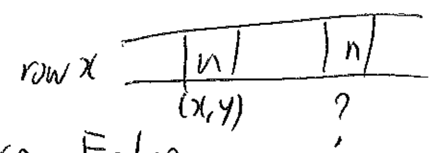
Key elements of Sudoku solving program

- find_empty (board) # Finds an empty position
- valid(board , (x,y), n) # validity of $n=1,2,\cdots,9$ at (x, y) position (row x, column y)

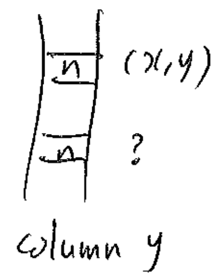
is composed of

i) check row x for duplicates,



if dup is found, return False

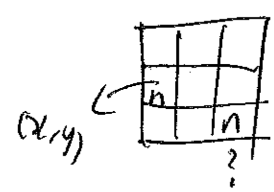
ii) check column y for duplicates



if dup is found, return False

iii) check box for duplicates

if dup is found in the box containing (x, y)
return False



If no dup, return True.

- solve (board)

is a recursive function that first finds an empty position and try $n=1,2,\cdots,9$ for valid (board , (x,y) , n)

if it is valid , fill (x,y) with n, run solve(board).

if solve(board) is False, set (x,y) back to 0 and try again with a different number, repeat on all the empty positions.

Nim (normal play: last person taking the last stone wins)

Winning strategy.

Definition (Nim-sum) Let $x_1, \cdots, x_n$ be the number of stones in each heap $1, \cdots, n$. The Nim-sum is the bitwise xor of $x_1, \cdots, x_n$, i.e. $s = \bigoplus_{i=1}^{n} x_i$

Main Lemma Let $x_1, \cdots, x_n$ be the number of stones in each heap and the Nim-sum is s (old Nim-sum). Let $y_1, \cdots, y_n$ be the number of stones in each heap after a valid move. (kth heap is changed) and the Nim-sum $t = \bigoplus_{i=1}^{n} y_i$ (New Nim-sum). Then $t = s \oplus x_k \oplus y_k$.

proof. $t = s \oplus s \oplus t$ (by $s \oplus s = 0$)

$$= s \oplus \left(\bigoplus_{i=1}^{n} x_i \oplus \bigoplus_{i=1}^{n} y_i \right)$$

$$= s \oplus \left(\bigoplus_{\substack{i=1 \\ i \neq k}}^{n} (x_i \oplus x_i) \right) \oplus (x_k \oplus y_k) \quad \left(\begin{array}{l} y_i = x_i \text{ if } i \neq k. \\ y_k < x_k \end{array} \right)$$

$$= s \oplus x_k \oplus y_k \quad (\text{by } x_i \oplus x_i = 0 \text{ if } i \neq k)$$

Lemma 1. If $s = 0$, a valid move results in $t \neq 0$.

proof. A valid move on the kth heap gives $t = s \oplus x_k \oplus y_k$ and $t = 0 \oplus x_k \oplus y_k = x_k \oplus y_k$. Since $y_k < x_k$, $x_k \oplus y_k \neq 0$. we have $t \neq 0$.

Lemma 2. If $s \neq 0$, there is a valid move that makes $t = 0$.

proof. Let x_k be the heap k such that the leading binary digit of s at dth bit is identical to dth bit of x_k. Take $y_k = s \oplus x_k$. Then $y_k < x_k$ and $t = s \oplus x_k \oplus y_k = s \oplus x_k \oplus s \oplus x_k = 0$.

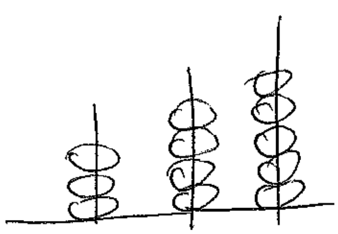
Theorem 1 Let $x_1, \cdots, x_n$ be the number of stones in each heap. with Nim-sum $s = \bigoplus_{i=1}^{n} x_i \neq 0$. Then the first player has a winning strategy.

proof. 1st player can make the Nim-sum 0 after a valid move. 2nd player's valid move gives a nonzero Nim-Sum. Repeating this, 1st player wins.

