1 Derived Parameters

Student number: z5059915
$h_{max1} = 0.2(9+9+3) = 4.2$ m
$h_{max2} = 0.1(1+5+1) = 0.7$ m
$b = 10(10-5) = 50$ Vs
$d = 0.01(5+0+1) = 0.06$ m

2 Pre-task

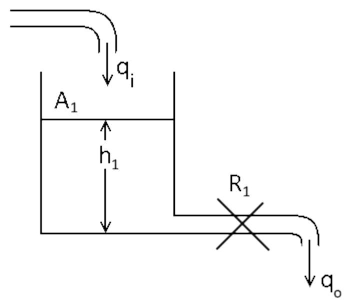

Figure 1: Diagram of single-tank system

The flow equations described by this system are:

$$q_i - q_o = A_1\dot{h_1} \tag{1}$$

$$q_o = \frac{h_1}{R_1} \tag{2}$$

Substituting (2) into (1), the following equation can be found:

$$\begin{aligned} q_i - \frac{h_1}{R_1} &= A_1\dot{h_1} \\ \rightarrow q_i &= A_1\dot{h_1} + \frac{h_1}{R_1} \end{aligned} \tag{3}$$

Taking the Laplace Transform of Eq. (3) and assuming initial conditions are zero, we have:

$$\begin{aligned} \mathcal{L}\{q_i\} &= \mathcal{L}\{A_1\dot{h_1}\} + \mathcal{L}\{\frac{h_1}{R_1}\} \\ \rightarrow Q_i(s) &= A_1 s H_1(s) + \frac{1}{R_1}H_1(s) \\ \rightarrow Q_i(s) &= H_1(s)[A_1 s + \frac{1}{R_1}] \\ \therefore G_1(s) &= \frac{H_1(s)}{Q_i(s)} = \frac{R_1}{A_1 R_1 s + 1} \end{aligned} \tag{4}$$

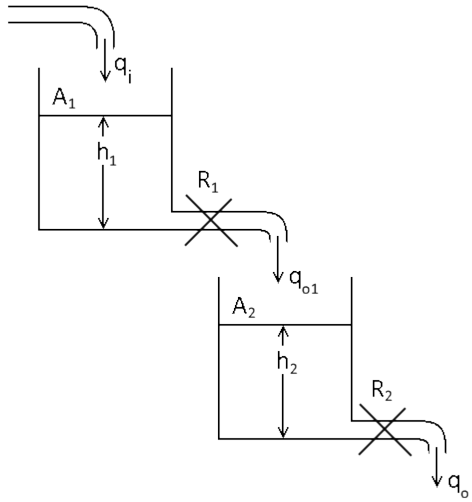

Figure 2: Diagram of two-tank cascade system

$$q_i - q_{o1} = A_1\dot{h_1} \tag{5}$$

$$q_{o1} = \frac{h_1}{R_1} \tag{6}$$

$$q_o = \frac{h_2}{R_2} \tag{7}$$

$$q_{o1} - q_o = A_2\dot{h_2} \tag{8}$$

Substituting Eq. (6) and (7) into (8), rearranging for h_1 we get:

$$h_1 = A_2R_1\dot{h_2} + \frac{R_1}{R_2}h_2 \tag{9}$$

Differentiating Eq. (9) we then have:

$$\dot{h_1} = A_2R_1\ddot{h_2} + \frac{R_1}{R_2}\dot{h_2} \tag{10}$$

Substituting Eq. (6) into (5) and using results (9) and (10):

$$q_i - A_2\dot{h_2} + \frac{1}{R_2}h_2 = A_1A_2R_1\ddot{h_2} + \frac{A_1R_1}{R_2}\dot{h_2} \tag{11}$$

Rearranging Eq. (11) and taking the Laplace Transform:

$$\begin{gathered} Q_i(s) = A_1A_2R_1s^2H_2(s) + [\frac{A_1R_1}{R_2} + A_2]sH_2(s) + \frac{1}{R_2}H_2(s) \\ \rightarrow Q_i(s) = H_2(s)[A_1A_2R_1s^2 + [\frac{A_1R_1}{R_2} + A_2]s + \frac{1}{R_2}] \end{gathered} \tag{12}$$

