# IDEALS AND QUOTIENTS OF DIAGONALLY QUASISYMMETRIC FUNCTIONS

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ABSTRACT. In 2004, J-C. Aval, F. Bergeron and N. Bergeron studied the algebra of diagonally quasi-symmetric functions DQSym in the ring  $\mathbb{Q}[\mathbf{x}, \mathbf{y}]$  with two sets of variables. They made conjectures on the structure of the quotient  $\mathbb{Q}[\mathbf{x}, \mathbf{y}]/\langle \mathsf{DQSym}^+\rangle$ , which is a quasi-symmetric analogue of the diagonal harmonic polynomials. In this paper, we construct a Hilbert basis for this quotient when there are infinitely many variables i.e.  $\mathbf{x} = x_1, x_2, \ldots$  and  $\mathbf{y} = y_1, y_2, \ldots$  Then we apply this construction to the case where there are finitely many variables, and compute the second column of its Hilbert matrix.

#### 1. Introduction

In the polynomial ring  $\mathbb{Q}[\mathbf{x}_n] = \mathbb{Q}[x_1, \dots, x_n]$  with n variables, the ring of symmetric polynomials,  $\mathsf{Sym}_n$ , (cf. [Mac] or [Sagan]) is the subspace of invariants under the symmetric group  $S_n$  action

$$\sigma \cdot f(x_1, x_2, \dots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}).$$

The quotient space  $\mathbb{Q}[\mathbf{x}_n]/\langle \mathsf{Sym}_n^+\rangle$  over the ideal generated by symmetric polynomials with no constant term is thus called the coinvariant space of symmetric group. Classic results by [Artin] and [Steinberg] asserts that this quotient forms an  $S_n$ -module that is isomorphic to the left regular representation. Moreover, considering the natural scalar product

$$\langle f, g \rangle = (f(\partial x_1, \dots, \partial x_n)(g(x_1, \dots, x_n)))(0, 0, \dots, 0),$$

this quotient is equal to the orthogonal complement of  $\operatorname{\mathsf{Sym}}_n$ . In particular, the coinvariant space is killed by Laplacian operator  $\Delta = \partial x_1^2 + \cdots + \partial x_n^2$ . Hence, it is also known as the harmonic space.

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One can show that  $\{h_k(x_k,\ldots,x_n): 1 \leq k \leq n\}$  forms a Gröbner basis of  $\langle \mathsf{Sym}_n^+ \rangle$  with respect to the usual order  $x_1 > \cdots > x_n$ , where  $h_k$  is the complete homogeneous basis of degree k. As a result, the dimension of  $\mathbb{Q}[\mathbf{x}_n]/\langle \mathsf{Sym}_n^+ \rangle$  is n!.

One generalization is the diagonal harmonic space. In the context of  $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n] = \mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ , the diagonally symmetric functions,  $\mathsf{DSym}_n$ , is the space of invariants under the diagonal action of  $S_n$ 

$$\sigma \cdot f(x_1, \dots, x_n, y_1, \dots, y_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}, y_{\sigma(1)}, \dots, y_{\sigma(n)}).$$

The diagonal harmonics,  $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n]/\langle \mathsf{DSym}_n^+ \rangle$ , was studied in [GH] and [Haiman] where it was used to prove the n! conjecture and Macdonald positivity. In particular, its dimension turns out to be  $(n+1)^{n-1}$ . More interesting results and applications can be found in [BBGHT], [BGHT] and [Haglund].

The ring of quasi-symmetric functions, QSym, was introduced by [Gessel] as generating function for P-partitions [Stanley]. It soon shows great importance in algebraic combinatorics e.g. [ABS], [GKLLRT]. In our context, QSym<sub>n</sub> can be defined as the space of invariants in  $\mathbb{Q}[\mathbf{x}_n]$  under the  $S_n$ -action of Hivert

$$\sigma \cdot \left( x_{i_1}^{a_1} \cdots x_{i_k}^{a_k} \right) = x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}$$
 where  $i_1 < \cdots < i_k, \ j_1 < \cdots < j_k \ \text{and} \ \{ j_1, \dots, j_k \} = \{ \sigma(i_1), \dots, \sigma(i_k) \}.$ 

In [AB] and [ABB], the authors studied the quotient  $\mathbb{Q}[\mathbf{x}_n]/\langle \mathsf{QSym}_n^+\rangle$  over the ideal generated by quasi-symmetric polynomials with no constant term, which they called the super-covariant space of  $S_n$ . Their main result is that a basis of this quotient corresponds to Dyck paths, and the dimension of the quotient space is the n-th Catalan number  $C_n$ .

After that, in [ABB2], they extended QSym to diagonal setting, called diagonally quasi-symmetric functions, DQSym. They described a Hopf algebra structure on DQSym, and made a conjecture about the linear structure of  $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n]/\langle \mathsf{DQSym}_n^+ \rangle$ .

In this paper, we continue the study of the linear structure. We start with the case where there are infinitely many variables i.e.  $R = \mathbb{Q}[[\mathbf{x}, \mathbf{y}]]$  is the ring of formal power series where  $\mathbf{x} = x_1, x_2, \ldots$  and  $\mathbf{y} = y_1, y_2, \ldots$ . The main result is that we give a description of a Hilbert basis for the quotient space R/I where  $I = \overline{\mathsf{DQSym}^+}$  is the closure of the ideal generated by  $\mathsf{DQSym}$  without constant terms. This Hilbert basis gives an upper bound on the degree of  $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n]/\langle \mathsf{DQSym}_n^+ \rangle$ . We then use it to compute the second column of the Hilbert matrix, which coincides with the conjecture in [ABB2].

#### 2. Definitions

2.1. **Bicompositions.** An element  $\tilde{\alpha} = \begin{pmatrix} \tilde{\alpha}_{11} & \tilde{\alpha}_{12} & \cdots \\ \tilde{\alpha}_{21} & \tilde{\alpha}_{22} & \cdots \end{pmatrix} \in \mathbb{N}^{2\mathbb{N}}$  is called a generalized bicomposition if all but finitely many  $(\tilde{\alpha}_{1k}, \tilde{\alpha}_{2k})$  are (0,0). Let k be the maximum number such that  $(\tilde{\alpha}_{1k}, \tilde{\alpha}_{2k}) \neq (0,0)$ . The length of  $\tilde{\alpha}$ , denoted by  $\ell(\tilde{\alpha})$ , is k. The size of  $\tilde{\alpha}$ , denoted by  $|\tilde{\alpha}|$ , is the sum of all its entries. For simplicity, we usually write  $\tilde{\alpha}$  as  $\begin{pmatrix} \tilde{\alpha}_{11} & \cdots & \tilde{\alpha}_{1k} \\ \tilde{\alpha}_{21} & \cdots & \tilde{\alpha}_{2k} \end{pmatrix}$ . There also exists a generalized bicomposition with length 0 and size 0, called the zero bicomposition, denoted by  $\binom{0}{0}$ .

Every monomial in R can be expressed as  $\mathbf{X}^{\tilde{\alpha}} = x_1^{\tilde{\alpha}_{11}} y_1^{\tilde{\alpha}_{21}} \cdots x_k^{\tilde{\alpha}_{1k}} y_k^{\tilde{\alpha}_{2k}}$  for some generalized bicomposition  $\tilde{\alpha}$ . A generalized bicomposition  $\alpha$  is called a bicomposition if  $\ell(\alpha) = 0$  or  $(\alpha_{1j}, \alpha_{2j}) \neq (0, 0)$  for all  $1 \leq j \leq \ell(\alpha)$ .

In this paper, we use Greek letters to denote bicompositions, and Greek letters with tilde to denote generalized bicompositions.

Let  $\tilde{\alpha}$ ,  $\tilde{\beta}$  and  $\tilde{\gamma}$  be non-zero generalized bicompositions. We write  $\tilde{\alpha} = \tilde{\beta}\tilde{\gamma}$  if  $\tilde{\alpha}_{ij} = \tilde{\beta}_{ij}$  for all  $1 \leq j \leq \ell(\tilde{\alpha}) - \ell(\tilde{\gamma})$ ,  $\tilde{\beta}_{ij} = 0$  for all  $j > \ell(\tilde{\alpha}) - \ell(\tilde{\gamma})$  and  $\tilde{\alpha}_{i(j+\ell(\tilde{\alpha})-\ell(\tilde{\gamma}))} = \tilde{\gamma}_{ij}$  for all  $j \geq 1$ . We write  $\tilde{\alpha} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tilde{\beta}$  if  $\tilde{\alpha}_{11} = \tilde{\alpha}_{21} = 0$  and  $\tilde{\alpha}_{i(j+1)} = \tilde{\beta}_{ij}$  for all  $j \geq 2$ .

Note that for each generalized bicomposition  $\tilde{\alpha}$  that is not a bicomposition, there is a unique way to decompose it into  $\tilde{\alpha} = \tilde{\beta} \binom{0}{0} \gamma$  for some generalized bicomposition  $\tilde{\beta}$  and bicomposition  $\gamma$ .

2.2. **Diagonally quasi-symmetric functions.** The algebra of diagonally quasi-symmetric functions, DQSym, is a subalgebra of  $\mathbb{Q}[[\mathbf{x}, \mathbf{y}]]$  spanned by monomials indexed by bicompositions

$$M_{\alpha} = \sum_{i_1 < \dots < i_k} x_{i_1}^{\alpha_{11}} y_{i_1}^{\alpha_{21}} \cdots x_{i_k}^{\alpha_{1k}} y_{i_k}^{\alpha_{2k}}.$$

As a graded algebra,  $\mathsf{DQSym} = \bigoplus_{n \geq 0} \mathsf{DQSym}_n$  where  $\mathsf{DQSym}_n = \mathrm{span} - \{M_\alpha : |\alpha| = n\}$  is the degree n component. The algebra structure is defined in [ABB2].

2.3. The F basis. We define a partial ordering  $\leq$  on bicompositions:  $\alpha \leq \beta$  and  $\beta$  covers  $\alpha$  if there exists a  $1 \leq k \leq \ell(\alpha) - 1$  such that either  $\alpha_{2k} = 0$  or  $\alpha_{1(k+1)} = 0$ ,

and

$$\beta = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1(k-1)} & \alpha_{1k} + \alpha_{1(k+1)} & \alpha_{1(k+2)} & \cdots & \alpha_{1\ell(\alpha)} \\ \alpha_{21} & \cdots & \alpha_{2(k-1)} & \alpha_{2k} + \alpha_{2(k+1)} & \alpha_{2(k+2)} & \cdots & \alpha_{2\ell(\alpha)} \end{pmatrix}.$$

By triangularity,  $\left\{F_{\alpha} = \sum_{\alpha \prec \beta} M_{\beta}\right\}$  forms a basis for DQSym. For example,

$$F_{\binom{2}{2}} = M_{\binom{2}{2}} + M_{\binom{2}{0} 2} + M_{\binom{1}{0} 2} + M_{\binom{1}{0} 2} + M_{\binom{1}{0} 0 2} + M_{\binom{2}{0} 0 2} + M_{\binom{2}{0} 1 1} + M_{\binom{2}{0} 0 1} + M_{\binom{1}{0} 1 1} + M_{\binom{1$$

For convenience, we set  $F_{\binom{0}{0}} = 1$ .

This basis has the following easy but important properties

If  $\alpha_{11} \geq 1$  and  $\alpha_{11} + \alpha_{21} \geq 2$ , then

(2.1) 
$$F_{\alpha} = x_1 F_{\begin{pmatrix} \alpha_{11} - 1 & \alpha_{12} & \cdots & \alpha_{1\ell(\alpha)} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2\ell(\alpha)} \end{pmatrix}} + F_{\alpha}(x_2, x_3, \dots, y_2, y_3, \dots);$$

If  $\alpha_{11} = 1$  and  $\alpha_{21} = 0$ , then

(2.2) 
$$F_{\alpha} = x_1 F_{\left( \frac{\alpha_{12} \dots \alpha_{1\ell(\alpha)}}{\alpha_{22} \dots \alpha_{2\ell(\alpha)}} \right)} (x_2, x_3, \dots, y_2, y_3, \dots) + F_{\alpha}(x_2, x_3, \dots, y_2, y_3, \dots);$$

If  $\alpha_{11} = 0$  and  $\alpha_{21} \geq 2$ , then

(2.3) 
$$F_{\alpha} = y_1 F_{\left( \begin{smallmatrix} 0 & \alpha_{12} & \cdots & \alpha_{1\ell(\alpha)} \\ \alpha_{21} - 1 & \alpha_{22} & \cdots & \alpha_{2\ell(\alpha)} \end{smallmatrix} \right)} + F_{\alpha}(x_2, x_3, \dots, y_2, y_3, \dots);$$

If  $\alpha_{11} = 0$  and  $\alpha_{21} = 1$ , then

(2.4) 
$$F_{\alpha} = y_1 F_{\binom{\alpha_{12} \dots \alpha_{1\ell(\alpha)}}{\alpha_{22} \dots \alpha_{2\ell(\alpha)}}}(x_2, x_3, \dots, y_2, y_3, \dots) + F_{\alpha}(x_2, x_3, \dots, y_2, y_3, \dots).$$

### 3. The G basis

In this section, we define a basis  $\{G_{\tilde{\epsilon}}\}$  indexed by generalized bicompositions for  $\mathbb{Q}[[\mathbf{x},\mathbf{y}]]$ .

Base cases:  $G_{\binom{0}{0}} = 1$  and  $G_{\tilde{\epsilon}} = F_{\tilde{\epsilon}}$  if  $\tilde{\epsilon}$  is a bicomposition. Otherwise, let  $\tilde{\epsilon} = \tilde{\alpha} \binom{0}{0} \beta$  where  $\beta$  is a non-zero bicomposition. Let  $k = \ell(\tilde{\epsilon}) - \ell(\beta) - 1$ .

If  $\beta_{11} > 0$ ,

(3.1) 
$$G_{\tilde{\epsilon}} = G_{\tilde{\alpha}\beta} - x_{k+1} G_{\tilde{\alpha} \begin{pmatrix} \beta_{11} - 1 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{pmatrix}}.$$

If  $\beta_{11} = 0$ ,

(3.2) 
$$G_{\tilde{\epsilon}} = G_{\tilde{\alpha}\beta} - y_{k+1} G_{\tilde{\alpha} \begin{pmatrix} 0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21} - 1 & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{pmatrix}}.$$

Inductively,  $\{G_{\tilde{\epsilon}}\}$  is defined for all generalized bicomposition  $\tilde{\epsilon}$ . Clearly  $G_{\tilde{\epsilon}}$  is homogeneous in degree  $|\tilde{\epsilon}|$ . Hence, we have a notion of leading monomial of  $G_{\tilde{\epsilon}}$ ,  $LM(G_{\tilde{\epsilon}})$  with respect to the lexicographic denoted by  $x_1 > y_1 > x_2 > y_2 > \cdots$ . To show that  $\{G_{\tilde{\epsilon}}\}$  form a basis, it suffices to prove the leading monomial of  $G_{\tilde{\epsilon}}$  is  $\mathbf{X}^{\tilde{\epsilon}}$ .

**Lemma 3.1.** Let  $\tilde{\alpha} = \binom{a}{b} \tilde{\beta}$  be a generalized bicomposition,

- (1) if a = b = 0, then  $G_{\tilde{\alpha}} = G_{\tilde{\beta}}(x_2, x_3, \dots, y_2, y_3, \dots)$ ,
- (2) if a > 0, then  $G_{\tilde{\alpha}} = x_1 G_{\binom{a-1}{b}\tilde{\beta}} + P(x_2, x_3, \dots, y_2, y_3, \dots)$ , (3) if a = 0 and b > 0, then  $G_{\tilde{\alpha}} = y_1 G_{\binom{0}{b-1}\tilde{\beta}} + P(x_2, x_3, \dots, y_2, y_3, \dots)$

for some  $P \in \mathbb{Q}[[\mathbf{x}, \mathbf{y}]]$ .

*Proof.* We prove by induction on the length of  $\tilde{\alpha}$ .

- (1) If  $\tilde{\alpha} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ , then  $G_{\tilde{\alpha}} = 1$  and we are done.
- (2) If  $\tilde{\beta} = \beta$  is a bicomposition,
  - (a) if a = b = 0 and  $\beta$  non-zero,
    - (i) if  $\beta_{11} \geq 1$  and  $\beta_{11} + \beta_{21} \geq 2$ , using (2.1) and (3.1), we get

$$G_{\tilde{\alpha}} = G_{\beta} - x_1 G_{\begin{pmatrix} \beta_{11} - 1 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{pmatrix}}$$

$$= F_{\beta} - x_1 F_{\begin{pmatrix} \beta_{11} - 1 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{pmatrix}}$$

$$= F_{\beta}(x_2, x_3, \dots, y_2, y_3, \dots) = G_{\beta}(x_2, x_3, \dots, y_2, y_3, \dots)$$

and the lemma follows.

(ii) if  $\beta_{11} = 1$  and  $\beta_{21} = 0$ , using (2.2), (3.1) and induction on  $\ell(\tilde{\beta})$ , we get

$$G_{\tilde{\alpha}} = G_{\beta} - x_{1}G_{\begin{pmatrix} 0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ 0 & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{pmatrix}}$$

$$= G_{\beta} - x_{1}G_{\begin{pmatrix} \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{pmatrix}}(x_{2}, x_{3}, \dots, y_{2}, y_{3}, \dots)$$

$$= F_{\beta} - x_{1}F_{\begin{pmatrix} \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{pmatrix}}(x_{2}, x_{3}, \dots, y_{2}, y_{3}, \dots)$$

$$= F_{\beta}(x_{2}, x_{3}, \dots, y_{2}, y_{3}, \dots) = G_{\beta}(x_{2}, x_{3}, \dots, y_{2}, y_{3}, \dots)$$

and the lemma follows.

(iii) if  $\beta_{11} = 0$  and  $\beta_{21} \ge 2$ , using (2.3) and (3.2), we get

$$G_{\tilde{\alpha}} = G_{\beta} - y_1 G_{\begin{pmatrix} 0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21} - 1 & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{pmatrix}}$$

$$= F_{\beta} - y_1 F_{\begin{pmatrix} 0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21} - 1 & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{pmatrix}}$$

$$= F_{\beta}(x_2, x_3, \dots, y_2, y_3, \dots) = G_{\beta}(x_2, x_3, \dots, y_2, y_3, \dots)$$

and the lemma follows.

(iv) if  $\beta_{11} = 0$  and  $\beta_{21} = 1$ , using (2.4), (3.2) and induction on  $\ell(\tilde{\beta})$ , we get

$$G_{\tilde{\alpha}} = G_{\beta} - y_{1}G_{\binom{0 \beta_{12} \cdots \beta_{1\ell(\beta)}}{0 \beta_{22} \cdots \beta_{2\ell(\beta)}}}$$

$$= G_{\beta} - y_{1}G_{\binom{\beta_{12} \cdots \beta_{1\ell(\beta)}}{\beta_{22} \cdots \beta_{2\ell(\beta)}}}(x_{2}, x_{3}, \dots, y_{2}, y_{3}, \dots)$$

$$= F_{\beta} - y_{1}F_{\binom{\beta_{12} \cdots \beta_{1\ell(\beta)}}{\beta_{22} \cdots \beta_{2\ell(\beta)}}}(x_{2}, x_{3}, \dots, y_{2}, y_{3}, \dots)$$

$$= F_{\beta}(x_{2}, x_{3}, \dots, y_{2}, y_{3}, \dots) = G_{\beta}(x_{2}, x_{3}, \dots, y_{2}, y_{3}, \dots)$$

and the lemma follows.

(b) if  $a \ge 1$  and  $a + b \ge 2$ , by definition  $G_{\tilde{\alpha}} = F_{\tilde{\alpha}}$ . Using (2.1), we get

$$G_{\tilde{\alpha}} = F_{\tilde{\alpha}} = x_1 F_{\binom{a-1}{b}\beta} + F_{\tilde{\alpha}}(x_2, x_3, \dots, y_2, y_3, \dots)$$

and the lemma follows, with  $P = F_{\tilde{\alpha}}(x_2, x_3, \dots, y_2, y_3, \dots)$ .

(c) if a=1 and b=0, by definition  $G_{\tilde{\alpha}}=F_{\tilde{\alpha}}$ . Using (2.2) and (2a). we get

$$G_{\tilde{\alpha}} = F_{\tilde{\alpha}} = x_1 F_{\beta}(x_2, x_3, \dots, y_2, y_3, \dots) + F_{\tilde{\alpha}}(x_2, x_3, \dots, y_2, y_3, \dots)$$
  
=  $x_1 G_{\binom{0}{0}\beta} + F_{\tilde{\alpha}}(x_2, x_3, \dots, y_2, y_3, \dots)$ 

and the lemma follows with  $P = F_{\tilde{\alpha}}(x_2, x_3, \dots, y_2, y_3, \dots)$ .

(d) if a = 0 and  $b \ge 2$ , by definition  $G_{\tilde{\alpha}} = F_{\tilde{\alpha}}$ . Using (2.3), we get

$$G_{\tilde{\alpha}} = F_{\tilde{\alpha}} = y_1 F_{\binom{a}{b-1}\beta} + F_{\tilde{\alpha}}(x_2, x_3, \dots, y_2, y_3, \dots)$$

and the lemma follows, with  $P = F_{\tilde{\alpha}}(x_2, x_3, \dots, y_2, y_3, \dots)$ . (e) if a = 0 and b = 1, by definition  $G_{\tilde{\alpha}} = F_{\tilde{\alpha}}$ . Using (2.4) and (2a). we get

$$G_{\tilde{\alpha}} = F_{\tilde{\alpha}} = y_1 F_{\beta}(x_2, x_3, \dots, y_2, y_3, \dots) + F_{\tilde{\alpha}}(x_2, x_3, \dots, y_2, y_3, \dots)$$
  
=  $y_1 G_{\binom{0}{0}\beta} + F_{\tilde{\alpha}}(x_2, x_3, \dots, y_2, y_3, \dots)$ 

and the lemma follows with  $P = F_{\tilde{\alpha}}(x_2, x_3, \dots, y_2, y_3, \dots)$ .

- (3) In the general case, let  $\tilde{\alpha} = \tilde{\gamma} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \beta$  where  $\beta$  is a non-empty bicomposition and  $k = \ell(\tilde{\alpha}) \ell(\beta) 1$ . We prove by induction on k. If k = 1, then we are back in case (2a) above. Hence, we assume k > 1 and  $\tilde{\gamma} = \binom{a}{b} \tilde{\mu}$ .
  - (a) If a = b = 0,
    - (i) if  $\beta_{11} \geq 1$ , by induction and (3.1), we have

$$G_{\tilde{\alpha}} = G_{\binom{0}{0}\tilde{\mu}\binom{0}{0}\beta} = G_{\binom{0}{0}\tilde{\mu}\beta} - x_k G_{\binom{0}{0}\tilde{\mu}\binom{\beta_{11}-1}{\beta_{21}}\frac{\beta_{12}}{\beta_{22}}\frac{\dots \beta_{1\ell(\beta)}}{\beta_{2\ell(\beta)}}}$$

$$= G_{\tilde{\mu}\beta}(x_2, x_3, \dots, y_2, y_3, \dots)$$

$$- x_{(k-1)+1} G_{\tilde{\mu}\binom{\beta_{11}-1}{\beta_{21}}\frac{\beta_{12}}{\beta_{22}}\frac{\dots \beta_{1\ell(\beta)}}{\beta_{2\ell(\beta)}}}(x_2, x_3, \dots, y_2, y_3, \dots)$$

$$= G_{\tilde{\mu}\binom{0}{0}\beta}(x_2, x_3, \dots, y_2, y_3, \dots)$$

and the lemma follows.

