1 Question 1

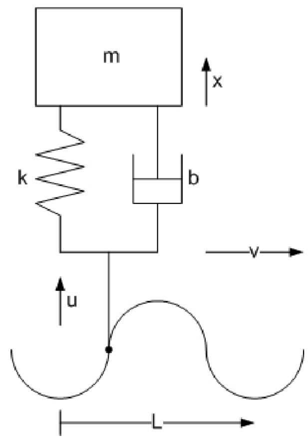

Figure 1: Diagram of spring-mass-damper system[1]

From Fig. 1, the free-body diagram for mass m can be determined as:

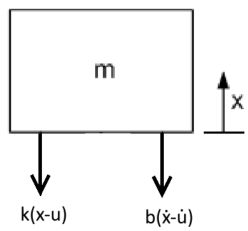

Figure 2: Free-body diagram of mass m

From the free-body diagram, the following equation can be used to describe the balance of forces:

$$m\ddot{x} = -k(x-u) - b(\dot{x} - \dot{u}) \tag{1}$$

Treating x as an incremental variable, and taking the Laplace transform of Eq. (1), we get:

$$\begin{gathered} ms^2X(s) = -k(X(s) - U(s)) - b(sX(s) - sU(s)) \\ \rightarrow ms^2X(s) + bsX(s) + kX(s) = bsU(s) + kU(s) \\ \rightarrow X(s)[ms^2 + bs + k] = U(s)[bs + k] \end{gathered}$$

Thus, the transfer function of the system can be found to be:

$$G(s) = \frac{X(s)}{U(s)} = \frac{bs + k}{ms^2 + bs + k} \tag{2}$$

Letting $s = j\omega$, the transfer function becomes:

$$\begin{aligned} G(j\omega) &= \frac{bj\omega + k}{m(j\omega)^2 + bj\omega + k} \\ &= \frac{k + bj\omega}{(k - m\omega^2) + bj\omega} \end{aligned} \tag{3}$$

Dividing through by $\frac{k}{k}$ and finding the magnitude of the transfer function:

$$\boxed{|G(\omega)| = \frac{\sqrt{1 + (\frac{b}{k}\omega)^2}}{\sqrt{(1 - \frac{m}{k}\omega^2)^2 + (\frac{b}{k}\omega)^2}}} \tag{4}$$

1.1 System Parameters

With m = 100, b = 500, and k = 5000 as per the specification:
1. Since spring-mass-damper system is a second-order system, the denominator satisfies the form:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 \tag{5}$$

From Eq. (2), we have the denominator:

$$s^2 + \frac{b}{m}s + \frac{k}{m} \tag{6}$$

So matching the coefficients of Eq. (5) and (6):

$$\omega_n^2 = \frac{k}{m} \rightarrow \omega_n = \sqrt{\frac{k}{m}} \tag{7}$$

$$\boxed{\therefore \omega_n = \sqrt{\frac{5000}{100}} = \sqrt{50} = 7.071 \text{ rad/s}} \tag{8}$$

2. Applying the same matching of coefficients from **1.** we have:

$$2\zeta\omega_n = \frac{b}{m} \rightarrow 2\zeta\sqrt{\frac{k}{m}} = \frac{b}{m} \tag{9}$$

$$\boxed{\therefore \zeta = \frac{b}{2\sqrt{km}} = \frac{500}{2\sqrt{5000 * 100}} = 0.3536} \tag{10}$$

3. In order to find the frequency when the response is maximum, ω_p, we take the derivative of Eq. (4) with respect to ω and set it to zero, this was done using the MATLAB code found in Appendix A, by only taking the positive real solutions for ω, it was found that:

$$\boxed{\omega_p = 6.436 \text{ rad/s}} \tag{11}$$

4. To find the magnitude at which the response is maximum we determine from Eq. (4), letting $\omega = \omega_p$:

$$|G(\omega_p)| = \frac{\sqrt{1 + (\frac{b}{k}\omega_p)^2}}{\sqrt{(1 - \frac{m}{k}\omega_p^2)^2 + (\frac{b}{k}\omega_p)^2}} = 20\log_{10}(\sqrt{1 + (\frac{b}{k}\omega_p)^2}) - 20\log_{10}(\sqrt{(1 - \frac{m}{k}\omega_p^2)^2 + (\frac{b}{k}\omega_p)^2}) \tag{12}$$

Substituting all known values, this becomes:

$$\boxed{|G(\omega_p)| = 20\log_{10}(1.1892) - 20\log_{10}(0.6661) = 5.035 \text{ dB}} \tag{13}$$

5. To find the phase angle at which the response is maximum we determine from Eq. (3), with $\omega = \omega_p$:

$$\phi(j\omega_p) = \angle\{\frac{k + bj\omega_p}{(k - m\omega_p^2) + bj\omega_p}\} = \tan^{-1}[\frac{b\omega_p}{k}] - \tan^{-1}[\frac{b\omega_p}{(k - m\omega_p^2)}] \tag{14}$$

Substituting all known parameters, the phase angle can be found to be:

$$\boxed{\phi(j\omega_p) = \tan^{-1}[0.6436] - \tan^{-1}[3.7515] = 32.765° - 75.074° = -42.309°} \tag{15}$$

6. The bandwidth of the system is the range $0 \leq \omega \leq \omega_b$ where the magnitude is $\geq 20\log_{10}|G(j0)| + 20\log_{10}(\frac{1}{\sqrt{2}})$. For this system, $|G(j0)|$ can be found by letting $\omega = 0$ in Eq. (4):

$$|G(0)| = \frac{\sqrt{1 + (\frac{b}{k}(0))^2}}{\sqrt{(1 - \frac{m}{k}(0)^2)^2 + (\frac{b}{k}(0))^2}} = \frac{\sqrt{1}}{\sqrt{1}} = 1 \tag{16}$$