Rearranging to find the transfer function:

$$\begin{gathered} \frac{H_2(s)}{Q_i(s)} = \frac{R_2}{A_1A_2R_1R_2s^2 + (A_1R_1 + A_2R_2)s + 1} \\ \rightarrow G_{2C}(s) = \frac{H_2(s)}{Q_i(s)} = \frac{R_2}{(A_1R_1s+1)(A_2R_2s+1)} \end{gathered} \tag{13}$$

3 Tasks

3.1 Task 1

To determine R_1, the steady-state value of the system needs to be approximated, from Appendix A, it can be approximated to be $h_{1,SteadyState} \approxeq 0.082154$ m, since the graph does not start at $h_1 = 0$ cm but approximately $h_1 = -0.2154$ cm, this offset needs to be applied since the Laplace transform was taken assuming $h_1 = 0$ at $t = 0$. Then the final value theorem can be applied as below:

$$\lim_{t\to\infty}[h_1(t)] = \lim_{s\to 0}[sH_1(s)] \tag{14}$$

The input to the system is a step-input of magnitude $3V$ and it is known that $q_i = \eta V$, with $\eta = 2.4$ x $10^{-6} m^3/Vs$, therefore $Q_i(s) = 7.2$ x $10^{-6}/s$. So we find that:

$$H_1(s) = \frac{7.2 \text{ x } 10^{-6} R_1}{s(A_1 R_1 s + 1)} \tag{15}$$

Using the above result, and applying it to the limit equation:

$$\begin{gathered} 0.082154 = \lim_{s\to 0}[\frac{7.2 \text{ x } 10^{-6} R_1}{A_1 R_1 s + 1}] = 7.2 \text{ x } 10^{-6} R_1 \\ \boxed{\therefore R_1 = \frac{0.082154}{7.2 \text{ x } 10^{-6}} = 11410.28 \text{ s/m}^2} \end{gathered} \tag{16}$$

To determine A_1, a value must be taken directly from the graph in Appendix A, exponential smoothing was used on the graphs to reduce the noise and improve the accuracy of the result, however there will be unavoidable error in the result, thus it is only an approximation for A_1. The value chosen was at time $t = \tau$, where τ is the time constant of the system, in this case $\tau = A_1 R_1$. At this time, the water level $h_1 \approxeq 0.63$ x $\lim_{t\to\infty}[h_1(t)] = 0.63$ x $0.082154 = 0.05176$ m. It was found from the graph that the closest value to this (0.051759...) was at time $t = 66.3$ s. So we have:

$$\boxed{\tau = A_1 R_1 \approxeq 66.3 \to A_1 \approxeq \frac{66.3}{11410.28} = 0.00581 \text{ m}^2} \tag{17}$$

3.2 Task 2

Applying the same process of using the final value theorem from 3.1, R_2 can be found, the steady-state value of the water level of tank 2 can be approximated as $h_{2,SteadyState} \approxeq 0.263 - 0.01712 = 0.24588$ m, since the height of tank 2, $h_2 \approxeq 1.712$ cm at $t = 0$ s, this was done for the same reason mentioned in Task 1. Using both the transfer function $G_{2C}(s)$, and the result for $Q_i(s)$ found above, and applying the final value theorem, R_2 was found:

$$H_2(s) = \frac{7.2 \text{ x } 10^{-6} R_2}{s(A_1 R_1 s + 1)(A_2 R_2 s + 1)} \tag{18}$$

$$\lim_{t\to\infty}[h_2(t)] = \lim_{s\to 0}[sH_2(s)] \tag{19}$$

$$\to 0.24588 = \lim_{s\to 0}[\frac{7.2 \text{ x } 10^{-6} R_2}{(A_1 R_1 s + 1)(A_2 R_2 s + 1)}] = 7.2 \text{ x } 10^{-6} R_2 \tag{20}$$

$$\boxed{\therefore R_2 \approxeq \frac{0.24588}{7.2 \text{ x } 10^{-6}} = 34150 \text{ s/m}^2} \tag{21}$$

