Theorem. $F_{1,r} \cdot F_{1,s} = c\, F_{1,r+s}$ for some c.

Lemma 1. $\forall m \leq n$, $\forall A, B \subseteq [m]$, $x \in [m] / A \cup B$,

$$sgn(\theta_A \cdot \theta_{B \cup \{x\}}) = -1^{|\{a \in A \mid a > x\}|} sgn(\theta_A \cdot \theta_B)$$

Proof. easy to check

Lemma 2. $\forall m \leq n$, $\forall A, B \subseteq [m]$ s.t. $A \cap B = \phi$

and for any increasing bijection $g: [m] \to D \subseteq [n]$,

$$sgn(\theta_A \cdot \theta_B) = sgn(\theta_{g(A)} \cdot \theta_{g(B)}).$$

Proof. Induction on $|B|$.

$|B| = 0 \Rightarrow sgn(\theta_A) = sgn(\theta_{g(A)}) = 1$.

suppose that the statement is true for $|B| = k-1$.

Let $|B| = k$ and take $b \in B$ arbitrary.

we have $sgn(\theta_A \cdot \theta_B) = sgn(\theta_A \cdot \theta_{(B/b) \cup \{b\}})$

$$\overset{\text{Lemma 1}}{=} -1^{|\{a \in A \mid a > b\}|} sgn(\theta_A \cdot \theta_{B/b})$$

$$\overset{\text{Ind. HYP.}}{=} -1^{|\{a \in A \mid a > b\}|} \cdot sgn(\theta_{g(A)} \cdot \theta_{g(B/b)})$$

$$\underset{\text{bijection}}{\overset{g \text{ increasing}}{=}} -1^{|\{g(a) \in g(A) \mid g(a) > g(b)\}|} \cdot sgn(\theta_{g(A)} \cdot \theta_{g(B/b)})$$

$$\overset{\text{Lemma 1}}{=} sgn(\theta_{g(A)} \cdot \theta_{g(B/b) \cup g(b)})$$

$$= sgn(\theta_{g(A)} \cdot \theta_{g(B)}) \quad \blacksquare$$

Proof of the theorem:

we have $F_{1,r} F_{1,s} = \sum_{\substack{A \subseteq [n] \\ |A| = r+s}} C_A \theta_A$.

Fix $A = [r+s]$. observe that $C_{[r+s]} = \sum_{\substack{B_1 \subseteq [r+s] \\ |B_1| = r \\ B_2 = [r+s]/B_1}} sgn(\theta_{B_1} \cdot \theta_{B_2})$.

Now, let $D = \{i_1, i_2, \dots, i_{r+s}\} \subseteq [n]$ s.t. $i_1 < i_2 < \dots < i_{r+s}$.

Define $i: [r+s] \to D$ which is an increasing bijection.
$i(k) = i_k$

now, by Lemma 2, we have

$$C_{[r+s]} = \sum_{\substack{B_1 \subseteq [r+s] \\ |B_1| = r \\ B_2 = [r+s]/B_1}} sgn(\theta_{i(B_1)} \cdot \theta_{i(B_2)})$$

$$\xlongequal{i \text{ bijection}} \sum_{\substack{i(B_1) \subseteq i([r+s]) \\ |i(B_1)| = r \\ i(B_2) = i([r+s])/i(B_1)}} sgn(\theta_{i(B_1)} \cdot \theta_{i(B_2)})$$

$$= \sum_{\substack{E_1 \subseteq \{i_1, \dots, i_{r+s}\} \\ |E_1| = r \\ E_2 = \{i_1, i_2, \dots, i_{r+s}\}/E_1}} sgn(\theta_{E_1} \cdot \theta_{E_2})$$

Again, observe that above is just $C_{\{i_1, i_2, \dots, i_{r+s}\}}$.

$\Rightarrow \forall \{i_1, i_2, \dots, i_{r+s}\} \subseteq n$, $C_{\{i_1, \dots, i_{r+s}\}} = C_{[r+s]} = C \Rightarrow F_{1r} F_{1s} = C \sum_{\substack{A \subseteq [n] \\ |A| = r+s}} \theta_A = C F_{1,r+s}$