(ii) if  $\beta_{11} = 0$ , by induction and (3.2), we have

$$G_{\tilde{\alpha}} = G_{\binom{0}{0}\tilde{\mu}\binom{0}{0}\beta} = G_{\binom{0}{0}\tilde{\mu}\beta} - y_k G_{\binom{0}{0}\tilde{\mu}\binom{0}{0}\tilde{\mu}\binom{0}{\beta_{21}-1}} \overset{\beta_{12} \dots \beta_{1\ell(\beta)}}{\beta_{22} \dots \beta_{2\ell(\beta)}})$$

$$= G_{\tilde{\mu}\beta}(x_2, x_3, \dots, y_2, y_3, \dots)$$

$$- y_{(k-1)+1} G_{\tilde{\mu}\binom{0}{\beta_{21}-1}} \overset{\beta_{12} \dots \beta_{1\ell(\beta)}}{\beta_{22} \dots \beta_{2\ell(\beta)}}) (x_2, x_3, \dots, y_2, y_3, \dots)$$

$$= G_{\tilde{\mu}\binom{0}{0}\beta}(x_2, x_3, \dots, y_2, y_3, \dots)$$

and the lemma follows.

(b) If  $a \geq 1$ ,

(i) if  $\beta_{11} \geq 1$ , by induction and (3.1), we have

$$G_{\tilde{\alpha}} = G_{\binom{a}{b}}\tilde{\mu}\binom{0}{0}\beta = G_{\binom{a}{b}}\tilde{\mu}\beta} - x_k G_{\binom{a}{b}}\tilde{\mu}\binom{\beta_{11}-1}{\beta_{21}}\frac{\beta_{12}}{\beta_{22}}\cdots\frac{\beta_{1\ell(\beta)}}{\beta_{2\ell(\beta)}})$$

$$= x_1 G_{\binom{a-1}{b}}\tilde{\mu}\beta + P_1(x_2, x_3, \dots, y_2, y_3, \dots)$$

$$- x_k \left(x_1 G_{\binom{a-1}{b}}\tilde{\mu}\binom{\beta_{11}-1}{\beta_{21}}\frac{\beta_{12}}{\beta_{22}}\cdots\frac{\beta_{1\ell(\beta)}}{\beta_{2\ell(\beta)}}\right)$$

$$+ P_2(x_2, x_3, \dots, y_2, y_3, \dots)$$

$$= x_1 \left(G_{\binom{a-1}{b}}\tilde{\mu}\beta - x_k G_{\binom{a-1}{b}}\tilde{\mu}\binom{\beta_{11}-1}{\beta_{21}}\frac{\beta_{12}}{\beta_{22}}\cdots\frac{\beta_{1\ell(\beta)}}{\beta_{2\ell(\beta)}}\right)$$

$$+ P(x_2, x_3, \dots, y_2, y_3, \dots)$$

$$= x_1 G_{\binom{a-1}{b}}\tilde{\mu}\binom{0}{0}\beta + P(x_2, x_3, \dots, y_2, y_3, \dots)$$

and the lemma follows with  $P = P_1 - x_k P_2$ .

(ii) if  $\beta_{11} = 0$ , by induction and (3.2), we have

$$G_{\tilde{\alpha}} = G_{\binom{a}{b}}\tilde{\mu}\binom{0}{0}\beta = G_{\binom{a}{b}}\tilde{\mu}\beta - y_kG_{\binom{a}{b}}\tilde{\mu}\binom{0}{\beta_{21}-1} \frac{\beta_{12} \cdots \beta_{1\ell(\beta)}}{\beta_{22} \cdots \beta_{2\ell(\beta)}})$$

$$= x_1G_{\binom{a-1}{b}}\tilde{\mu}\beta + P_1(x_2, x_3, \dots, y_2, y_3, \dots)$$

$$- y_k\left(x_1G_{\binom{a-1}{b}}\tilde{\mu}\binom{0}{\beta_{21}-1} \frac{\beta_{12} \cdots \beta_{1\ell(\beta)}}{\beta_{22} \cdots \beta_{2\ell(\beta)}}\right)$$

$$+ P_2(x_2, x_3, \dots, y_2, y_3, \dots)$$

$$= x_1\left(G_{\binom{a-1}{b}}\tilde{\mu}\beta - y_kG_{\binom{a-1}{b}}\tilde{\mu}\binom{0}{\beta_{21}-1} \frac{\beta_{12} \cdots \beta_{1\ell(\beta)}}{\beta_{22} \cdots \beta_{2\ell(\beta)}}\right)$$

$$+ P(x_2, x_3, \dots, y_2, y_3, \dots)$$

$$= x_1G_{\binom{a-1}{b}}\tilde{\mu}\binom{0}{0}\beta + P(x_2, x_3, \dots, y_2, y_3, \dots)$$

and the lemma follows with  $P = P_1 - y_k P_2$ . (c) If a = 0 and  $b \ge 1$ , (i) if  $\beta_{11} \geq 1$ , by induction and (3.1), we have

$$G_{\tilde{\alpha}} = G_{\binom{0}{b}\tilde{\mu}\binom{0}{0}\beta} = G_{\binom{0}{b}\tilde{\mu}\beta} - x_k G_{\binom{0}{b}\tilde{\mu}\binom{\beta_{11}-1}{\beta_{21}}\frac{\beta_{12}}{\beta_{22}}\frac{\dots\beta_{1\ell(\beta)}}{\beta_{2\ell(\beta)}})}$$

$$= y_1 G_{\binom{0}{b-1}\tilde{\mu}\beta} + P_1(x_2, x_3, \dots, y_2, y_3, \dots)$$

$$- x_k \left(y_1 G_{\binom{0}{b-1}\tilde{\mu}\binom{\beta_{11}-1}{\beta_{21}}\frac{\beta_{12}}{\beta_{22}}\frac{\dots\beta_{1\ell(\beta)}}{\beta_{2\ell(\beta)}}\right)}$$

$$+ P_2(x_2, x_3, \dots, y_2, y_3, \dots)$$

$$= y_1 \left(G_{\binom{0}{b-1}\tilde{\mu}\beta} - x_k G_{\binom{0}{b-1}\tilde{\mu}\binom{\beta_{11}-1}{\beta_{21}}\frac{\beta_{12}}{\beta_{22}}\frac{\dots\beta_{1\ell(\beta)}}{\beta_{2\ell(\beta)}}}\right)$$

$$+ P(x_2, x_3, \dots, y_2, y_3, \dots)$$

$$= y_1 G_{\binom{0}{b-1}\tilde{\mu}\binom{0}{0}\beta} + P(x_2, x_3, \dots, y_2, y_3, \dots)$$

and the lemma follows with  $P = P_1 - x_k P_2$ .