In order to find a good approximation for A_2, points had to be chosen from the graph and substituted into the time-domain solution for $H_2(s)$. To find the time-domain solution, the inverse Laplace transform was taken of $H_2(s)$. Every point of the graph was then sampled and all positive results for A_2 were taken and averaged (in order to reduce the error margin), this was all done using MATLAB (refer to Appendix for code). It was found that:

$$\boxed{A_2 \approxeq 0.001597 \text{ m}^2} \tag{22}$$

3.3 Task 3

From the block diagram of Figure 3 in the MMAN3200 Lab Assignment document, the feedback loop transfer function can be found to be (using block diagram rule for a feedback loop):

$$\frac{H_{2,FB}(s)}{H_{2d,FB}(s)} = \frac{K_{FB}G_{2C}(s)}{1 + K_{FB}G_{2C}(s)} \tag{23}$$

Substituting all known parameters into Eq. (13):

$$G_{2C}(s) = \frac{34150}{(66.29s + 1)(54.538s + 1)} \tag{24}$$

Applying this result to Eq. (18), with $H_{2d,FB}(s)$ being a step-input of magnitude 0.16m:

$$H_{2,FB}(s) = \frac{0.16 * 34150K_{FB}}{s * ((66.29s + 1)(54.538s + 1) + 34150K_{FB})} \tag{25}$$

The steady-state value of the $h_{2,FB}(t)$ graph (Appendix D), can be approximated to be 0.135m, since this is a feedback system, the offset of $h_{2,FB}$ at $t = 0$ is already considered by the sensor, so no offset needs to be applied. Applying the final value theorem, the value for K_{FB} can be found:

$$\lim_{t\to\infty}[h_{2,FB}(t)] = \lim_{s\to 0}[sH_{2,FB}(s)] \tag{26}$$

$$\begin{gathered} 0.135 = \lim_{s\to 0}[\frac{0.16 * 34150K_{FB}}{(66.29s + 1)(54.538s + 1) + 34150K_{FB}}] = \frac{0.16 * 34150K_{FB}}{1 + 34150K_{FB}} \\ \rightarrow 0.135 + 4610.25K_{FB} = 5464K_{FB} \\ \boxed{\therefore K_{FB} = 158.126 \text{ x } 10^{-6} \text{ m}^2\text{/s}} \end{gathered} \tag{27}$$

The steady-state error of the closed-loop system can be found by the following:

$$\text{Steady-state error } = \frac{|h_{SS,theoretical} - h_{SS,experimental}|}{h_{SS,theoretical}} * 100\% \tag{28}$$

Where $h_{SS,theoretical} = h_{d,FB} = 0.16$ and $h_{SS,experimental} \approxeq 0.135$ from the graph in Appendix D:

$$\boxed{\text{Steady-state error } = \frac{|0.16 - 0.135|}{0.16} * 100\% = 15.625\%} \tag{29}$$

3.4 Task 4

Using the previously determined value for R_1, the valve characteristic K' can be found. By realising that:

$$q_o = K'[(p_a + \rho g h) - p_a] = K'\rho g h \tag{30}$$

a relationship between R_1 and K' can be resolved. Applying Eq. (2) to (17) we obtain:

$$K'\rho g h = \frac{h_1}{R_1} \tag{31}$$

In the above case, $h \equiv h_1$, the density of water is known to be approximately 1000 kg/m^3, and gravity, g is approximately 9.81 m/s^2:

$$\boxed{\therefore K' = \frac{1}{\rho g R_1} = \frac{1}{1000 \text{ x } 9.81 \text{ x } 11410.28} = 8.934 \text{ x } 10^{-9} \text{ m}^4\text{s/kg}} \tag{32}$$

3.5 Task 5

a) From Figure 4 in the Lab Assignment document, the following relationships were derived (note that q_{o1} is the flow from Tank 1 to Tank 2):

$$q_i - q_{o1} = A_1\dot{h_1} \tag{33}$$

$$q_{o1} = \frac{h_1 - h_2}{R_1} \tag{34}$$

$$q_o = \frac{h_2}{R_2} \tag{35}$$

$$q_{o1} - q_o = A_2\dot{h_2} \tag{36}$$

Substituting (35) into (36):

$$q_{o1} - \frac{h_2}{R_2} = A_2\dot{h_2} \rightarrow q_{o1} = A_2\dot{h_2} + \frac{h_2}{R_2} \tag{37}$$