So, the zero frequency magnitude is $20\log_{10}(1) = 0$ dB, therefore bandwidth frequency can be found as the frequency when the magnitude is $0 + 20\log_{10}(\frac{1}{\sqrt{2}}) = -3.01$ dB. This occurs when:

$$20\log_{10}(\sqrt{1+(\frac{b}{k}\omega)^2}) - 20\log_{10}(\sqrt{(1-\frac{m}{k}\omega^2)^2+(\frac{b}{k}\omega)^2}) = 20\log_{10}(\frac{1}{\sqrt{2}}) \tag{17}$$

$$\rightarrow 20\log_{10}(\frac{\sqrt{1+(\frac{b}{k}\omega)^2}}{\sqrt{(1-\frac{m}{k}\omega^2)^2+(\frac{b}{k}\omega)^2}}) = 20\log_{10}(\frac{1}{\sqrt{2}})$$

$$\rightarrow \frac{\sqrt{1+(\frac{b}{k}\omega)^2}}{\sqrt{(1-\frac{m}{k}\omega^2)^2+(\frac{b}{k}\omega)^2}} = \frac{1}{\sqrt{2}}$$

$$\rightarrow 2(1+(\frac{b}{k}\omega)^2) = (1-\frac{m}{k}\omega^2)^2+(\frac{b}{k}\omega)^2$$

Rearranging the above equation, the following was found:

$$m^2\omega^4 - (2km+b^2)\omega^2 - k^2 = 0 \tag{18}$$

Substituting known parameters and applying the quadratic formula to find ω^2:

$$\omega^2 = \frac{1250000 \pm \sqrt{(-1250000)^2 - 4(10000)(-25000000)}}{2(10000)}$$

$$\omega^2 = 142.539 \text{ rad/s}, -17.539 \text{ rad/s}$$

$$\boxed{\therefore \omega_b = \omega = 11.94 \text{ rad/s}} \tag{19}$$

7. The gain margin of a system can be found to be the reciprocal of the magnitude $|G(j\omega)|$ of the system at the frequency when the phase is -180°. By simplifying Eq. (3), the frequency at the phase crossover can be found:

$$G(j\omega) = \frac{k+bj\omega}{(k-m\omega^2)^2+bj\omega} * \frac{(k-m\omega^2)^2-bj\omega}{(k-m\omega^2)^2-bj\omega} = \frac{k^2+(b^2-mk)\omega^2-bmj\omega^3}{k^2+(b^2-2km)\omega^2+m^2\omega^4} \tag{20}$$

$$\therefore \phi(j\omega) = \tan^{-1}[\frac{-bm\omega^3/k^2+(b^2-2km)\omega^2+m^2\omega^4}{k^2+(b^2-mk)\omega^2/k^2+(b^2-2km)\omega^2+m^2\omega^4}] = \tan^{-1}[\frac{-bm\omega^3}{k^2+(b^2-mk)\omega^2}] \tag{21}$$

To find the gain margin, the frequency when the phase $\phi(j\omega) = -180°$ must be found, so we have:

$$\phi(j\omega) = \tan^{-1}[\frac{-bm\omega^3}{k^2+(b^2-mk)\omega^2}] = -180 \tag{22}$$

$$\rightarrow \frac{-bm\omega^3}{k^2+(b^2-mk)\omega^2} = 0$$

$$\boxed{\therefore \omega = 0 \text{ rad/s}} \tag{23}$$

However, at $\omega = 0$, the phase is:

$$\phi(j0) = \tan^{-1}[\frac{-bm(0)^3}{k^2+(b^2-mk)(0)^2}] = 0° \tag{24}$$

So the phase never crosses -180°, therefore the gain margin, K_g:

$$\boxed{K_g = \infty} \tag{25}$$

To find the phase margin, the frequency when the magnitude of the gain is 0 dB must be found, that is when $|G(j\omega)| = 1$ (since $20\log_{10}(1) = 0$ dB). So the gain crossover frequency can be found from Eq. (4) as the following:

$$\frac{\sqrt{1+(\frac{b}{k}\omega)^2}}{\sqrt{(1-\frac{m}{k}\omega^2)^2+(\frac{b}{k}\omega)^2}} = 1 \tag{26}$$

$$\rightarrow 1+(\frac{b}{k}\omega)^2 = (1-\frac{m}{k}\omega^2)^2+(\frac{b}{k}\omega)^2$$

$$\rightarrow 1 = (1-\frac{m}{k}\omega^2)^2$$

$$\rightarrow 2\frac{m}{k}\omega^2 = m^2\omega^4$$

$$\boxed{\rightarrow \omega = \sqrt{\frac{2k}{m}} = \sqrt{\frac{2*5000}{100}} = 10 \text{ rad/s}} \tag{27}$$

Substituting the above result for ω and all known parameters into Eq. (21) to find the phase angle at the gain crossover frequency:

$$\phi(j10) = \tan^{-1}[\frac{-(500)(100)(10)^3}{(5000)^2+((500)^2-(100)(5000))(10)^2}] = -90° \tag{28}$$

Then it follows that the phase margin, γ, is:

$$\boxed{\gamma = 180° - 90° = 90°} \tag{29}$$

1.2 Bode Plot

To determine a straight-line approximation of the bode plot of the system, known parameters must be substituted into Eq. (3) and rearranged:

$$G(j\omega) = \frac{500j\omega+5000}{5000+500j\omega+100(j\omega)^2} = \frac{5(j\omega+10)}{(50+5j\omega+(j\omega)^2)} = \frac{\frac{j\omega}{10}+1}{1+\frac{j\omega}{10}+\frac{(j\omega)^2}{50}} \tag{30}$$

