FREE, GRADED, CONNECTED, CO-COMMUTATIVE HOPF ALGEBRAS AS INTEGER SEQUENCES

Abstract. Just putting something here to take up space.

1. Introduction

This paper is motivated by a question about the Hopf algebra of symmetric functions in noncommutative variables, NCSym. This is a graded Hopf algebra with dimensions given by the Bell numbers, so bases of NCSym are naturally indexed by set partitions. The Catalan numbers enumerate several distinguished families of partitions, including the nonnesting and noncrossing set partitions. We recognized that some well-known bases of NCSym had product and coproduct formulas that were closed when restricted to the nonnesting or noncrossing subsets.

"cite multiple papers"
— Lucas

Problem 1.1. Classify all Hopf subalgebras of NCSym with graded dimensions equal to the Catalan numbers.

We realized that we could resolve Problem 1.1 by applying a recent result of Aliniaeifard and Thiem [3]. This is a consequence of the the category in which NCSym lies, and applies to a much larger class of Hopf algebras. This category is the full subcategory of free, graded, connected, and cocommutative Hopf algebras, which we will denote by FGCCHA. In this context, [3, Theorem ?.?] show that that the isomorphism class of an FGCCHA H can be fully determined by the *dimension sequence*

$$\overrightarrow{\dim}(H) = \big(\dim(H_1), \dim(H_2), \dim(H_3), \dots\big) \qquad \text{where} \qquad H = \bigoplus_{n\geq 0} H_n.$$

We answer Problem 1.1 by showing that all Hopf subalgebras of NCSym are FGCCHAs, which follows from the Shirshov–Witt Theorem (see Lemma 6.6).

Properties of the category of FGCCHAs (or its dual) have been considered in several references: [17], [6], [20], [4], and [8]. Going back to the Milnor–Moore theorem, the structure of FGCCHAs have been characterized by the number of free generators and the dimension of the subspace of primitives. In this paper, we take the perspective that these can be thought of simply as sequences of integers satisfying certain relations.

In addition to NCSym, much work has gone into understanding specific examples including the FGCCHAs of noncommutative symmetric functions (see Example 3.4), permutations (see Example 4.3), and trees [1, 10]. From our perspective, proving that two FGCCHAs are isomorphic amounts to showing properties of their dimension sequences.

This paper considers what we can learn about a single Hopf algebra in the category of FGCCHAs from the sequence of the graded dimensions of the Hopf algebra. Building on known results (Theorem 4.1, Proposition 4.8, Proposition 4.9, Lemma 5.4, and Proposition 6.6), this perspective allows us determine whether maps between FGCCHAs exist by checking properties of the sequences.

The sections of this paper are structured as follows. Sections 2 and 3 state the relationship between the dimension of an FGCCHA, the number of free generators, and the dimension of the Lie algebra of primitives. Section 2 establishes preliminary notation about sequences, while Section 3 focuses on the algebraic content.

In Section 4 we interpret a result by Aliniaeifard and Thiem as a characterization of FGCCHAs by their sequence of graded dimensions, and Theorem 4.2 states that a sequence of nonnegative integers is the graded dimensions of an FGCCHA if and only if the INVERTi transform [13] of the sequence is nonnegative. Theorem 4.6 describes an explicit construction of the isomorphisms between FGCCHAs and a canonical representative of each isomorphism class.

In the last two sections we use the association between FGCCHAs and sequences from Section 4 to describe the structure of the category of FGCCHAs. In Section 5, Theorem 5.1 shows that there exists a surjective homomorphism from an FGCCHA with dimension sequence $\vec{h}$ to an FGCCHA with dimension sequence $\vec{k}$ if and only if the INVERTi transform of $\vec{k}$ is dominated by if the INVERTi transform of $\vec{h}$. In the next section, Theorem 6.1 states that an FGCCHA with dimension sequence $\vec{h}$ occurs as a Hopf subalgebra of an FGCCHA with dimension sequence $\vec{k}$ if and only if the inverse Euler transform of $\vec{k}$ dominates the inverse Euler transform of $\vec{h}$. This classifies all Hopf subalgebras of FGCCHAs.

2. Combinatorics of sequences of numbers

Let $\mathbb{Q}^{\mathbb{Z}_+}$ (respectively $\mathbb{Z}^{\mathbb{Z}_+}$ and $\mathbb{N}^{\mathbb{Z}_+}$) denote the space of infinite sequences $(c_n)_{n\geq 1} = (c_1, c_2, \ldots)$ of rational (respectively, integer and natural) numbers, i.e. $c_n \in \mathbb{Q}$ for all $n \geq 1$. Given two sequences $\vec{c} = (c_1, c_2, \ldots)$ and $\vec{d} = (d_1, d_2, \ldots)$, we write

$$\vec{c} \leq \vec{d} \qquad \text{if and only if} \qquad c_n \leq d_n \text{ for all } n \geq 1.$$

Let $\vec{0} \in \mathbb{Q}^{\mathbb{Z}_+}$ denote the zero sequence, so that $\vec{c} \geq \vec{0}$ if and only if $\vec{c}$ consists of entirely nonnegative entries.

Our main results make use of three interrelated sequences that we will denote by $\vec{h}$, $\vec{a}$, and $\vec{p}$. In this section we take a purely enumerative perspective to these sequences, assuming only that they satisfy the relation given in Equation (2.1) and may have values from a field of characteristic 0.

However, in later sections these sequences come from an FGCCHA H and have an interpretation when they are non-negative integers. For a graded vector space $V = \bigoplus_{n\geq 0} V_n$

with $V_0 = \mathbb{C}$, let

$$\overrightarrow{\dim}(V) = (\dim(V_n))_{n\geq 1}\,.$$

Similarly, for a graded set $X = \biguplus_{n\geq 1} X^{(n)}$, let

$$\overrightarrow{\mathrm{card}}(X) = \left(|X^{(n)}|\right)_{n\geq 0}\,.$$

This interpretation will provide useful motivation here:

- $\vec{h} = (h_1, h_2, \ldots)$ will be the graded dimensions of H,
- $\vec{a} = (a_1, a_2, \ldots)$ will be the graded numbers of free generators of the algebra, and
- $\vec{p} = (p_1, p_2, \ldots)$ will be the graded dimension of the Lie algebra of primitives $\mathcal{P}(H)$.

To define our sequences, we make use of the fact—recorded in Proposition 2.2 below—that any formal power series in $\mathbb{Q}[\![t]\!]$ with constant term 1 can be expressed in three equivalent ways,

$$1 + \sum_{k\geq 1} h_k t^k = \frac{1}{1 - \sum_{m\geq 1} a_m t^m} = \prod_{d\geq 1} \frac{1}{(1-t^d)^{p_d}}\,. \tag{2.1}$$

which determines a triple of sequences: $(\vec{h}, \vec{a}, \vec{p})$ with $\vec{h} = (h_1, h_2, \ldots)$, $\vec{a} = (a_1, a_2, \ldots)$, and $\vec{p} = (p_1, p_2, \ldots)$.

Example 2.1. Take $f(t) = 1 + 2t + 3t^2 + \cdots \in \mathbb{Q}[\![t]\!]$, so that $\vec{h} = (2, 3, 4, \ldots)$. Then $f(t)$ is the power series,

$$f(t) = \frac{1}{1 - 2t^2 + t^2} = \frac{1}{(1-t)^2}$$

so the remaining sequences are $\vec{a} = (2, -1, 0, \ldots)$ and $\vec{p} = (2, 0, 0, \ldots)$.

We will make extensive use of explicit formulas relating the sequences $\vec{h}$, $\vec{p}$, and $\vec{a}$. While these are well-known, we include a proof for the sake of completeness.

A *composition* of n is a sequence $\beta = (\beta_1, \beta_2, \ldots, \beta_\ell)$ of positive integers such that $\beta_1 + \beta_2 + \cdots + \beta_\ell = n$. We refer to the integers β_i as the parts of β and write $\ell(\beta)$ for the length of β, which is the number of parts. We will use the notation $\beta \vDash n$ to indicate that β is a composition of n.

We say that λ is a *partition* of n, if $\lambda \vDash n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)}$. We will indicate that λ is a partition of n with the notation $\lambda \vdash n$. We will also use the notation $m_d(\lambda)$ to be the number of times that d appears as a part in λ.

For a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \vDash n$, define

$$c_\alpha = c_{\alpha_1} c_{\alpha_2} \cdots c_{\alpha_\ell}.$$

Note that $c_\alpha = c_\beta$ whenever $\beta \vDash n$ is a composition with the same parts as α in a possibly different order, i.e. $\beta_i = \alpha_{\sigma(i)}$ for some permutation σ of the integers $\{1, 2, \ldots, \ell(\alpha)\}$.

Proposition 2.2. *Any one sequence $\vec{h}$, $\vec{a}$, or $\vec{p} \in \mathbb{Q}^{\mathbb{Z}_+}$ belongs to a unique triple $(\vec{h}, \vec{a}, \vec{p})$ of sequences that satisfy Equation (2.1), given by:*

(i) $h_n = \sum_{\beta \vDash n} a_\beta = \sum_{\lambda \vdash n} \prod_{d \geq 1} \binom{p_d + m_d(\lambda) - 1}{m_d(\lambda)}$,

(ii) $a_n = \sum_{\beta \vDash n} (-1)^{\ell(\beta)-1} h_\beta = \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)-1} \prod_{d \geq 1} \binom{p_d}{m_d(\lambda)}$, *and*

(iii) $p_n = \sum_{d|n} \sum_{\beta \vDash d} \frac{d \cdot \mu(n/d)}{n \cdot \ell(\beta)} a_\beta = \sum_{d|n} \sum_{\beta \vDash d} \frac{d \cdot \mu(n/d)(-1)^{\ell(\beta)-1}}{n \cdot \ell(\beta)} h_\beta$.

Remark 2.3. Although not necessarily obvious from the form of the equation in the proposition, an induction argument implies that if $\vec{a} \in \mathbb{Z}^{\mathbb{Z}_+}$ or $\vec{h} \in \mathbb{Z}^{\mathbb{Z}_+}$, then $\vec{p} \in \mathbb{Z}^{\mathbb{Z}_+}$.

Proof. In order to see (i), take the series expansion of the second and third expressions in Equation (2.1) to obtain

$$1 + \sum_{k \geq 1} h_k t^k = \sum_{k \geq 0} \left(\sum_{m \geq 1} a_m t^m \right)^k = \prod_{d \geq 1} \left(\sum_{k \geq 0} \binom{p_d + k - 1}{k} t^{kd} \right). \tag{2.2}$$

Expanding each product of sums and isolating the coefficient of t^n in each expression gives the desired equation.

Now we prove (ii). Take the reciprocal of Equation (2.1) and isolate the sum over m to obtain

$$1 - \frac{1}{1 + \sum_{k \geq 1} h_k t^k} = \sum_{m \geq 1} a_m t^m = 1 - \prod_{d \geq 1} (1 - t^d)^{p_d}.$$

Expanding the left- and rightmost expressions yields

$$\sum_{r \geq 1} (-1)^{r-1} \left(\sum_{k \geq 1} h_k t^k \right)^r = \sum_{m \geq 1} a_m t^m = 1 - \prod_{d \geq 1} \sum_{k \geq 0} (-1)^k \binom{p_d}{k} t^{dk}.$$

Isolating the coefficient of t^n, we obtain equation (ii).

Finally, we deduce (iii). Beginning with Equation (2.1) as above, take the logarithm of each term and Taylor expand about 1 to obtain

$$\sum_{r \geq 1} \frac{(-1)^{r-1} (\sum_{k \geq 1} h_k t^k)^r}{r} = \sum_{r \geq 1} \frac{(\sum_{m \geq 1} a_m t^m)^r}{r} = \sum_{d \geq 1} \sum_{j \geq 1} p_d \frac{t^{jd}}{j}.$$

Now we isolate the coefficient of t^n in each of the expressions:

$$\sum_{\beta \vDash n} \frac{(-1)^{\ell(\beta)-1} h_\beta}{\ell(\beta)} = \sum_{\beta \vDash n} \frac{a_\beta}{\ell(\beta)} = \frac{1}{n} \sum_{d|n} d p_d.$$

Lastly, multiply the equation by n and apply Möbius inversion to solve for np_n. $\square$

Proposition 2.2 can be stated as biconditional theorem in the sense that any one equality in the proposition implies, by reversing the calculations, that the corresponding generating function relations in Equation (2.1) will hold.

Definition 2.4. The *sequence transfer maps* relate sequences $(\vec{a}, \vec{h}, \vec{p})$ satisfying Equation (2.1):

$$\begin{array}{rcl} \phi_{p,a} : \mathbb{Q}^{\mathbb{Z}_+} & \to & \mathbb{Q}^{\mathbb{Z}_+} \\ \vec{p} & \mapsto & \vec{a} \end{array}, \qquad \begin{array}{rcl} \phi_{h,a} : \mathbb{Q}^{\mathbb{Z}_+} & \to & \mathbb{Q}^{\mathbb{Z}_+} \\ \vec{h} & \mapsto & \vec{a} \end{array}, \qquad \text{and} \qquad \begin{array}{rcl} \phi_{p,h} : \mathbb{Q}^{\mathbb{Z}_+} & \to & \mathbb{Q}^{\mathbb{Z}_+} \\ \vec{p} & \mapsto & \vec{h} \end{array}$$

as well as their inverses

$$\phi_{a,p} = \phi_{p,a}^{-1}, \qquad \phi_{a,h} = \phi_{h,a}^{-1}, \qquad \text{and} \qquad \phi_{h,p} = \phi_{p,h}^{-1}.$$

Following the conventions of [13], the map $\phi_{p,h}$ is known as the Euler transform of the sequence (and consequently $\phi_{h,p}$ is the inverse Euler transform). Also following the conventions of [13], the map $\phi_{a,h}$ is known as the INVERT transform and $\phi_{h,a}$ is the INVERTi transform.

Example 2.5. Take $f(t) = 1 + t + 2t^2 + 5t^3 + 14t^4 + \cdots \in \mathbb{Q}[[t]]$, so that h_n is equal to the n^{th} Catalan number $\frac{1}{n+1}\binom{2n}{n}$. It is well known that $f(t)$ satisfies the functional equation

$$f(t) = \frac{1}{1 - t - t^2 - 2t^3 - 5t^4 - 14t^5 - \cdots} = \frac{1}{1 - tf(t)}$$

so we have that $\phi_{h,a}(1, 2, 5, 14, 42, 132, \ldots) = \vec{a} = (1, 1, 2, 5, 14, 42, \ldots)$ (that is, a_n is the $n - 1^{st}$ Catalan number). We can use Proposition 2.2 to calculate the first few values of $\vec{p} = (1, 1, 3, 8, 25, \ldots)$. It follows from [18, §5.1.3] that $\vec{p}$ is equal to sequence [13, A022553].

If the sequence $\vec{a}$ consists of entirely non-negative integers, then $\vec{h} = \phi_{a,h}(\vec{a})$ and $\vec{p} = \phi_{a,p}(\vec{a})$ will also have combinatorial interpretation in terms of words. Recall that a word in a set X is a finite sequence $w = w_1 w_2 \ldots w_\ell$ of "letters" $w_i \in X$. If $X = \bigcup_{n \geq 1} X^{(n)}$ is a graded set, then define the *degree* of a word to be

$$\mathsf{deg}(w_1 w_2 \ldots w_\ell) = \sum_{i=1}^{\ell} \mathsf{deg}(w_i) \qquad \text{where } \mathsf{deg}(x) = n \text{ for all } x \in X^{(n)}.$$

For a fixed order on X, we order the words on X lexicographically. The *rotation* of a word $w = w_1 w_2 \ldots w_\ell$ is the word

$$\mathsf{cyc}(w) = w_2 \ldots w_\ell w_1.$$

This defines an operation of order ℓ on words of with ℓ letters. A word is *Lyndon* if it is strictly smaller than each of $\mathsf{cyc}(w), \mathsf{cyc}^2(w), \ldots, \mathsf{cyc}^{\ell-1}(w)$. For instance, if $X = \{x < y\}$, then $xyxyy$ is a Lyndon word, but neither $xyxy$ nor xyx are: $xyxy = \mathsf{cyc}^2(xyxy)$, while $xyx > xxy = \mathsf{cyc}^2(xyx)$.

If the sequence $\vec{a}$ is of non-negative integers rather than any values from a field, there is a combinatorial interpretation for $\phi_{a,h}(\vec{a})$ and $\phi_{a,p}(\vec{a})$. The proposition below is well known, but the presentation of the formulae in Proposition 2.2 does not match a reference that we could find in the literature. Hence we include the details of the following result for completeness.

Proposition 2.6. *Let $\vec{a} \in \mathbb{N}^{\mathbb{Z}_+}$ so that there exists a graded set $X = \bigcup_{n\geq 1} X^{(n)}$ with $\overrightarrow{\mathrm{card}}(X) = \vec{a}$. If $\vec{h} = \phi_{a,h}(\vec{a})$ and $\vec{p} = \phi_{a,p}(\vec{a})$, then for each $n \geq 1$, we have:*

(1) h_n is equal to the number of words of degree n in the alphabet X for all $n \geq 1$, and

(2) p_n is equal to the number of Lyndon words of degree n in the alphabet X for all $n \geq 1$.

Proof. To see (1), note that any degree n word $w = w_1 w_2 \cdots w_\ell$ in X defines a unique composition $(\mathsf{deg}(w_1), \mathsf{deg}(w_2), \ldots, \mathsf{deg}(w_\ell))$, and for a particular composition $\beta \vDash n$, every $w \in X^{(\beta_1)} \times X^{(\beta_2)} \times \cdots \times X^{(\beta_\ell)}$ has this property. Thus, the number of degree n words in X agrees with the formula for h_n given in Proposition 2.2 (i).

For the second point the reader may refer to [20, Theorem 4.9, Theorem 5.1] for a more complete exposition.

