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CATALAN PATHS AND QUASI-SYMMETRIC FUNCTIONS

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ABSTRACT. We investigate the quotient ring R of the ring of formal power series $\mathbb{Q}[[x_1, x_2, \dots]]$ over the closure of the ideal generated by non-constant quasi-symmetric functions. We show that a Hilbert basis of the quotient is naturally indexed by Catalan paths (infinite Dyck paths). We also give a filtration of ideals related to Catalan paths from $(0,0)$ and above the line $y = x - k$. We investigate as well the quotient ring R_n of polynomial ring in n variables over the ideal generated by non-constant quasi-symmetric polynomials. We show that the dimension of R_n is bounded above by the nth Catalan number.

1. Introduction

The ring $Qsym$ of quasi-symmetric functions was introduced by Gessel [11] as a source of generating functions for P-partitions [16]. Since then, quasi-symmetric functions have appeared in many combinatorial contexts [6, 16, 17]. The relation of $Qsym$ to the ring of symmetric functions was first clarified by Malvenuto and Reutenauer [15] via a graded Hopf duality to the Solomon descent algebras, and then Gelfand *et. al.* [10] defined the graded Hopf algebra NC of non-commutative symmetric functions and identified it with the Solomon descent algebra. In recent literature, we see a growing interest in quasi-symmetric functions and non-commutative symmetric functions as refinements of the ring of symmetric functions.

One unexplored avenue is as an analogue of the (symmetric) harmonic spaces. A classic combined result of Artin and Steinberg [1, 18] shows that the quotient ring of the polynomial ring $\mathbb{Q}[x_1, x_2, \dots, x_n]$ in n variables over the ideal $\mathcal{I}_n$ generated by non-constant symmetric polynomials has dimension $n!$. In fact, this space is a graded symmetric group module that affords the left regular representation. Refinement and generalization of this result has lead to an explosion of incredible results and conjectures; for example see [4, 5, 7, 9, 12] for a small portion of this.

Here we are interested in the quotient ring R of the ring of formal power series $\mathbb{Q}[[x_1, x_2, x_3, \dots]]$ over the closure of the homogeneous ideal $\mathcal{J}$ generated by all non-constant quasi-symmetric functions. That is, the quotient

$$R = \mathbb{Q}[[x_1, x_2, x_3, \dots]] \Big/ \overline{\mathcal{J}}.$$

To every monomial $x_1^{\tilde{\alpha}_1} x_2^{\tilde{\alpha}_2} x_3^{\tilde{\alpha}_3} \cdots$ (of finite total degree) in $\mathbb{Q}[[x_1, x_2, x_3, \dots]]$ we associate a path in the plane as follows: $(0,0) \to (\tilde{\alpha}_1, 0) \to (\tilde{\alpha}_1, 1) \to (\tilde{\alpha}_1 + \tilde{\alpha}_2, 1) \to$

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$(\tilde{\alpha}_1+\tilde{\alpha}_2, 2) \to (\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_3, 2) \to \cdots$. If this path remains above the line $y = x$ we say that the path is a *Catalan path* (or infinite Dyck path). Our main result is

Theorem 1.1. *A monomial Hilbert basis of R is given by the monomials of $\mathbb{Q}[[x_1, x_2, x_3, \dots]]$ corresponding to Catalan paths.*

We also consider a special filtration of ideals $\mathcal{J}^{(e)}$ and their respective quotients, such that $\mathcal{J} = \mathcal{J}^{(0)}$ and $\mathcal{J}^{(e)} \subseteq \mathcal{J}^{(e+1)}$. The Hilbert basis of each quotient is indexed by paths above the line $y = x - e$.

Let R_n denote the quotient of the polynomial ring $\mathbb{Q}[x_1, x_2, \dots, x_n]$ over the ideal $\mathcal{J}_n$ generated by all non-constant quasi-symmetric polynomials. Recall that $C_n = \frac{1}{n+1}\binom{2n}{n}$ are the famous Catalan numbers. The passage from infinitely many variables to finitely many variables is non-trivial and requires more work.

Theorem 1.2. $\dim R_n \le C_n$.

In fact the equality holds in Theorem 1.2 and we have a complete proof of this together with François Bergeron [2], who has discovered the spaces R_n completely independently and in the same period that we did. The results for R_n are extremely interesting, and are the object of an ongoing collaboration with F. Bergeron and A. Garsia. We plan to write at least one more paper [3] to investigate further properties of R_n and its generalization. In particular, the finite version of the successive quotients by the ideals $\mathcal{J}_n^{(e)}$ is related to the work of [5]. Much of these results can be explained in a more general framework and will be the object of further study. We are convinced that these results are but the tip of a new iceberg. In particular, we would like to find any natural algebras acting on these spaces. What are the possible generalizations and specializations of the super-harmonics?

We underline here that F. Hivert [13] has developed an action of the Hecke algebra for which a polynomial is invariant if and only if it is quasi-symmetric. Unfortunately Hivert's action is not compatible with multiplication and does not preserve the ideal $\mathcal{J}_n$, hence it does not induce the desired action on the quotient. It is still interesting to note that Hivert's action is also related to Catalan numbers.

In Section 2 we recall appropriate definitions. In Section 3 we introduce a special family of generators for the ideal $\mathcal{J}$ and the associated filtration $\mathcal{J}^{(e)}$. In Section 4 we use these generators to show that the monomials corresponding to Catalan paths span our quotient, as well as the analogous result for $\mathcal{J}^{(e)}$. Theorem 1.2 follows from this section. To complete the proof of Theorem 1.1, we use a Gröbner basis argument in Section 5 to show independence.

2. Basic definitions

A composition $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_k]$ of a positive integer d is an ordered list of positive integers whose sum is d. We denote this by $\alpha \models d$. We call the integers α_i the *parts* of α, and denote the number of parts in α by $\ell(\alpha)$. Given two compositions $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_k]$ and $\beta = [\beta_1, \beta_2, \dots, \beta_\ell]$, we denote by $\alpha\beta$ the concatenation product $[\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_\ell]$. Also, there exists a natural one-to-one correspondence between compositions of d and subsets of $\{1, 2, \dots, d-1\}$. If $A = \{a_1, a_2, \dots, a_{k-1}\} \subset [d-1]$, where $a_1 < a_2 < \dots < a_{k-1}$, then A corresponds to the composition, $\alpha = [a_1 - a_0, a_2 - a_1, \dots, a_k - a_{k-1}]$, where $a_0 = 0$ and $a_k = d$. For ease of notation, we shall denote the set corresponding to a given composition α by $D(\alpha)$. For compositions α and β we say that α is a *refinement* of β if $D(\beta) \subset D(\alpha)$, and denote this by $\alpha \preccurlyeq \beta$.

