Khovanov homology, knot Floer homology, and spectral sequences

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Smith College

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Everything in this talk is joint work with Samuel Tripp.

Khovanov homology

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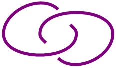

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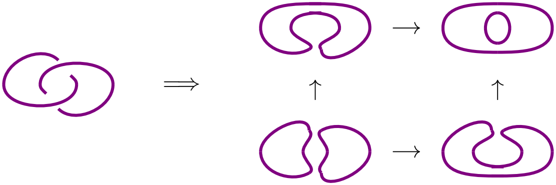

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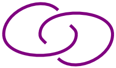
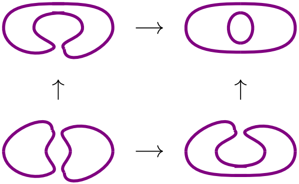


$\Longrightarrow$



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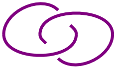
    - We replace vertices by modules and edges by linear maps to get a chain complex $\mathrm{CKh}(D)$.
    - The Khovanov homology $\mathrm{Kh}(D)$ is the homology of this chain complex.



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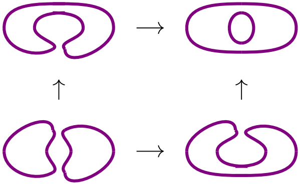

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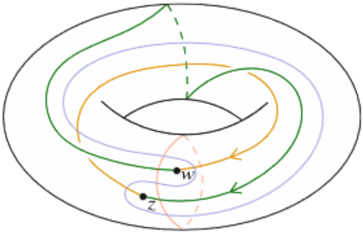

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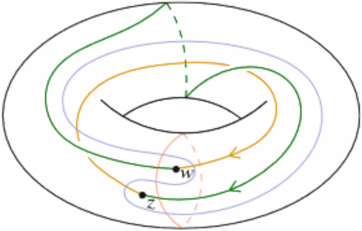

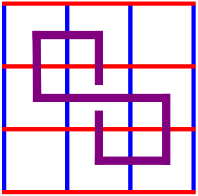

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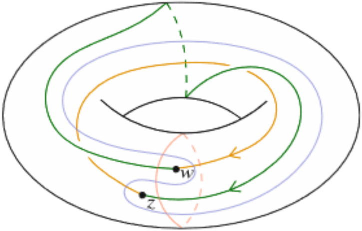
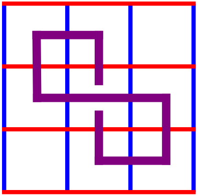
    - Later defined using grid diagrams.
    - In either case, we get knot Floer homology $\mathrm{HFK}(D)$ as the homology of a complex $\mathrm{CFK}(D)$ associated to some kind of diagram D.





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- Both theories also have transverse invariants: $\psi(K) \in \mathrm{Kh}(K)$ and $\theta(K) \in \mathrm{HFK}(K)$.

The Dowlin spectral sequence

- In [Dow18], Dowlin defined a spectral sequence with $E_2 \cong \overline{\mathrm{Kh}}(K)$ and $E_\infty \cong \widehat{\mathrm{HFK}}(K)$.

Spectral sequences

- A **spectral sequence** is, essentially, just a sequence of chain complexes $E_0, E_1, E_2, \ldots$ called "pages" and isomorphisms $H^*(E_i) \to E_{i+1}$.

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\begin{array}{ccc}
E_0^{0,2} & E_0^{1,2} & E_0^{2,2} \\
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- In many cases, the differentials on the higher pages will all be trivial; when this happens, we say that the sequence "converges".
- Many spectral sequences exist between different knot homologies, e.g. $\mathrm{Kh}(K) \to \mathrm{Kh}_{\mathrm{Lee}}(K)$ and $\mathrm{HFK}(K) \to \mathrm{HF}(S^3)$.

