Counting Principles:

Product Rule: A procedure consisting of two tasks

If there are n_1 ways to do the first task

and for each of these ways of doing the first task,

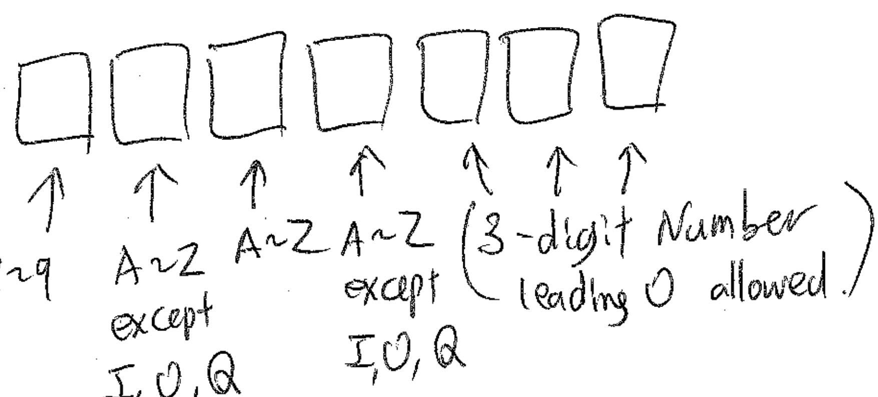
there are n_2 ways to do the second task

then there are $n_1 n_2$ ways to do the procedure.

The extended product rule:

$$\left.\begin{array}{l} n_1 \text{ ways to do the first task} \\ \vdots \\ n_k \text{ ways to do the } k^{th} \text{ task} \end{array}\right\} \to n_1 \cdots n_k \text{ ways to do the procedure}$$

Ex. a) License plates. How many?



b) Recently on Sept 30, 2025, 9VTH090 is observed

Assuming lexicographic order, how many license plates are observed until then?

Sum Rule: If a task can be done either in n_1 ways in Case 1 or n_2 ways in Case 2, with Case 1 and Case 2 are mutually disjoint and exhaustive. Then the task can be done in $n_1 + n_2$ ways.

Extended Sum Rule:

$$\left.\begin{array}{l} \text{Case } 1 : n_1 \text{ ways} \\ \vdots \\ \text{Case } k : n_k \text{ ways} \end{array}\right) \rightarrow \begin{array}{l} n_1 + \cdots + n_k \text{ ways to} \\ \text{do the procedure.} \end{array}$$

$$|A_1 \cup \cdots \cup A_m| = |A_1| + \cdots + |A_m| \quad \text{when } A_i \cap A_j = \emptyset$$

for all $i \neq j$.

Inclusion-Exclusion

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Ex. How many positive integers not exceeding 100 are divisible by 4 or by 6?

Extended Inclusion-Exclusion

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\underbrace{1 \le i_1 < \cdots < i_k \le n}_{\text{if } k=1,\ i_1 \text{ only}}} \underbrace{|A_{i_1} \cap \cdots A_{i_k}|}_{k\text{-fold intersection}}$$

Ex.
$$\begin{aligned} |A_1 \cup A_2 \cup A_3| = {} & |A_1| + |A_2| + |A_3| \\ & - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| \\ & + |A_1 \cap A_2 \cap A_3| \end{aligned}$$

Pigeon Hole Principle.

If k is a positive integer and $k+1$ or more objects are placed into k boxes, then at least one box contains two or more objects.

Ex. Let n be a positive integer. There is a multiple of n that has only 0s and 1s in its decimal expansion.

Sol. Consider $n+1$ integers

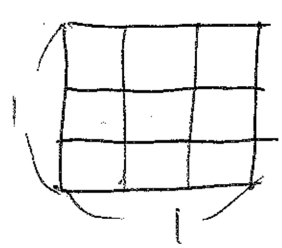
$$\begin{array}{r} 1 \\ 11 \\ 111 \\ \vdots \\ 1\cdots1 \quad (n+1 \text{ 1's}) \end{array}$$

By PHP, two of these have the same remainder when divided by n. Take the difference of these. The larger − the smaller is divisible by n, and its decimal expansion consists of 1's in the beginning, 0's in the end.

The Generalized PHP.

If more than kn objects are placed into k boxes, at least one box contains more than n objects

Ex. 10 points are placed inside $1m \times 1m$ square. Then there are two of these points whose distance is at most $\frac{\sqrt{2}}{3}$ m. Hint:



Definition (Subsequence) Given a sequence $a_1, \dots, a_n$, a subsequence is $a_{i_1}, \dots, a_{i_k}$ where $1 \le i_1 < \dots < i_k \le n$. (if $k=1$, only consider a_{i_1}). The number k is called the length of the subsequence.

Definition (increasing)

A sequence such that $a_n \le a_{n+1}$ for each n.

Definition (strictly increasing)

A sequence such that $a_n < a_{n+1}$ for each n.

Note: decreasing, strictly decreasing are similarly defined.
$\ge$, $>$

Erdős - Szekeres Theorem.

Suppose $a_1, a_2, \dots, a_{n^2+1}$ be a sequence of real numbers. There is a subsequence of length $n+1$ that is either increasing or decreasing.

Proof. Let $a_1, \dots, a_{n^2+1}$ be the given sequence. For each term a_k associate (i_k, d_k) where i_k is the length of longest increasing subsequence starting at a_k, and d_k is the length of longest decreasing subsequence. If the conclusion is false, than there are n^2 possible (i_k, d_k). Since we have n^2+1 terms, two terms a_s, a_t with $s<t$ are associated with the same pair. That is, $i_s = i_t$, $d_s = d_t$.

If $a_s \le a_t$, then a_s followed by an increasing subsequence of length i_t will have length i_s+1. If $a_s > a_t$, then a_s followed by a decreasing subsequence of length d_t will have length d_s+1. Contradiction of maximality of i_s and d_s.