The proof will make use of the set

$$I_n = \{(i, w) \mid \text{degree } n \text{ words } w \text{ in } X \text{ and } 1 \leq i \leq \mathsf{deg}(w_1)\},$$

which has size $|I_n| = \sum_{\beta \vDash n} \beta_1 a_\beta$. Note that since $a_\beta = a_{\mathsf{cyc}^k(\beta)}$, then we have

$$|I_n| = \sum_{\beta \vDash n} \frac{1}{\ell(\beta)} \sum_{k=1}^{\ell(\beta)} \beta_1 a_{\mathsf{cyc}^k(\beta)} = \sum_{\beta \vDash n} \frac{1}{\ell(\beta)} \sum_{k=1}^{\ell(\beta)} \mathsf{cyc}^{-k}(\beta)_1 a_\beta = \sum_{\beta \vDash n} \frac{n a_\beta}{\ell(\beta)} \ .$$

We will show that

$$\sum_{\beta \vDash n} \frac{n a_\beta}{\ell(\beta)} = |I_n| = \sum_{d|n} d\ |\{\text{degree } d \text{ Lyndon words in } X\}|, \tag{2.3}$$

from which Möbius inversion shows that the number of degree n Lyndon words is equal to the formula for p_n in terms of $\vec{a}$ given in Proposition 2.2 (iii).

To start, we define a "faux-cycling" operation on I_n by

$$\mathsf{fcyc}(i, w) = \begin{cases} \big(i+1, w\big) & \text{if } i < \mathsf{deg}(w_1) \\ \big(1, \mathsf{cyc}(w)\big) & \text{otherwise.} \end{cases}$$

For any degree n word $w = w_1 w_2 \cdots w_\ell$ and $1 \leq a \leq \ell$,

$$(0, \mathsf{cyc}^a(w)) = \mathsf{fcyc}^{\mathsf{deg}(w_1) + \cdots + \mathsf{deg}(w_{a-1})}(0, w).$$

Thus, fcyc is periodic of order n. Moreover, if k is the minimal positive integer for which $\mathsf{cyc}^k(w) = w$, then the fcyc orbit of $(0, w)$—or (i, w) for any $1 \leq i \leq \mathsf{deg}(w_1)$—has size

$$d = \mathsf{deg}(w_1 w_2 \cdots w_k).$$

Finally, the cyclic shifts of $w' = w_1 w_2 \cdots w_k$ are distinct, so taking the unique minimal one, we obtain a Lyndon word of degree d. Since w is the (ℓ/k)-fold concatenation of w' with itself, the d-elements of the fcyc-orbit of $(0, w)$ are uniquely determined by w'. $\square$

Example 2.7. For specific examples of sequences, $\vec{p}$ may have better-known interpretations than the one given in Proposition 2.6. Continuing with Example 2.5, we have that if $h_n = a_{n+1} = \frac{1}{n+1}\binom{2n}{n}$, Proposition 2.6 says that p_n is equal to the number of Lyndon words in an alphabet $X = \bigcup_{n\geq 1} X^{(n)}$ with $a_n = |X^{(n)}|$. However, [13, A022553] states that p_n is also equal to the number of length $2n$ Lyndon words in the alphabet $\{0 < 1\}$ with an equal number of 0s and 1s.

Corollary 2.8. *If $\vec{a} \in \mathbb{N}^{\mathbb{Z}_+}$, then $\vec{h} \geq \vec{p} \geq \vec{a} \geq \vec{0}$.*

Proof. Let $X = \biguplus_{n\geq 1} X^{(n)}$ be a graded set with $|X^{(n)}| = a_n$. Considered as a word with one letter, each element $x \in X$ is a Lyndon word, so by Proposition 2.6 the inequalities above correspond to the set inclusions

$$\{\text{degree } n \text{ words in } X\} \supseteq \{\text{degree } n \text{ Lyndon words in } X\} \supseteq X^{(n)}. \qquad \square$$

3. Sequences and Tensor algebras

We now give the algebraic context for the results of Section 2. As in the introduction, we use the abbreviation FGCCHA to mean Free Graded Connected Cocommutative Hopf Algebra, which we now formally define.

In order to define an FGCCHA, we must first recall the tensor algebra of a graded vector space $V = \bigoplus_{n\geq 1} V_i$:

$$\mathsf{T}(V) = \bigoplus_{n\geq 0} \Big(\bigoplus_{k\geq 0} V_k\Big)^{\otimes n}, \tag{3.1}$$

with a graded multiplication given by $\otimes$, where elements of V_k have degree k.

Definition 3.1. An FGCCHA is a graded cocommutative Hopf algebra H which is freely generated by some graded subset $X = \biguplus_{n\geq 1} X^{(n)}$ of H, so that *as algebras*

$$H \cong \mathsf{T}(\mathbb{C}X) \qquad \text{where} \qquad \mathbb{C}X = \bigoplus_{k\geq 0} \mathbb{C}\text{-span}\{x \in X^{(k)}\}.$$

Under this isomorphism, the coproduct of H corresponds to some graded algebra homomorphism $\tilde{\Delta} : \mathsf{T}(\mathbb{C}X) \to \mathsf{T}(\mathbb{C}X) \otimes \mathsf{T}(\mathbb{C}X)$.

We will also consider Lie subalgebras of $\mathsf{T}(V)$ and FGCCHAs under the commutator bracket $[x, y] = xy - yx$.

Definition 3.2 (see [20, §0.2])**.** The free Lie algebra $\mathfrak{L}(X)$ on a graded set $X = \biguplus_{n\geq 1} X^{(n)}$ is the smallest graded subspace of $\mathsf{T}(\mathbb{C}X)$ that contains X and is closed under the commutator bracket. This is a Lie algebra under the commutator bracket, but not a Hopf algebra.

Given any FGCCHA H, the primitive elements of H are elements of the graded subspace

$$\mathcal{P}(H) = \{x \in H \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}.$$

To be clear, the generating set X may not be contained in $\mathcal{P}(H)$. The bracket operation $[x, y] = xy - yx$ makes $\mathcal{P}(H)$ into a Lie algebra.

Given a graded Lie algebra L, the (graded) derived subalgebra of L is

$$[L, L] = \bigoplus_{n \geq 1} [L, L]_n \qquad \text{where} \qquad [L, L]_n = \mathbb{C}\text{-span}\{\text{degree-}n \text{ commutators of } L\}. \tag{3.2}$$

The main result of this section relates the graded dimensions of an FGCCHA to its generating set and primitives. This sort of result is well-known to experts, but we have not found a statement which incorporates grading in the literature.

Proposition 3.3. *Let H be a FGCCHA with generating set X. The triple of sequences*

$$\left(\overrightarrow{\dim}(H),\, \overrightarrow{\mathrm{card}}(X),\, \overrightarrow{\dim}\left(\mathcal{P}(H)\right)\right)$$

satisfies Equation (2.1).

The proof of Proposition 3.3 occupies the remainder of the section and makes use of the Cartier-Milnor-Moore and Poincare–Birkhoff–Witt theorems, which we state below. First, however, we give an example to illustrate the result.

Example 3.4. The Hopf algebra of noncommutative symmetric functions NSym [9] is freely generated by $X = \{\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3, \ldots\}$ where $\mathsf{deg}(\mathbf{h}_i) = i$ for all $i \geq 1$ so that $\overrightarrow{\mathrm{card}}(X) = (1, 1, 1, \ldots)$. The bases of the degree n homogeneous components of NSym are indexed by compositions of n and hence $\overrightarrow{\dim}(\mathsf{NSym}) = (1, 2, 4, 8, \ldots)$. The dimensions of the space pf primitives is the sequence $\overrightarrow{\dim}(\mathcal{P}(\mathsf{NSym})) = (1, 1, 2, 3, 6, 9, 18, 30, 56, \ldots)$ whose n^{th} term is equal to the number of compositions of n that are Lyndon [11].

We now begin to assemble our proof of Proposition 3.3. As stated above, this is a folklore result which we prove using well-known theorems for the sake of completeness.

Given a graded Lie algebra $L = \bigoplus_{n \geq 1} L^{(n)}$, let the *universal enveloping algebra* of L be

$$\mathcal{U}(L) = \mathsf{T}(L) / \langle x \otimes y - y \otimes x - [x, y] \mid x, y \in L \rangle.$$

This is a cocommutative Hopf algebra where the coproduct on $\mathcal{U}(L)$ is defined so that every element of L is primitive. By [20, Theorem 1.4] we have $\mathcal{P}(\mathcal{U}(L)) = L$.

The Poincaré–Birkhoff–Witt Theorem [20, §0.1] states for any given ordered homogeneous basis $\{z_i\}$ of L, the enveloping algebra $\mathcal{U}(L)$ has a homogeneous basis

$$\{z_{i_1} z_{i_2} \cdots z_{i_k} \mid i_1 \leq i_2 \leq \cdots \leq i_k\}.$$

This statement carries important information about the dimension of $\mathcal{U}(L)$, namely that

$$\overrightarrow{\dim}(\mathcal{U}(L))_n = \#\{\text{multisubsets } S \text{ of } \{z_i\} \mid \sum_{z \in S} m_z(S) \mathsf{deg}(z) = n\},$$

where $m_z(S)$ denotes the multiplicity of z in S.

For any FGCCHA H, the space $\mathcal{P}(H)$ is a Lie algebra under the bracket $[x,y] = xy - yx$. The Milnor-Moore theorem [17, Theorem 5.18] states there is a Hopf algebra isomorphism

$$\begin{array}{rcl} \mathcal{U}(\mathcal{P}(H)) & \to & H \\ \mathcal{P}(H) \ni x & \mapsto & x. \end{array}$$

Proof of Proposition 3.3. Let $\vec{h} = \overrightarrow{\dim}(H)$, $\vec{a} = \overrightarrow{\mathrm{card}}(X)$, and $\vec{p} = \overrightarrow{\dim}(\mathcal{P}(H))$. We will show that $\vec{h} = \phi_{a,h}(\vec{a})$ and $\vec{p} = \phi_{h,p}(\vec{h})$, from which the claim follows.

