EPFL
Spring 2021

Multivariate Statistics
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25.02.2021 **_Week 1_**

Exercise 1 Consider a bivariate Pareto density :

$$f(x,y) = c(x+y-1)^{-p-2}, \text{ for } x,y > 1, \text{ and } p > 2.$$

1. Show that c is equal to $p(p+1)$.
2. Determine the marginal laws of this density and compute $\mathbb{E}[X]$.
3. Calculate the variance-covariance matrix Σ.
4. Consider a sample $(X_1, Y_1)', \ldots, (X_n, Y_n)'$ of independent and identically distributed random vectors following the Pareto density with parameters p. Estimate the parameter p using the maximum likelihood method.

Exercise 2 Let X_1 and X_2 be two independent Gamma random variables with common scale parameters : $X_1 \sim \text{Gamma}(\alpha, \lambda)$ and $X_2 \sim \text{Gamma}(\beta, \lambda)$. Define

$$\begin{aligned} Y_1 &= X_1 + X_2 \\ Y_2 &= \frac{X_1}{X_1 + X_2} \end{aligned}$$

1. Write the joint density of (X_1, X_2).
2. Determine the joint density of (Y_1, Y_2).
3. Deduce the marginal distributions of Y_1 and Y_2.

Exercise 3 Suppose that $\mathbf{X}_1, \ldots, \mathbf{X}_n \in \mathbb{R}^p$ are independent and identically distributed random vectors following a multivariate Gaussian distribution $N_p(\boldsymbol{\mu}, \Sigma)$. We consider the sample mean

$$\bar{\mathbf{X}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{X}_i$$

and the sample variance-covariance matrix

$$S = \frac{1}{n} \sum_{i=1}^{n} \left(\mathbf{X}_i - \bar{\mathbf{X}}\right) \left(\mathbf{X}_i - \bar{\mathbf{X}}\right)'$$

1. Show that $\bar{\mathbf{X}}$ is an unbiased estimate of $\boldsymbol{\mu}$. (i.e. $E\left[\bar{\mathbf{X}}\right] = \boldsymbol{\mu}$).
2. Show that $E[S] = \dfrac{n-1}{n}\Sigma$. Propose another estimate of Σ which is not biased.

Exercise 4 We consider a matrix $\Sigma \in \mathbb{R}^{p \times p}$ and we write

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \text{ and } \Sigma^{-1} = \Psi = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix}.$$

Show the following equations :
(a) $\Sigma_{12}\Sigma_{22}^{-1} = -\Psi_{11}^{-1}\Psi_{12}$
(b) $\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} = \Psi_{11}^{-1}$