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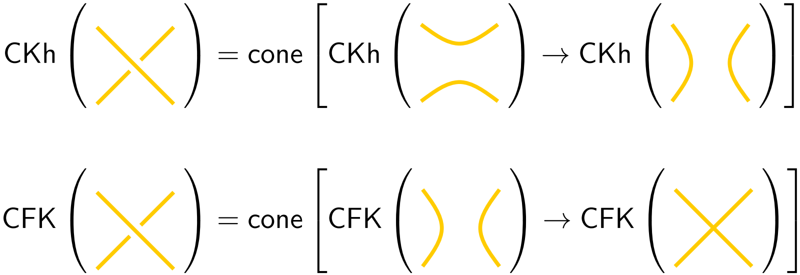
- It has also been used to prove that $\mathrm{Kh}(K)$ detects several more knots, like the figure-eight knot [Bal+21], the cinquefoil [BHS21], and the $(2, 6)$-torus knot [Mar20].

Cubes of resolutions

One important development for comparing Kh(K) and HFK(K) was a cube of resolutions description of HFK(K) that mirrors Kh(K) [OS09].



Diagrams

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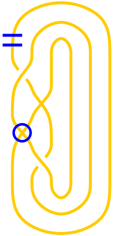

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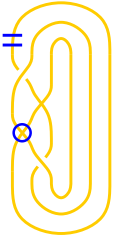

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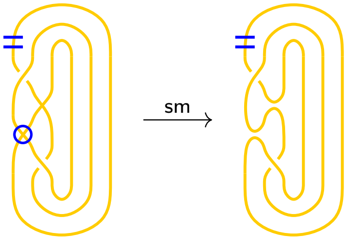
- The knot represented by a partially-singular braid diagram can be drawn by taking the **unoriented smoothing** of all the singular vertices.



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- The dimension of the module is $O(2^n)$ for a diagram with n crossings as was the case with $\mathrm{CKh}(D)$.
- Constructing the spectral sequences requires that the diagram D have "enough" singular vertices to satisfy certain algebraic conditions.

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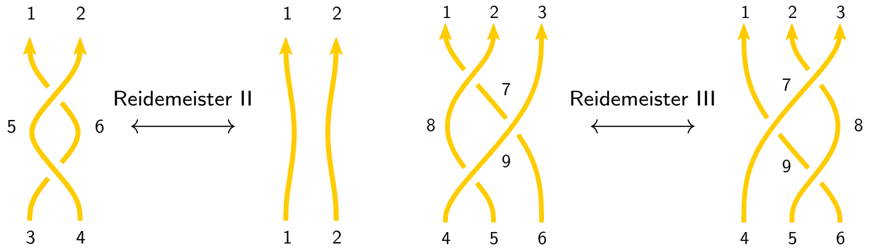
- How do we prove that every page is a knot invariant?
    - Find (filtered) chain maps $C_2^-(D) \to C_2^-(D)$ that correspond to Reidemeister moves. . .
    - . . . or Markov moves if you want to represent your knots as braids.