For any composition $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_k]$ of d we denote by M_α the *monomial quasi-symmetric function* [11]: $M_\alpha(x_1, x_2, \dots) = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} \dots x_{i_k}^{\alpha_k}$. This is a homogeneous infinite series of degree d. We define $M_0 = 1$, where 0 denotes the unique empty composition of 0. It is known from the work of Gessel that the monomial quasi-symmetric functions form a linear basis of a ring (in fact a Hopf algebra) $Qsym$ of quasi-symmetric functions.

Another useful basis of the ring $Qsym_n$ is given by the *fundamental quasi-symmetric function* [11]:

$$F_\alpha(x_1, x_2, \dots) = \sum_{\alpha \succcurlyeq \beta} M_\beta(x_1, x_2, \dots) = \sum_{\substack{j_1 \le j_2 \le \cdots \le j_d \\ i \in D(\alpha) \Rightarrow j_i < j_{i+1}}} x_{j_1} x_{j_2} \cdots x_{j_d}.$$

Fundamental quasi-symmetric functions satisfy the following obvious, but crucial, relations. For $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_k] \models d$,

(2.1)

$$F_\alpha(x_1, x_2, \dots) = \begin{cases} x_1 F_{[\alpha_1 - 1, \alpha_2, \dots, \alpha_k]}(x_1, x_2, \dots) + F_\alpha(x_2, x_3, \dots) & \text{if } \alpha_1 > 1, \\ x_1 F_{[\alpha_2, \alpha_3, \dots, \alpha_k]}(x_2, x_3, \dots) + F_\alpha(x_2, x_3, \dots) & \text{if } \alpha_1 = 1. \end{cases}$$

Here $F_\alpha(x_2, x_3, \dots)$ is the function $F_\alpha(x_1, x_2, \dots)$ in which the variable x_i is replaced by x_{i+1}. These relations are the key ingredients in our proof.

In the following we have to consider generalized (infinite) compositions. That is, a sequence $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2, \dots)$ such that the parts $\tilde{\alpha}_j \ge 0$ for $j \ge 1$ (we allow some parts to be zero) and the sum of the parts $d(\tilde{\alpha}) = \sum \tilde{\alpha}_i < \infty$. We say that $\tilde{\alpha}$ is a generalized composition of $d(\tilde{\alpha}) < \infty$. We use a " ˜ " to indicate that we have a generalized composition and no " ˜ " if the composition is *standard*, that is, without zeros. We also consider generalized compositions of finite length and denote by $\ell(\tilde{\alpha})$ the number of parts of $\tilde{\alpha}$. The concatenation of a finite length generalized composition $\tilde{\alpha}$ with an infinite one $\tilde{\beta}$ is denoted by $\tilde{\alpha}\tilde{\beta}$. We also write $\tilde{\alpha}+\tilde{\beta}$ and $\tilde{\alpha} \le \tilde{\beta}$ to denote the componentwise sum and componentwise inequalities, respectively. For an infinite generalized composition $\tilde{\alpha}$, since $d(\tilde{\alpha}) < \infty$, only finitely many parts of $\tilde{\alpha}$ are non-zero. Thus there is always a finite generalized composition $\tilde{\nu}$ such that $\tilde{\alpha} = \tilde{\nu}\, 0\, 0 \cdots$.

In this paper, we devote our attention to the ideal $\mathcal{J} = \langle F_\alpha(x_1, x_2, \dots) \rangle_{\alpha \models d > 0}$ of $\mathbb{Q}[[x_1, x_2, \dots]]$ generated by the non-constant quasi-symmetric functions, and consider the quotient

$$R = \mathbb{Q}[[x_1, x_2, \dots]] \Big/ \overline{\mathcal{J}}, \tag{2.2}$$

where $\overline{\mathcal{J}}$ denotes the closure (with respect to the standard topology with formal power series) of $\mathcal{J}$ in $\mathbb{Q}[[x_1, x_2, \dots]]$.

3. The generators $G_{\tilde{\alpha}}$

We exploit the relation (2.1) to construct a set $\{G_{\tilde{\alpha}}\} \subseteq \mathcal{J}$ indexed by the generalized (infinite) composition $\tilde{\alpha}$ such that there exists a factorization $\tilde{\alpha} = \tilde{\pi}\tilde{\rho}$ where $d(\tilde{\pi}) - \ell(\tilde{\pi}) \ge 0$. We first define recursively the functions $G_{\tilde{\alpha}}$ for all infinite generalized composition $\tilde{\alpha}$. Then in Lemma 3.1 we characterize the $\tilde{\alpha}$ obtained from the transitive closure of the $G_{\tilde{\alpha}} \in \mathcal{J}$. Let $\tilde{\alpha} = \tilde{\nu}\, 0\, 0 \cdots$ where $\ell(\tilde{\nu}) < \infty$ and the last part of $\tilde{\nu}$ is non-zero, or $\tilde{\alpha} = 0\, 0 \cdots$. Our definition is recursive on $n = \ell(\tilde{\nu})$. If

$\tilde{\nu} = \nu$ is a standard composition, then let

$$G_{\tilde{\alpha}} = F_\nu(x_1, x_2, \dots). \tag{3.1}$$

If $\ell(\tilde{\nu}) = 0$, then this formula gives $G_{0\,0\,\dots} = 1$. Assume now that $\tilde{\nu}$ is non-standard and let $\tilde{\nu} = \tilde{\gamma}\, 0\, a\, \beta$ be the unique factorization of $\tilde{\nu}$ such that $a > 0$ is a positive integer, β is a (possibly empty) standard composition and $\tilde{\gamma}$ is a (possibly empty) generalized composition. For $\tilde{\alpha} = \tilde{\gamma}\, 0\, a\, \beta\, 0\, 0 \cdots$ and $k = \ell(\tilde{\gamma}\, 0) = \ell(\tilde{\gamma}) + 1$, we define

$$G_{\tilde{\alpha}} = G_{\tilde{\gamma}\, a\, \beta\, 0 \cdots} - x_k G_{\tilde{\gamma}\,(a-1)\,\beta\, 0 \cdots}. \tag{3.2}$$

Both terms on the right are well defined by induction since $\ell(\tilde{\gamma}\, a\, \beta) = \ell(\tilde{\nu}) - 1 < n$.

