MMAN2300:
Dynamics of a single degree-of-freedom spring-mass-damper system

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Contents

1 Introduction

The purpose of this experiment was to investigate the dynamics of a damped single degree-of-freedom system that was subjected to both free and forced vibration. By measuring the system's response under free vibration after applying an initial displacement, an experimental value for the natural frequency of the system, ω_n, was determined, as well as the logarithmic decrement of the response, δ, which was used to find the damping ratio of the system, ζ. From these values of ζ, ω_n and an analytically determined ω_n, it became possible to plot theoretical and experimental non-dimensionalised amplitudes of the steady-state response of the system under forced vibration against the frequency ratio $\frac{\omega}{\omega_n}$. These plots give an insight into how the system responds as it approaches resonance (as the excitation frequency applied to the system approaches the natural frequency [that is, as $\omega \rightarrow \omega_n$]).

2 Apparatus

The apparatus that was used consisted of an aluminium beam with an out-of-balance mass at the centre that was connected to a motor creating forced vibration. This motor was connected to a power supply. An accelerometer was placed on the beam, in which the signal was passed through a charge-conditioning amplifier, and then to the oscilloscope or frequency analyser. The required apparatuses are listed below:

- Accelerometer
- Power supply
- Motor
- Oscilloscope or other frequency analyser
- Charge-conditioning amplifier
- Aluminium beam fixed at both ends by supports with a rotating, out-of-balance mass at the centre

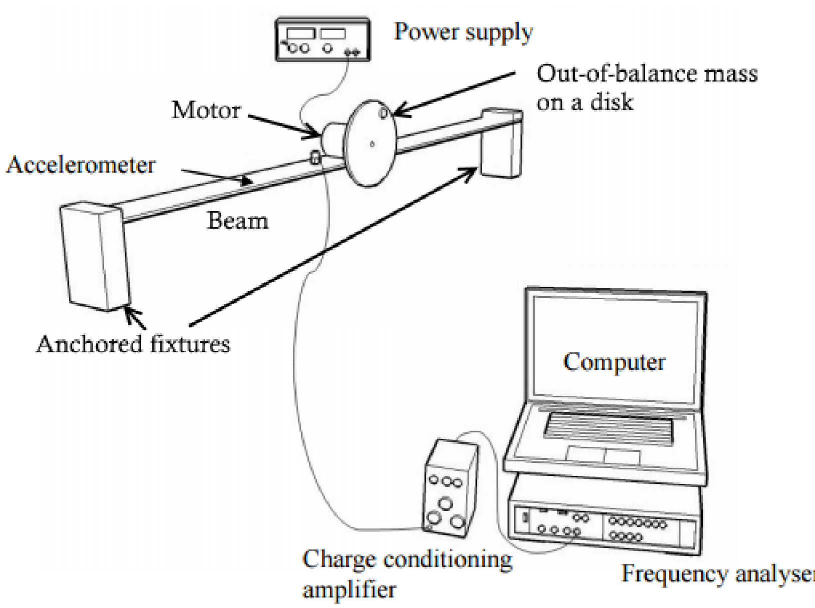

Fig 1. Schematic diagram of the rotating out-of-balance test rig and instrumentation[1]

3 Experimental Procedure

1. Displaced the beam by some distance x, and recorded the free response of the system with an oscilloscope/frequency analyser as a time, Δt, between two peaks in ms, and the amplitude of each peak, X, in mV. The number of periods, $N > 1$, between the two peaks was also recorded.

2. Repeated (1) for $N = 1$.

3. The motor was set to rotate at some frequency, $0 \leq \omega \leq 3\omega_n$, by turning on the power supply, creating a forced vibration of the beam. The time between two peaks in ms, Δt, the amplitude of the first peak in mV, X, and the number of periods between the peaks, N, were recorded using the oscilloscope/frequency analyser.

4. Repeated (3) 8 more times with different frequencies by varying the voltage supplied to the motor by the power supply.

4 Results

Undertaking steps (1) and (2) of the above procedure, the following data was recorded and tabulated into Table (1) and (2).