Finding diagrams for knots

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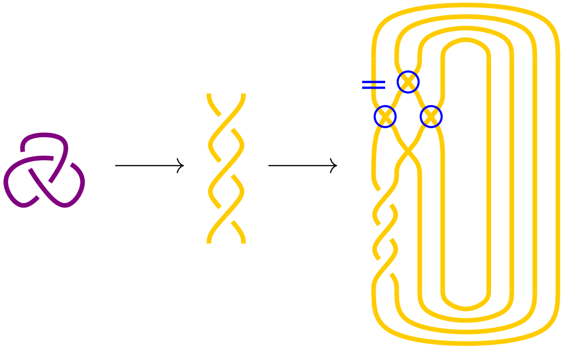
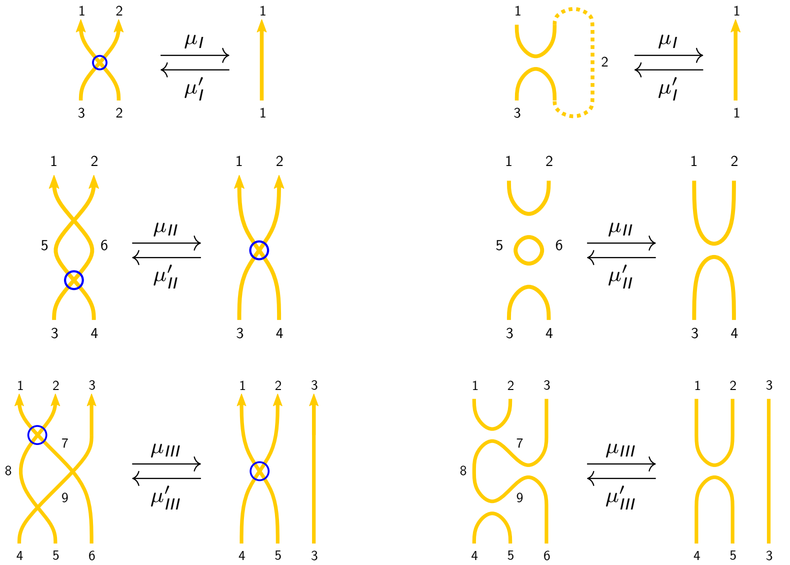
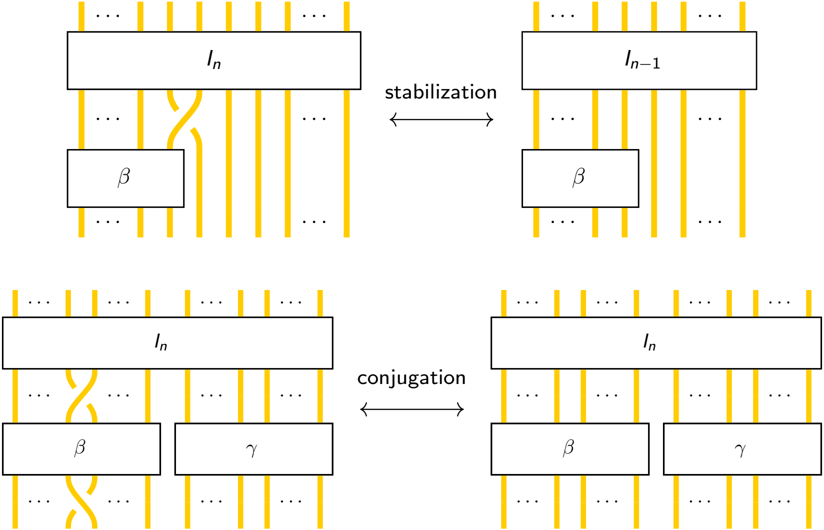
- Then, we embed braids into a partially-singular diagram with "enough singularities" that the spectral sequence construction works.



MOY Moves

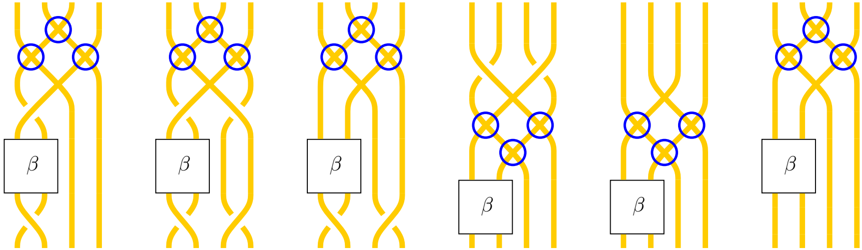


Stabilization and Conjugation



Example Conjugation Invariance

The steps to prove conjugation invariance for $n = 2$ and $\alpha = \sigma_1$.



Conclusion

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- We proved that the Dowlin spectral sequence from $\mathrm{Kh}(K)$ to $\mathrm{HFK}(K)$ is itself an invariant, which proves that each page E_k is its own new knot invariant.
- Possible future directions:
  - Compare transverse invariants $\psi(K)$ and $\theta(K)$ using the spectral sequence.
  - Also look at concordance invariants $s(K)$ and $\tau(K)$.
  - Efficiently compute pages of the spectral sequence for more knots.
  - Find an annular version of the spectral sequence?

The End

Thanks for listening!

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