(ii) if  $\beta_{11} = 0$ , by induction and (3.2), we have

$$G_{\tilde{\alpha}} = G_{\binom{0}{b}\tilde{\mu}\binom{0}{0}\beta} = G_{\binom{0}{b}\tilde{\mu}\beta} - y_k G_{\binom{0}{b}\tilde{\mu}\binom{0}{b}\tilde{\mu}\binom{0}{\beta_{21}-1} \beta_{22} \cdots \beta_{2\ell(\beta)}}$$

$$= y_1 G_{\binom{0}{b-1}\tilde{\mu}\beta} + P_1(x_2, x_3, \dots, y_2, y_3, \dots)$$

$$- y_k \left( y_1 G_{\binom{0}{b-1}}\tilde{\mu}\binom{0}{\beta_{21}-1} \beta_{22} \cdots \beta_{1\ell(\beta)} \right)$$

$$+ P_2(x_2, x_3, \dots, y_2, y_3, \dots)$$

$$= y_1 \left( G_{\binom{0}{b-1}\tilde{\mu}\beta} - y_k G_{\binom{0}{b-1}\tilde{\mu}\binom{0}{\beta_{21}-1} \beta_{22} \cdots \beta_{2\ell(\beta)}} \right)$$

$$+ P(x_2, x_3, \dots, y_2, y_3, \dots)$$

$$= y_1 G_{\binom{0}{b-1}\tilde{\mu}\binom{0}{0}\beta} + P(x_2, x_3, \dots, y_2, y_3, \dots)$$

and the lemma follows with  $P = P_1 - y_k P_2$ .

Corollary 3.2. Let  $\tilde{\epsilon}$  be a generalized bicomposition, then the leading monomial of  $G_{\tilde{\epsilon}}$  is  $\mathbf{X}^{\tilde{\epsilon}}$ . Hence,  $\{G_{\tilde{\alpha}}\}$  forms a Hilbert basis for R.

*Proof.* We prove by induction on  $\ell(\tilde{\epsilon})$  and  $|\tilde{\epsilon}|$ . If  $\tilde{\epsilon} = {0 \choose 0}$ , by definition  $G_{\tilde{\epsilon}} = 1 = X^{\tilde{\epsilon}}$ . Otherwise, let  $\tilde{\epsilon} = {a \choose b}\tilde{\beta}$ .

- (1) If a = b = 0 and  $\tilde{\beta}$  non-zero, by induction on  $\ell(\tilde{\epsilon})$  and Lemma 3.1, we have  $LM(G_{\tilde{\epsilon}}) = LM(G_{\tilde{\beta}}(x_2, x_3, \dots, y_2, y_3, \dots)) = (x_2, x_3, \dots, y_2, y_3, \dots)^{\tilde{\beta}} = \mathbf{X}^{\tilde{\epsilon}}.$
- (2) If  $a \ge 1$ , by induction on  $|\tilde{\epsilon}|$  and Lemma 3.1, we have

$$LM(G_{\tilde{\epsilon}}) = LM\left(x_1 G_{\binom{a-1}{b}\tilde{\beta}}\right) = \mathbf{X}^{\tilde{\epsilon}}.$$

(3) If a = 0 and  $b \ge 1$ , by induction on  $|\tilde{\epsilon}|$  and Lemma 3.1, we have

$$LM(G_{\tilde{\epsilon}}) = LM\left(y_1G_{\binom{0}{b-1}\tilde{\beta}}\right) = \mathbf{X}^{\tilde{\epsilon}}.$$

## 4. The Hilbert Basis

The set  $\{x^{\tilde{\alpha}}F_{\beta}\}$  is a spanning set of the ideal I. For each  $\tilde{\alpha}$  and  $\beta$ , we write  $x^{\tilde{\alpha}}F_{\beta}$  in terms of the G basis by the following rules.

- (1) We reorder the product  $x^{\tilde{\alpha}}F_{\beta}$  as  $\cdots \left(x_2^{\tilde{\alpha}_{21}}\left(y_2^{\tilde{\alpha}_{22}}\left(x_1^{\tilde{\alpha}_{11}}\left(y_1^{\tilde{\alpha}_{21}}F_{\beta}\right)\right)\right)\right)$ .
- (2) We reduce the above product recursively using (3.1)

$$(4.1) x_i G_{\tilde{\gamma}} = x_i G_{\left( \dots \tilde{\gamma}_{1i} \dots \right)} = G_{\left( \dots \tilde{\gamma}_{1i}+1 \dots \right)} - G_{\left( \dots 0 \tilde{\gamma}_{1i}+1 \dots \right)};$$

or using (3.2) when  $\tilde{\gamma}_{1i} = 0$  for some i,

$$(4.2) y_i G_{\tilde{\gamma}} = y_i G_{\left( \dots \tilde{\gamma}_{1i} \dots \right)} = G_{\left( \dots \tilde{\gamma}_{1i} \dots \right)} - G_{\left( \dots 0 \tilde{\gamma}_{1i} \dots \right)}.$$

(3) When  $\tilde{\gamma}_{1i} = a > 0$ , we reduce  $y_i G_{\tilde{\gamma}}$  as

$$(4.3) y_1 G_{\tilde{\gamma}} = y_i G_{\left( \dots a_{\tilde{\gamma}_{2i}} \dots \right)} = y_i \left( G_{\left( \dots a_{\tilde{\gamma}_{2i}} \dots \right)} + x_i G_{\left( \dots a_{\tilde{\gamma}_{2i}} \dots \right)} \right)$$

$$= y_i G_{\left( \dots a_{\tilde{\gamma}_{2i}} \dots \right)} + x_i \left( y_i G_{\left( \dots a_{\tilde{\gamma}_{2i}} \dots \right)} \right) = \cdots$$

$$= \sum_{k=0}^{a-1} x_i^k \left( y_i G_{\left( \dots a_{\tilde{\gamma}_{2i}} \dots \right)} \right) + x_i^a \left( y_i G_{\left( \dots a_{\tilde{\gamma}_{2i}} \dots \right)} \right).$$

The "···" above means  $\tilde{\gamma}_{11}$  ···  $\tilde{\gamma}_{1(i-1)}$ ,  $\tilde{\gamma}_{1(i+1)}$  ···  $\tilde{\gamma}_{1\ell(\tilde{\gamma})}$ ,  $\tilde{\gamma}_{21}$  ···  $\tilde{\gamma}_{2(i-1)}$  or

 $\tilde{\gamma}_{2(i+1)} \cdots \tilde{\gamma}_{1\ell(\tilde{\gamma})}$  with respect to their positions in the generalized bicomposition.

For example,

$$y_{1}F_{\binom{1}{0}} = y_{1}\left(G_{\binom{0}{0}} + x_{1}G_{\binom{0}{0}}\right) = y_{1}G_{\binom{0}{0}} + x_{1}y_{1}G_{\binom{0}{0}}$$

$$=G_{\binom{0}{1}} - G_{\binom{0}{0}} + x_{1}\left(G_{\binom{0}{1}} - G_{\binom{0}{0}}\right)$$

$$=G_{\binom{0}{1}} - G_{\binom{0}{0}} + G_{\binom{0}{0}} + G_{\binom{0}{0}} + G_{\binom{0}{0}} + G_{\binom{0}{0}}$$

$$=G_{\binom{0}{1}} - G_{\binom{0}{0}} + G_{\binom{0}{0}} + G_{\binom{0}{0}} + G_{\binom{0}{0}} + G_{\binom{0}{0}}$$

For each of the above rule, we choose one  $G_{\tilde{\eta}}$  as leading basis element. We define a function  $\phi$  from  $(\{x_i\} \times \{G_{\tilde{\gamma}}\}) \cup (\{y_i\} \times \{G_{\tilde{\gamma}}\})$  to  $\{G_{\tilde{\gamma}}\}$  as follows. In the case of rules (4.1), (4.2), we choose  $\phi(x_i, G_{\tilde{\gamma}}) = G_{\left( \begin{subarray}{c} \dots \ 0 \ \tilde{\gamma}_{1i} + 1 \ \dots \ 0 \ \tilde{\gamma}_{2i} \ \dots \ 0$ 

**Lemma 4.1.** The above process of choosing is invertible, i.e.  $\phi$  is injective.

*Proof.* Since each time we multiply  $x_i$  or  $y_i$ , the chosen term contains a  $\binom{0}{0}$  at position i. Combining this fact with the rule that we have to multiply  $y_i$  before  $x_i$ , we have the following inverse function.