From Eq. (34), Eq. (37) can be rewritten as:

$$\frac{h_1 - h_2}{R_1} = A_2\dot{h_2} + \frac{h_2}{R_2} \tag{38}$$

Rearranging Eq. (38) for h_1:

$$h_1 = A_2R_1\dot{h_2} + (\frac{R_1}{R_2} + 1)h_2 \tag{39}$$

Differentiating to find $\dot{h_1}$:

$$\dot{h_1} = A_2R_1\ddot{h_2} + (\frac{R_1}{R_2} + 1)\dot{h_2} \tag{40}$$

Substituting Eq. (39) into (34) and simplifying we get:

$$q_{o1} = A_2\dot{h_2} + \frac{1}{R_2}h_2 \tag{41}$$

Now, applying Eq. (41) and (40) to Eq. (33) and simplifying gives the following:

$$q_i = A_1A_2R_1\ddot{h_2} + [\frac{A_1R1}{R_2} + A_1 + A_2]\dot{h_2} + \frac{1}{R_2}h_2 \tag{42}$$

Taking the Laplace Transform of the above equation, assuming zero initial conditions and rearranging:

$$\boxed{G_o(s) = \frac{H_2(s)}{Q_i(s)} = \frac{R_2}{A_1A_2R_1R_2s^2 + [A_1R_1 + A_1R_2 + A_2R_2]s + 1}} \tag{43}$$

Substituting known parameters into the above equation:

$$\boxed{G_o(s) = \frac{34150}{3615.497s^2 + 319.243s + 1}} \tag{44}$$

b) Applying a unity negative feedback to (44) as seen in Figure 5 of the MMAN3200 Lab document, the root locus can be plotted using the rlocus() function of MATLAB as:

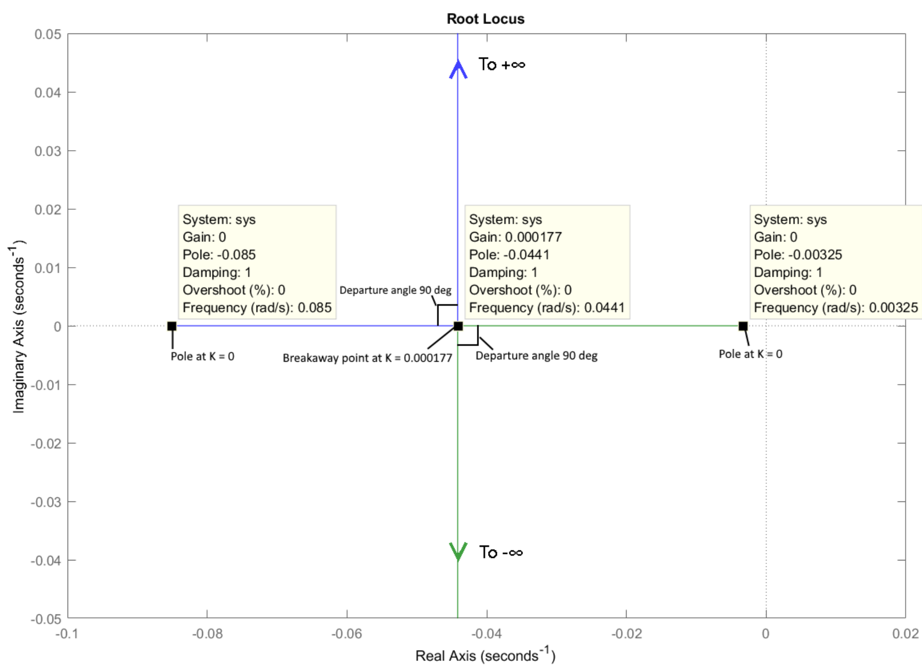

Figure 3: Root locus plot of $G_o(s)$

c) At the breakaway point, which can be found by determining $\frac{d(-K)}{ds} = 0 = 1 + KG_o(s)$ as below:

$$KG_o(s) = -1 \tag{45}$$

$$\rightarrow \frac{34150K}{3615.497s^2 + 319.243s + 1} = -1 \tag{46}$$

$$\therefore K = -\frac{3615.497}{34150}s^2 - \frac{319.243}{34150}s - \frac{1}{34150} \tag{47}$$

