A formal theory of sets define sets via ZFC axioms

Intuitively, sets are collection of objects.

Notation $\in$: belongs to

$\notin$: does not belong to.

Empty set $\emptyset = \{\}$

Roster method $\{1, 2, 3\}$

Set builder notation $\{x \in \mathbb{Z}^+ \mid 1 \le x \le 3\}$

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ the set of natural numbers

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ the set of all integers

$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ the set of positive integers

$\mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$ the set of rational numbers

$\mathbb{R}$. the set of all real numbers

Intervals
$[a, b] = \{x \mid a \le x \le b\}$
$[a, b) = \{x \mid a \le x < b\}$
$(a, b] = \{x \mid a < x \le b\}$
$(a, b) = \{x \mid a < x < b\}$

Subsets

$A \subseteq B$ if and only if $\forall x (x \in A \to x \in B)$

$A = B$ if and only if $(A \subseteq B) \wedge (B \subseteq A)$

Remark) $A = B$ if and only if $\forall x (x \in A \leftrightarrow x \in B)$

Cartesian Product

$A \times B = \{ (a,b) \mid a \in A \wedge b \in B \}$, $A^2 = A \times A$

Power set

Given a set A, $\mathcal{P}(A) = \{ B \mid B \subseteq A \}$, set of all subsets of A.

Truth sets

The truth set of $P(x)$ is $\{ x \in D \mid P(x) \}$, the set of $x \in D$ such that $P(x)$ holds.

Ex. $D = \mathbb{Z} \rightarrow$ The truth set of $|x| \leq 2$ is $\{-2, -1, 0, 1, 2\}$

$D = \mathbb{R} \rightarrow$ The truth set of $|x| \leq 2$ is $[-2, 2]$

Intersection $A \cap B = \{ x \mid x \in A \wedge x \in B \}$

Union $A \cup B = \{ x \mid x \in A \vee x \in B \}$

set difference $A - B = \{ x \mid x \in A \wedge x \notin B \}$

Commutative Laws $A \cup B = B \cup A$, $A \cap B = B \cap A$

Associative laws $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$

Distributive laws $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Ex. $(A \subseteq B) \equiv (A \cap B = A) \equiv (A \cup B = B) \equiv (A - B = \emptyset)$

Extensions $\bigcup_{i=1}^{n} A_i$, $\bigcup_{i=1}^{\infty} A_i$, $\bigcap_{i=1}^{n} A_i$, $\bigcap_{i=1}^{\infty} A_i$, $\bigcup_{i \in I} A_i$, $\bigcap_{i \in I} A_i$

Multisets.

An unordered collection of elements where an element can occur more than once.

$\{m_1 \cdot a_1, m_2 \cdot a_2, \ldots, m_r \cdot a_r\}$

m_i's : multiplicities, a_i's : distinct elements

Ex21. $P = \{4 \cdot a, 1 \cdot b, 3 \cdot c\}$, $Q = \{3 \cdot a, 4 \cdot b, 2 \cdot d\}$

$P \cup Q = \{\max(4,3) \cdot a, \max(1,4) \cdot b, \max(3,0) \cdot c, \max(0,2) \cdot d\}$

$P \cap Q = \{\min(4,3) \cdot a, \min(1,4) \cdot b, \min(3,0) \cdot c, \min(0,2) \cdot d\}$

$P - Q = \{\max(4-3,0) \cdot a, \max(1-4,0) \cdot b, \max(3-0,0) \cdot c, \max(0-2,0) \cdot d\}$

$P + Q = \{(4+3) \cdot a, (1+4) \cdot b, (3+0) \cdot c, (0+2) \cdot d\}$

Functions.

domain of f codomain of f

A function f from a set A to a set B is an assignment of exactly one element of B to each element of A.

One-to-one (injection)

let f be a function $f : A \to B$

f is one-to-one if $\forall a \forall b \, (f(a) = f(b) \to a = b)$

Onto (surjection)

A function $f : A \to B$ is onto if $\forall b \in B, \exists a \in A, (f(a) = b)$

Bijection $\equiv$ (One-to-one) $\wedge$ (Onto)

Compositions

Let $g: A \to B$ and $f: B \to C$. The composition

$f \circ g : A \to C$ is defined by $f \circ g(a) = f(g(a))$

for all $a \in A$.

Ex. $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$

$g: \mathbb{R} \to \mathbb{R}$ by $g(x) = x^3 + 1$

$f \circ g(x) = (x^3+1)^2$, $g \circ f(x) = (x^2)^3 + 1$

Inverse Functions

$f: A \to B$ and $g: B \to A$ are inverses of each other

if $\left(\begin{array}{ll} f \circ g(b) = b & \text{for all } b \in B \\ \text{and } g \circ f(a) = a & \text{for all } a \in A. \end{array} \right)$. We write $g = f^{-1}$.

Ex. $\left. \begin{array}{l} f: \mathbb{R} \to \mathbb{R} \text{ by } f(x) = x^3 \\ g: \mathbb{R} \to \mathbb{R} \text{ by } g(x) = x^{\frac{1}{3}} \end{array} \right)$ are inverses of each other

Sequences

A sequence is a function from a subset of $\mathbb{Z}$ to a set S

Ex. $a_n = \frac{1}{n}$, $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$, $a_4 = \frac{1}{4}$, ...

Ex. (Fibonacci Sequence)

A sequence $(f_n)_{n \geq 0}$ defined by $f_0 = 0$, $f_1 = 1$ and

$f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$ (recurrence relation)

$f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 3$, $f_5 = 5$, $f_6 = 8$, $f_7 = 13$, ...

Summations

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \cdots + a_n$$

Geometric sequence $(r \neq 0)$

$$a_n = a r^n, \quad n \geq 0$$

$$\sum_{j=0}^{n} a_j = \begin{cases} \dfrac{a(r^{n+1}-1)}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

Geometric Series

If $x \in \mathbb{R}$ with $|x| < 1$, then

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

Taking derivatives with respect to x,

$$\sum_{k=1}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2}$$

Double Sums $\sum_i \sum_j a_{ij} = \sum_i \left(\sum_j a_{ij} \right)$

Question) Is it correct? $\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$

Ex.

$i \backslash j$	0	1	2	3	4	5
0	-1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\cdots$	
1		-1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\cdots$
2			-1	$\frac{1}{2}$	$\frac{1}{4}$	$\cdots$
3				-1	$\frac{1}{2}$	$\frac{1}{4} \cdots$

For each $i = 0, 1, \cdots$, we have $\sum_j a_{ij} = 0$

Thus, $\sum_i \left(\sum_j a_{ij} \right) = \sum_i 0 = 0$.

For each $j = 0, 1, \cdots$, we have $\sum_i a_{ij} = -\frac{1}{2^j}$

Thus, $\sum_j \left(\sum_i a_{ij} \right) = \sum_j \left(-\frac{1}{2^j} \right) = -2$

We see that $\sum_i \sum_j a_{ij} \neq \sum_j \sum_i a_{ij}$

Fubini-Tonelli Theorem for Counting Measures

$$\sum_i \sum_j |a_{ij}| = \sum_j \sum_i |a_{ij}| = \sum_{(i,j)} |a_{ij}| \quad \left(\begin{array}{l} \text{If one of the sums} \\ \text{is } \infty \text{, then all are } \infty \end{array} \right)$$

and if any one of these sums is finite, then

$$\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij} = \sum_{(i,j)} a_{ij}$$

Ex. (A special case) If a double sequence $(a_{ij})_{\substack{i \geq 0 \\ j \geq 0}}$ has finitely many negative terms, then

$$\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$$