For $n \geq 1$, h_n is the number of degree n monomials in X, which is also the number of degree n words in X. Applying Proposition 2.6, $\vec{h} = \phi_{a,h}(\vec{a})$.

On the other hand, the Milnor–Moore theorem states that H is isomorphic to the universal enveloping algebra of $\mathcal{P}(H)$. By the Poincare–Birkhoff–Witt theorem, for any fixed homogeneous basis $Y = \biguplus_{n\geq 0} Y^{(n)}$ of $\mathcal{P}(H)$, h_n counts the multisets of Y whose elements have degree summing (with repetition) to n. Since $|Y^{(n)}| = p_n$ for all $n \geq 1$, the number of these multisets is exactly

$$\sum_{\lambda \vdash n} \prod_{d\geq 1} \binom{p_d + m_d(\lambda) - 1}{m_d(\lambda)},$$

so by Proposition 2.2 we have $\vec{p} = \phi_{h,p}(\vec{h})$. □

4. Characterization of FGCCHAs by graded dimension

This section gives a classification of isomorphism types of FGCCHAs via the associated integer sequences. This begins with Aliniaeifard and Thiem's result.

Theorem 4.1 ([4, Theorem 4.2])**.** *Let H and K be FGCCHAs. Then $H \cong K$ if and only if $\overrightarrow{\dim}(H) = \overrightarrow{\dim}(K)$.*

This result may not be surprising to experts in the field. However, we have not found it recorded elsewhere in the literature, and by stating it [4] generalize a recurrent approach to comparing Hopf algebras; see Example 4.3.

We consider here the question of determining which sequences $\vec{h} \in \mathbb{N}^{\mathbb{Z}_+}$ appear as the graded dimension of an FGCCHA. In fact, we go further by identifying an explicit representative of each isomorphism class of FGCCHA. Recall the free graded Lie algebra from Definition 3.2 and for each sequence $\vec{a} \in \mathbb{N}^{\mathbb{Z}_+}$, let

$$\mathfrak{L}(\vec{a}) = \mathfrak{L}(X_{\vec{a}}) \qquad \text{where} \qquad X_{\vec{a}} = \biguplus_{n\geq 1} \{x_i^{(n)} \mid 1 \leq i \leq a_n\}. \tag{4.1}$$

Further recall the map $\phi_{h,a}$ from Definition 2.4 and the universal enveloping algebra map $\mathcal{U}(-)$ defined in Section 3.

Theorem 4.2. *There is a bijection*

$$\begin{array}{rcl} \left\{ \textit{sequences } \vec{a} \in \mathbb{N}^{\mathbb{Z}_+} \right\} & \to & \left\{ \begin{array}{c} \textit{Isomorphism classes} \\ \textit{of } \mathsf{FGCCHA} \end{array} \right\} \\ \vec{a} & \mapsto & \mathcal{U}(\mathfrak{L}(\vec{a}))) \\ \phi_{h,a}(\overrightarrow{\dim}(H)) & \leftarrow & H \end{array}.$$

In particular, $\vec{h} \in \mathbb{N}^{\mathbb{Z}_+}$ is the dimension sequence of an FGCCHA *H if and only if $\phi_{h,a}(\vec{h}) \in \mathbb{N}^{\mathbb{Z}_+}$. Similarly, $\vec{p} \in \mathbb{N}^{\mathbb{Z}_+}$ is the dimension sequence for $\mathcal{P}(H)$ of an* FGCCHA *H if and only if $\phi_{p,a}(\vec{p}) \in \mathbb{N}^{\mathbb{Z}_+}$.*

We prove the theorem in Section 4.1 following additional examples and results.

Example 4.3. *A cocommutative Hopf algebra on permutations.* Let $\vec{h} = (n!)_{n\geq 1}$, and $\vec{a} = \phi_{h,a}(\vec{h}) = (1, 1, 3, 13, 71, \ldots)$. Let $\dot{\mathfrak{S}}$ denote the set of connected permutations [2] (also sometimes called indecomposable or irreducible permutations in the literature). Then $\vec{a}$ is equal to $\overrightarrow{\mathrm{card}}(\dot{\mathfrak{S}})$ [13, A003319]. By Theorem 4.2 there exists a FGCCHA $\mathcal{U}(\mathfrak{L}(\dot{\mathfrak{S}}))$ of graded dimension $n!$. Viewing each permutation as a word in the alphabet $\mathbb{Z}_+$, the product is shifted concatenation of permutations, and the coproduct is defined such that every indecomposable permutation is primitive.

There are numerous FGCCHAs with graded dimension $n!$ in the literature that have superficially different presentations. This includes the Hopf algebras of heap ordered trees and permutation in [10], the dual of the associated graded of the Malvenuto–Reutenauer Hopf algebra of permutations [1], the Hopf algebra of permutations $\mathfrak{S}\mathbf{Sym}$ defined in [12, §3], and the Hopf algebra of permutations ($\mathbb{K}\mathfrak{S}$) from [15, §5]. The papers [1, 10, 12, 15] construct explicit isomorphisms between these Hopf algebras, and indeed by Theorem 4.1, they are all isomorphic to $\mathcal{U}(\mathfrak{L}(\dot{\mathfrak{S}}))$.

Example 4.4. Many combinatorially interesting sequences correspond to FGCCHAs, but it is not obvious when this is the case; we illustrate this by considering two closely-related sequences.

(1) $(1, 1, 2, 3, 5, 8, 13, \ldots)$, the Fibonacci sequence
(2) $(2, 1, 3, 4, 7, 11, 18, \ldots)$, the Lucas sequence

Let us consider the Fibonacci sequence $\vec{f} = (1, 1, 2, 3, 5, 8, 13, \ldots)$ as if it could be the graded dimensions of some Hopf algebra. We compute first that $\phi_{h,a}(\vec{f}) = (1, 0, 1, 0, 1, 0, \ldots)$ and by Theorem 4.2 conclude that there is a Hopf algebra H with a basis indexed by compositions with only odd parts and with one generator at each odd degree. This Hopf algebra H has graded dimension equal to $\overrightarrow{\dim}(H) = \vec{f}$. This Hopf algebra is sometimes referred to as the peak algebra [5, §2].

Next consider the Lucas sequence $\vec{\ell} = (2, 1, 3, 4, 7, 11, 18, \ldots)$ as if it could also be the graded dimensions of a Hopf algebra. We compute that $\phi_{h,a}(\vec{\ell})$ and determine that the sequence is alternating and begins $(2, -3, 7, -13, \ldots)$ and so we must conclude that there does not exist a Hopf algebra with graded dimensions equal to $\vec{\ell}$.

The rest of the section concerns the technical question of explicitly realizing the Aliniaeifard–Thiem isomorphism in every possible way; this has further applications in later sections.

Definition 4.5. For an FGCCHA H, an *ordered primitive generating set* is a sequence of tuples

$$\vec{A} = \left(\vec{A}^{(n)} = (\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_{a_n}^{(n)})\right)_{n=1}^{\infty} \qquad \text{with} \qquad \begin{array}{l}(1) \text{ each } \alpha_i^{(n)} \in \mathcal{P}(H)_n \text{ , and} \\ (2)\ H \text{ is freely generated by } \vec{A}.\end{array}$$

Let

$$\mathrm{OPG}(H) = \{\text{ordered primitive generating sets of } H\}\,.$$

We now know that every FGCCHA is isomorphic to $\mathcal{U}(\mathfrak{L}(\vec{a}))$ for some $\vec{a} \in \mathbb{N}^{\mathbb{Z}_+}$, so it is natural to ask how many isomorphisms there are between these two objects. We noticed that the isomorphisms between $\mathcal{U}(\mathfrak{L}(\vec{a}))$ and H are in bijection with ordered sequences of primitive generators of H. The following theorem uses this relationship to quantify the number of isomorphisms $\mathcal{U}(\mathfrak{L}(\vec{a})) \cong H$.

Theorem 4.6. *Let H be an FGCCHA with $\vec{h} = \overrightarrow{\dim}(H)$. Then writing $\vec{a} = \phi_{h,a}(\vec{h})$ and $\vec{p} = \phi_{h,p}(\vec{h})$, we have bijections*

$$\begin{array}{rcl}\Gamma : \{\textit{graded Hopf algebra isomorphisms}\ \mathcal{U}(\mathfrak{L}(\vec{a})) \to H\} & \to & \mathrm{OPG}(H) \\ \phi & \mapsto & \phi(X_{\vec{a}})\end{array}$$

and

$$\Xi : \mathrm{OPG}(H) \quad \to \quad \left\{ \begin{array}{c} \textit{Sequences } \left(M^{(n)} \in \mathrm{Mat}_{a_n \times p_n}(\mathbb{C})\right)_{n=1}^{\infty} \\ \textit{with } \det\left((M_{i,j}^{(n)})_{1 \le i,j \le a_n}\right) \neq 0 \textit{ for } n \geq 1 \end{array} \right\}.$$

The proof of Theorem 4.6 is given in Section 4.2. As we do not explicitly describe our bijection, the second part of Theorem 4.6 should be read as a statement about how "big" the set $\mathrm{OPG}(H)$ is. Realizing our bijection requires one to make a rather complicated choice of basis for $\mathcal{P}(H)$, which we explain fully in the proof. However, we can illustrate the bijection concretely with a small example.

