EPFL
ENAC TRANSP-OR
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Mathematical Modeling of Behavior
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EXERCISE SESSION 9 (Solution)

Exercise 1 Consider a logit model for mode choice with three alternatives: car (c), public transportation (p) and slow modes (s). The utility specifications are the following:

$$\begin{aligned}
U_{c,n} &= \beta_{c,1} + \beta_{c,2} \cdot \text{cost}_{c,n} + \beta_{c,3} \cdot \text{tt}_{c,n} + \varepsilon_{c,n}, \\
U_{p,n} &= \beta_{p,1} + \beta_{p,2} \cdot \text{cost}_{p,n} + \beta_{p,3} \cdot \text{tt}_{p,n} + \varepsilon_{p,n}, \\
U_{s,n} &= \beta_{s,1} + \beta_{s,3} \cdot \text{tt}_{s,n} + \varepsilon_{s,n},
\end{aligned}$$

where $\text{cost}_{i,n}$ is the cost associated by customer n with alternative $i \in \{c, p\}$, and $\text{tt}_{i,n}$ is the travel time associated with alternative $i \in \{c, p, s\}$ by customer n. We denote by $\mathbb{E}[\varepsilon_{i,n}] = \alpha_i$ the mean of the distribution of the error terms $\varepsilon_{i,n}$ $\forall i \in \{c, p, s\}, n$.

Show that it is possible to rewrite the utility functions in order to have $\mathbb{E}[\varepsilon_{i,n}] = 0$ $\forall i \in \{c, p, s\}, n$ and the same probabilities.

Solution: First we need to perform a change of variable by adding and subtracting α_i to each utility function:

$$\begin{aligned}
U_{c,n} &= (\beta_{c,1} + \alpha_c) + \beta_{c,2} \cdot \text{cost}_{c,n} + \beta_{c,3} \cdot \text{tt}_{c,n} + (\varepsilon_{c,n} - \alpha_c), \\
U_{p,n} &= (\beta_{p,1} + \alpha_p) + \beta_{p,2} \cdot \text{cost}_{p,n} + \beta_{c,3} \cdot \text{tt}_{p,n} + (\varepsilon_{p,n} - \alpha_p), \\
U_{s,n} &= (\beta_{s,1} + \alpha_s) + \beta_{c,3} \cdot \text{tt}_{s,n} + (\varepsilon_{s,n} - \alpha_s),
\end{aligned}$$

In this way, $\mathbb{E}[\varepsilon_{i,n} - \alpha_i] = 0$.

With respect to the choice probabilities, we define a new notation for the changes of variable: $U'_{i,n} = V'_{i,n} + \varepsilon'_{i,n}$, where $V'_{i,n} = V_{i,n} + \alpha_i$ and $\varepsilon'_{i,n} = \varepsilon_{i,n} - \alpha_i$, $\forall i \in \{c, p, s\}, n$. Then, the following proves that both models provide the same probabilities:

$$\begin{aligned}
\text{P}(U'_{i,n} \geq U'_{j,n}) &= \text{P}(V'_{i,n} + \varepsilon'_{i,n} \geq V'_{j,n} + \varepsilon'_{j,n}) \\
&= \text{P}(V_{i,n} + \alpha_i + \varepsilon_{i,n} - \alpha_i \geq V_{j,n} + \alpha_j + \varepsilon_{j,n} - \alpha_j) \\
&= \text{P}(V_{i,n} + \varepsilon_{i,n} \geq V_{j,n} + \varepsilon_{j,n}) \\
&= \text{P}(U_{i,n} \geq U_{j,n})
\end{aligned}$$

Exercise 2 Recall the red bus/blue bus paradox that has been seen during the lectures. Travelers initially face a decision between two modes of transportation: car and blue bus. Travel time is the only variable considered in the utility functions and is equal for both modes. We then suppose that a third mode, namely the red bus, is introduced and that all travelers consider it to be exactly the same as the blue bus.

Assume that the error terms for the red and blue bus are correlated and that the correlation is 95%. Derive the scale parameter (μ_m) and calculate the probabilities of choosing car and bus[1].