Breaking Eq. (30) up into factors (Note: ω^* is the corner frequency):
1) $1+\frac{j\omega}{10} \rightarrow \omega^* = 10$ rad/s
2) $\frac{1}{1+\frac{j\omega}{10}+\frac{(j\omega)^2}{50}} \rightarrow \omega^* = \sqrt{50}$ rad/s
For 1) the magnitude can be expressed as:

$$|1+\frac{j\omega}{10}| = \sqrt{1^2+(\frac{\omega}{\omega^*})^2} \tag{31}$$

Considering the following three cases:
For $\omega << \omega^*$

$$|1+\frac{j\omega}{10}| \approxeq \sqrt{1^2+0^2} = 20\log_{10}(1) = 0 \text{ dB} \tag{32}$$

For $\omega >> \omega^*$

$$|1+\frac{j\omega}{10}| \approxeq \sqrt{(\frac{\omega}{\omega^*})^2} = 20\log_{10}(\frac{\omega}{\omega^*}) \text{ dB} \tag{33}$$

Which is a straight line with a slope of 20 dB/decade.
Finally $\omega = \omega^*$

$$|1+\frac{j\omega}{10}| = \sqrt{1^2+1^2} = 20\log_{10}(\sqrt{2}) = 3.01 \text{ dB} \tag{34}$$

Now considering the phase of 1), which is:

$$\angle(1 + \frac{j\omega}{\omega^*}) = \tan^{-1}(\frac{\omega/\omega^*}{1}) = \tan^{-1}(\omega/\omega^*) \tag{35}$$

Considering the same three cases as above:
For $\omega << \omega^*$

$$\angle(1 + \frac{j\omega}{\omega^*}) \approxeq \tan^{-1}(0) = 0^\circ \tag{36}$$

For $\omega >> \omega^*$

$$\angle(1 + \frac{j\omega}{\omega^*}) \approxeq \tan^{-1}(\infty) = 90^\circ \tag{37}$$

Finally $\omega = \omega^*$

$$\angle(1 + \frac{j\omega}{\omega^*}) = \tan^{-1}(1) = 45^\circ \tag{38}$$

Following the same process as above for 2):

$$\left|\frac{1}{1 + \frac{j\omega}{10} + \frac{(j\omega)^2}{50}}\right| = \frac{1}{\sqrt{(1 - (\frac{\omega}{\omega^*})^2)^2 + (\frac{1}{\sqrt{2}}\frac{\omega}{\omega^*})^2}} \tag{39}$$

For $\omega << \omega^*$

$$\left|\frac{1}{1 + \frac{j\omega}{10} + \frac{(j\omega)^2}{50}}\right| \approxeq \frac{1}{\sqrt{1^2 + 0^2}} = -20\log_{10}(1) = 0 \text{ dB} \tag{40}$$

For $\omega >> \omega^*$

$$\left|\frac{1}{1 + \frac{j\omega}{10} + \frac{(j\omega)^2}{50}}\right| \approxeq \frac{1}{\sqrt{(\frac{\omega}{\omega^*})^2}} = -20\log_{10}((\frac{\omega}{\omega^*})^2) = -40\log_{10}(\frac{\omega}{\omega^*}) \text{ dB} \tag{41}$$

Which is a straight line with a slope of -40 dB/decade.
Finally $\omega = \omega^*$

$$\left|\frac{1}{1 + \frac{j\omega}{10} + \frac{(j\omega)^2}{50}}\right| = \frac{1}{\sqrt{(\frac{1}{\sqrt{2}})^2}} = -20\log_{10}(\frac{1}{\sqrt{2}}) = 3.01 \text{ dB} \tag{42}$$

Now, for the phase of 2):

$$\angle(\frac{1}{1 + \frac{j\omega}{10} + \frac{(j\omega)^2}{50}}) = -\tan^{-1}(\frac{\frac{1}{\sqrt{2}}\frac{\omega}{\omega^*}}{1 - (\frac{\omega}{\omega^*})^2}) \tag{43}$$

For $\omega << \omega^*$

$$\angle(\frac{1}{1 + \frac{j\omega}{10} + \frac{(j\omega)^2}{50}}) \approxeq -\tan^{-1}(0) = 0^\circ \tag{44}$$

For $\omega >> \omega^*$

$$\angle(\frac{1}{1 + \frac{j\omega}{10} + \frac{(j\omega)^2}{50}}) \approxeq -\tan^{-1}(0) = -180^\circ \tag{45}$$

Finally $\omega = \omega^*$

$$\angle(\frac{1}{1 + \frac{j\omega}{10} + \frac{(j\omega)^2}{50}}) \approxeq -\tan^{-1}(\frac{\frac{1}{\sqrt{2}}}{0}) = -90^\circ \tag{46}$$

