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Homework 1
Problem 6 (10 Points)

Theorem:

$$(\forall i \in \{1, ..., n\} : x_i \leq \frac{2}{3}) \Leftrightarrow (\exists S \subseteq \{1, ..., n\} : \frac{1}{3}) \leq (\sum_{i \in S} x_i) \leq \frac{2}{3}$$

Proof:

$$p = (\forall i \in \{1, ..., n\} : x_i \leq \frac{2}{3})$$

$$q = (\exists S \subseteq \{1, ..., n\} : \frac{1}{3} \leq (\sum_{i \in S} x_i) \leq \frac{2}{3})$$

$p \Rightarrow q$:

Proof by Cases:
Let $X = i \in \{1, ..., n\} : x_i$
Case 1:
$\exists x \in X : \frac{1}{3} \leq x \leq \frac{2}{3}$
Then there exists a subset S containing x s.t. $\frac{1}{3} \leq (\sum_{i \in S} x_i) \leq \frac{2}{3}$
Namely the set containing element x between $\frac{1}{3}$ and $\frac{2}{3}$ as described above.

Case 2:
$\forall x \in X : x < 1/3$
Let B be the largest subset of X s.t. $\sum_{i \in B} x_i < 1/3$.
Let $Y = X - B$ and let y be some element of Y.
Then, $\frac{1}{3} \leq (\sum_{i \in B} x_i) + y \leq \frac{2}{3}$
Prove the above proposition by contradiction:
Assume that $\frac{1}{3} > (\sum_{i \in B} x_i) + y > \frac{2}{3}$
If $(\sum_{i \in B} x_i) + y < 1/3$ then y would be an element of B, not Y; a contradiction.
If $(\sum_{i \in B} x_i) + y > \frac{2}{3}$ then $y > \frac{2}{3}$, but there is no element in X s.t. $x > \frac{1}{3}$, another contradiction.

$q \Rightarrow p$:

Direct Proof:
Let $X = i \in \{1, ..., n\} : x_i$
Take the smallest subset of X, call it V, s.t. $\sum_{i \in v} = 1/3$.
Take $C = X - V$, and minimize C s.t. it contains one and only one element.
This element can, at most be equal to $\frac{2}{3}$, thereby showing that $\forall i \in \{1, ..., n\} : x_i \leq \frac{2}{3}$