$$\frac{d(-K)}{ds} = \frac{3615.497}{17075}s + \frac{319.243}{34150} = 0 \tag{48}$$

$$\rightarrow 3615.497s = -159.6215 \tag{49}$$

$$\therefore s = -0.04415 \tag{50}$$

So it can be seen that the breakaway point is at s = -0.04415, and by the characteristic equation $1 + KG_o(s) = 0$, we find that the gain $K = 0.000177$ at this point. So the optimal value of K for this application is:

$$\boxed{K = 0.000177}$$

d) The requirements of the performance of the system will directly influence the choice of K, if faster performance is required, choosing the breakaway value of K is optimal. If time is not a concern but obtaining the exact desired value is paramount, than K = 0 is an optimal choice, it is the only value for which there is no error. The K value that would give a decent trade-off between response time and steady-state error would be at the breakaway point of the root locus plot (that is, K = 0.000177) as this is the point at which the system is critically damped (hence the fast response time), and the system would exponentially stable, meaning it does not overshoot the desired value. The higher gain compared to K = 0 means there will be a greater steady-state error however since $SSE = \frac{h_d}{1+KR_2}$, but it has a good trade-off between this and speed, making it an optimal choice.

3.6 Task 6

To obtain a desired height of $d = 0.06$ m as per the derived parameters, the final value theorem must once again be taken of Eq. (13):

$$\lim_{t\to\infty}[h_2(t)] = \lim_{s\to 0}[\frac{\eta V R_2}{(A_1R_1s+1)(A_2R_2s+1)}] \tag{51}$$

$$\to 0.06 = \eta V R_2 \tag{52}$$

$$\therefore V = 0.7321V \tag{53}$$

So the magnitude of the step-input required to obtain a desired height of 0.06m is:

$$\boxed{\eta V = 0.7321 \text{ x } 2.4 \text{ x } 10^{-6} = 1.757 \text{ x } 10^{-6}\ m^3/s} \tag{54}$$

3.7 Task 7

It was found that the system described in Task 6 above settled towards its steady-state value at approximately $t = 500$ s, and so this was the time chosen to apply the impulse of 50 Vs. The affect this impulse has on the system can be seen in Appendix E, where at t = 500 s there is a 'bump' that decays back to the original steady-state value of the system. From the determined parameters, b = 50Vs, so the magnitude of the impulse is 50 x 2.4 x 10^{-6} = 0.00012 m^3, since this impulse is applied at t = 500 s, we have:

$$q_i(t) = 0.00012\delta(t-500)u(t-500) \tag{55}$$

$$\therefore Q_{i,impulse}(s) = 0.00012e^{-500s} \tag{56}$$

From Task 6, the magnitude of the step-input was determined to be 1.757 x 10^{-6}, so we have:

$$Q_{i,step}(s) = \frac{1.757 \text{ x } 10^{-6}}{s} \tag{57}$$

Finding the transfer functions of the step-input and impulse separately:

$$H_{2,step}(s) = \frac{1.757 \text{ x } 10^{-6}R_2}{s(A_1R_1s+1)(A_2R_2s+1))} = \frac{0.06}{s(66.3s+1)(54.538s+1)} \tag{58}$$

$$H_{2,impulse}(s) = \frac{0.00012e^{-500s}R_2}{(A_1R_1s+1)(A_2R_2s+1))} = \frac{4.098e^{-500s}}{(66.3s+1)(54.538s+1)} \tag{59}$$

Now by adding $H_{2,step}(s)$ and $H_{2,impulse}(s)$ together to make a single function (since Laplace Transform is a linear function this is possible) and taking the inverse Laplace using MATLAB (ilaplace) to determine $h_2(t)$, the plot in Appendix E was obtained.

Note that the water level at steady state + the highest point of the impulse does not exceed $h_{1,max}$ or $h_{2,max}$ and so nothing more needs to be considered.

4 Appendix

Note: Exponential smoothing was applied to the graphs to reduce noise and improve accuracy of results taken from the graph.