Table 1: Response of the system in free vibration for $N > 1$

	N	Δt (ms)	X_1 (mV)	X_{N+1} (mV)
Trial 1	2	101	255	240
Trial 2	6	303	334	281
Trial 3	3	151	323	299

Table 2: Response of the system in free vibration for $N = 1$

	Δt (ms)	X_1 (mV)	X_2 (mV)
Trial 1	50	255	248
Trial 2	50	334	324
Trial 3	50	323	314

The average free response of the system in the time domain was plotted using the equation:

$$x(t) = Xe^{-\zeta\omega_n t} sin(\omega_d t + \Phi) \tag{1}$$

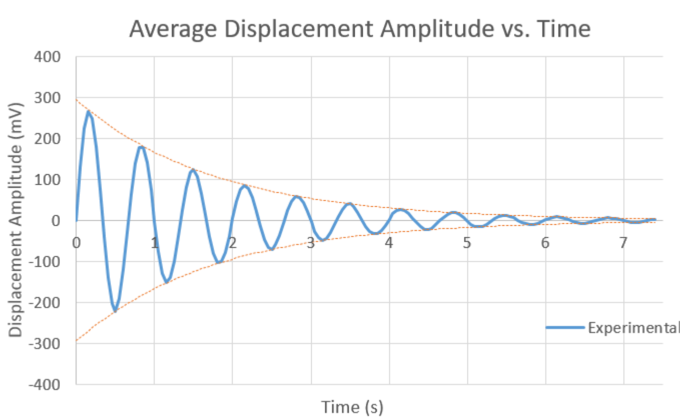

Figure 1: Average response of the system under free vibration

By applying the logarithmic decrement equation:

$$\delta = \frac{1}{N} ln(\frac{X_1}{X_{N+1}}) \tag{2}$$

The following table of logarithmic decrements was calculated, along with an average log decrement.

Table 3: Logarithmic decrement of the system in free vibration

	δ			
Table	Trial 1	Trial 2	Trial 3	δ_{avg}
1	0.0303	0.0288	0.0257	0.0283
2	0.0278	0.0304	0.0283	0.0288

From Table (3), the total average logarithmic decrement may be obtained by:

$$\delta_{avg} = \frac{\delta_{avg,1} + \delta_{avg,2}}{2} = \frac{0.0283 + 0.0288}{2} = 0.02855 \tag{3}$$

Applying the following equation with the result from Eq. (2), the damping ratio, ζ, was determined:

$$\zeta = \frac{\delta_{avg}}{\sqrt{4\pi^2 + \delta_{avg}^2}} = \frac{0.02855}{\sqrt{4\pi^2 + 0.02855^2}} = 0.0045438 \tag{4}$$

From Tables (1) and (2), the damped natural frequency was calculated from the relationship that $T_d = \frac{\Delta t}{N}$ x 10^{-3} s and $\omega_d = \frac{2\pi}{T_d}$ rad/s.

Table 4: Damped natural frequency of the system

	ω_d (rad/s)			
Table	Trial 1	Trial 2	Trial 3	$\omega_{d,avg}$ (rad/s)
1	124.42	124.42	124.83	124.56
2	125.66	125.66	125.66	125.66

Now, the total average damped natural frequency, $\omega_{d,avg}$ was calculated as:

$$\omega_{d,avg} = \frac{\omega_{d,avg,1} + \omega_{d,avg,2}}{2} = \frac{124.56 + 125.66}{2} = 125.110 \text{ rad/s} \tag{5}$$

Using the results from Eq. (3) and (4), it becomes possible to determine the experimental natural frequency of the system.

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{125.11}{\sqrt{1-0.0045438^2}} = 125.112 \text{ rad/s} \tag{6}$$

Undertaking steps (3) and (4) of the experimental procedure, the response of the system under forced vibration was recorded and tabulated into Table (5).

Table 5: Response of the system under forced vibration, for $N \geq 1$

Trial	Δt (ms)	N	X_1 (mV)
1	166	2	204
2	212	3	386
3	169	3	1360
4	53	1	2460
5	52.5	1	3820
6	58.2	2	519
7	45.8	2	462
8	41.4	2	406
9	40.5	2	439

Applying the equation $\omega = \frac{2\pi}{T}$ rad/s, where $T = \frac{\Delta t}{N}$ x 10^{-3} s to the data in Table (5), the excitation frequency generated by the out-of-balance mass was determined.