Example 4.7. We construct the set $\mathrm{OPG}(H)$ for the FGCCHA $H = \mathbb{C}\langle x, y, z\rangle$ freely generated by two primitive generators x and y in degree one and one primitive generator z in degree two. Thus, in the notation of Theorem 4.6, we have $\vec{a} = (2, 1, 0, \ldots)$ and $\vec{p} = \phi_{a,p}(\vec{a}) = (2, 2, \ldots)$. The theorem then states that $\mathrm{OPG}(H)$ is determined by a choice of matrices

$$M^{(1)} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ with } ac - bd \neq 0 \qquad \text{and} \qquad M^{(2)} = \begin{bmatrix} e & f \end{bmatrix} \text{ with } e \neq 0$$

to the ordered primitive generator set

$$\Big(\big(ax + by, cx + dy\big), \big(ez + f(xy - yx)\big), \emptyset, \ldots \Big) \in \mathrm{OPG}(H).$$

4.1. **Proof of Theorem 4.2.** We will make use the following established results about free Lie algebras from [20].

Proposition 4.8 ([20, Theorem 0.5])**.** *For a sequence $\vec{a}$ of a nonnegative integers, the enveloping algebra $\mathcal{U}(\mathfrak{L}(\vec{a}))$ is an* FGCCHA *which is generated in degree n by the a_n-many primitive generators of $\mathfrak{L}(\vec{a})$.*

We now prove Theorem 4.2; recall Proposition 3.3.

Proof of Theorem 4.2. Given a sequence $\vec{a} \in \mathbb{N}^{\mathbb{Z}_+}$, Proposition 4.8 states that $\mathcal{U}(\mathfrak{L}(\vec{a}))$ is an FGCCHA freely generated by a set with graded cardinality $\vec{a}$, so the map which sends $\vec{a}$ to $\mathcal{U}(\mathfrak{L}(\vec{a}))$ is well defined. Now consider the injectivity of this map. By Proposition 3.3,

$$\overrightarrow{\dim}\big(\mathcal{U}(\mathfrak{L}(\vec{a}))\big) = \phi_{a,h}(\vec{a}),$$

which by Aliniaeifard and Thiem's result (Theorem 4.1) is an invariant of the isomorphism type. The map $\phi_{a,h}$ is invertible and hence injective, for any two sequences $\vec{a}, \vec{b}$ in the domain,

$$\mathcal{U}(\mathfrak{L}(\vec{a})) \cong \mathcal{U}(\mathfrak{L}(\vec{b})) \qquad \text{if and only if} \qquad \vec{a} = \vec{b}.$$

Finally for surjectivity, fix an isomorphism class of FGCCHA, let H be a representative of that class, and let $\vec{h} = \overrightarrow{\dim}(H)$. By Proposition 3.3, $\vec{a} = \phi_{h,a}(\vec{h})$ is the graded cardinality of the free generating set of H and so $\vec{a}$ belongs to the domain. Moreover, we claim that the image of $\vec{a}$ is in the isomorphism class with representative H. As we deduced in the previous paragraph, the dimension sequence of $\mathcal{U}(\mathfrak{L}(\vec{a}))$ is $\phi_{a,h}(\vec{a})$, which by definition is the dimension sequence of H. Thus, Aliniaeifard and Thiem's result (Theorem 4.1) implies that $H \cong \mathcal{U}(\mathfrak{L}(\phi_{h,a}(\vec{h})))$. $\square$

4.2. **Proof of Theorem 4.6.** We now consider the problem of explicitly constructing isomorphisms between FGCCHAs via the set $\mathrm{OPG}(H)$ defined at the beginning of the section. Our proofs in this section and the next will make use of the subspaces

$$H_+ = \bigoplus_{n\geq 1} H_n \qquad \text{and} \qquad H_+^2 = \mu(H_+ \otimes H_+).$$

The space H_+^2 is a graded subspace of H with

$$(H_+^2)_n = \bigoplus_{\substack{\alpha \vDash n \\ \alpha \neq (n)}} H_{\alpha_1} H_{\alpha_2} \cdots H_{\alpha_\ell} \subseteq H_n. \tag{4.2}$$

By definition (see the beginning of Section 3), the degree n generators of H descend to a basis of $H_n/(H_+^2)_n$, and therefore for any graded generating set $\vec{A} = \biguplus_{n\geq 1} \vec{A}^{(n)}$ of H,

$$\dim(H_n/(H_+^2)_n) = |\vec{A}^{(n)}|.$$

We also use an additional relation between H_+/H_+^2 and generators of H which appears in [7].

Proposition 4.9 ([7, Proposition 2.2 and 2.4])**.** *If H is a free graded connected algebra and $\vec{A} \subseteq H$ is a graded set with $H_+ = \mathbb{C}\vec{A} \oplus H_+^2$, then $\vec{A}$ freely generates H. If moreover H is a Hopf algebra, then $\mathcal{P}(H) = \mathbb{C}\vec{A} \oplus [\mathcal{P}(H), \mathcal{P}(H)]$.*

Proof of Theorem 4.6. Let $\vec{a} = \phi_{h,a}(\overrightarrow{\dim}(H))$ and $X_{\vec{a}}$ be an element of $\mathrm{OPG}(\mathcal{U}(\mathfrak{L}(\vec{a})))$. We first consider the map

$$\begin{array}{rcl}\Gamma : \{\text{graded Hopf algebra isomorphisms } \mathcal{U}(\mathfrak{L}(\vec{a})) \to H\} & \to & \mathrm{OPG}(H) \\ \phi & \mapsto & \phi(X_{\vec{a}})\end{array},$$

where $\phi(X_{\vec{a}})$ is the ordered set

$$\phi(X_{\vec{a}}) = \biguplus_{n\geq 0} \left(\phi(x_1^{(n)}), \phi(x_2^{(n)}), \ldots, \phi(x_{a_n}^{(n)})\right) \in \mathrm{OPG}(H).$$

We show that Γ is a bijection. The first step is to observe that Γ is well-defined: if $\phi : \mathcal{U}(\mathfrak{L}(\vec{a})) \to H$ is an isomorphism then the set $\phi(X_{\vec{a}})$ must freely generate H. Moreover as ϕ is a graded isomorphism and $x_i^{(n)}$ is primitive of degree n for all $n \geq 1$ and $1 \leq i \leq a_n$, it follows that $\phi(x_i^{(n)})$ is also primitive of degree n. The injectivity of Γ follows from the fact that any two algebra morphisms which agree on a generating set must be equal. For surjectivity, consider an arbitrary $\vec{A} = \left((\alpha_i^{(n)})_{i=1}^{a_n}\right)_{n\geq 1} \in \mathrm{OPG}(H)$ and define $\phi_{\vec{A}} : \mathcal{U}(\mathfrak{L}(\vec{a})) \to H$ by algebraically extending the mapping of generators $\phi_{\vec{A}} : x_i^{(n)} \mapsto \alpha_i^{(n)}$. Then $\phi_{\vec{A}}$ is an isomorphism and $\phi_{\vec{A}}(X_{\vec{a}}) = \vec{A} \in \mathrm{OPG}(H)$.

We now define a bijection

$$\Xi : \mathrm{OPG}(H) \quad \to \quad \left\{ \begin{array}{c} \text{Sequences } \left(M^{(n)} \in \mathrm{Mat}_{a_n \times p_n}(\mathbb{C})\right)_{n=1}^{\infty} \\ \text{with } \det\left((M_{i,j}^{(n)})_{1\leq i,j\leq a_n}\right) \neq 0 \text{ for } n \geq 1 \end{array} \right\}.$$

Our definition depends on a choice of $\vec{A} \in \mathrm{OPG}(H)$ and a homogeneous basis

$$\biguplus_{n\geq 1} \{v_1^{(n)}, v_2^{(n)}, \ldots, v_{p_n - a_n}^{(n)}\}$$

of the derived subalgebra $[\mathcal{P}(H), \mathcal{P}(H)]$ (see Equation (3.2)), which we now fix. By Proposition 4.9 $\mathcal{P}(H) = \mathbb{C}\vec{A} \oplus [\mathcal{P}(H), \mathcal{P}(H)]$ so the set $\{\alpha_1^{(n)}, \ldots, \alpha_{a_n}^{(n)}, v_1^{(n)}, \ldots, v_{p_n-a_n}^{(n)}\}$ is a basis for the degree n component of $\mathcal{P}(H)$. For any $\vec{B} = \left((\beta_i^{(n)})_{i=1}^{a_n}\right)_{n\geq 1} \in \mathrm{OPG}(H)$, we define $M^{(n)}$ to be the matrix whose rows express each $\beta_i^{(n)}$ in this basis, and set

$$\Xi(\vec{B}) = (M^{(n)})_{n\geq 1}.$$

The matrix $(M_{i,j}^{(n)})_{1\leq i,j\leq a_n}$ gives the change of basis from $\vec{A}_n$ to $\vec{B}_n$ in the quotient $\mathcal{P}(H)_n/[\mathcal{P}(H), \mathcal{P}(H)]_n$. Since this is invertible, it has a nonzero determinant.