4.1 Appendix A

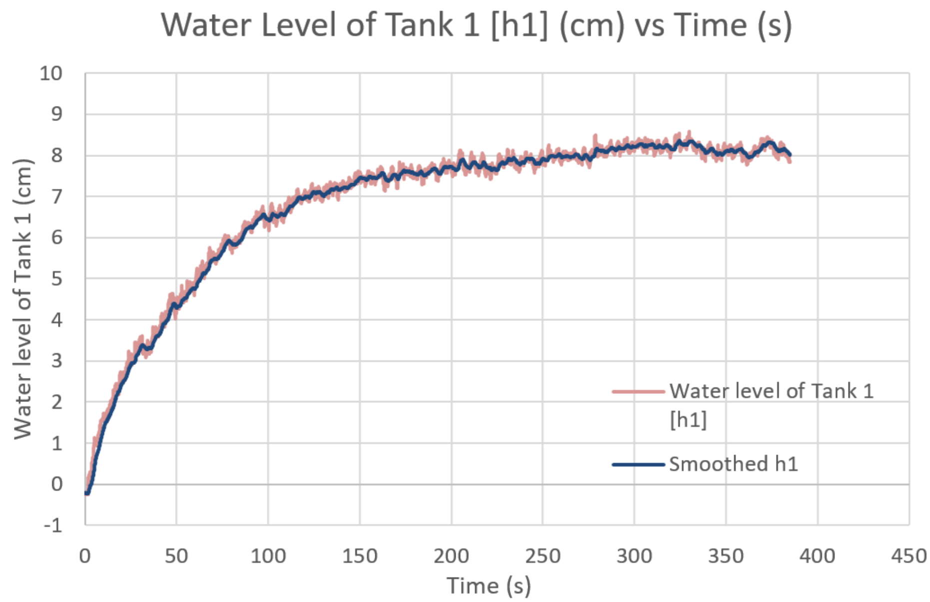

Figure 4: Water level of tank 1 over time

4.2 Appendix B

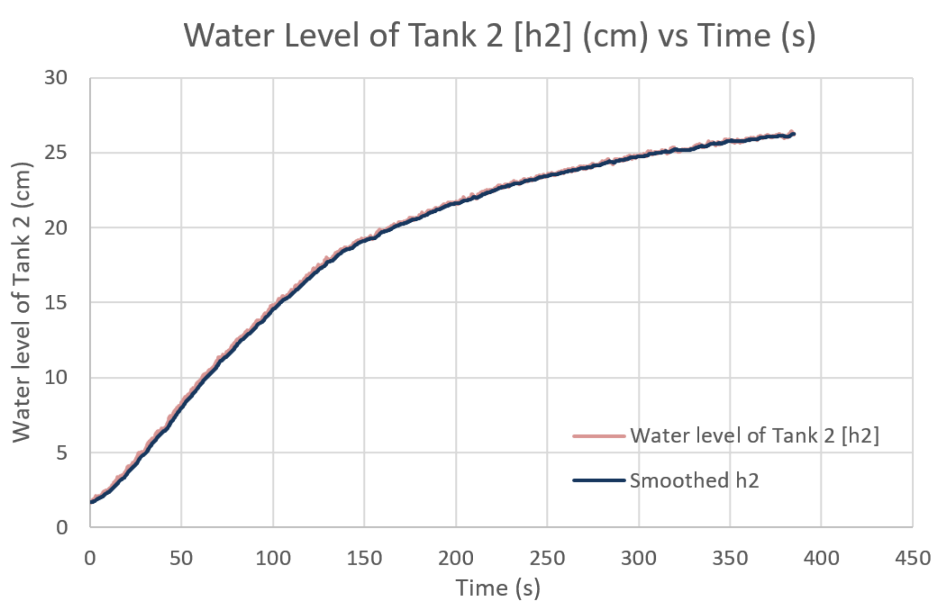

Figure 5: Water level of tank 2 over time

4.3 Appendix C

```
[NUM,TXT,RAW]=xlsread('DataSheet_2T_20170424_Group1200.xls');
time = NUM(:,1);
h2 = NUM(:,6);
syms s;
syms A2;
%Known parameters%
R2 = 34150;
R1 = 11410.28;
A1 = 0.00581;
A2tot = 0;
i = 0;

%start from 2 because smoothing function has a NaN in the first entry
for k = 2:length(time)
    t = time(k);

    %Transfer function H2(s)%
    H2s = (7.2*10^(-6))/(A2*A1*R1*s*(s+(1/(A1*R1)))*(s+(1/(A2*R2))));

    %Taking inverse laplace of transfer function at t%
    il = ilaplace(H2s,t);

    h2graph = h2(k)/100;