Table 6: Excitation frequency generated by the out-of-balance mass

Trial	ω (rad/s)
1	75.70
2	88.91
3	111.54
4	118.55
5	119.68
6	215.92
7	274.37
8	303.54
9	310.28

From the results of X_1 in Table (5), and ω in Table (6), the equation $r = \frac{\omega}{\omega_n}$ is applied to obtain a plot of the steady-state amplitude of the response of the system under forced vibration, X, against the frequency ratio, r.

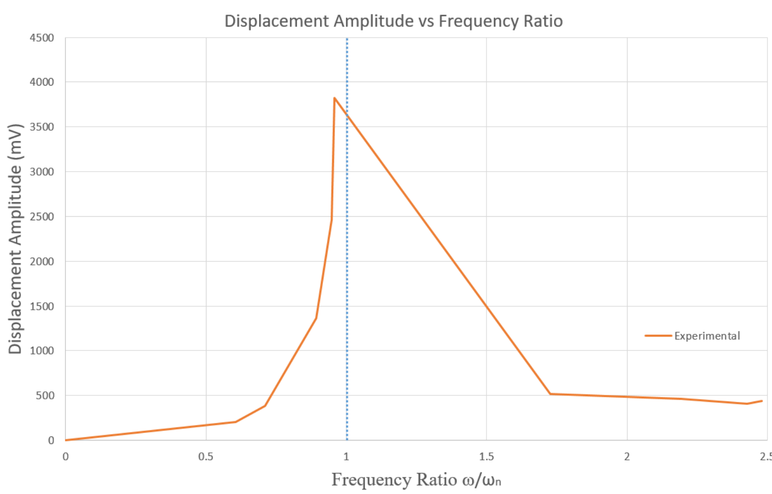

Figure 2: Experimental displacement amplitude against frequency ratio

To determine the accuracy of the results, theoretical values of ω_n and ω_d were found, and an analytical plot of the non-dimensionalised steady-state amplitude of the response of the system under forced vibration against the frequency ratio, r, was ascertained. The natural frequency of the system, ω_n

was calculated using:

$$\omega_n = \sqrt{\frac{k}{m}} \tag{7}$$

With k being the equivalent spring stiffness for a beam fixed at both ends in transverse vibration, equated to:

$$k = 192\frac{EI}{l^3} \tag{8}$$

Where l is the length of the beam, E is the Young's Modulus of the beam, and I is the second moment of inertia, known to be:

$$I = \frac{bh^3}{12} \tag{9}$$

Where b is the width of the beam, and h is the height.
In Eq. (6), m is the effective combined mass of the motor, out-of-balance weight and the beam (for a beam fixed at both ends and carrying a central mass, M), and is determined to be:

$$m = M + \frac{13}{35}m_{beam} \tag{10}$$

With M being the combined mass of the motor, disk, and out-of-balance weight, and m_{beam} being the mass of the beam, known to be:

$$m_{beam} = \rho_{Al}V = \rho_{Al}bhl \tag{11}$$

The following data is known about the aluminium beam:

$$\begin{gathered} h = 3 \text{ x } 10^{-3} \text{ m} \\ b = 40 \text{ x } 10^{-3} \text{ m} \\ l = 0.57 \text{ m} \\ E_{Al} = 7.1 \text{ x } 10^{10} \text{ N/m}^2 \\ \rho_{Al} = 2750 \text{ kg/m}^3 \\ M = 0.248 \text{ kg} \end{gathered}$$

Combining Eq. (7) and (8), and substituting the required values from above, k can be calculated as:

$$k = 192\frac{bh^3}{12l^3}E = 192\frac{(0.04)(0.003)^3}{12(0.57)^3}(7.1\text{x}10^{10}) = 6624.872 \text{ N/m} \tag{12}$$

Combining Eq. (9) and (10), and once again substituting the required values from above, m can be determined as:

$$m = M + \frac{13}{35}\rho_{Al}bhl = (0.248) + \frac{13}{35}(2750)(0.003)(0.04)(0.57) = 0.31787 \text{ kg} \tag{13}$$

From the results of Eq. (11) and (12), the theoretical natural frequency of the system can be determined:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{6624.872}{0.31787}} = 144.3657 \text{ rad/s} \tag{14}$$

Since the damping coefficient of the system, c, is unknown, the experimental damping ratio had to be used instead to determine the theoretical damped natural frequency, ω_d. Using Eq. (5) the results from (3) and (13), ω_d was determined to be:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 144.3657\sqrt{1-(0.0045438)^2} = 144.3642 \text{ rad/s} \tag{15}$$

