04.03.2021 **Week 2**

Exercise 1 Let $\mathbf{X} \sim N_p(\boldsymbol{\mu}, \Sigma)$ a multivariate Gaussian random vector in $\mathbb{R}^p$. We consider the partition $\mathbf{X} = (\mathbf{X}_1', \mathbf{X}_2')'$ where $\mathbf{X}_1 \in \mathbb{R}^k$, for $1 \leq k < p$ and $\mathbf{X}_2 \in \mathbb{R}^{p-k}$.

1. Derive the conditional density of $\mathbf{X}_1 \mid \mathbf{X}_2 = \mathbf{x}_2$.

   Hint : It might be easier to consider the precision matrix $\Sigma^{-1} = \Psi = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix}$ when writing the Gaussian densities above.
2. Derive the marginal density of $\mathbf{X}_1$.

Exercise 2 We consider $\mathbf{X} \sim N_p(\boldsymbol{\mu}_X, \Sigma_X)$ and $\mathbf{Y} \mid \mathbf{X} = \mathbf{x} \sim N_q(\boldsymbol{\alpha} + \beta\mathbf{x}, \Sigma)$ where $\boldsymbol{\mu} \in \mathbb{R}^p$, $\Sigma_X \in R^{p\times p}$, $\boldsymbol{\alpha} \in \mathbb{R}^q$, $\beta \in \mathbb{R}^{q\times p}$ and $\Sigma \in \mathbb{R}^{q\times q}$.

1. Prove that $(\mathbf{X}', \mathbf{Y}')' \sim N_{p+q}$, compute its mean and show that

$$\mathrm{V}\,(\mathbf{X}', \mathbf{Y}')' = \begin{pmatrix} \Sigma_X & \Sigma_X\beta' \\ \beta\Sigma_X & \Sigma + \beta\Sigma_X\beta' \end{pmatrix}$$

   Hint : Start by setting $\mathbf{U} = \mathbf{Y} - \boldsymbol{\alpha} - \beta\mathbf{X}$ and showing that $\mathbf{X}$ and $\mathbf{U}$ are independent, then find a matrix A and a vector $\mathbf{c}$ such that $(\mathbf{X}', \mathbf{Y}')' = A(\mathbf{X}', \mathbf{U}')' + \mathbf{c}$.
2. Show that the conditional distribution of $\mathbf{X} \mid \mathbf{Y} = \mathbf{y}$ is Gaussian with

$$\begin{aligned} \mathrm{E}\,[\mathbf{X} \mid \mathbf{Y} = \mathbf{y}] &= \boldsymbol{\mu}_X + \beta\Sigma_X(\Sigma + \beta\Sigma_{\mathbf{X}}\beta')^{-1}(\mathbf{y} - \boldsymbol{\alpha} - \boldsymbol{\mu}_X\beta), \\ \mathrm{V}\,[\mathbf{X} \mid \mathbf{Y} = \mathbf{y}] &= \Sigma_X - \Sigma_X\beta'(\Sigma + \beta\Sigma_X\beta')^{-1}\beta\Sigma_X, \end{aligned}$$

   assuming that the matrices $\Sigma, \Sigma_X, (\Sigma + \beta\Sigma_X\beta')$ are invertible.

Exercise 3 Let $\mathbf{X}_1, \cdots, \mathbf{X}_n$ be i.i.d. random vectors in $\mathbb{R}^p$ with mean $\boldsymbol{\mu}$ and covariance matrix Σ. Consider the sample mean $\bar{\mathbf{X}}$ and the sample variance-covariance matrix S and show that $T_0^2 = n(\bar{\mathbf{X}} - \boldsymbol{\mu})'\, S^{-1}(\bar{\mathbf{X}} - \boldsymbol{\mu})$ is invariant under all the afine transformations $\mathbf{Y}_i = A\mathbf{X}_i + \mathbf{b}$, where A is an invertible matrix of constants and $\mathbf{b}$ is a vector of constants.

Exercise 4 (Simulation of Gaussian random vectors in `R`)
We consider the random vector $\mathbf{X}$ in $\mathbb{R}^2$ from a standard gaussian distribution $N_2(\mathbf{0}, \mathbf{1})$.

1. Generate $N = 1000$ independent random replicates $\mathbf{X}_1, \ldots, \mathbf{X}_n$ of $\mathbf{X}$. *Hint :* The function `rnorm(n)` generates n independent replicates of a univariate random distribution.
   a) Estimate the sample mean and covariance matrix.
   b) plot the sample and draw diagnostic plots of the marginal distributions.

2. We define another Gaussian vector $\mathbf{Y}$ satisfying : $\mathbb{E}\,[Y_1] = 5$, $\mathbb{E}\,[Y_2] = 5$, $\mathrm{Var}(Y_1) = 1$, $\mathrm{Var}(Y_2) = 1$ and $\mathrm{Cov}(Y_1, Y_2) = 0.9$.
   a) Generate $N = 1000$ independent replicates $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ starting from $\mathbf{X}_1, \ldots, \mathbf{X}_n$.
   Hint : The function `chol` computes the Choleski factorization of Σ and returns $\mathbf{A}$ such that $\mathbf{A}'\mathbf{A} = \Sigma$.
   b) plot the sample and draw visual diagnostic plots of the marginal distributions.