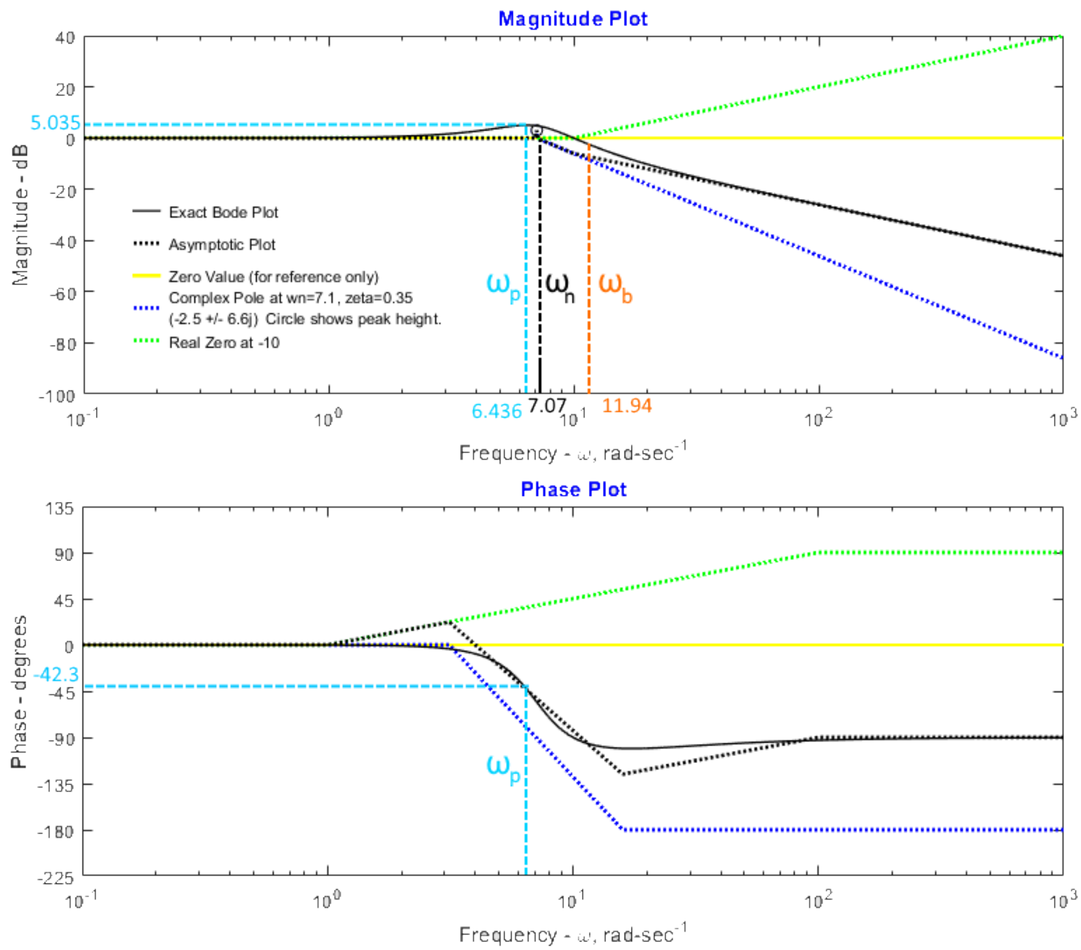

Figure 3: Straight-line asymptote approximation of Bode plot of $G(j\omega)$

Plotting these asymptotes in MATLAB (BodePlotGui, 2014), a straight-line approximation for the Bode plot of the system can be obtained:

For the system to have a damping factor of $\zeta = \frac{1}{\sqrt{2}}$, it is required from Eq. 10 that:

$$\zeta = \frac{b}{2\sqrt{km}} = \frac{1}{\sqrt{2}} \tag{47}$$

$$\rightarrow \frac{b}{\sqrt{k}} = 10\sqrt{2} \tag{48}$$

i) Keeping the damper value b the same:

$$\boxed{\sqrt{k} = \frac{b}{10\sqrt{2}} = \frac{50}{\sqrt{2}} \rightarrow k = \frac{50^2}{2} = 1250 \text{ N/m}} \tag{49}$$

ii) Keeping the spring value k the same:

$$\boxed{b = 10\sqrt{2k} = 10\sqrt{10000} = 1000 \text{ kg/s}} \tag{50}$$

The Bode plots for the original system, and the systems in i) and ii) were generated in MATLAB:

From the above figure it can be seen that the phase margin for the original system is significantly less than that of the systems with the adjusted parameters. This implies the system is closer to instability than the other two systems and so it has a higher frequency of oscillation in its response in the time domain.

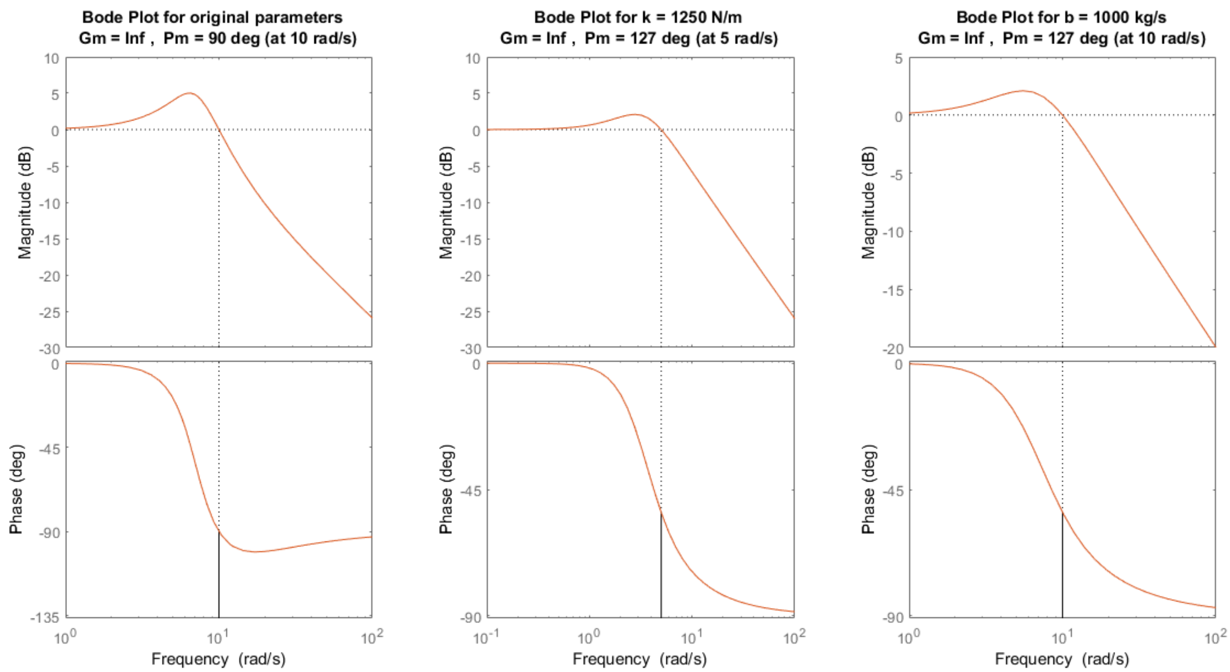

Figure 4: Bode plots of systems with varying parameters

Looking at the two adjusted parameter systems, the system of adjusted k has its phase margin at a lower frequency than the system of adjusted b. This means the response of the adjusted b system will be faster in the time-domain.

