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CS 70, Summer 2013
Homework 1
Problem 5 (7 Points)

5a. [3 points]

Theorem: For all natural numbers n, if n is odd then $n^2 + 2n$ is odd.

Direct Proof:

It can be shown that n^2 is odd (for odd n), and $2n$ is even, and that the operation $odd + even$ always produces odd output.

Proof by contradiction (n^2 is odd, for odd n):

Assume n^2 is even, for odd n, then n^2 is divisible by only one even number, 2. It follows that if n^2 is divisible by 2, then n must be divisible by $\sqrt{2}$, a contradiction since $n \in \mathbb{N}$

Direct proof($2n$ is even):

For any choice of n, $2n$, is divisible by 2. Therefore $2n$ is even.

Direct proof ($even + odd = odd$):

For some numbers $a, b \in \mathbb{N}$, $(2a) + (2b + 1)$ is of the form $even + odd$.

$$(2a) + (2b + 1) \tag{1}$$
$$= 2(a + b) + 1 \tag{2}$$
$$= 2n + 1,\ n \in \mathbb{N} \tag{3}$$
$$\tag{4}$$

and, by definition, 2n+1 is odd.

5b. [3 points]

Theorem: For all natural numbers n, $n^2 + 7n + 1$ is odd.

Proof by cases:

Case 1(n is odd):

For n odd, it can be shown that $n^2 + 7n + 1$ is of the form $odd + odd + 1$ which is odd.

Proof by contradiction (n^2 is odd, for odd n):

Assume n^2 is even, for odd n, then n^2 is divisible by only one even number, 2. It follows that if n^2 is divisible by 2, then n must be divisible by $\sqrt{2}$, a contradiction since $n \in \mathbb{N}$

Direct proof($7n$ is odd):

For any choice of n, $7n$, is not divisible by 2. Therefore $7n$ is odd.

Direct proof ($odd + odd = even$):

For some numbers $a, b \in \mathbb{N}$, $(2a+1)+(2b+1)$ is of the form $odd + odd$.

$$\begin{aligned}
&(2a+1)+(2b+1) && (5)\\
&= 2(a+b+1) && (6)\\
&= 2n,\ n \in \mathbb{N} && (7)\\
& && (8)
\end{aligned}$$

and, by definition, 2n is even.

Following directly from above, if $odd + odd = even$, then $odd + odd + 1 = even + 1$, and $odd + odd + 1 = odd$.

It was shown above that $n^2 + 7n + 1$ is of the form $odd + odd + 1$, proving the theorem for n *odd*.

Case 2 (n is even):

It can be shown that $n^2 + 7n + 1$ is of the form $even + even + 1$, for odd n.

Direct proof (n^2 is even):

If n is even, then for some $m \in \mathbb{N}$ $n = 2m$

$$\begin{aligned}
n^2 &= (2m)^2 && (9)\\
&= 4 * m^2 && (10)
\end{aligned}$$

, and 4 is divisble by 2, so n^2 is even.

Direct proof (7n is even):

If n is even, then for some $m \in \mathbb{N}$ $n = 2m$

$$\begin{aligned}
7n &= 7(2m) && (11)\\
& && (12)
\end{aligned}$$

which is divisible by 2, and is therefore even.

Direct Proof ($even + even + 1 = odd$):

For some numbers $a, b \in \mathbb{N}$, $(2a)+(2b)+1$ is of the form $even + even + 1$.

$$\begin{aligned}
&(2a)+(2b)+1 && (13)\\
&= 2(a+b)+1 && (14)\\
&= 2n+1,\ n \in \mathbb{N} && (15)\\
& && (16)
\end{aligned}$$

and, by definition, 2n+1 is odd.

It was shown above that $n^2 + 7n + 1$ is of the form $even + even + 1$, proving the theorem for n *even*.

5c. [3 points]

Theorem: For all real numbers a,b, if $a + b \leq 10$ then $a \leq 7$ or $b \leq 3$

Proof by contradiction:

Assume, if $a + b \leq 10$ then $a > 7$ or $b > 3$.

5d. [3 points]

Theorem: For all real numbers r, if r is is irrational then r+1 is irrational.

Proof:

5e. [3 points]

Theorem: For all natural numbers n, $10n^2 > n!$.

Proof: