{x_1^{(n)}, \dots, x_{b_n}^{(n)}\} \oplus [L^{(n)}, L^{(n)}]_n/[L^{(n)}, L^{(n)}]_n\right) \\ &= b_n. \end{aligned}$$

□

Example 7.3. Building on Example 7.1, we construct an isomorphic copy of NSym inside of $H = \mathbb{C}\langle x, y\rangle$ in the manner of the above proof. This means that for each degree $n \geq 1$, we will choose a primitive element $x_1^{(n)}$ in degree n that is linearly independent of all commutators in generators $x_1^{(i)}$ for $i < n$, and then define

$$L = \langle x_1^{(1)}, x_1^{(2)}, x_1^{(3)}, \dots\rangle.$$

In degree 1, there are no constrains and we choose

$$x_1^{(1)} = x.$$

In higher degrees, we let

$$x_1^{(n)} = \underbrace{[y, [y, \cdots [y}_{n-1 \text{ times}}, x] \cdots]].$$

To see that $x_1^{(n)}$ is linearly independent of all commutators in $x_1^{(i)}$ for $i < n$, we consider the number of times x occurs in any generator or bracket of generators. This will be 1 if we have a generator and k for a bracket of k generators, and moreover remains constant on terms in any Jacobi relation or anti-commutation relation. Thus, L_n is graded by this statistic, with the degree 1 component being the span of $x_1^{(n)}$, and the sum of components of degree 2 or greater comprising all commutators of the $x_1^{(i)}$ for $i < n$.

Examples from earlier versions....do we keep? — Lucas

Example 7.4. The Lie algebra of primitives $\mathcal{P}(H)$ is the free Lie algebra $\mathfrak{L}_{\{x,y,z,w\}}$. Now consider the Lie subalgebra

$$M = \mathbb{C}\text{-span}\{[x, y]\} \subseteq \mathfrak{L}_{\{x,y,z,w\}},$$

as well as its universal enveloping algebra $\mathcal{U}(M) \subseteq H$. By Theorem **??**, this is a free Lie algebra generated by one element, and by dimension consideration we can take the free generator to be $[x, y]$. Therefore by Theorem **??**, the universal enveloping algebra $\mathcal{U}(M)$ is freely generated (as an algebra) by $[x, y]$. Now $[x, y] = xy - yx$ is a homogeneous element of degree 2 in H, so as the subalgebra generated by $[x, y]$, the universal enveloping algebra $\mathcal{U}(M)$ is a graded sub-Hopf algebra and has Hilbert series

$$\frac{1}{1 - 0t - t^2 - 0t^3 - \cdots}.$$

Since H has no free generators in degree 2, this shows that sub-Hopf algebras may have more free generators in a given degree than the Hopf algebra which contains them.

Is is also possible to obtain a sub-Hopf algebra which has more free generators (in total) than the Hopf algebra which contains it. Consider the Lie subalgebra

$$N = \langle [x,y],[x,z],[x,w],[y,z],[y,w],[z,w] \rangle \subseteq \mathfrak{L}_{\{x,y,z,w\}}.$$

This is free on the set $\{[x,y],[x,z],[x,w],[y,z],[y,w],[z,w]\}$. Repeating the argument we made for $\mathcal{U}(M)$, we have that $\mathcal{U}(N)$ is a graded sub-Hopf algebra of H with six free generators in degree two, so that its Hilbert series is

Definitely true, but details needed. — Lucas

$$\frac{1}{1-0t-6t^2-0t^3-\cdots}.$$

8. Example

I did not know where we wanted those examples — Félix

Example 8.1. Given two Hopf algebras A and B respectively associated with $\vec{a}$ sequences $\vec{a_A} = (0,1,0,0,\dots)$ and $\vec{a_B} = (1,0,0,0,\dots)$, let us see that the $\vec{h}$ sequence $\vec{h_A}$ associated to $\vec{a_A}$ is dominated by the $\vec{h}$ sequence $\vec{h_B}$ associated to $\vec{a_B}$ but corresponding $\vec{p}$ sequence $\vec{p_A}$ and $\vec{p_B}$ are not comparable. Following the computation of the $\vec{h}$ sequence from the $\vec{a}$ sequence given by

$$h_n = \sum_{\beta \models n} a_{\beta_1} a_{\beta_2} \cdots a_{\beta_{\ell(\beta)}}$$

We can see that

$$(\vec{h_1})_n = 1$$

This comes from the fact that the only set composition β that impact the summation is the composition with n parts given by $(1,\dots,1)$. On the other side we have that

$$(\vec{h_2})_n = \begin{cases} 0 & \textit{if } n \textit{ is odd} \\ 1 & \textit{if } n \textit{ is even} \end{cases}$$

This comes from the fact that the only set composition β that impact the summation is the even compositions with $n/2$ parts given by $(2,\dots,2)$. This gives us that $\vec{h_1}$ is domintated by $\vec{h_2}$. However, after computation we have that $(\vec{p_1})_1 = 1$ and $(\vec{p_2})_1 = 0$ but we have $(\vec{p_1})_2 = 0$ and $(\vec{p_2})_2 = 1$. Hence, those two sequences are not comparable.

Example 8.2. Let us consider the Fibonacci sequence $\vec{f} = (1,1,2,3,5,8,13,\dots)$ as if it could be the graded dimensions of some Hopf algebra. We compute first that $\phi_{ha}(\vec{f}) = (1,0,1,0,1,0,\dots)$ and by Theorem/Proposition/Corollary **??** conclude that there is a Hopf algebra H with a basis indexed by compositions with only odd parts and with one generator at each odd degree. This Hopf algebra H has graded dimension equal to $\overrightarrow{\dim}(H) = \vec{f}$.

Next consider the Lucas sequence $\vec{\ell} = (2,1,3,4,7,11,18,\dots)$ as if it could also be the graded dimensions of a Hopf algebra. We compute that $\phi_{ha}(\vec{\ell})$ and determine that the sequence is alternating and begins $(2,-3,7,-13,\dots)$ and so we must conclude that there does not exist a Hopf algebra with graded dimensions equal to $\vec{\ell}$.

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Sequence	h sequence	a sequence	p sequence
Catalan	$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862$	$1, 0, 1, 2, 6, 18, 57, 186, 622, 2120$	$1, 0, 1, 3, 9, 27, 87, 282, 946, 3$
Ternary trees	$1, 1, 3, 12, 55, 273, 1428, 7752, 43263$	$1, 0, 2, 7, 34, 171, 905, 4952, 27802$	$1, 0, 2, 9, 43, 215, 1137, 6193,$
Motzkin	$1, 1, 2, 4, 9, 21, 51, 127, 323, 835$	$1, 0, 1, 1, 3, 6, 15, 36, 91, 232$	$1, 0, 1, 2, 5, 11, 28, 68, 174, 4$

Table 1. Sequences