We now characterize the transitive closure of the definition (3.1) and (3.2) within $\mathcal{J}$. At this point it is useful to introduce the following family of ideals. For any $e \geq 0$, let $\mathcal{J}^{(e)} = \langle F_\alpha : \exists\, \pi\rho = \alpha,\ d(\pi) - \ell(\pi) \geq e \rangle$. This is a filtration $\mathcal{J}^{(e)} \subseteq \mathcal{J}^{(e+1)}$ such that $\mathcal{J} = \mathcal{J}^{(0)}$. For a generalized composition $\tilde{\alpha}$, we say that it *reaches level* e if there exists a factorization $\tilde{\alpha} = \tilde{\pi}\tilde{\rho}$ such that $d(\tilde{\pi}) - \ell(\tilde{\pi}) \geq e$.

Lemma 3.1.

1. *If $\tilde{\alpha}$ reaches level e, then $G_{\tilde{\alpha}} \in \mathcal{J}^{(e)}$.*
2. *Conversely, in* (3.2)*, if $\tilde{\gamma}\,(a-1)\,\beta\, 0 \cdots$ reaches level e, then $\tilde{\alpha}$ reaches level e.*

Proof. For the first statement we proceed by induction on $\ell(\tilde{\nu})$ where $\tilde{\alpha} = \tilde{\nu}\, 0\, 0 \cdots$ and the last part of $\tilde{\nu}$ is non-zero. If $\ell(\tilde{\nu}) = 0$, then $G_{0\,0 \cdots} = 1$ is not in any of the ideals $\mathcal{J}^{(e)}$. Assume that $\ell(\tilde{\nu}) > 0$. We first consider the case when $\tilde{\nu} = \nu$ is a standard composition. If $\tilde{\alpha}$ reaches level e, then so is ν and we have $G_{\tilde{\alpha}} = F_\nu(x_1, x_2, \dots) \in \mathcal{J}^{(e)}$. If $\tilde{\nu}$ is a non-standard generalized composition, then let $\tilde{\alpha} = \tilde{\pi}\tilde{\rho}$ be the factorization such that $d(\tilde{\pi}) - \ell(\tilde{\pi}) \geq e$, and let $\tilde{\alpha} = \tilde{\nu}\, 0\, 0 \cdots = \tilde{\gamma}\, 0\, a\, \beta\, 0\, 0 \cdots$ be the factorization used in (3.2). If $\tilde{\pi}$ is an initial factor of $\tilde{\gamma}$, then it is clearly an initial factor of both $\tilde{\gamma}\, a\, \beta\, 0\, 0 \cdots$ and $\tilde{\gamma}\,(a-1)\,\beta\, 0\, 0 \cdots$ and they both reach level e. By the induction hypothesis, both $G_{\tilde{\gamma}\, a\, \beta\, 0 \cdots}$ and $G_{\tilde{\gamma}\,(a-1)\,\beta\, 0 \cdots}$ are in $\mathcal{J}^{(e)}$ and in turn $G_{\tilde{\alpha}} \in \mathcal{J}^{(e)}$. If we now assume that $\tilde{\pi} = \tilde{\gamma}\, 0$, then $d(\tilde{\gamma}) - \ell(\tilde{\gamma}) = d(\tilde{\pi}) - \big(\ell(\tilde{\pi}) - 1\big) \geq e + 1 > e$ and again the induction hypothesis can be applied to (3.2) to show that $G_{\tilde{\alpha}} \in \mathcal{J}^{(e)}$.

We are left to check the case where $\tilde{\pi} = \tilde{\gamma}\, 0\, a\, \tilde{\mu}$. For the first term in (3.2), $\tilde{\gamma}\, a\, \tilde{\mu}$ is an initial factor and $d(\tilde{\mu}\, a\, \tilde{\gamma}) - \ell(\tilde{\gamma}\, a\, \tilde{\mu}) = d(\tilde{\pi}) - \big(\ell(\tilde{\pi}) - 1\big) \geq e + 1 > e$. The induction hypothesis gives that $G_{\tilde{\gamma}\, a\, \beta\, 0 \cdots} \in \mathcal{J}^{(e)}$. For the second term indexed by $\tilde{\gamma}\,(a-1)\,\beta\, 0\, 0 \cdots$ we have $d(\tilde{\gamma}(a-1)\tilde{\mu}) - \ell(\tilde{\gamma}(a-1)\tilde{\mu}) = \big(d(\tilde{\pi}) - 1\big) - \big(\ell(\tilde{\pi}) - 1\big) \geq e$. Again the induction hypothesis gives us that $G_{\tilde{\gamma}\,(a-1)\,\beta\, 0 \cdots} \in \mathcal{J}^{(e)}$, concluding the proof that $G_{\tilde{\alpha}} \in \mathcal{J}^{(e)}$.

For the second statement of the lemma let $\tilde{\gamma}\,(a-1)\,\beta\, 0\, 0 \cdots = \tilde{\pi}\tilde{\rho}$ be a factorization such that $d(\tilde{\pi}) - \ell(\tilde{\pi}) \geq e$. If $\tilde{\pi}$ is an initial factor of $\tilde{\gamma}$, then it is clear that $\tilde{\alpha}$ reaches level e. On the other hand if $\tilde{\pi} = \tilde{\gamma}\,(a-1)\,\tilde{\mu}$, then we have $d(\tilde{\gamma}\, 0\, a\, \tilde{\mu}) - \ell(\tilde{\gamma}\, 0\, a\, \tilde{\mu}) = \Big(d\big(\tilde{\gamma}\,(a-1)\,\tilde{\mu}\big) + 1\Big) - \Big(\ell\big(\tilde{\gamma}\,(a-1)\,\tilde{\mu}\big) + 1\Big) \geq e$. Thus $\tilde{\alpha}$ reaches level e. ∎