    %Using a value of h2 (from point k on the graph)%
    eqn = il == h2graph;

    %solve equation for A2%
    A2soln = vpasolve(eqn, A2);

    %there are negative solutions for A2 at the times closer to zero and
    %closer to steady-state, these can be ignored.
    if (A2soln > 0)
        A2tot = A2tot + A2soln;
        i = i + 1;
    end
end

%get all positive solutions and average them out
AvgA2 = A2tot / i;
display(AvgA2);
```

MATLAB Code for processing the data from the $h_2(t)$ graph

4.4 Appendix D

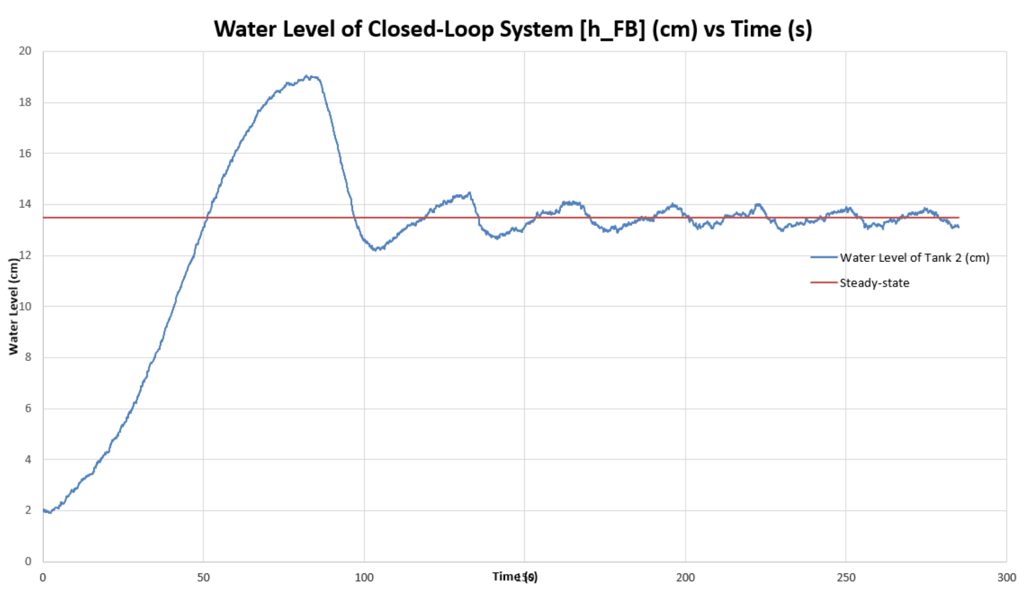

Figure 6: Water level of tank 2 in a closed-loop system over time

4.5 Appendix E

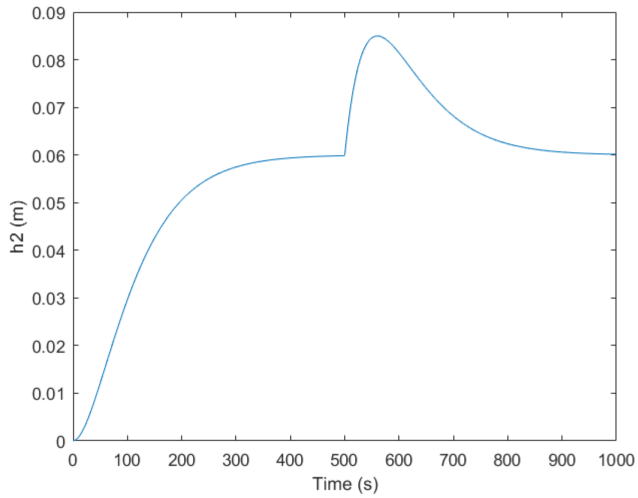

Figure 7: Water level in Tank 2 vs. Time when subjected to a step-input at t=0 and an impulse at t=500s