On the other hand, given a sequence $\left(M^{(n)}\right)_{n\geq 1}$ with $M^{(n)} \in \mathrm{Mat}_{a_n \times p_n}$ and

$$\det\left((M_{i,j}^{(n)})_{1\leq i,j\leq a_n}\right) \neq 0,$$

set

$$\beta_i^{(n)} = \sum_{j=1}^{a_n} M_{i,j}^{(n)} \alpha_j^{(n)} + \sum_{j=1}^{p_n - a_n} M_{i,j+a_n}^{(n)} v_j^{(n)}.$$

Then the ordered set

$$\vec{B} = \biguplus_{n \geq 1} \left(\beta_1^{(n)}, \beta_2^{(n)}, \ldots, \beta_{a_n}^{(n)} \right)$$

descends to a basis of H_+/H_+^2 and therefore generates H by Proposition 4.9. Since $\vec{B} \subseteq \mathcal{P}(H)$, $\vec{B} \in \mathrm{OPG}(H)$, and moreover $\Xi(\vec{B}) = (M^{(n)})_{n \geq 0}$. Therefore we have constructed an inverse of Ξ, showing that it is a bijection. □

Remark 4.10. Given two FGCCHAs H and K with $\overrightarrow{\dim}(H) = \overrightarrow{\dim}(K)$, Theorem 4.1 gives $H \cong K$. However, this isomorphism is abstract. As an explicit isomorphism may be desired, we present a method of constructing one.

Given graded sets X and Y generating H and K respectively, we apply Lemma 5.4 to obtain primitive generators $\mathbf{e}(X)$ and $\mathbf{e}(Y)$. Choosing bijections $f_n : \mathbf{e}(X)_n \to \mathbf{e}(Y)_n$ for all $n \geq 1$ we obtain a bijection of primitive generators $f : \mathbf{e}(X) \to \mathbf{e}(Y)$ which extends to an isomorphism $\tilde{f} : H \to K$.

5. Classifying surjections between FGCCHAs

We now consider the problem of constructing surjective homomorphisms between two FGCCHAs, H and K, beginning with a naive approach to this problem. Let $\vec{A} \in \mathrm{OPG}(H)$ and $\vec{B} \in \mathrm{OPG}(K)$ be ordered primitive generating sets as in Definition 4.5, and let $\vec{a} = \overrightarrow{\mathrm{card}}(\vec{A})$ and $\vec{b} = \overrightarrow{\mathrm{card}}(\vec{B})$. If $\vec{a} \geq \vec{b}$, projecting from $\vec{A}$ to $\vec{B}$ gives a surjective homomorphism:

$$\begin{aligned} f_{\vec{B}}^{\vec{A}} : H &\to K \\ \alpha_i^{(n)} &\mapsto \begin{cases} \beta_i^{(n)} & \text{for } 1 \leq i \leq b_n \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{5.1}$$

where $\alpha_i^{(n)}$ or $\beta_i^{(n)}$ denotes the ith element of the degree n parts of $\vec{A}$ or $\vec{B}$, respectively. We find that this construction actually accounts for all surjections between FGCCHAs.

Theorem 5.1. *Let H and K be FGCCHAs which are generated by sets of graded cardinality $\vec{a}$ and $\vec{b}$, respectively.*

(1) There is a surjective homomorphism $f : H \to K$ if and only if $\vec{a} \geq \vec{b}$.
(2) For every surjective homomorphism $f : H \to K$, there exists a (non-unique) choice of ordered primitive generating sets $\vec{A} \in \mathrm{OPG}(H)$ and $\vec{B} \in \mathrm{OPG}(K)$ for which $f = f_{\vec{B}}^{\vec{A}}$.

The proof of Theorem 5.1 is given after Lemma 5.4, following some intermediate examples and results.

Example 5.2. Recall the FGCCHA from Example 4.7, $H = \mathbb{C}\langle x, y, z\rangle$. Also let $K = \mathbb{C}\langle u, v\rangle$ denote an FGCCHA which is freely generated by two degree-one primitive elements u and v. By Theorem 5.1 (1), since $(2, 1, 0, \ldots) \geq (2, 0, 0, \ldots)$, there is a surjective homomorphism.

We define $f : H \to K$ by

$$f(x) = u, \qquad f(y) = v, \qquad \text{and} \qquad f(z) = uv - vu,$$

which does not obviously match the format of Equation (5.1). However, if we write f using a slightly different set of ordered primitive generators,

$$\vec{A} = \Big((x, y), (z - xy + yx), \emptyset, \ldots\Big) \qquad \text{and} \qquad \vec{B} = \Big((u, v), \emptyset, \ldots\Big),$$

then we have

$$f(x) = u, \qquad f(y) = v, \qquad \text{and} \qquad f(z - xy + yx) = 0,$$

so we have found the $\vec{A}$ and $\vec{B}$ for which $f = f^{\vec{A}}_{\vec{B}}$ as predicted by the theorem.

Corollary 5.3. *Given a surjective homomorphism $f : H \to K$ between two FGCCHAs,*

(1) K is isomorphic to a Hopf subalgebra of H.

(2) Every other surjective homomorphism $g : H \to K$ has the form $g = \gamma \circ f \circ \tau$ for some automorphims γ of K and τ of H.

Proof. Let f be a surjective homomorphism from H to K. By Theorem 5.1, there exist $\vec{A} = \big((\alpha_i^{(n)})_{i=1}^{a_n}\big)_{n\geq 0} \in \mathrm{OPG}(H)$ and $\vec{B} = \big((\beta_i^{(n)})_{i=1}^{b_n}\big)_{n\geq 0} \in \mathrm{OPG}(K)$ such that $f = f^{\vec{A}}_{\vec{B}}$. To see (1), we observe that the homomophism $\kappa : K \to H$ defined by $\beta_i^{(n)} \mapsto \alpha_i^{(n)}$ is a left inverse of $f^{\vec{A}}_{\vec{B}}$. Therefore κ must be injective, so the Hopf subalgebra $\kappa(K) \subseteq H$ is isomorphic to K.

For (2), let f be as in the previous paragraph and let g be a surjective homomorphism from H to K. We first note that by Theorem 5.1, $g = f^{\vec{C}}_{\vec{D}}$ for some $\vec{C} \in \mathrm{OPG}(H)$ and $\vec{D} \in \mathrm{OPG}(K)$. We then take $\tau : H \to H$ to be the extension of the order-preserving map which sends $\vec{C}$ to $\vec{A}$, and similarly $\gamma : K \to K$ as the extension of the order-preserving mapping of $\vec{B}$ to $\vec{D}$. Then $g = \gamma \circ f \circ \tau$. □

Our proof makes use of the *Eulerian idempotent* (see [1, §1.4] or [16, §4.5.2])

$$\mathbf{e}(x) = x - \frac{1}{2}\mu \circ \Delta_+(x) + \frac{1}{3}\mu^{(2)} \circ \Delta_+^{(2)}(x) - \cdots \tag{5.2}$$

where Δ_+ denotes the positive coproduct, $\Delta_+(x) = \Delta(x) - x \otimes 1 - 1 \otimes x$, and we write $\mu^{(k)}$ and $\Delta^{(k)}$ for the k^{th} composition of the multiplication and comultiplication maps, respectively. By associativity and coassociativity of the Hopf algebra, these maps are well defined.

Aguiar and Sottile use the Eulerian idempotent to produce primitive generators for the Grossman-Larson Hopf algebra of heap-ordered trees in [1], as do Novelli and Thibon for a Hopf algebra of dimension Catalan at each degree in [18, §5]. Lauve and Mastnak use a related primitive idempotent to produce primitive generators for the symmetric functions in noncommuting variables in [14]. Foissy and Patras recently showed that this method holds more generally for any FGCCHA and Lie idempotent family [8], generalizing the following result of Patras and Reutenauer [19].

Lemma 5.4 ([19, Lemma 22]). *Let H be an FGCCHA freely generated by a graded set $X = \biguplus_{n\geq 1} X_n$. Then the graded set*

$$\mathbf{e}(X) = \biguplus_{n\geq 1} \{\mathbf{e}(x) \mid x \in X_n\}$$

is a complete set of primitive generators of H.

Proof of Theorem 5.1. Let H and K be FGCCHAs which are generated by sets of graded cardinality $\vec{a}$ and $\vec{b}$, respectively. If $\vec{a} \geq \vec{b}$, then Equation (5.1) defines a surjective homomorphism between H and K, giving the "if" part of (1).

To prove the converse and (2), suppose that we have surjective homomorphism

$$f : H \to K.$$

For the next part of the argument, fix $n \geq 1$. Recall the subspaces $(H_+^2)_n \subseteq H_n$ and $(K_+^2)_n \subseteq K_n$ defined in Equation (4.2), and that

$$\dim(H_n/(H_+^2)_n) = a_n \qquad \text{and} \qquad \dim(K_n/(K_+^2)_n) = b_n.$$

As f is a graded surjective algebra homomorphism, it must map H_n onto K_n and $(H_+^2)_n$ onto $(K_+^2)_n$. Therefore f induces a surjective linear transformation from $H_n/(H_+^2)_n$ to $K_n/(K_+^2)_n$. Surjectivity implies that $a_n = \dim(H_n/(H_+^2)_n) \geq \dim(K_n/(K_+^2)_n) = b_n$.