In light of the previous lemma, let $\mathcal{G}^{(e)}$ denote the set of all generalized infinite compositions $\tilde{\alpha}$ reaching level e, that satisfy $d(\tilde{\pi}) - \ell(\tilde{\pi}) \geq e$. We remark that the set $\{G_{\tilde{\alpha}}\}_{\tilde{\alpha} \in \mathcal{G}^{(e)}}$ constructed above is contained in $\mathcal{J}^{(e)}$ and contains $\{F_\alpha(x_1, x_2, \dots) : \exists\, \alpha = \pi\rho,\ d(\pi) - \ell(\pi) \geq e\}$. Hence we have

Lemma 3.2. $\mathcal{J}^{(e)} = \langle G_{\tilde\alpha}\rangle_{\tilde\alpha\in\mathcal{G}^{(e)}}$. $\square$

Our next task is to characterize the leading monomial of each function $G_{\tilde\alpha}$. Before this we need to specify which monomial order we use. Let $X^{\tilde\alpha} = x_1^{\tilde\alpha_1}x_2^{\tilde\alpha_2}\cdots$ and $X^{\tilde\beta} = x_1^{\tilde\beta_1}x_2^{\tilde\beta_2}\cdots$ be any two monomials where $\tilde\alpha$ and $\tilde\beta$ are two generalized infinite compositions. We say that $X^{\tilde\alpha} \le_{lex} X^{\tilde\beta}$ if and only if $d(\tilde\alpha) > d(\tilde\beta)$, or $d(\tilde\alpha) = d(\tilde\beta)$ and the leftmost non-zero entry in $[\tilde\beta_1-\tilde\alpha_1, \tilde\beta_2-\tilde\alpha_2,\dots,]$ is positive. The order $\le_{lex}$ is a classical monomial order in the sense that it is a total order and if $X^{\tilde\alpha} \le_{lex} X^{\tilde\beta}$, then $X^{\tilde\alpha}X^{\tilde\gamma} = X^{\tilde\alpha+\tilde\gamma} \le_{lex} X^{\tilde\beta+\tilde\gamma} = X^{\tilde\beta}X^{\tilde\gamma}$. Here the sum of generalized compositions is componentwise.

For any formal power series $P = P(x_1,x_2,\dots) \in \mathbb{Q}[[x_1,x_2,\dots]]$ we let $LM(P)$ denote the leading monomial of P. That is, $LM(P)$ is the monomial of P with non-zero coefficient of smallest degree and largest in lexicographic order. In other words, the leading monomial for the order $\le_{lex}$. We let $LC(P)$ denote the coefficient of $LM(P)$ in P. Remark that for any two functions P and Q, we have $LM(PQ) = LM(P)LM(Q)$. We need the following result which is the extension for the G-functions of relation (2.1) for the F-functions.

Lemma 3.3. *Let $\tilde\alpha = b\,\tilde\rho$ be any generalized infinite composition and $b \ge 0$.*

$$G_{\tilde\alpha}(x_1,x_2,\dots) = \begin{cases} G_{\tilde\rho}(x_2,x_3,\dots) & \text{if } b=0,\\ x_1G_{(b-1)\,\tilde\rho} \;+\; R_{\tilde\alpha}(x_2,x_3,\dots) & \text{if } b>0. \end{cases} \tag{3.3}$$

Proof. We remark that the function $G_{\tilde\rho}(x_2,x_3,\dots)$ is the function $G_{\tilde\rho}(x_1,x_2,\dots)$ in which the variable x_i is replaced by x_{i+1}. We let $\tilde\alpha = \tilde\nu\,0\,0\cdots$ and proceed by induction on $\ell(\tilde\nu)$. If $\ell(\tilde\nu)=0$, then we have $G_{0\,0\dots}=1$ and (3.3) is valid. In the following we extensively use the relation (2.1). If $\tilde\alpha = b\,\rho\,0\,0\cdots$, then (3.1) gives us $G_{\tilde\alpha} = F_{b\,\rho}(x_1,x_2,\dots)$. If $b>1$, then we get

$$\begin{aligned} G_{b\,\rho\,0\dots} &= F_{b\,\rho}(x_1,x_2,\dots) \;=\; x_1F_{(b-1)\,\rho}(x_1,x_2,\dots) \;+\; F_{b\,\rho}(x_2,x_3,\dots)\\ &= x_1G_{(b-1)\,\rho\,0\dots} \;+\; R_{b\,\rho\,0\dots}(x_2,x_3,\dots), \end{aligned}$$

and (3.3) follows for this case with $R_{\tilde\alpha}(x_2,x_3,\dots) = F_{b\,\rho}(x_2,x_3,\dots)$.

For $b=1$, we first need to understand (3.3) in the case $G_{0\,\rho\,0\dots}$. For this assume that $\rho = a\beta$. If $a>1$, then the definitions (3.1) and (3.2) give

$$\begin{aligned} G_{0\,a\,\beta\,0\dots} &= G_{a\,\beta\,0\dots} - x_1G_{(a-1)\,\beta\,0\dots} \;=\; \big(F_{a\,\beta} - x_1F_{(a-1)\,\beta}\big)(x_1,x_2,\dots)\\ &= F_{a\,\beta}(x_2,x_3,\dots) = G_{a\,\beta\,0\dots}(x_2,x_3,\dots). \end{aligned}$$

If $a=1$, then we use the induction hypothesis on $\ell(0\,\beta) = \ell(0\,1\,\beta)-1 < \ell(\tilde\nu)$ to get

$$\begin{aligned} G_{0\,1\,\beta\,0\dots} &= G_{1\,\beta\,0\dots} - x_1G_{0\,\beta\,0\dots} \;=\; F_{1\,\beta}(x_1,x_2,\dots) - x_1F_{\beta}(x_2,x_3,\dots)\\ &= F_{1\,\beta}(x_2,x_3,\dots) = G_{1\,\beta\,0\dots}(x_2,x_3,\dots). \end{aligned}$$

Now we can go back to the case of $\tilde\alpha = 1\,\rho\,0\cdots$:

$$\begin{aligned} G_{1\,\rho\,0\dots} &= F_{1\,\rho}(x_1,x_2,\dots) \;=\; x_1F_{\rho}(x_2,x_3,\dots) \;+\; F_{1\,\rho}(x_2,x_3,\dots)\\ &= x_1G_{0\,\rho\,0\dots} \;+\; R_{1\,\rho\,0\dots}(x_2,x_3,\dots), \end{aligned}$$

where $R_{1\,\rho\,0\dots}(x_2,x_3,\dots) = F_{1\,\rho}(x_2,x_3,\dots)$.