All three systems having an infinite gain margin with a positive phase margin implies these systems are all stable.

In order to attenuate the vibrations of the system by more than 10 dB, it is required that the frequency at -10 dB is found as below:

$$20\log_{10}(x) = -10 \text{ dB}$$
$$x = 0.31628$$
$$\rightarrow |G(\omega)| = 0.31628$$

Solving the above using Eq. (4) and solving for ω the following was found:

$$\boxed{\omega = 19.587 \text{ rad/s}} \tag{51}$$

From the system, there is the relationship:

$$\omega = \frac{2\pi v}{L}$$
$$\rightarrow v = \frac{\omega L}{2\pi} = \frac{(19.587)(0.5)}{2\pi}$$
$$\therefore v = 1.56 \text{ m/s}$$

For all $\omega > 19.587$ rad/s the vibration will be attenuated by more than 10 dB, so the comfortable range of riding speed is:

$$\boxed{v > 1.56 \text{ m/s}} \tag{52}$$

2 Question 2

Considering the following system:

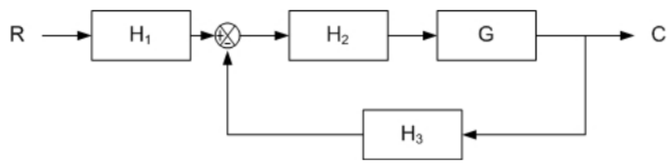

Figure 1: Diagram of system from MMAN3200 Bode Plot Assignment Document[1]

Where

$$G(s) = \frac{20}{(s+0.1)(s+1)(s+2)} = \frac{20}{s^3+3.1s^2+2.3s+0.2} \tag{1}$$

$$H_1(s) = H_2(s) = H_3(s) = 1 \tag{2}$$

Using MATLAB to draw the Bode Plot for $G(s)$ (Appendix B) we have the following plot:

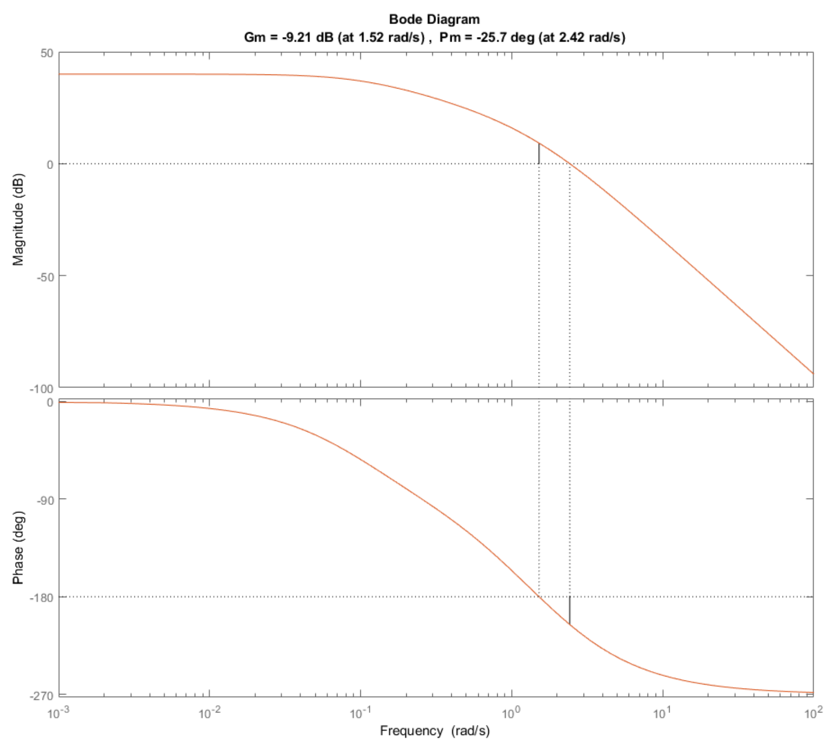

Figure 2: Bode Plot for transfer function $G(s)$

Determining the feedback system transfer function from Fig. 1 above:

$$G_{FB}(s) = \frac{C(s)}{R(s)} = H_1(s)\frac{H_2(s)G(s)}{1+H_3(s)H_2(s)G(s)} \tag{3}$$

$$\rightarrow G_{FB}(s) = (1)\frac{(1)G(s)}{1+(1)(1)G(s)} = \frac{G(s)}{1+G(s)} = \frac{20}{s^3+3.1s^2+2.3s+20.2} \tag{4}$$

Simulating the step response of $G_{FB}(s)$ using MATLAB (Appendix D), the following plot was found:

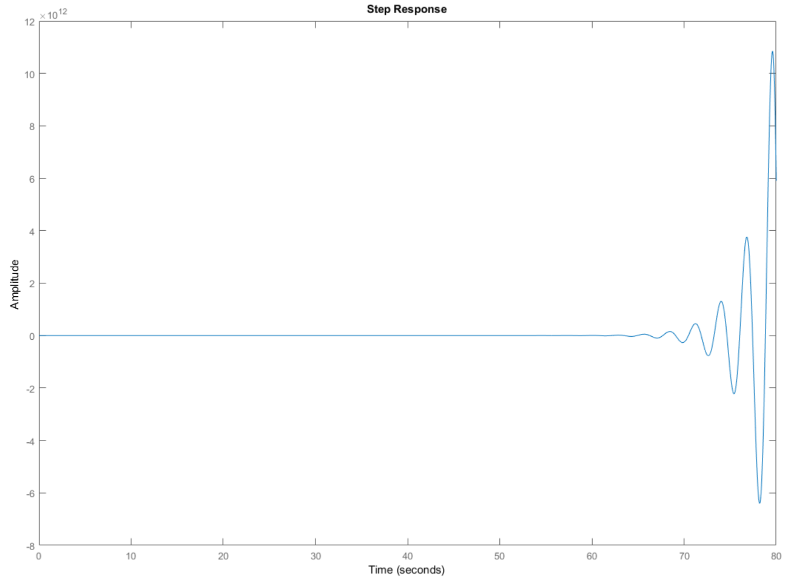

Figure 3: Step response of $G_{FB}(s)$

As can be seen from the above figure, the response of the system increases exponentially with oscillatory behaviour, thus it can be said that the system is **oscillatory unstable**. For the system to become stable, it can be seen from Fig. 2 that the gain has to be increased (the gain margin in Fig. 2 is negative, this causes instability). To determine what value to assign to $H_2(s)$, the first step is to find the frequency at which the phase crossover occurs for $G(s)$, this can be read off of Fig 2. but it is not quite accurate enough, so to derive it manually from Eq. (1), letting $s = j\omega$:

$$G(j\omega) = \frac{20}{(-3.1\omega^2 + 0.2) + j(2.3\omega - \omega^3)} \tag{5}$$

To get an expression for Eq. (5) without complex numbers in the denominator:

$$\begin{aligned} G(j\omega) &= \frac{20}{(-3.1\omega^2 + 0.2) + j(2.3\omega - \omega^3)} * \frac{(-3.1\omega^2 + 0.2) - j(2.3\omega - \omega^3)}{(-3.1\omega^2 + 0.2) - j(2.3\omega - \omega^3)} \\ &= \frac{20(0.2 - 3.1\omega^2) + 20j(2.3\omega - \omega^3)}{(0.2 - 3.1\omega^2)^2 + (2.3\omega - \omega^3)^2} \end{aligned} \tag{6}$$

Therefore, an expression for the phase $\phi(j\omega)$ can be obtained:

$$\phi(j\omega) = \tan^{-1}[\frac{20(2.3\omega - \omega^3)}{20(0.2 - 3.1\omega^2)}] \tag{7}$$

To find the frequency ω at which the phase crosses $-180°$:

$$\phi(j\omega) = \tan^{-1}[\frac{20(2.3\omega - \omega^3)}{20(0.2 - 3.1\omega^2)}] = -180° \rightarrow \frac{20(2.3\omega - \omega^3)}{20(0.2 - 3.1\omega^2)} = 0 \tag{8}$$

$$\rightarrow \omega^3 = 2.3\omega$$

$$\therefore \omega = \sqrt{2.3} \text{ rad/s} \tag{9}$$

Finding an expression for the magnitude of $G(j\omega)$ from Eq. (5):

$$|G(j\omega)| = \frac{20}{\sqrt{(0.2 - 3.1\omega^2)^2 + (2.3\omega - \omega^3)^2}} \tag{10}$$

So at $\omega = \sqrt{2.3}$, then the magnitude can be found to be:

$$|G(j\sqrt{2.3})| = \frac{20}{\sqrt{(0.2 - 3.1(\sqrt{2.3})^2)^2 + (2.3\sqrt{2.3} - \sqrt{2.3}^3)^2}} = 2.886 \tag{11}$$

The value of $H_2(s)$ will increase or decrease the gain, to find the value of $H_2(s)$ where the system is on the border between stability and instability, the gain margin at $\omega = \sqrt{2.3}$ has to be 0dB, the magnitude for this is $10^{\frac{0}{20}} = 1$ (from $20\log_{10}(x) = 0$) so from Eq. (3), letting $H_2(s) = K$, where K is some constant:

$$K * |G(j\sqrt{2.3})| = 1 \tag{12}$$
$$\rightarrow 2.886K = 1$$

$$\boxed{\therefore H_2(s) = K = \frac{1}{2.886} = 0.3465} \tag{13}$$

Plotting the step response of the system with this new value of $H_2(s)$ yields:

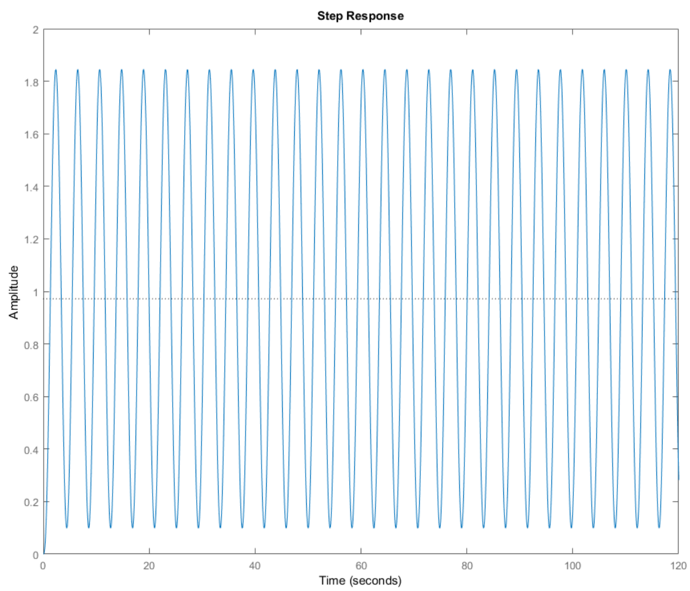

Figure 4: Step response of $G_{FB}(s)$ with $H_2(s) = 0.3465$

It can be seen from the above figure that the response no longer increases exponentially but it doesn't decay either, so it can be seen that it is on the exact border between stability and instability. So for the system to be stable, the value of $H_2(s)$ must be chosen so that it satisfies:

$$\boxed{0 \leq H_2(s) < 0.3465} \tag{14}$$

Using a value of $H_2(s) = 0.1$ gives a nice system response to unit-step input (Appendix E), where the response decays towards the unit-step input magnitude with some steady-state error and so this will be used as the adjusted value for $H_2(s)$.

Oscillatory behaviour occurs in a system when the poles of that system are complex, so in order to reduce or eliminate the oscillations, the imaginary component of the poles of the system must be reduced to made to equal 0. However, in a third-order system the dynamic behaviour is quite different when there are systems with a complex conjugate pair of poles and a single real pole as in the above system. In this case, the dominant pole/pair of poles (that is the pole/pair of poles that has a larger value in their real component) generally decides the behaviour of the system in terms of oscillatory or exponential motion.

For oscillatory motion to be reduced, the distance between the real pole and complex conjugate poles must be reduced if the complex conjugate poles are dominant, and for it to be essentially non-existent the real pole must dominate the complex conjugate pair of poles, this idea comes from what is known as the Dominant Pole Approximation of a system (Cheever, 2005)[2].

Looking at the Pole-Zero maps in MATLAB (using pzplot command) for $G_{FB}(s)$ when $H_3(s) = s + 0.1$, $s + 1$ and $s + 2$, and the corresponding step responses (Fig 5. below), it can be seen that only when the complex conjugate pair dominates the real pole the system exhibits oscillatory behaviour.

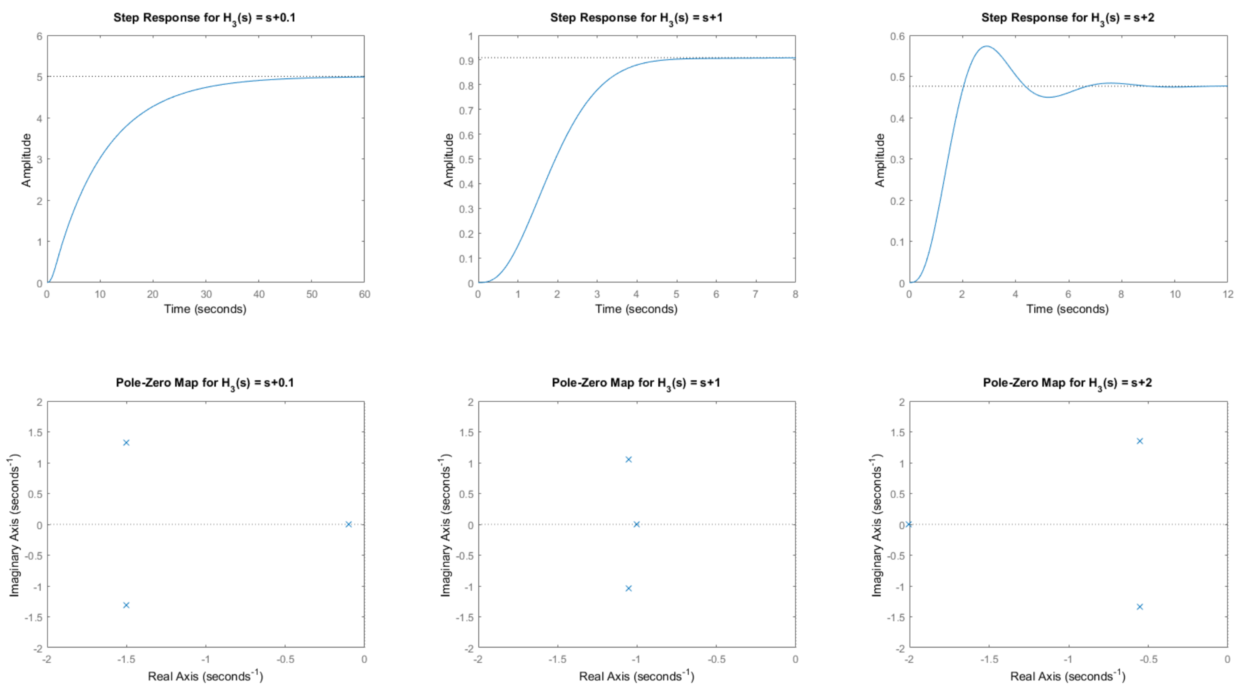

Figure 5: Step Responses and Pole-Zero Maps for 3 variations of $H_3(s)$

It can also be seen from the above figure that when $H_3(s) = s + 1$ the real pole does not dominate the complex conjugate pair by much, therefore, for the system to have exponential behaviour, we choose $H_3(s) = s + A$, where A is a constant, such that:

$$0 \leq A \leq 1 \tag{15}$$

Since this is the approximate range in which the real pole will dominate the complex conjugate poles (and still remain stable [for $A < 0$ the system becomes unstable]).
Note that if oscillations are not problematic, A can be greater than 1.