Let i be the largest number that  $(\tilde{\gamma}_{1i}, \tilde{\gamma}_{2i}) = (0, 0)$  and  $0 < i < \ell(\tilde{\gamma})$ .

(1) If 
$$\tilde{\gamma}_{1(i+1)} > 0$$
, then  $\phi^{-1} \left( G_{\left( \dots 0 \ \tilde{\gamma}_{1(i+1)} \ \dots \right)} \right) = x_i G_{\left( \dots \ \tilde{\gamma}_{1(i+1)}^{-1} \ \dots \right)}$ .

(2) If  $\tilde{\gamma}_{1(i+1)} = 0$  and,  $\tilde{\gamma}_{1(i+2)} = 0$  or  $\tilde{\gamma}_{2(i+1)} > 1$ , then

$$\phi^{-1}\left(G_{\left(\begin{smallmatrix}\cdots&0&\tilde{\gamma}_{1(i+1)}&\cdots\\\cdots&0&\tilde{\gamma}_{2(i+1)}&\cdots\end{smallmatrix}\right)}\right)=y_iG_{\left(\begin{smallmatrix}\cdots&\tilde{\gamma}_{1(i+1)}&\cdots\\\cdots&\tilde{\gamma}_{2(i+1)}-1&\cdots\end{smallmatrix}\right)}.$$

(3) If  $\tilde{\gamma}_{1(i+1)} = 0$ ,  $\tilde{\gamma}_{2(i+1)} = 1$  and  $\tilde{\gamma}_{1(i+2)} > 0$ , then

$$\phi^{-1}\left(G_{\left( \begin{subarray}{ccccc} \cdots & 0 & 0 & \tilde{\gamma}_{1(i+2)} & \cdots \\ \cdots & 0 & 1 & \tilde{\gamma}_{2(i+2)} & \cdots \end{subarray}}\right) = y_i G_{\left( \begin{subarray}{ccccc} \cdots & \tilde{\gamma}_{1(i+2)} & \cdots \\ \cdots & \tilde{\gamma}_{2(i+2)} & \cdots \end{subarray}}\right).$$

Then, we can construct a map  $\Phi: \{X^{\tilde{\alpha}}F_{\beta}: |\beta| \geq 1\} \to \{G_{\tilde{\gamma}}\}$  that is defined by "composing"  $\phi$  with itself  $(|\tilde{\alpha}|-1)$  times. By the above Lemma, we also have  $\Phi$  is injective. For simplicity, we define  $\phi^{-1}(G_{\tilde{\gamma}})$  (or  $\Phi^{-1}(G_{\tilde{\gamma}})$ ) to be  $X^{\tilde{\alpha}}G_{\tilde{\beta}}$  (or  $X^{\tilde{\alpha}}F_{\beta}$ ) if  $\phi(X^{\tilde{\alpha}}G_{\tilde{\beta}}) = G_{\tilde{\gamma}}$  (or  $\Phi(X^{\tilde{\alpha}}F_{\beta}) = G_{\tilde{\gamma}}$  respectively).

**Lemma 4.2.** In the expansion of  $X^{\tilde{\alpha}}F_{\beta}$  in the G basis using the rules above, the term  $\Phi(X^{\tilde{\alpha}}F_{\beta})$  appears only once. In particular, it has coefficients 1 or -1.

*Proof.* We begin with the claim that if  $\tilde{\mu} \neq \tilde{\nu}$ , then  $\phi(x_i G_{\tilde{\mu}})$  and  $\phi(y_i G_{\tilde{\mu}})$  do not appear in the expansion of  $x_i G_{\tilde{\nu}}$  and  $y_i G_{\tilde{\nu}}$  respectively.

Let k be the smallest integer such that  $(\tilde{\mu}_{k1}, \tilde{\mu}_{k2}) \neq (\tilde{\nu}_{k1}, \tilde{\nu}_{k2})$ . In rules (4.1), (4.2) and (4.3), for all  $G_{\tilde{\gamma}}$  in the expansion of  $x_i G_{\tilde{\mu}}$  or  $y_i G_{\tilde{\mu}}$ , the first i-1 columns of  $\tilde{\gamma}$  is the same as that of  $\tilde{\mu}$ . Hence, the claim follows if k < i.

If k = i, and if we are multiplying  $x_i$  using rules (4.1) or (4.2), then the claim holds because either the i-th or the i+1-th columns of  $x_iG_{\tilde{\mu}}$  will be different from terms in expansions of  $x_iG_{\tilde{\nu}}$ . If we are multiplying by  $y_i$ , then note that if the i-th column of  $\mu$  is (0,0), then  $\mu_{(i+1)1}$  must be 0 because otherwise, that means we multiplied an  $x_i$  or  $x_j$  or  $y_j$  with j > i before  $y_i$ , which violates our rule. And the same condition applies to  $\nu$ . With this restriction, it is easy to check that the claim holds.

If k > i, in both cases, if we choose any term in the expansion that is not  $\phi(x_i G_{\tilde{\nu}})$  or  $\phi(y_i G_{\tilde{\nu}})$ , then the i or i+1 column of its index must be different from that of  $\phi(x_i G_{\tilde{\nu}})$ 

or  $\phi(y_iG_{\tilde{\mu}})$ . If we choose  $\phi(x_iG_{\tilde{\nu}})$  or  $\phi(y_iG_{\tilde{\nu}})$ , we also have  $\phi(x_iG_{\tilde{\mu}}) \neq \phi(x_iG_{\tilde{\nu}})$  and  $\phi(y_iG_{\tilde{\mu}})\phi(y_iG_{\tilde{\nu}})$  because  $\mu \neq \nu$ .

Since each term in the expansion of  $X^{\tilde{\alpha}}F_{\beta}$  corresponds to a sequence of choice using rules (4.1), (4.2) or (43), if at some point, we choose a term that is different from the choice in  $\Phi$ , then a recursive use of the claim asserts that  $\Phi(X^{\tilde{\alpha}}F_{\beta})$  will not appear again.

We now define an order  $(<_G)$  on the set of generalized bicompositions as follows

- (1) If  $\tilde{\alpha}$  and  $\tilde{\beta}$  are bicompositions, then  $\tilde{\alpha} <_G \tilde{\beta}$  if  $\tilde{\alpha} <_{lex} \tilde{\beta}$ .
- (2) If  $\tilde{\alpha}$  is a bicomposition and  $\tilde{\beta}$  is not, then  $\tilde{\alpha} <_G \tilde{\beta}$ .
- (3) If  $\tilde{\alpha} = \tilde{\mu}\binom{0}{0}\alpha'$ ,  $\tilde{\beta} = \tilde{\nu}\binom{0}{0}\beta'$  where  $\alpha'$  and  $\beta'$  are bicompositions, let  $u = \ell(\tilde{\alpha}) \ell(\alpha') 1$ ,  $v = \ell(\tilde{\alpha}) \ell(\beta') 1$ , then  $\tilde{\alpha} <_G \tilde{\beta}$  if
  - (a) u < v, or
  - (b) u = v,  $\alpha'_{11} > 0$  and  $\beta'_{11} = 0$ , or
  - (c) u = v,  $\alpha'_{11} > 0$ ,  $\beta'_{11} > 0$  (or  $\alpha'_{11} = 0$ ,  $\beta'_{11} = 0$ ) and  $\phi(G_{\tilde{\alpha}}) <_G \phi(G_{\tilde{\beta}})$  where we define  $\phi(G_{\tilde{\delta}})$  to be  $\tilde{\gamma}$  if  $\phi(x_i G_{\tilde{\gamma}}) = G_{\tilde{\delta}}$  or  $\phi(y_i G_{\tilde{\gamma}}) = G_{\tilde{\delta}}$  for some i.