Solution: Given that the correlation is 95%, μ_m is computed as:

$$1 - \mu^2/\mu_m^2 = 0.95 \Leftrightarrow \underline{\mu_m = \sqrt{1/0.05}} \tag{1}$$

The probabilities of choosing car and bus can be obtained by using the NL model. The expected maximum utility of bus is:

$$\begin{aligned} V_{\text{bus}} &= \frac{1}{\mu_m} \ln(e^{\mu_m \beta T} + e^{\mu_m \beta T}) \\ &= \beta T + \frac{1}{\mu_m} \ln 2 \end{aligned} \tag{2}$$

where T is the travel time, and β is its coefficient. The probability of choosing car is:

$$\begin{aligned} P(\text{car}) &= \frac{e^{\mu V_{\text{car}}}}{e^{\mu V_{\text{car}}} + e^{\mu V_{\text{bus}}}} \\ &= \frac{e^{\beta T}}{e^{\beta T} + e^{\beta T + \frac{1}{\mu_m} \ln 2}} \\ &= \frac{1}{1 + 2^{1/\mu_m}} \end{aligned} \tag{3}$$

Given the value of μ_m, finally we obtain:

$$\underline{P(\text{car})} = \frac{1}{1 + 2^{\sqrt{0.05}}} = \underline{0.461} \tag{4}$$

and

$$\underline{P(\text{bus})} = 1 - P(\text{car}) = \underline{0.539} \tag{5}$$

Exercise 3 Consider the cross-nested logit model whose nesting structure is represented in Figure 1. Let $a_i, i \in \{1, \ldots, 5\}$ be the nodes representing alternative i and $n_m, m \in \{1, 2, 3\}$ be the nodes representing nest m. Let μ_m be the scale parameter of nest m, and α_{im} be the membership parameter of alternative i to nest m. The scale parameter μ is normalized to 1. We ask you to answer the following questions:

[1] Note that μ is normalized to one.

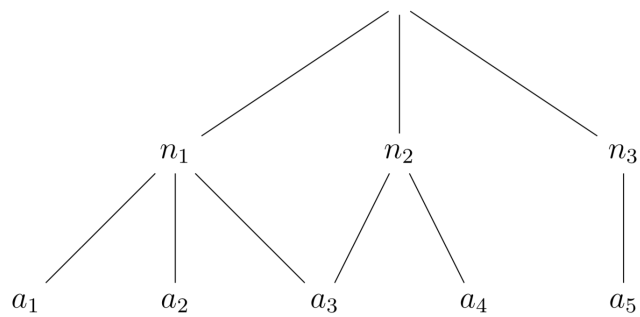

Figure 1: Cross-nested structure.

1. Which α_{im} and μ_m are fixed, to which value and why? Give an exhaustive list.
2. Which α_{im} and μ_m can be estimated? Give an exhaustive list.
3. What properties do the estimated μ_m verify in order to be consistent with random utility?
4. What properties do the estimated α_{im} verify to be consistent with random utility?
5. What is the associated MEV function?

Solution:

1. - Some alternatives do not belong to some nests: $\alpha_{12} = \alpha_{13} = \alpha_{22} = \alpha_{23} = \alpha_{33} = \alpha_{41} = \alpha_{43} = \alpha_{51} = \alpha_{52} = 0$.
   - Some alternatives belong only to one nest: $\alpha_{11} = \alpha_{21} = \alpha_{42} = \alpha_{53} = 1$.
   - Nest 3 contains only one alternative: $\mu_3 = 1$.
2. Those that are not fixed: α_{31}, α_{32}, μ_1, μ_2.
3. $\mu_2, \mu_3 \geq \mu = 1$.
4. $\alpha_{31} + \alpha_{32} = 1$.
5. $G(y) = (y_1^{\mu_1} + y_2^{\mu_1} + \alpha_{31}^{\mu_1} y_3^{\mu_1})^{\frac{1}{\mu_1}} + (\alpha_{32}^{\mu_2} y_3^{\mu_2} + y_4^{\mu_2})^{\frac{1}{\mu_2}} + y_5$.

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