Recurrence Relations

$$\begin{cases} a_0, \cdots, a_{r-1} \text{ are given and} \\ a_n = f(a_{n-1}, \cdots, a_{n-r}), \quad n \geq r \end{cases}$$

A sequence $(a_n)_{n\geq 0}$ is uniquely determined by above

Arithmetic Sequence

$$\begin{cases} a_0 = a \\ a_n = a_{n-1} + d, \quad n \geq 1. \end{cases}$$

First few terms: $a, \ a+d, \ a+2d, \cdots$

Generally, $a_n = a + nd$

$$\sum_{k=0}^{n} a_k = \sum_{k=0}^{n} (a+kd) = \frac{\overset{\text{1st}}{a} + \overset{\text{last}}{(a+nd)}}{2} \cdot \underset{\text{number of terms}}{(n+1)}$$

proof)

$$\begin{array}{lccccc} & a & a+d & a+2d & \cdots & a+nd \\ + & a+nd & a+(n-1)d & & \cdots & a \\ \hline & \multicolumn{5}{l}{(2a+nd)(n+1)} \end{array}$$

Ex. Find the sum of years when FIFA World Cup are held from 2000 to 2050.

sol.

$$\frac{2002+2050}{2} \cdot 13$$

Notation) $\prod_{i=0}^{n} a_i = a_0 a_1 \cdots a_n$

Cardinality

Denote $|A|$ by the cardinality of A

If A is a finite set, $|A|$ is the number of elements

Definition 1 Two sets A and B have the same cardinality if and only if there is a bijection $f: A \to B$.

Definition 2 If there is a one-to-one function $f: A \to B$ we write $|A| \leq |B|$.

Remark) Both $=$ and $\leq$ are transitive. That is, if three sets A, B, and C satisfy $|A| = |B| \wedge |B| = |C|$ then $|A| = |C|$. Also, if three sets A, B, and C satisfy $|A| \leq |B| \wedge |B| \leq |C|$, then $|A| \leq |C|$.

Definition (Proper subset) A set B is a proper subset of A if $B \subseteq A \wedge B \neq A$.

Theorem 1 If B is a proper subset of a finite set A, then $|B| < |A|$ (that is $|B| \leq |A| \wedge |A| \neq |B|$)

Theorem 2. If A is an infinite set, then there is a proper subset B such that $|B| = |A|$.

Countably infinite sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \overbrace{\mathbb{Z}[x], \mathbb{F}_2[x]}^{\text{polynomial rings}}, \underbrace{\mathbb{Z}(x), \mathbb{F}_2(x)}_{\text{rational functions}}, \cdots$

Uncountably infinite sets $\mathbb{R}, \mathbb{C}, \mathbb{R}[x], \mathbb{R}(x), \mathbb{C}[x], \mathbb{C}(x), \cdots$

More results on Cardinality

Cantor's Theorem.. For any set A, $|A| < |P(A)|$

Schröder - Bernstein Theorem, For two sets A and B,

$$\left(|A| \leq |B| \ \wedge \ |B| \leq |A| \right) \equiv \left(|A| = |B| \right)$$

That is, if there are injections $f: A \to B$ and $g: B \to A$ then there is a bijection $h: A \to B$.

Applications ① $|\mathbb{Z}^+| < |P(\mathbb{Z}^+)| = |\mathbb{R}| = |\mathbb{C}|$

② $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| = |\mathbb{Z}[x]| = |\mathbb{F}_2[x]|$
$= |\mathbb{Z}(x)| = |\mathbb{F}_2(x)|$ (countably infinite)

③ $|\mathbb{R}| = |\mathbb{C}| = |\mathbb{R}[x]| = |\mathbb{R}(x)| = |\mathbb{C}[x]| = |\mathbb{C}(x)|$

④ $|\mathbb{R}| < |P(\mathbb{R})| = |\{ f \mid f: \mathbb{R} \to \mathbb{R} \text{ is a function} \}|$

Matrices

Definition. Let m and n be positive integers and

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The ith row of A is $[a_{i1} \cdots a_{in}]$

The jth column of A is $\begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}$

Arithmetic. (+) Let $A=[a_{ij}]$ and $B=[b_{ij}]$ be $m \times n$ matrices. The sum $A+B$ is a matrix that has $a_{ij}+b_{ij}$ as (i,j)th entry.

$(\cdot)$ Let $A=[a_{ij}]$ be $m \times k$ matrix and $B=[b_{ij}]$ be $k \times n$ matrix. The product of A and B is denoted by AB and its ijth entry is

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj} = \sum_{p=1}^{k} a_{ip}b_{pj}$$

Ex. If A is 3×4 matrix and B is 5×6 matrix then AB is undefined.

Ex. $AB \neq BA$ with $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Definition (Transpose) Let $A = [a_{ij}]$ be an $m \times n$ matrix. The transpose of A, denoted by A^T is the $n \times m$ matrix obtained by $A^t = [b_{ij}]$ with $b_{ij} = a_{ji}$ for $1 \le i \le n$, $1 \le j \le m$

Definition (Symmetric) Let $A = [a_{ij}]$ be a square matrix. If $A^T = A$, then A is called symmetric

Ex. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Tip) $A = \begin{bmatrix} - \text{Row1} - \\ - \text{Row2} - \\ \vdots \\ - \text{Row } m - \end{bmatrix}$, $A^T = \begin{bmatrix} | & | & & | \\ \text{Row 1} & \text{Row 2} & \cdots & \text{Row } m \\ | & | & & | \end{bmatrix}$

Ex. $A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$ is symmetric.

Definition (Zero-One matrix)

A matrix whose entries are either 0 or 1.

Boolean operations

$$b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{(and)}$$

$$b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{(or)}$$

$A \vee B$: $\vee$ entrywise

$A \wedge B$: $\wedge$ entrywise

Boolean Product.

Let $A = (a_{ij})$ be $m \times k$ zero-one matrix and $B = [b_{ij}]$ be $k \times n$ zero-one matrix. The Boolean product of A and B, denoted by $A \odot B$ is the $m \times n$ matrix whose $(i,j)^{th}$ entry c_{ij} is

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{ik} \wedge b_{kj})$$

Ex. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

$$A \odot B = \begin{bmatrix} (1\wedge1)\vee(0\wedge0) & (1\wedge1)\vee(0\wedge1) & (1\wedge0)\vee(0\wedge1) \\ (0\wedge1)\vee(1\wedge0) & (0\wedge1)\vee(1\wedge1) & (0\wedge0)\vee(1\wedge1) \\ (1\wedge1)\vee(0\wedge0) & (1\wedge1)\vee(0\wedge1) & (1\wedge0)\vee(0\wedge1) \end{bmatrix}$$

$$= \begin{bmatrix} 1\vee0 & 1\vee0 & 0\vee0 \\ 0\vee0 & 0\vee1 & 0\vee1 \\ 1\vee0 & 1\vee0 & 0\vee0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Notations) $A^n = \underbrace{A \cdot A \cdot \cdots \cdot A}_{n \text{ times}}$

$$A^{[n]} = \underbrace{A \odot A \odot \cdots \odot A}_{n \text{ times.}}$$

$A^0 = I$, $A^{[0]} = I$.