Zackery Field
ID: 23031734
CS 70, Summer 2013
Homework 1
Problem 2 (9 Points)

2a. Theorem: $\forall x \exists y \, (xy \geq x^2)$

Proof by Cases:

Case 1 (where $x > 0$):
Let $x > 0$,
$xy \geq x^2$
$y \geq x$
$xy \geq x^2$ holds for $y \geq x$

Case 2 (where $x < 0$):
Let $x < 0$,
$xy \geq x^2$
$y \leq x$
$xy \geq x^2$ holds for $y \leq x$

Case 3 (where $x = 0$):
Let $x = 0$,
$xy \geq x^2$
$0 \geq 0$
$xy \geq x^2$ holds for all $y \in \mathbb{R}$

Let $y = x$,
$xy \geq x^2$
$x^2 \geq x^2$
Choosing $y = x$ validates $\forall x \exists y \, (xy \geq x^2)$, $x \in \mathbb{R}$, regardless of choice for x.

2b. Theorem: $\exists y \forall x \, (xy \geq x^2)$
Proof by contradiction:
Assume for some $a \in \mathbb{R}$ that $y = a$ satisfies $\forall x \, (xy \geq x^2)$
Take $x = 1$,

$$xa \geq x^2 \quad (1)$$
$$-a \geq 1 \quad (2)$$
$$a \leq -1 \quad (3)$$

Take $x = -1$

$$xa \geq x^2 \quad (4)$$
$$a \geq 1 \quad (5)$$

It has been shown that in order to satisfy $\forall x\ (xy \geq x^2)$, a must be both ≥ 1 and ≤ -1, a contradiction. Therefore, theorem is false.

2c. Theorem: $\neg\forall x \exists y\ (xy > 0 \rightarrow y > 0)$

Proof by cases:

$$\neg\forall x \exists y\ (xy > 0 \rightarrow y > 0) \quad (6)$$
$$\exists x \neg\exists y\ (xy > 0 \rightarrow y > 0) \quad (7)$$
$$\exists x \forall y \neg\ (xy > 0 \rightarrow y > 0) \quad (8)$$
$$\exists x \forall y \neg\ (xy \leq 0 \vee y > 0) \quad (9)$$
$$\exists x \forall y\ (xy > 0 \vee y \leq 0) \quad (10)$$

Take $x = 1$,

Case 1 ($y > 0$):

$y \leq 0$ causes $xy > 0 \vee y \leq 0$ to hold, validating the theorem.

Case 2 ($y \leq 0$):

$y \leq 0$ causes $xy > 0 \vee y \leq 0$ to hold, validating the theorem.

For both cases, $y \in \mathbb{R}$, it has been shown that there exists an x s.t. $\exists x \forall y\ (xy > 0 \vee y \leq 0$