To obtain a plot of the theoretical non-dimensionalised amplitude of response of the system against the frequency ratio, the following relationship is required:

$$\frac{MX}{me} = \frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \tag{16}$$

where $r = \dfrac{\omega}{\omega_n}$. An alternative method can be seen in Appendix A. By supplying a sufficient number of values for ω, the following plot was constructed:

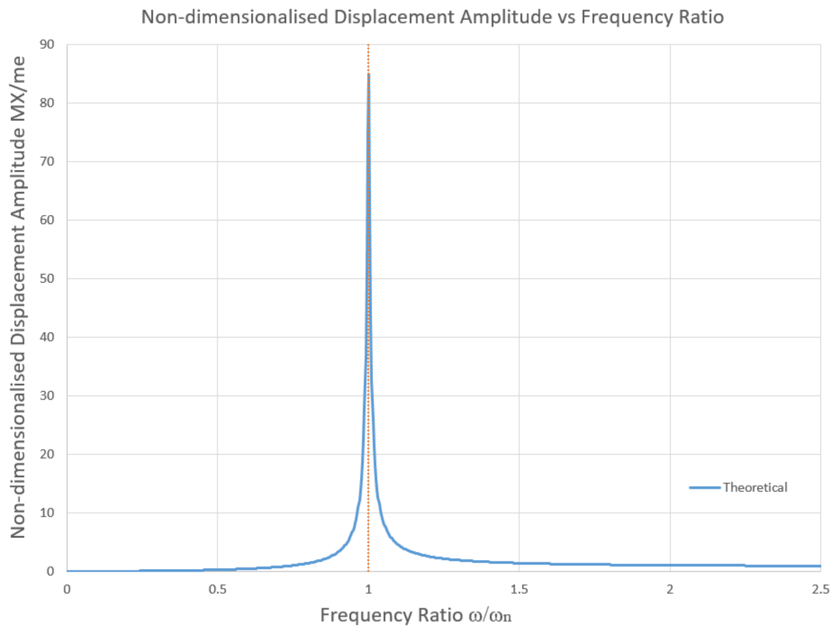

Figure 3: Theoretical non-dimensionalised displacement amplitude against frequency ratio

As is obvious, Fig. (1) and (2) differ somewhat, for reasons that will be discussed in the section below.

5 Discussion

There were many inconsistencies that occurred throughout the experimental and theoretical stages for this experiment. These errors in question can be attributed to a variety of factors; ranging from human error, to rounding. The first major error that perhaps impacted the results the most was the error present in the determination of the logarithmic decrement of the system under free vibration, δ. This δ was not only taken as an average of multiple δ values (see Eq. 2 and Table 3) since $\ln\left(\frac{X_1}{X_2}\right)$ was not experimentally equal to $\frac{1}{N}\ln\left(\frac{X_1}{X_{N+1}}\right)$, but also has the contributing factor that there was the inability to set the exact point required on the oscilloscope, this can be credited to both human error and lack of specific accuracy of the oscilloscope, as it only measured to 1mV and 1ms accuracy. The result was a significant margin of error that carried on to several other computations.

The value this error directly impacted was the experimental damping ratio of the system, ζ (see Eq. 3). The inability to set the exact point on the oscilloscope was an issue that arose once again when finding the time difference, Δt, between two peaks of the free response of the system. From this, an inaccuracy in the experimental damped natural frequency, ω_d, of the system ensued, which was amplified due to the fact that this was taken as an average of multiple values, much like δ was (see Eq. 4 and Table 4). These discrepancies in ζ and ω_d traversed through into the calculations for the experimental natural frequency of the system (see Eq. 5), with the error in ζ also affecting the analytically obtained damped natural frequency ω_d, and the theoretical plot in Fig. 3 (see Eq. 15). The experimental natural frequency of the system differed from the theoretical value by $\left|\frac{\omega_{n,theo}-\omega_{n,exp}}{\omega_{n,theo}}\right| \approxeq 13.3\%$, and the damped natural frequency differed by $\left|\frac{\omega_{d,theo}-\omega_{d,exp}}{\omega_{d,theo}}\right| \approxeq 13.3\%$, generally speaking, these percentage errors are quite large and more care should have been taken in the experimental phase.