**Lemma 4.3.** The order defined above is a total order on the set of generalized bicompositions such that if  $G_{\tilde{\gamma}} = \Phi(X^{\tilde{\alpha}}F_{\beta})$ , then for all  $G_{\tilde{\delta}}$  that appears in the expansion of  $X^{\tilde{\alpha}}F_{\beta}$ , we have  $\tilde{\gamma} \geq_G \tilde{\delta}$ .

*Proof.* Clearly this is a total order. If  $\tilde{\alpha} < \tilde{\beta}$  by (1) or (2), then  $\tilde{\beta}$  cannot appear in the expansion of  $\Phi^{-1}(\tilde{\alpha}) = \tilde{\alpha}$ .

If  $\tilde{\alpha} < \tilde{\beta}$  by (3a), that means  $\phi^{-1}(\tilde{\alpha}) = x_{u+1}G_{\tilde{\gamma}}$  or  $y_{u+1}G_{\tilde{\gamma}}$  for some  $\tilde{\gamma}$ . Hence,  $\tilde{\beta}$  cannot appear in the expansion of  $\Phi^{-1}(\tilde{\alpha})$  because  $\tilde{\beta}_{(v+1)1} = \tilde{\beta}_{(v+1)2} = 0$  cannot be created.

If  $\tilde{\alpha} < \tilde{\beta}$  by (3b), that means  $\phi^{-1}(\tilde{\alpha}) = x_{u+1}G_{\tilde{\gamma}}$  for some  $\tilde{\gamma}$ . Hence,  $\tilde{\beta}$  cannot appear in the expansion of  $\Phi^{-1}(\tilde{\alpha})$  because it is not in that of  $x_{u+1}G_{\tilde{\delta}}$  for any  $\tilde{\delta}$ .

With this ordering, there is a unique leading  $G_{\tilde{\delta}}$  for each expansion of  $X^{\tilde{\alpha}}F_{\beta}$ .

**Theorem 4.4.** The set  $A = \{G_{\tilde{\alpha}} \mid G_{\tilde{\alpha}} \notin Img(\Phi)\}$  forms a Hilbert basis for the quotient space R/I.

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*Proof.* For any polynomial p in R, we write p in terms of the G basis with  $<_G$  order. For each term  $G_{\tilde{\alpha}} \in Img(\Phi)$ , we subtract p by  $\Phi^{-1}(G_{\tilde{\alpha}}) \in I$  and  $G_{\tilde{\alpha}}$  is cancelled. If we repeat this process (possibly countably many times), we can express p as a series of A.

### 5. Finitely many variables case

In the case that there are finitely many variables,  $R_n = \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ , the above constructions of  $\mathsf{DQSym}(x_1, \ldots, x_n, y_1, \ldots, y_n)$ , the F, G bases and the ideal  $I_n = < \mathsf{DQSym}^+(x_1, \ldots, x_n, y_1, \ldots, y_n) > \text{remain the same by taking } x_i = y_i = 0 \text{ for } i > n$ . In this case,  $LM(G_{\tilde{\alpha}}) = X^{\tilde{\alpha}}$  whenever  $\ell(\tilde{\alpha}) \leq n$  and hence  $\{G_{\tilde{\alpha}} : \ell(\tilde{\alpha}) \leq n\}$  spans  $R_n$ .

Let  $R_n^{i,j}$  be the span of  $\{X^{\tilde{\alpha}}: \ell(\tilde{\alpha}) \leq n, \sum_k \tilde{\alpha}_{1k} = i, \sum_k \tilde{\alpha}_{2k} = j\}$ . Since  $I_n$  is bihomogeneous in  $\mathbf{x}$  and  $\mathbf{y}$ ,  $I_n = \bigoplus_{i,j} I_n^{i,j}$  where  $I_n^{i,j} = I_n \cap R_n^{i,j}$ , and  $R_n/I_n = \bigoplus_{i,j} V_n^{i,j}$  where  $V_n^{i,j} = R_n/I_n \cap R_n^{i,j}$ .

The Hilbert matrix corresponding to  $R_n/I_n$  is the matrix  $M_n(i,j) = \dim(V_n^{i-1,j-1})$ .

The goal of this section is to compute the second column of the Hilbert matrix. The proof is slight generalization of the one in [ABB].

**Lemma 5.1.** The set  $\{G_{\tilde{\alpha}} \mid G_{\tilde{\alpha}} \notin Img(\Phi), \ell(\tilde{\alpha}) \leq n\}$  spans the quotient  $R_n/I_n$ .

*Proof.* Among all  $\tilde{\alpha}$  such that  $G_{\tilde{\alpha}} \in Img(\Phi)$ ,  $\ell(\tilde{\alpha}) \leq n$  and  $G_{\tilde{\alpha}}$  cannot be reduced to 0, let  $\tilde{\beta}$  be the smallest one with respect to the  $<_G$  order. Then,

$$G_{\tilde{\beta}} = G_{\tilde{\beta}} - \Phi^{-1}(G_{\tilde{\beta}}) + \Phi^{-1}(G_{\tilde{\beta}})$$
  
$$\equiv G_{\tilde{\beta}} - \Phi^{-1}(G_{\tilde{\beta}}) \mod I_n$$

But since  $G_{\tilde{\beta}}$  is the leading term in  $\Phi^{-1}(G_{\tilde{\beta}})$ , terms in  $G_{\tilde{\beta}} - \Phi^{-1}(G_{\tilde{\beta}})$  are strictly smaller than  $G_{\tilde{\beta}}$ , and thus they reduce to 0. This contradicts to our assumption on  $\tilde{\beta}$ .

Let  $B_n$  be the set of generalized bicompositions  $\{\tilde{\alpha}\}$  such that  $\sum_{i=1}^k (\tilde{\alpha}_{1i} + \tilde{\alpha}_{2i}) < k$  for all  $1 \le k \le n$  and  $\ell(\tilde{\alpha}) \le n$ . Clearly from the definition of G basis, if  $\tilde{\alpha} \notin B_n$ , then

 $G_{\tilde{\alpha}} \in I_n$ . Therefore, the set  $\{X^{\tilde{\alpha}} : \tilde{\alpha} \in B_n\}$  spans  $R_n/I_n$ , the proof is essentially the same as Lemma 5.1. In particular,  $X^{\tilde{\alpha}} \in I_n$  for all  $|\tilde{\alpha}| \geq n$ .

**Lemma 5.2.** The set  $\{X^{\tilde{\alpha}}F_{\beta}: \tilde{\alpha} \in B_n, |\beta| \geq 0\}$  spans  $R_n$ .

Proof. We already have  $X^{\tilde{\epsilon}} \equiv \sum_{\tilde{\alpha} \in B_n} X^{\tilde{\alpha}} \mod I_n$ , which means  $X^{\tilde{\epsilon}} = \sum_{\tilde{\alpha} \in B_n} X^{\tilde{\alpha}} + \sum_{|\beta| \geq 1} P_{\beta} F_{\beta}$  for some polynomial  $P_{\beta}$ . If we reduce each monomial  $P_{\beta}$  using the above rule, and write the product of F basis in terms of F basis, the claim will be satisfied in a finite number of steps.

For a generalized bicomposition  $\tilde{\alpha}$  with  $\ell(\tilde{\alpha}) \leq n$ , we define its reverse  $\overline{\alpha}$  to be the generalized bicomposition such that  $\overline{\alpha}_{1i} = \tilde{\alpha}_{1(n-i+1)}$  and  $\overline{\alpha}_{2i} = \tilde{\alpha}_{2(n-i+1)}$  for all  $1 \leq i \leq n$ .