We then consider when $\tilde\alpha = \tilde\gamma\,0\,a\,\beta\,0\,0\cdots$. This is the factorization needed to use (3.2) with $k = \ell(\tilde\gamma)+1$. If $\tilde\gamma$ is empty, then we have $b=0$ and we are in the case considered above. Assume that $\tilde\gamma = b\,\tilde\mu$. If $b=0$, then applying the induction hypothesis we have

$$\begin{aligned} G_{0\,\tilde\mu\,0\,a\,\beta\,0\cdots}(x_1,x_2,\dots) &= G_{0\,\tilde\mu\,a\,\beta\,0\cdots}(x_1,x_2,\dots) - x_k G_{0\,\tilde\mu\,(a-1)\,\beta\,0\cdots}(x_1,x_2,\dots) \\ &= G_{\tilde\mu\,a\,\beta\,0\cdots}(x_2,x_3,\dots) - x_{(k-1)+1} G_{\tilde\mu\,(a-1)\,\beta\,0\cdots}(x_2,x_3,\dots) \\ &= G_{\tilde\mu\,0\,a\,\beta\,0\cdots}(x_2,x_3,\dots). \end{aligned}$$

Here we remark that even though $\ell(\tilde\mu)+1 = k-1$, we have to replace x_{k-1} by $x_{(k-1)+1} = x_k$ in the recurrence for $G_{\tilde\mu\,0\,a\,\beta\,0\cdots}(x_2,x_3,\dots)$. For $b>0$,

$$\begin{aligned} G_{b\,\tilde\mu\,0\,a\,\beta\,0\cdots} &= G_{b\,\tilde\mu\,a\,\beta\,0\cdots} - x_k G_{b\,\tilde\mu\,(a-1)\,\beta\,0\cdots} \\ &= x_1 G_{(b-1)\,\tilde\mu\,a\,\beta\,0\cdots} + R_{b\,\tilde\mu\,a\,\beta\,0\cdots}(x_2,x_3,\dots) \\ &\quad - x_k\big(x_1 G_{(b-1)\,\tilde\mu\,(a-1)\,\beta\,0\cdots} + R_{b\,\tilde\mu\,(a-1)\,\beta\,0\cdots}(x_2,x_3,\dots)\big) \\ &= x_1\big(G_{(b-1)\,\tilde\mu\,a\,\beta\,0\cdots} - x_k G_{(b-1)\,\tilde\mu\,(a-1)\,\beta\,0\cdots}\big) \\ &\quad + R_{b\,\tilde\mu\,a\,\beta\,0\cdots}(x_2,x_3,\dots) - x_k R_{b\,\tilde\mu\,(a-1)\,\beta\,0\cdots}(x_2,x_3,\dots) \\ &= x_1 G_{(b-1)\,\tilde\mu\,0\,a\,\beta\,0\cdots} + R_{b\,\tilde\mu\,0\,a\,\beta\,0\cdots}(x_2,x_3,\dots), \end{aligned}$$

where $R_{b\,\tilde\mu\,0\,a\,\beta\,0\cdots}(x_2,x_3,\dots) = \big(R_{b\,\tilde\mu\,a\,\beta\,0\cdots} - x_k R_{b\,\tilde\mu\,(a-1)\,\beta\,0\cdots}\big)(x_2,x_3,\dots)$. ☐

Corollary 3.4. *For any generalized composition $\tilde\alpha$ we have $LM(G_{\tilde\alpha}) = X^{\tilde\alpha}$.*

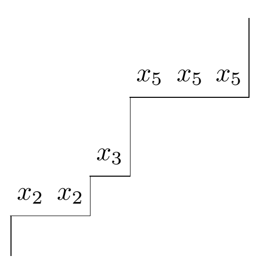
Proof. Let $\tilde\alpha = \tilde\nu\,0\,0\cdots$. We proceed by induction on $\ell(\tilde\nu)$ and the degree $d = d(\tilde\alpha) = \sum \tilde\alpha_i$. If $\ell(\tilde\nu) = 0$ we have $G_{0\,0\cdots} = 1 = X^{0\,0\cdots}$. If $\ell(\tilde\nu)\geq 1$, then let $\tilde\alpha = b\,\tilde\rho$ as in Lemma 3.3. If $b=0$, then the induction hypothesis on $\ell(\tilde\nu)$ gives $LM(G_{0\,\tilde\rho}) = LM\big(G_{\tilde\rho}(x_2,x_3,\dots)\big) = x_1^0 x_2^{\tilde\rho_1} x_3^{\tilde\rho_2}\cdots = X^{\tilde\alpha}$. Now if $b>0$ we use the second part of Lemma 3.3 and the induction hypothesis on d, and get $LM\big(x_1 G_{(b-1)\,\tilde\mu} + R_{\tilde\alpha}(x_2,x_3,\dots)\big) = x_1 LM(G_{(b-1)\,\tilde\mu}) = X^{\tilde\alpha}$. ☐

Remark 3.5. From the above corollary, by triangularity, it is clear that the set $\{G_{\tilde\alpha}\}$ for all $\tilde\alpha$ forms a Hilbert basis of $\mathbb{Q}[[x_1,x_2\dots]]$. We will see in Section 5 that in fact $\{G_{\tilde\alpha}\}_{\tilde\alpha\in\mathcal{G}^{(e)}}$ forms a Hilbert basis of $\overline{\mathcal{J}^{(e)}}$.