A comparison of Figure 2 and 3 yielded a graphical approximation of the accuracy of the experiment conducted. Previously mentioned errors aside, there were additional factors that contributed to the discrepancy in the shape of Figure 2 relative to Figure 3. Such factors being that a low number of samples (9) were taken experimentally under forced vibration in Figure 2, with less samples being taken after $\frac{\omega}{\omega_n} = 1$ than before, this is apparent by looking at the shape of the plot on either side of $\frac{\omega}{\omega_n} = 1$, the plot on the left-hand side is much smoother and more true to Figure 3's description of the shape. This inconsistency causes a disparity in the shape of the graph around the aforementioned point. Another aspect that resulted in an error in the plot is that under experimental conditions it was difficult to get the system to resonance, and so the peak of the graph is not accurate to what would be expected theoretically (though the peak magnitude cannot directly be compared here due to the nature of the two plots [one is non-dimensionalised]).

6 Conclusion

Conclusively, it is obvious from the all of the errors detailed above, that the overall results were not within an acceptable error range that can be deemed "accurate", as the experimental natural frequency, ω_n, was incorrect by a considerable 13.3% and the experimental plot differed from the theoretical plot quite significantly. A greater effort should have been taken to minimise this by increasing the number of samples taken in both the free response and forced response procedures to reduce the potential impact that outlier values or random error would have on the experimental results. The general trend of Fig. 2 does somewhat prove the efficacy of the experiment, but since the theoretical plot itself wasn't entirely accurate either (since the experimentally determined damping ratio was used in all necessary calculations) it made it slightly more difficult to determine the validity of the comparison. The experimental results were able to, however, show the relationships between various vibration equations. Thus, loosely proving the general dynamics of the single degree-of-freedom spring-mass-damper system under forced excitation due to a rotating out-of-balance mass.

7 Appendix

7.1 Appendix A

As discussed in the Results, an alternative method to calculating the theoretical plot of the amplitude of vibration of a system under forced vibration against the frequency ratio is to utilise the relationship of:

$$F_0 = me\omega^2 \tag{1}$$

where m is the mass of the out-of-balance weight and is equal to 4g, e is the radius that the mass rotates at and is equal to 30mm, and ω is the frequency that the disk is rotating at. This relationship can then be substituted into the following equation:

$$\frac{kX}{F_0} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \tag{2}$$

to get:

$$X = \frac{me\omega^2/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \tag{3}$$

Now, k is known from Eq. 11, ζ is known from Eq. 3, ω_n is known from Eq. 13, and the values of m and e are listed above. Thus, the only unknown is ω, and so amplitude becomes a function of ω, $X(\omega)$. Substituting a sufficient number of values of ω, the following plot may be obtained:

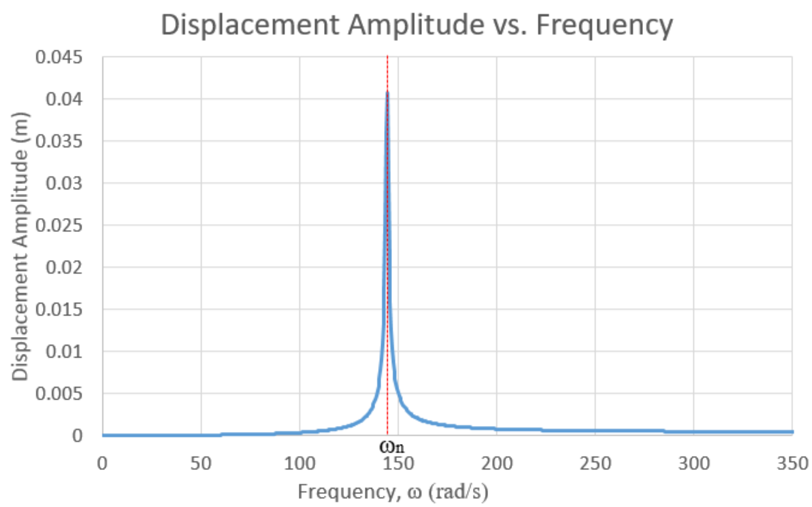

Figure 4: Theoretical displacement amplitude against excitation frequency

As is immediately obvious, the trend in Fig. 4 is exactly the same as that of Fig. 3.

References

[1] UNSW Australia School of Mechanical and Manufacturing Engineering,. Schematic diagram of the rotation out-of-balance test rig and instrumentation. 2016. Web. Aug. 2016.