1. Logic and Proofs.

$p, q, r, \cdots$: statements

T, F : truth value (True, False)

$\neg p$: negation of p, read as "not p"

Truth Table (list of truth values for each case)

p	$\neg p$
T	F
F	T

$\neg(\neg p) \equiv p$

$\wedge$: and, read "$p \wedge q$" as "p and q"

$\vee$: or, read "$p \vee q$" as "p or q"

$\oplus$: xor, exclusive or

$\rightarrow$: implies, read "$p \rightarrow q$" as "p implies q"

$\leftrightarrow$: if and only if,

Consider a statement $p \rightarrow q$

Converse of $p \rightarrow q$ is $q \rightarrow p$

Contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$, (equivalent to $p \rightarrow q$)

Two statements of the same truth value are equivalent. ($\equiv$)

Ex) $\neg(p \rightarrow q) \equiv p \wedge \neg q$, $(p \rightarrow q) \equiv (\neg p) \vee q$

Ex) $(p \rightarrow q) \equiv (\neg q \rightarrow \neg p)$

p	q	$p \vee q$	$p \wedge q$	$p \oplus q$	$p \to q$	$q \to p$	$p \leftrightarrow q$
T	T	T	T	F	T	T	T
T	F	T	F	T	F	T	F
F	T	T	F	T	T	F	F
F	F	F	F	F	T	T	T

De Morgan's Laws

p	q	$p \vee q$	$\neg p \wedge \neg q$	$p \wedge q$	$\neg p \vee \neg q$
T	T	T	F	T	F
T	F	T	F	F	T
F	T	T	F	F	T
F	F	F	T	F	T

- $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$
- $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$

Bit operations T: 1, F: 0

Ex)

	01 1011 0110
	11 0001 1101
Bitwise or	11 1011 1111
Bitwise and	01 0001 0100
Bitwise xor	10 1010 1011

Tautology: Always T
Contradiction: Always F

Commutativity

$$(p \vee q) \equiv (q \vee p)$$
$$(p \wedge q) \equiv (q \wedge p)$$

Associativity

$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$
$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

Distributive Laws

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$
$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

De Morgan's Laws

$$\neg (p \vee q) \equiv \neg p \wedge \neg q$$
$$\neg (p \wedge q) \equiv \neg p \vee \neg q$$

Extension of Distributive Laws

$$p \vee \left(\bigwedge_{i=1}^{n} q_i \right) \equiv \bigwedge_{i=1}^{n} (p \vee q_i)$$

$$p \wedge \left(\bigvee_{i=1}^{n} q_i \right) \equiv \bigvee_{i=1}^{n} (p \wedge q_i)$$

Extension of De Morgan's Laws

$$\neg \left(\bigvee_{i=1}^{n} p_i \right) \equiv \bigwedge_{i=1}^{n} (\neg p_i)$$

$$\neg \left(\bigwedge_{i=1}^{n} p_i \right) \equiv \bigvee_{i=1}^{n} (\neg p_i)$$

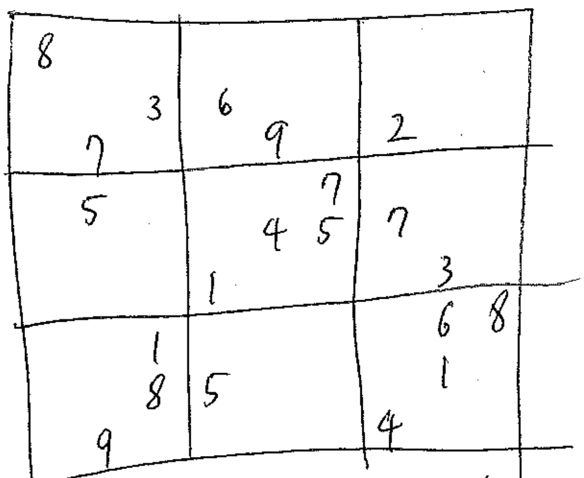
Satisfiability: There exist assignments of truth values that makes the statement true.

Ex. (Sudoku)

9x9 puzzle, made of nine 3x3 subgrids

each cell is assigned one of 1,2,..., 9. Some cells are given

The puzzle is solved if every row, every column, every subgrids contain each of 9 possible numbers.

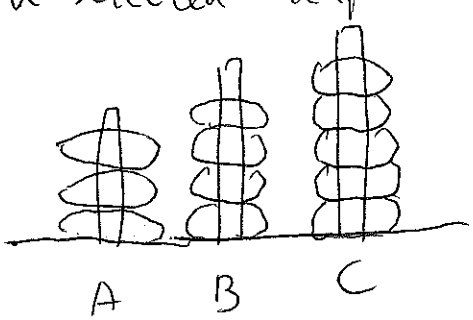


"World's hardest Sudoku"
by Arto Inkala, 2012

Ex. (Nim)

Two players take turns removing objects from distinct heaps.

On each turn, a player must remove at least one object from a selected heap. A player who takes the last object wins.



In this setup, the first player has a winning strategy.