4. It is at most Catalan

Let $\mathcal{Q}^{(e)} = \{G_{\tilde\alpha}\}_{\tilde\alpha\in\mathcal{G}^{(e)}}$ be the generating set of $\mathcal{J}^{(e)}$ constructed in Lemma 3.2. In this section we show that after reduction, at most the monomials corresponding to Catalan paths form a Hilbert basis of $R = \mathbb{Q}[[x_1,x_2,\dots]]\big/\overline{\mathcal{J}}$. For this we reduce every other monomial to these. In fact for $R^{(e)} = \mathbb{Q}[[x_1,x_2,\dots]]\big/\overline{\mathcal{J}^{(e)}}$, we show that at most the monomials corresponding to paths above the line $y = x-e$ form a Hilbert basis of $R^{(e)}$, for all $e\geq 0$. We conclude this section with the corresponding result for finitely many variables, $R_n^{(e)} = \mathbb{Q}[x_1,x_2,\dots,x_n]\big/\mathcal{J}_n^{(e)}$, which is a generalization of Theorem 1.2.

Given any generalized infinite composition $\tilde\alpha$ we associate a unique path in the plane with steps going north or east. More precisely, for $\tilde\alpha = (\tilde\alpha_1,\tilde\alpha_2,\tilde\alpha_3,\dots)$, we construct the path $(0,0)\to(\tilde\alpha_1,0)\to(\tilde\alpha_1,1)\to(\tilde\alpha_1+\tilde\alpha_2,1)\to(\tilde\alpha_1+\tilde\alpha_2,2)\to(\tilde\alpha_1+\tilde\alpha_2+\tilde\alpha_3,2)\to\cdots$. For example, for $\tilde\alpha = 0\,2\,1\,0\,3\,0\,0\cdots$ we have the path



For every east step at height $i-1$ we associate a variable x_i. The product of all the variables associated to a path encoded by $\tilde{\alpha}$ is denoted $X^{\tilde{\alpha}}$. We now remark that for any factorization $\tilde{\alpha} = \tilde{\pi}\tilde{\rho}$, the rightmost coordinate of the path at height $\ell(\tilde{\pi}) - 1$ is $\big(d(\tilde{\pi}), \ell(\tilde{\pi}) - 1\big)$.

Definition 4.1. For an integer $e \geq 0$, we say that a generalized composition $\tilde{\alpha}$ is of type e-Catalan if its associated path remains above the line $y = x - e$. That is, every coordinate (x_i, y_i) of the path is such that $x_i - y_i \leq e$.

Lemma 4.2. *The monomials of* $\mathbb{Q}[[x_1, x_2, \dots]]$ *corresponding to paths remaining above the line* $y = x - e$ *contain a Hilbert basis of the quotient* $R^{(e)}$.

Proof. Let $X^{\tilde{\alpha}}$ be any monomial of degree d. If the path corresponding to $\tilde{\alpha}$ goes under the line $y = x - e$, then let $\tilde{\alpha} = \tilde{\pi}\tilde{\rho}$ be any factorization such that the coordinate $\big(d(\tilde{\pi}), \ell(\tilde{\pi}) - 1\big)$ is under the line $y = x - e$. That is, $d(\tilde{\pi}) - \ell(\tilde{\pi}) \geq e$. From Lemma 3.1 we conclude that the function $G_{\tilde{\alpha}}$ with leading monomial $X^{\tilde{\alpha}}$ is in $\mathcal{J}^{(e)}$. This monomial can thus be replaced by monomials of degree d but strictly smaller with respect to $<_{lex}$. Repeating this step (possibly countably many times) with the next largest monomial going under the line $y = x - e$, any monomial $X^{\tilde{\alpha}}$ can be reduced modulo the ideal $\overline{\mathcal{J}^{(e)}}$ to a series containing only monomials $X^{\tilde{\beta}}$ where $\tilde{\beta}$ is of type e-Catalan. □

We are now in a position to generalize Theorem 1.2 and prove it. For this, note that the quasi-symmetric polynomials in n variables are defined by setting $0 = x_{n+1} = x_{n+2} = \cdots$ in the quasi-symmetric functions. That is, $F_\alpha(x_1, x_2, \dots, x_n) = F_\alpha(x_1, x_2, \dots, x_n, 0, 0, \dots)$. We then define $J_n^{(e)} = \langle F_\alpha(x_1, x_2, \dots, x_n) : \alpha$ *reaches level* $e\rangle$ and $R_n^{(e)} = \mathbb{Q}[x_1, x_2, \dots, x_n]/J_n^{(e)}$. Similarly we set $G_{\tilde{\alpha}}(x_1, x_2, \dots, x_n) = G_{\tilde{\alpha}}(x_1, x_2, \dots, x_n, 0, 0, \dots)$. It is clear that Lemma 3.1 holds for $J_n^{(e)}$ in the same way. More over if $\tilde{\alpha} = \tilde{\nu}\, 0\, 0 \cdots$ for $\ell(\tilde{\nu}) = n$, then

$$LM(G_{\tilde{\alpha}}(x_1, x_2, \dots, x_n)) = LM(G_{\tilde{\alpha}}(x_1, x_2, \dots, x_n, 0, 0, \dots)) = x_1^{\tilde{\nu}_1} x_2^{\tilde{\nu}_2} \cdots x_n^{\tilde{\nu}_n}.$$

Let $C_n^{(e)}$ denote the number of generalized compositions $\tilde{\alpha}$ of type e-Catalan such that $\tilde{\alpha} = \tilde{\nu}\, 0\, 0 \cdots$ and $\ell(\tilde{\nu}) = n$. These are in bijection with the paths from $(0,0)$ to $(n+e, n)$ that remain above the line $y = x - e$. Indeed, if we have a path of type e-Catalan, it suffices to add a horizontal line from $(\tilde{\nu}_1 + \tilde{\nu}_2 + \cdots + \tilde{\nu}_n, n)$ to $(n+e, n)$. When $e = 0$, we have $C_n^{(0)} = C_n$, the nth Catalan number. This enumerates the classical Dyck paths from $(0,0)$ to (n,n) remaining above the line $y = x$. See [16] for an extensive account on Catalan numbers. We have the following generalization to our Theorem 1.2.

Corollary 4.3. $\dim(R_n^{(e)}) \leq C_n^{(e)}$.

Proof. We use the same argument as in Lemma 4.2. For this we use the fact that for any monomial $x_1^{\tilde{\nu}_1} x_2^{\tilde{\nu}_2} \cdots x_n^{\tilde{\nu}_n}$ in $\mathbb{Q}[x_1, x_2, \dots, x_n]$, if $\tilde{\nu}$ reaches level e, then $G_{\tilde{\nu}\, 0 \cdots}(x_1, x_2, \dots, x_n) \in \mathcal{J}_n^{(e)}$. Hence a basis of $R_n^{(e)}$ is contained in the monomials corresponding to paths of type e-Catalan and our result follows. □

Again the equality holds and we address this question in [2].