To determine the required value for $H_1(s)$ for the system to reach a steady-state magnitude of 1 for a unit step input, assuming $c(t)$ is the response of the system in the time domain, we determine:

$$\lim_{t\to\infty}[c(t)] = \lim_{s\to 0}[sC(s)] = \lim_{s\to 0}[s\frac{R(s)H_1(s)H_2(s)G(s)}{1 + H_2(s)H_3(s)G(s)}] = 1 \tag{16}$$

Since the input is a unit step, $R(s) = \frac{1}{s}$, assuming $H_1(s)$ is a constant, and taking $H_2(s) = 0.1$ as found previously:

$$0.1H_1(s)\lim_{s\to 0}[\frac{20/(s^3+3.1s^2+2.3s+0.2)}{1+2H_3(s)/(s^3+3.1s^2+2.3s+0.2)}]=1 \tag{17}$$

$$\to H_1(s)[\frac{100}{1+10\lim\limits_{s\to 0}H_3(s)}]=10 \tag{18}$$

$$\boxed{\therefore H_1(s)=0.1+\lim_{s\to 0}H_3(s)} \tag{19}$$

Therefore the required $H_1(s)$ is dependant on the chosen $H_3(s)$. A good value for $H_3(s)$ would be:

$$H_3(s)=s+1 \tag{20}$$

This is approximately on the verge between oscillatory and exponential behaviour (as seen in Fig. 5 and explained above) and so essentially exhibits critically-damped behaviour, which means the response is fast and without oscillation, this satisfies result (15) and assumes oscillations are undesirable for this application.
Note that the desirable choice for $H_3(s)$ would be when the real pole and complex conjugate poles lie collinearly on the Pole-Zero map, this value is roughly $H_3(s)=s+1.018$, this value provides the fastest non-oscillatory response.

For the choice of $H_3(s)$ in result (20), substituting into Eq. (19), we have:

$$\boxed{H_1(s)=0.1+\lim_{s\to 0}[s+1]=0.1+1=1.1} \tag{21}$$

This result can be graphically verified by looking at the step response graph from Fig. 5, it can be seen that the response settles just above 0.9, so to bring it to a steady-state value of 1, we require:

$$0.9H_1(s)=1\to H_1(s)=\frac{1}{0.9}\approxeq 1.1 \tag{22}$$

It is not quite 1.1 since a graphical approximation was taken for the steady-state of 0.9, even though it settles slightly above this. Taking that error into account, the value for $H_1(s)$ is verified to be 1.1.

3 Appendix

3.1 Appendix A

```
%ensure frequency can only be positive
syms w positive;
b = 500;
k = 5000;
m = 100;

%|G(w)|
num = sqrt(1+((b/k)*w)^2);
den = sqrt((1-((m/k)*w^2))^2 + ((b/k)*w)^2);
Gw = num/den;

dGw = diff(Gw, w);
soln = double(solve(dGw == 0, w));
display(soln);
```

MATLAB Code for differentiating $|G(\omega)|$ and finding the values of ω when it is 0

3.2 Appendix B

```
close all;
num = [20];
den = [1 3.1 2.3 0.2];
sys = tf(num,den);
bode(sys); hold on; grid on;
margin(sys);
```

MATLAB Code for plotting $G(s)$

3.3 Appendix C

```
close all;
num = [20];
den = [1 3.1 2.3 20.2];
sys = tf(num,den);
bode(sys); hold on; grid on;
margin(sys);
```

MATLAB Code for plotting $G_{FB}(s)$

3.4 Appendix D

```
close all;
%system parameters
H1 = 1;
H2 = 1;
H3 = 1;

%Transfer function of G(s)
Gnum = [20];
Gden = [1 3.1 2.3 0.2];
G = tf(Gnum,Gden);

%H2 is in series with G
sys1 = series(H2, G);
%The above has a negative feedback of magnitude H3
sys2 = feedback(sys1, H3);
%The above is in series with H1
sys = series(H1, sys2);
%simulate the step response of the system
step(sys);
```

MATLAB Code for simulating the step response of the system in Fig. 1 of Section 2

3.5 Appendix E

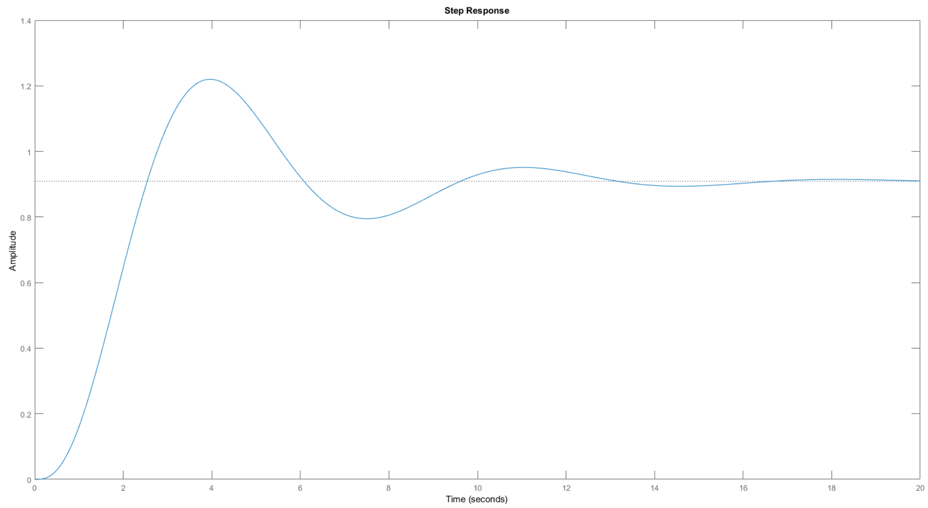

Figure 1: Step response of $G_{FB}(s)$ with $H_2(s) = 0.1$

[1] Assignment − Bode Plot. (2017). [PDF]
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[Accessed 21 May 2017].

[2] Cheever, E. (2005). Unit Step Response − Step Response of higher order systems. [online]
Swarthmore College − Linear Physical Systems Analysis.
Available at: http://lpsa.swarthmore.edu/Transient/TransInputs/TransStep.html#higher
[Accessed 21 May 2017].

[3] BodePlotGui. (2014). echeever.
Available at: https://github.com/echeever/BodePlotGui