We continue with our fixed n for (2). First choose a tuple of elements $\left(y_1^{(n)}, \ldots, y_{b_n}^{(n)}\right)$ from K_n which descend to a basis of $K_n/(K_+^2)_n$. Let $\beta_i^{(n)} = \mathsf{e}(y_i^{(n)})$ where e is the Eulerian idempotent. By Lemma 5.4 and Proposition 4.9 the tuple $\left(\beta_1^{(n)}, \ldots, \beta_{b_n}^{(n)}\right)$ also descends to a basis of $K_n/(K_+^2)_n$. Since f induces a surjection $H_n/(H_+^2)_n \to K_n/(K_+^2)_n$, we may choose an ordered subset $\left(x_1^{(n)}, \ldots, x_{a_n}^{(n)}\right)$ of H_n which descends to a basis of $H_n/(H_+^2)_n$ and moreover has the property that

$$f(x_i^{(n)}) = \begin{cases} \beta_i^{(n)} & \text{if } i \leq b_n \\ 0 & \text{otherwise.} \end{cases}$$

We can then take $\alpha_i^{(n)} = \mathsf{e}(x_i^{(n)})$, and as above the set $\left(\alpha_1^{(n)}, \ldots, \alpha_{a_n}^{(n)}\right)$ will still form a basis of $H_n/(H_+^2)_n$. Moreover, as f is a Hopf morphism

$$f(\alpha_i^{(n)}) = f(\mathsf{e}(x_i^{(n)})) = \mathsf{e}(f(x_i^{(n)})) = \mathsf{e}(\beta_i^{(n)}) = \beta_i^{(n)}.$$

Finally, repeat this construction for all $n \geq 1$. By Proposition 4.9 the sequences

$$\vec{A} = \left((\alpha_1^{(n)}, \ldots, \alpha_{a_n}^{(n)})\right)_{n\geq 1} \qquad \text{and} \qquad \vec{B} = \left((\beta_1^{(n)}, \ldots, \beta_{b_n}^{(n)})\right)_{n\geq 1}$$

belong to $\mathrm{OPG}(H)$ and $\mathrm{OPG}(K)$ respectively, and by construction $f = f_{\vec{B}}^{\vec{A}}$. $\square$

6. Classifying Hopf subalgebras by primitive dimensions

Let H be an FGCCHA. In this section we construct all the isomorphism classes of Hopf subalgebras of H; in particular we show that every Hopf subalgebra is an FGCCHA satisfying a simple condition. To begin, recall the function $\phi_{p,a}$ on $\mathbb{Q}^{\mathbb{Z}_+}$ from Definition 2.4, and recall that for any nonnegative sequence $\vec{a} = (a_1, a_2, \ldots)$, we write $\mathfrak{L}(\vec{a})$ for the free Lie algebra generated by the graded set $X_{\vec{a}}$ with a_n elements in degree n as in Equation (4.1).

Theorem 6.1. *Let H be a free graded connected cocommutative Hopf algebra and let $\vec{p} = \overrightarrow{\dim}(\mathcal{P}(H)_n)$. Then there is a bijection*

$$\left\{ \begin{array}{c} \textit{Isomorphism classes of} \\ \textit{Hopf subalgebras } K \subseteq H \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{Sequences } \vec{b} \in \mathbb{N}^{\mathbb{Z}_+} \\ \textit{for which } \phi_{a,p}(\vec{b}) \leq \vec{p} \end{array} \right\}$$

$$\begin{array}{rcl} K & \mapsto & \phi_{h,a}\big(\overrightarrow{\dim}(K)\big) \\ \mathcal{U}(\mathfrak{L}(\vec{b})) & \leftarrow\!\shortmid & \vec{b} \end{array}$$

Remark 6.2. Using the sequence transfer maps from Definition 2.4, Theorem 6.1 implies that isomorphisms of Hopf subalgebras of H are also indexed by sequences $\vec{q} \in \mathbb{N}^{\mathbb{Z}_+}$ for which $\vec{p} \geq \vec{q}$ and $\phi_{p,a}(\vec{q}) \in \mathbb{N}^{\mathbb{Z}_+}$.

The proof of Theorem 6.1 will construct a Hopf subalgebra in H isomorphic to each $\mathcal{U}(\mathfrak{L}(\vec{b}))$. Before completing the proof we will give several motivating examples and intermediate results.

Example 6.3. We illustrate a counterintuitive aspect of containment between FGCCHAs. Let H and K be FGCCHAs generated by graded sets with respective graded cardinalities $\vec{a} = (1, 0, 0, 0, \ldots)$ and $\vec{b} = (0, 1, 0, 0, \ldots)$, and further denote $\vec{h} = \overrightarrow{\dim}(H)$ and $\vec{k} = \overrightarrow{\dim}(K)$. Using Proposition 2.2, we can compute that

$$h_n = 1 \qquad \text{for all } n \geq 1.$$

We can also compute that

$$k_n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

as $b_\beta \neq 0$ implies that $\beta = (2, \ldots, 2)$. Therefore, $\vec{k} \leq \vec{h}$, so one might expect that H should contain a Hopf subalgebra isomorphic to K. However Theorem 6.1 tells us this is not the case: $\phi_{a,p}(\vec{a}) = \vec{a}$, $\phi_{a,p}(\vec{b}) = \vec{b}$, and $\vec{a}$ and $\vec{b}$ are incomparable.

Example 6.4. Recall the FGCCHA $K = \mathbb{C}\langle u, v\rangle$ from Example 5.2, so that u and v have degree one. Then $\vec{p} = \overrightarrow{\dim}(\mathcal{P}(K))$ is the sequence $\vec{p} = (2, 1, 2, 3, 6, 9, 18, 30, \ldots)$ [13, A001037] given by

$$p_n = \frac{1}{n} \sum_{d|n} \mu(n/d) 2^d.$$

Then we can take

$$\vec{q} = \vec{p} - (1, 0, 0, \ldots)$$

so that clearly $\vec{p} \geq \vec{q}$. The sequence $\vec{q}$ is identified in [13, A059966], and direct computation then gives that

$$\phi_{p,a}(\vec{q}) = (1, 1, 1, \ldots) \in \mathbb{N}^{\mathbb{Z}_+},$$

so this sequence determines a subalgebra isomorphic to $\mathcal{U}(\mathfrak{L}(1, 1, 1, \ldots))$ inside of K. In Example 3.4 we identify $\mathcal{U}(\mathfrak{L}(1, 1, 1, \ldots)) \cong \mathsf{NSym}$, so K has a Hopf subalgebra isomorphic to NSym.

We now state and prove an intermediate result in the proof of Theorem 6.1; the proof of the theorem follows. Recall the definition of the derived subalgebra $[L, L]$ of a graded Lie algebra L from Equation (3.2).

Lemma 6.5. *Let $\vec{a} = (a_1, a_2, \ldots) \in \mathbb{N}^{\mathbb{Z}_+}$. Then*

$$\overrightarrow{\dim}([\mathfrak{L}(\vec{a}), \mathfrak{L}(\vec{a})]) = \phi_{a,p}(\vec{a}) - \vec{a},$$

or equivalently $\vec{a} = \overrightarrow{\dim}\big(\mathfrak{L}(\vec{a})/[\mathfrak{L}(\vec{a}), \mathfrak{L}(\vec{a})]\big)$.

Proof. Using Proposition 4.9, $\mathfrak{L}(\vec{a}) = \mathbb{C}X_{\vec{a}} \oplus [\mathfrak{L}(\vec{a}), \mathfrak{L}(\vec{a})]$. Taking the dimension vector of each space, we obtain an equivalent equation. □

Lemma 6.6 (Shirshov–Witt Theorem [21, 22])**.** *Every Lie subalgebra of a free Lie algebra is free.*

Proof of Theorem 6.1. We first show the map $K \mapsto \phi_{h,a}\big(\overrightarrow{\dim}(K)\big)$ is injective. Suppose that J and K are Hopf subalgebras of H with the property that $\phi_{h,a}\big(\overrightarrow{\dim}(J)\big) = \phi_{h,a}\big(\overrightarrow{\dim}(K)\big)$. Since $\phi_{h,a}$ is invertible by Proposition 2.2 $\overrightarrow{\dim}(J) = \overrightarrow{\dim}(K)$. We therefore conclude by Theorem 4.1 that $J \cong K$.

Now suppose that $\vec{b} \in \mathbb{N}^{\mathbb{Z}_+}$ has the property that $\vec{q} = \phi_{a,p}(\vec{b}) \leq \vec{p}$. We will construct a Hopf subalgebra K of H which is isomorphic to $\mathcal{U}(\mathfrak{L}(\vec{b}))$; by Proposition 2.2 this will show that the map $K \mapsto \phi_{h,a}\big(\overrightarrow{\dim}(K)\big)$ is surjective, completing the proof.

There is some subtlety to defining K. Since we have not assumed that $a_n \geq b_n$, we cannot take K to be generated by some subset of the free generators of H, and may have to choose elements of H_+^2 as generators for K. These choices cannot always be expressed uniformly, so we resort to an inductive argument to demonstrate their existence. Specifically, we construct a tower of Lie subalgebras

$$0 = L^{(1)} \subseteq L^{(2)} \subseteq L^{(3)} \subseteq \cdots \subseteq \mathcal{P}(H)$$

such that

$$L^{(n)} \cong \mathfrak{L}(b_1, \ldots, b_{n-1}, 0, 0, \ldots).$$

Taking $L^{(\infty)}$ to be the union of the $L^{(n)}$, we obtain K as $\mathcal{U}(L^{(\infty)}) \subseteq \mathcal{U}(\mathcal{P}(H)) = H$.

We proceed by induction with case case $L^{(1)} = 0$. We then assume inductively that $L^{(n)}$ has been constructed. By Lemma 6.5 and our inductive hypothesis,

$$\overrightarrow{\dim}([L^{(n)}, L^{(n)}]) = \phi_{a,p}(b_1, \ldots, b_{n-1}, 0, 0, \ldots) - (b_1, \ldots, b_{n-1}, 0, 0, \ldots).$$

Thus $\overrightarrow{\dim}([L^{(n)}, L^{(n)}])_n$ is equal to the nth term of $\phi_{a,p}(b_1, \ldots, b_{n-1}, 0, 0, \ldots)$, which is also equal to $\overrightarrow{\dim}(L^{(n)})_n$. We can further deduce—after a careful examination of the definition of $\phi_{a,p}$—that

$$\phi_{a,p}(b_1, \ldots, b_{n-1}, 0, 0, \ldots)_n = \phi_{a,p}(\vec{b})_n - b_n = q_n - b_n. \tag{6.1}$$

Therefore,

$$\overrightarrow{\dim}\left(\mathcal{P}(H)/[L^{(n)}, L^{(n)}]\right)_n = p_n - (q_n - b_n) = b_n + (p_n - q_n) \geq b_n.$$

The preceding inequality shows that we can choose b_n linearly independent elements $\{x_1^{(n)}, \ldots, x_{b_n}^{(n)}\}$ of $\mathcal{P}(H)_n$ which remain linearly independent modulo $[L^{(n)}, L^{(n)}]$ and define $L^{(n+1)}$ to be the Lie algebra generated by elements of $L^{(n)}$ as well as $x_1^{(n)}, \ldots, x_{b_n}^{(n)}$.

Since $L^{(n+1)}$ is a sub-Lie algebra of a free Lie algebra, namely $\mathcal{P}(H)$, Lemma 6.6 states that $L^{(n+1)}$ is also free. As $L^{(n+1)}$ is generated by any generating set of $L^{(n)}$ and b_n-many homogeneous elements of degree n, we deduce that

$$L^{(n+1)} \cong \mathfrak{L}(b_1, \ldots, b_{n-1}, b_n', 0, \ldots) \qquad \text{for some } b_n' \leq b_n.$$

The final step of the proof is to show that $b_n' = b_n$. Using Lemma 6.5 and the definition of $L^{(n+1)}$,

$$b_n' = \dim\left(L_n^{(n+1)}/[L^{(n+1)}, L^{(n+1)}]_n\right)$$

Then $[L^{(n+1)}, L^{(n+1)}]_n = [L^{(n)}, L^{(n)}]_n$ as $L^{(n)}$ and $L^{(n+1)}$ contain the same homogeneous elements of degree $n-1$ and less. Therefore,

$$\begin{aligned}\dim\Big(L_n^{(n+1)}/[L^{(n+1)}, L^{(n+1)}]_n\Big) &= \dim\Big(\mathbb{C}\text{-span}\{x_1^{(n)}, \ldots, x_{b_n}^{(n)}\} \oplus [L^{(n)}, L^{(n)}]_n/[L^{(n)}, L^{(n)}]_n\Big) \\ &= b_n.\end{aligned}$$

□

References

[1] Marcelo Aguiar and Frank Sottile. Cocommutative Hopf algebras of permutations and trees. *Journal of Algebraic Combinatorics*, 22(4):451–470, 2005.

[2] Marcelo Aguiar and Frank Sottile. Structure of the Malvenuto–Reutenauer Hopf algebra of permutations. *Advances in Mathematics*, 191(2):225–275, 2005.

[3] Farid Aliniaeifard and Nathaniel Thiem. Pattern groups and a poset based Hopf monoid. *Journal of Combinatorial Theory, Series A*, 172:105187, 2020.

[4] Farid Aliniaeifard and Nathaniel Thiem. Hopf structures in the representation theory of direct products. *Electronic Journal of Combinatorics*, 29:Paper No. 4.39, 2022.

[5] Nantel Bergeron, Stefan Mykytiuk, Frank Sottile, and Stephanie van Willigenburg. Shifted quasi-symmetric functions and the Hopf algebra of peak functions. *Discrete Mathematics*, 246(1–3):57–66, March 2002.

[6] Richard E. Block. Commutative Hopf algebras, Lie coalgebras, and divided powers. *Journal of Algebra*, 96(1):275–306, 1985.

[7] Loïc Foissy. Primitive elements of a connected free bialgebra. arXiv preprint arXiv:2309.16199, 2023.

[8] Loïc Foissy and Frédéric Patras. Lie theory of free cocommutative and commutative cofree Hopf algebras. HAL preprint hal-04773293v1, 2024.

[9] IM Gelfand, D Krob, A Lascoux, B Leclerc, VS Retakh, and JY Thibon. Noncommutative symmetric functions. *Advances in Mathematics*, 112(2):218–348, 1995.

[10] Robert L Grossman and Richard G Larson. Hopf algebras of heap ordered trees and permutations. *Communications in Algebra*, 37(2):453–459, 2009.

[11] Michiel Hazewinkel. The primitives of the Hopf algebra of noncommutative symmetric functions. *São Paulo Journal of Mathematical Sciences*, 1:175–202, dez. 2007.

[12] Florent Hivert, Jean-Christophe Novelli, and Jean-Yves Thibon. Commutative combinatorial Hopf algebras. *Journal of Algebraic Combinatorics*, 28:65–95, 2008.

[13] OEIS Foundation Inc. The on-line encyclopedia of integer sequences, 2024. `https://oeis.org`.

[14] Aaron Lauve and Mitja Mastnak. The primitives and antipode in the Hopf algebra of symmetric functions in noncommuting variables. *Advances in Applied Mathematics*, 47(3):536–544, 2011.

[15] Teresa Xueshan Li. The monomial basis and the q-basis of the Hopf algebra of parking functions. *Journal of Algebraic Combinatorics*, 42(2):473–496, 2015.

[16] J.L. Loday. *Cyclic Homology*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2013.

[17] John W Milnor and John C Moore. On the structure of Hopf algebras. *Annals of Mathematics*, 81:211–264, 1965.

[18] Jean-Christophe Novelli and Jean-Yves Thibon. Hopf algebras and dendriform structures arising from parking functions. *Fundamenta Mathematicae*, 193(1):189–241, 2007.

[19] F. Patras and C. Reutenauer. On descent algebras and twisted bialgebras. *Moscow Mathematical Journal*, 4(1):199–216, 2004. Author webpage link.

[20] C. Reutenauer. *Free Lie Algebras*. London Mathematical Society Monographs New Series. Oxford University Press, 1993.

[21] A. I. Shirshov. *Subalgebras of Free Lie Algebras*, pages 3–13. Birkhäuser Basel, Basel, 2009.

[22] Ernst Witt. Die unterringe der freien lieschen ringe. *Mathematische Zeitschrift*, 64(1):195–216, 1956.

Sequence	h sequence	a sequence	p sequence
Catalan	$1, 2, 5, 14$	$1, 1, 2, 5, 14, 42, 132, 429, 1430$	$1, 1, 3, 8, 25, 75, 245, 800, 2700$
Motzkin	$1, 2, 4, 9, 21$	$1, 1, 1, 2, 4, 9, 21, 51, 127, 323$	$1, 1, 2, 4, 10, 22, 56, 136, 348, 890$
Fibonacci	$1, 1, 2, 3, 5$	$1, 0, 1, 0, 1, 0, 1, 0, 1, 0$	$1, 0, 1, 1, 2, 2, 4, 5, 8, 11$
Tribonacci	$0, 1, 1, 2, 4$	$0, 1, 1, 1, 2, 3, 4, 6, 9, 13$	$0, 1, 1, 1, 3, 4, 8, 13, 23, 38$
Lucas	$2, 1, 3, 4, 7$	$2, -3, 7, -13, 27, -53, 107, -213, 427, -853$	$2, -2, 3, -2, 8, -9, 22, -25, 64, -94$
Bell	$1, 2, 5, 15, 52$	$1, 1, 2, 6, 22, 92, 426, 2146, 11624, 67146$	$1, 1, 3, 9, 34, 135, 610, 2965, 15612, 87871$
Odd Part.	$1, 1, 2, 2, 3$	$1, 0, 1, -1, 1, -1, 1, -2, 2, -2$	$1, 0, 1, 0, 1, 0, 1, 0, 1, 0$
Park. Func.	$1, 3, 16, 125$	$1, 2, 11, 92, 1014, 13795, 223061$	$1, 2, 13, 106, 1145, 15204, 241630$

Table 1. Sequences

Rooted trees (Grossman-Larson, Connes-Kreimer), ordered trees (Grossman-Larson), heap-ordered trees (Grossman-Larson), planar binary trees (Loady-Ronco), permutations (Malvenuto-Reutenauer, Aguiar-Sottile), compositions (QSym/NSym), odd compositions (Peak algebra of Stembridge), parking functions ($(n+1)^{n-1}$). Set partitions.