5. It is a Hilbert basis

In the previous section, the generating set $\mathcal{Q}^{(e)} = \{G_{\tilde{\alpha}}\}_{\tilde{\alpha}\in\mathcal{G}^{(e)}}$ of $\mathcal{J}^{(e)}$ was used to reduce every monomial to e-Catalan type generalized compositions. It is in fact a Hilbert basis for the given ideal. We use here ideas of Gröbner basis theory on valuation rings. This is crucial to complete the proof of Theorem 1.1. Let us recall a few basic facts about Gröbner bases; see [8] for more details.

To show that a set S is a Gröbner basis it is enough to show that all polynomial syzygies of that set are reducible in S. The polynomial syzygy of P and Q is defined by $S(P,Q) = LC(Q)M_1P - LC(P)M_2Q$ where $\mathrm{lcm}\big(LM(P), LM(Q)\big) = M_1\cdot LM(P) = M_2\cdot LM(Q)$. This shows that the given set contains all the generators of the leading monomials of the ideal. We use the classic Buchberger's lemma [8]:

Lemma 5.1. *Assume $P, Q \in S$. If there is an $R \in S$ such that $LM(R)$ divides $lcm\big(LM(P), LM(Q)\big)$, and if both $S(R,Q)$ and $S(R,P)$ are reducible in S, then $S(P,Q)$ is reducible in S.*

This result is easily adapted to our context. We first remark that our sets $\mathcal{Q}^{(e)}$ satisfy the following useful property:

Lemma 5.2. *For all $\tilde{\alpha} \le \tilde{\rho}$ (componentwise),*

$$G_{\tilde{\alpha}} \in \mathcal{Q}^{(e)} \qquad \Longrightarrow \qquad G_{\tilde{\rho}} \in \mathcal{Q}^{(e)}.$$

Proof. If $\tilde{\alpha} = \tilde{\pi}\tilde{\nu}$ satisfies $d(\tilde{\pi}) - \ell(\tilde{\pi}) \ge e$, then let $r = \ell(\tilde{\pi})$ and consider $\tilde{\rho} = \tilde{\gamma}\tilde{\mu}$ where $\ell(\tilde{\gamma}) = r$. Since $d(\tilde{\pi}) \le d(\tilde{\gamma})$, we have $d(\tilde{\gamma}) - \ell(\tilde{\gamma}) \ge d(\tilde{\pi}) - \ell(\tilde{\pi}) \ge e$. By Lemma 3.1, $G_{\tilde{\rho}} \in \mathcal{Q}^{(e)}$. ∎

We can now adapt the proof (see [8]) of Lemma 5.1 to our situation. For any pair $\tilde{\alpha}, \tilde{\pi} \in \mathcal{G}^{(e)}$, we define $S(G_{\tilde{\alpha}}, G_{\tilde{\pi}})$ as before with our definition of LM and LC. We show in this section that any such $S(G_{\tilde{\alpha}}, G_{\tilde{\pi}})$ is reducible in $\mathcal{Q}^{(e)}$. Let $\tilde{\rho} \in \mathcal{G}^{(e)}$ be the unique element such that $X^{\tilde{\rho}} = \mathrm{lcm}\big(X^{\tilde{\alpha}}, X^{\tilde{\pi}}\big) = M_1X^{\tilde{\alpha}} = M_2X^{\tilde{\alpha}}$. We have

$$\begin{aligned} S(G_{\tilde{\alpha}}, G_{\tilde{\pi}}) &= M_1G_{\tilde{\alpha}} - M_2G_{\tilde{\pi}} \;=\; M_1G_{\tilde{\alpha}} - G_{\tilde{\rho}} + G_{\tilde{\rho}} - M_2G_{\tilde{\pi}} \\ &= S(G_{\tilde{\alpha}}, G_{\tilde{\rho}}) + S(G_{\tilde{\rho}}, G_{\tilde{\pi}}). \end{aligned} \tag{5.1}$$

If both $S(G_{\tilde{\alpha}}, G_{\tilde{\rho}})$ and $S(G_{\tilde{\rho}}, G_{\tilde{\pi}})$ are reducible in $\mathcal{Q}^{(e)}$, then so is $S(G_{\tilde{\alpha}}, G_{\tilde{\pi}})$. It is sufficient to show that all $S(G_{\tilde{\alpha}}, G_{\tilde{\rho}})$ are reducible in $\mathcal{Q}^{(e)}$ for $\tilde{\alpha} \le \tilde{\rho}$ componentwise.

We can further reduce our problem as follows. Assume that $\tilde{\alpha}$ and $\tilde{\rho}$ in $\mathcal{G}^{(e)}$ are generalized compositions of d_1 and d_2 respectively. If $\tilde{\alpha} \le \tilde{\rho}$, then $d_1 \le d_1$. If $d_2 - d_1 > 1$, we can select a generalized composition $\tilde{\alpha} \le \tilde{\pi} \le \tilde{\rho}$ and use (5.1) again. We can thus assume that $d_2 - d_1 = 1$. That is, the two generalized compositions differ on one part by one unit.

Lemma 5.3. *The set $\mathcal{Q}^{(e)}$ is a Gröbner basis of $\mathcal{J}^{(e)}$.*

Proof. From the discussion above it is sufficient to show that all the expressions of $\mathcal{J}^{(e)}$ of the form $S(G_{\tilde{\gamma}\,a\,\tilde{\beta}}, G_{\tilde{\gamma}\,(a-1)\,\tilde{\beta}}) = G_{\tilde{\gamma}\,a\,\tilde{\beta}} - x_kG_{\tilde{\gamma}\,(a-1)\,\tilde{\beta}}$ where $k = \ell(\tilde{\gamma}) + 1$, are reducible in $\mathcal{Q}^{(e)}$. Let us denote by $m_{\tilde{\gamma}\,a\,\tilde{\beta}}(x_1, x_2, \dots)$ the leading monomial of $S(G_{\tilde{\gamma}\,a\,\tilde{\beta}}, G_{\tilde{\gamma}\,(a-1)\,\tilde{\beta}})$.

Let $\tilde{\beta} = \tilde{\nu}\,0\,0\cdots$. We set up an induction on $\ell(\tilde{\nu})$. Assume first that $\tilde{\nu} = \nu$ is a (possibly empty) standard composition. The second part of Lemma 3.1 and the recursive definition (3.2) give $S(G_{\tilde{\gamma}\,a\,\nu\,0\cdots}, G_{\tilde{\gamma}\,(a-1)\,\nu\,0\cdots}) = G_{\tilde{\gamma}\,0\,a\,\nu\,0\cdots} \in \mathcal{Q}^{(e)}$. If

$\tilde\nu$ is not standard, then let $\tilde\beta = \tilde\pi\, 0\, b\, \mu\, 0\, 0 \cdots$ for $b > 0$ and μ a (possibly empty) standard composition. Let $\ell = \ell(\tilde\gamma\, a\, \tilde\pi) + 1$. Using (3.2), we have

$$\begin{aligned} S(G_{\tilde\gamma\, a\, \tilde\pi\, 0\, b\mu 0\cdots}, G_{\tilde\gamma (a-1)\tilde\pi\, 0\, b\mu 0\cdots}) &= G_{\tilde\gamma\, a\, \tilde\pi\, 0\, b\, \mu\, 0\cdots} - x_k G_{\tilde\gamma\, (a-1)\, \tilde\pi\, 0\, b\, \mu\, 0\cdots} \\ &= G_{\tilde\gamma\, a\, \tilde\pi\, b\, \mu\, 0\cdots} - x_\ell G_{\tilde\gamma\, a\, \tilde\pi\, (b-1)\, \mu\, 0\cdots} \\ &\quad - x_k \big(G_{\tilde\gamma (a-1) \tilde\pi\, b\, \mu 0\cdots} - x_\ell G_{\tilde\gamma (a-1)\tilde\pi (b-1)\mu 0\cdots} \big) \\ &= S(G_{\tilde\gamma\, a\, \tilde\pi\, b\, \mu\, 0\cdots}, G_{\tilde\gamma\, (a-1)\, \tilde\pi\, b\, \mu\, 0\cdots}) \\ &\quad - x_\ell\, S(G_{\tilde\gamma\, a\, \tilde\pi (b-1)\mu\, 0\cdots}, G_{\tilde\gamma (a-1)\tilde\pi (b-1)\mu\, 0\cdots}). \end{aligned}$$

We observe that $m_{\tilde\gamma\, a\, \tilde\pi\, 0\, b\mu 0\cdots}(x_1, x_2, \dots) = x_\ell\, m_{\tilde\gamma\, a\, \tilde\pi (b-1)\mu\, 0\cdots}(x_1, x_2, \dots)$ since $m_{\tilde\gamma\, a\, \tilde\pi\, b\, \mu\, 0\cdots}(x_1, x_2, \dots)$ is distinct and lexicographically smaller than $x_\ell\, m_{\tilde\gamma\, a\, \tilde\pi (b-1)\mu\, 0\cdots}(x_1, x_2, \dots)$. Thus $S(G_{\tilde\gamma\, a\, \tilde\pi\, 0\, b\mu 0\cdots}, G_{\tilde\gamma (a-1)\tilde\pi\, 0\, b\mu 0\cdots})$ is resolved by two expressions such that $\ell(\tilde\pi\, b\, \mu) = \ell(\tilde\pi (b-1)\mu) < \ell(\tilde\nu)$ and by the induction hypothesis $S(G_{\tilde\gamma\, a\, \tilde\pi\, 0\, b\mu 0\cdots}, G_{\tilde\gamma (a-1)\tilde\pi\, 0\, b\mu 0\cdots})$ is reducible in $\mathcal{Q}^{(e)}$. □

Corollary 5.4. *The set $\mathcal{Q}^{(e)}$ is a Hilbert basis of $\overline{\mathcal{J}^{(e)}}$.*

Proof. Given an element $P \in \overline{\mathcal{J}^{(e)}}$ let $X^{\tilde\beta} = LM(P)$. Since $\mathcal{Q}^{(e)}$ is a Gröbner basis it contains an element $G_{\tilde\alpha}$ such that $\tilde\alpha \le \tilde\beta$ componentwise. Lemma 5.2 gives us that $G_{\tilde\beta} \in \mathcal{Q}^{(e)}$. The element $P - LC(P) \cdot G_{\tilde\alpha} \in \overline{\mathcal{J}^{(e)}}$ is such that $LM(P - LC(P) \cdot G_{\tilde\alpha}) <_{lex} X^{\tilde\beta}$. If we repeat this process (possibly countably many times) we can express P as a series in the elements of $\mathcal{Q}^{(e)}$. □

We are now in a position to conclude our investigation and prove the general version of our Theorem 1.1.

Corollary 5.5. *The monomial Hilbert basis of $R^{(e)}$ is given by the monomials of $\mathbb{Q}[[x_1, x_2, x_3, \dots]]$ corresponding to the paths of type e-Catalan.*

Proof. As noted in Remark 3.5 above, the set $\{G_{\tilde\alpha}\}$ forms a Hilbert basis of $\mathbb{Q}[[x_1, x_2, x_3, \dots]]$, and Corollary 5.4 gives that the set $\mathcal{Q}^{(e)}$ is a Hilbert basis of $\overline{\mathcal{J}^{(e)}}$. Thus a Hilbert basis of the quotient $R^{(e)} = \mathbb{Q}[[x_1, x_2, x_3, \dots]] \big/ \overline{\mathcal{J}^{(e)}}$ is given by the set $\{G_{\tilde\alpha}\} \setminus \mathcal{Q}^{(e)} = \big\{ G_{\tilde\alpha} \;\big|\; \tilde\alpha \textit{ corresponds to a path of type e-Catalan} \big\}$. The result follows by triangularity. □

Remark 5.6. To show the equality in Theorem 1.2, it appears that the set $\{G_{\tilde\alpha} \mid \ell(\tilde\alpha) = n, \text{and } \tilde\alpha \text{ reaches level } e\}$ forms a linear basis of $J_n^{(e)}$. Unfortunately the argument of Section 5 is not sufficient to show this with finitely many variables and it requires more work. This is the object of our collaboration in [2].

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