Theorem 2 Let $x_1, \cdots, x_n$ be the number of stones in each heap. with Nim-sum $s = \bigoplus_{i=1}^{n} x_i = 0$. Then the second player has a winning strategy.

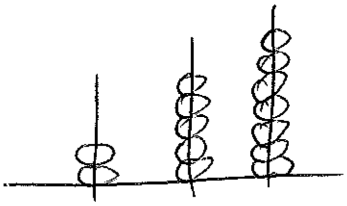
proof. 1st player's valid move makes the Nim-Sum nonzero. 2nd player can make the Nim-Sum 0 after a valid move. Repeating this, 2nd player wins.

Ex.



Assuming optimal plays for both players who wins?

Ex.



Assuming optimal plays for both players who wins?

Predicates and Quantifiers

Propositional function P at x : once a variable x is assigned, the statement $P(x)$ becomes a proposition and has a truth value.

Ex1. let $P(x)$ be the statement "$x > 3$"

$P(4)$ is T, $P(2)$ is F.

Ex3. let $Q(x,y)$ denote "$x = y+3$"

$Q(1,2)$ is F, $Q(3,0)$ is T.

Universal quantifier $\forall$

$\forall x\, P(x)$ is T if $P(x)$ is true for every x.

is F if there is x for which $P(x)$ is false

Existential quantifier $\exists$

$\exists x\, P(x)$ is T if $P(x)$ is true for some x.

is F if $P(x)$ is false for every x.

Ex. With the domain $\mathbb{R}$,

$\exists x\,(x^2+1 \geq 5)$ is T (because $2^2+1 \geq 5$)

$\forall x\,(x^2+1 \geq 5)$ is F (because $0^2+1 < 5$ and 0 is a counterexample

Ex. $\forall x\,(P(x) \wedge Q(x)) \equiv (\forall x\, P(x)) \wedge (\forall x\, Q(x))$

Negations (De Morgan's Laws)

$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$

Note) When we negate, $\left(\begin{array}{l} \wedge \text{ becomes } \vee \\ \vee \text{ becomes } \wedge \\ \exists \text{ becomes } \forall \\ \forall \text{ becomes } \exists \end{array}\right)$ and negate each proposition involved.

Ex. $\neg \forall x\, (x^2 > x) \equiv \exists x\, (x^2 \le x)$

Explanation: The negation of "$x^2 > x$ for all x" is "there exists x such that $x^2 \le x$."

Ex22. $\neg \forall x\, (P(x) \to Q(x)) \equiv \exists x\, (P(x) \wedge \neg Q(x))$

proof. Letting $H(x) = (P(x) \to Q(x))$, we have

$$\neg \forall x\, H(x) \equiv \exists x\, \neg H(x)$$

Recalling that $\neg(P(x) \to Q(x)) \equiv P(x) \wedge \neg Q(x)$

we obtain the result.

$\exists!$: There exist unique..

Ex4. $\forall x \exists y\, (x+y=0)$ is T in the domain $\mathbb{R}$.

$\exists y\, \forall x\, (x+y=0)$ is F ".

so distinct quantifiers cannot commute in general.

Proof Methods

Exhaustive Proof: Proof by cases.

$$\left[(p_1 \vee \cdots \vee p_n) \to q\right] \equiv \left[(p_1 \to q) \wedge \cdots \wedge (p_n \to q)\right]$$

Ex 1. $(n+1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$.

The proof exhausts all cases $n = 1, 2, 3, 4$.

Ex. Let $x \% y$ be the remainder when x is divided by y. Show that $n^3 \% 9 \in \{8, 0, 1\}$

Existence Proof.

Constructive

Ex. There exist irrational numbers x and y such that x^y is rational.

proof. Take $x = \sqrt{2}$, $y = \log_{\sqrt{2}} 3$ Then x, y are irrational and $x^y = 3$ is rational.

Nonconstructive

Ex. There exist irrational numbers x and y such that x^y is rational.

proof. If $\sqrt{2}^{\sqrt{2}}$ is irrational, then $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^2 = 2$ is rational, take $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$

If $\sqrt{2}^{\sqrt{2}}$ is rational, take $x = \sqrt{2}$, $y = \sqrt{2}$