We denote the set  $\{X^{\tilde{\alpha}} : \overline{\alpha} \in B_n\}$  by  $A_n$ . The endomorphism of  $R_n$  that sends  $x_i$  to  $x_{n-i+1}$  and  $y_i$  to  $y_{n-i+1}$  is clearly an algebra isomorphism that fixes  $\mathsf{DQSym}(\mathbf{x}, \mathbf{y})$ , in fact, it sends  $M_{\alpha}$  to  $M_{\alpha'}$  where  $\alpha'$  is the reversed bicomposition of  $\alpha$ . Therefore, by Lemma 5.2, the set  $\{X^{\tilde{\alpha}}F_{\beta} : \tilde{\alpha} \in A_n, |\beta| \geq 0\}$  spans  $R_n$ .

Hence,  $I_n = \langle F_\gamma : |\gamma| \geq 1 \rangle$  is spanned by  $\{X^{\tilde{\alpha}}F_\beta F_\gamma : \tilde{\alpha} \in A_n, |\beta| \geq 0, |\gamma| \geq 1 \}$ , which means it is spanned by  $\{X^{\tilde{\alpha}}F_\beta : \tilde{\alpha} \in A_n, |\beta| \geq 1 \}$ .

**Lemma 5.3.** For  $X^{\tilde{\alpha}}F_{\beta} \in R_n^{i,1}$  with  $\tilde{\alpha} \in A_n$ ,  $|\beta| \geq 1$  and  $|\tilde{\alpha}| + |\beta| < n$ , let  $G_{\tilde{\gamma}} = \Phi(X^{\tilde{\alpha}}F_{\beta})$ , then  $\ell(\tilde{\gamma}) \leq n$ .

*Proof.* First, rules (4.1) and (4.2) increase the length by 1 while (4.3) increase the length by 2. Now, we need to track  $\tilde{\gamma}_{\ell(\tilde{\gamma})}$ . If  $\tilde{\gamma}_{\ell(\tilde{\gamma})}$  comes from  $\beta_{\ell(\beta)}$  and gets shifted, since we can use (4.3) at most once, we can make at most  $|\tilde{\alpha}| + 1$  steps to the right. Therefore,  $\ell(\tilde{\gamma}) \leq |\tilde{\alpha}| + 1 + \ell(\beta) \leq |\tilde{\alpha}| + 1 + |\beta| \leq n$ .

If  $\tilde{\gamma}_{\ell(\tilde{\gamma})}$  is 1 which comes from multiplying  $x_k$  or  $y_k$  to  $G_{\tilde{\epsilon}}$  with  $k > \ell(\tilde{\epsilon})$ , since  $\tilde{\alpha} \in A_n$ , we have  $\sum_{i \geq k} (\tilde{\alpha}_{1i} + \tilde{\alpha}_{2i}) < n - k + 1$ . In this process, we use rules (4.1) and (4.2) only and each increases the length by 1. Therefore,  $\tilde{\gamma}_{\ell(\tilde{\gamma})}$  can be shifted to at most position k + n - k = n.

Corollary 5.4. Let  $M_n$  be the Hilbert matrix of  $R_n/I_n$ , then  $M_n(n-1,2) = \frac{1}{n} \binom{2n-2}{n-1}$ ,  $M_n(i,2) = \sum_{1 \le j \le i, 1 \le k \le 2} M_{n-1}(j,k)$  for  $1 \le i \le n-2$ , and  $M_n(2,1) = 0$  for  $i \ge n$ .

Proof. Lemma 5.1 shows that  $C_i = \{G_{\tilde{\alpha}} \in V_n^{i,1} : G_{\tilde{\alpha}} \notin Img(\Phi)\}$  spans  $V_n^{i,1}$ . Suppose there is a linear dependence  $P = \sum_{G_{\tilde{\alpha}} \in C_i} a_{\tilde{\alpha}} G_{\tilde{\alpha}} \in I_n^{i,1}$ . Since  $I_n^{i,1}$  is spanned by  $D = G_{\tilde{\alpha}} \in C_i$ 

$$\{X^{\tilde{\alpha}}F_{\beta}\in R_n^{i,1}: \tilde{\alpha}\in A_n, |\beta|\geq 1\}, \text{ we have } P=\sum_{X^{\tilde{\alpha}}F_{\beta}\in D}b_{\tilde{\alpha}\beta}X^{\tilde{\alpha}}F_{\beta}. \text{ This means the}$$

leading term of P when we expand in G basis is some  $G_{\tilde{\gamma}}$  such that  $\tilde{\gamma} \in Img(\Phi)$  and by Lemma 5.3  $\ell(\tilde{\gamma}) \leq n$ , which is absurd. Therefore,  $C_i$  is a linear basis for  $V_n^{i,1}$ .

Now,  $M_n(i,1) = \dim V_n^{i-1,1} = |C_{i-1}|$ . Let  $G_{\tilde{\gamma}} \in V_n^{i,1}$  and k be the unique number that  $\tilde{\gamma}_{k2} = 1$ . First, from definition of G,  $\tilde{\gamma} \notin B_n$  implies  $G_{\tilde{\gamma}} \in I_n$  and  $G_{\tilde{\gamma}} \in Img(\Phi)$ .

If 
$$i = n - 1$$
, then  $|\tilde{\gamma}| = n - 1$ . If  $k < \ell(\tilde{\gamma})$ , since  $\sum_{j=k+1}^{n} \tilde{\gamma}_{1j} \ge n - k$ , we will be

using rules (4.3) when applying  $\phi^{-1}$ . This reduces the length by 2 while the size by 1, which means  $G_{\tilde{\gamma}} \in Img(\Phi)$ . If  $k = \ell(\tilde{\gamma})$ , we only use rules (4.1) and (4.2) when applying  $\phi^{-1}$ . In this case,  $G_{\tilde{\gamma}} \notin Img(\Phi)$  whenever  $\tilde{\gamma} \in B_n$ . Therefore,  $|C_{n-2}|$  is the Catalan number  $\frac{1}{n} \binom{2n-2}{n-1}$ .

If  $1 \leq i \leq n-2$ ,  $|\tilde{\gamma}| \leq n-2$ . From the definition of  $\phi$ ,  $G_{\tilde{\gamma}} \notin Img(\Phi)$  if and only if  $G_{\left(\tilde{\gamma}_{11} \ \dots \ \tilde{\gamma}_{1(n-1)} \ \tilde{\gamma}_{21} \ \dots \ \tilde{\gamma}_{2(n-1)} \ \right)} \in V_{n-1}^{j,k} \setminus Img(\Phi)$  for some  $1 \leq j \leq i, 1 \leq k \leq 2$ . Therefore,  $M_n(i,2) = \sum_{1 \leq j \leq i, 1 \leq k \leq 2} M_{n-1}(j,k)$  for  $1 \leq i \leq n-2$ .

By the symmetry  $M_n(a,b) = M_n(b,a)$ , we obtain the first to rows of the Hilbert matrix, namely  $M_n(2,n-1) = \frac{1}{n} \binom{2n-2}{n-1}$ ,  $M_n(2,i) = \sum_{1 \leq j \leq i, 1 \leq k \leq 2} M_{n-1}(k,j)$  for  $1 \leq i \leq n-2$ , and  $M_n(2,i) = 0$  for  $i \geq n$ .

This method can be applied directly to some other terms. To be more specific, for  $2i+j \leq n$ , the set  $\{G_{\tilde{\alpha}} \mid G_{\tilde{\alpha}} \notin Img(\Phi), \ell(\tilde{\alpha}) \leq n\}$  is a linear basis in  $V_n^{i,j}$ . Therefore,

the formula for each column stabilizes when the number of variables is large enough. However, it fail in other terms and this set is not a linear basis in general.

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