04.03.2021 ***Week 2***

Exercise 1 Let $\mathbf{X} \sim N_p(\boldsymbol{\mu}, \Sigma)$ a multivariate Gaussian random vector in $\mathbb{R}^p$. We consider the partition $\mathbf{X} = (\mathbf{X}_1', \mathbf{X}_2')'$ where $\mathbf{X}_1 \in \mathbb{R}^k$, for $1 \leq k < p$ and $\mathbf{X}_2 \in \mathbb{R}^{p-k}$.

1. Derive the conditional density of $\mathbf{X}_1 \mid \mathbf{X}_2 = \mathbf{x}_2$.

   Hint : It might be easier to consider the precision matrix $\Sigma^{-1} = \Psi = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix}$ when writing the Gaussian densities above.
2. Derive the marginal density of $\mathbf{X}_1$.

Solution 1

1. The density of $\mathbf{X}_1 \mid \mathbf{X}_2 = x_2$ is proportional to

$$\begin{aligned} f_{\mathbf{X}_1 \mid \mathbf{X}_2 = \mathbf{x}_2}(\mathbf{x}_1) &= \frac{f_{\mathbf{X}_1, \mathbf{X}_2}(\mathbf{x}_1, \mathbf{x}_2)}{f_{\mathbf{X}_2}(\mathbf{x}_2)} \\ &\overset{\mathbf{x}_1}{\propto} \exp\left(-\frac{1}{2}\left(\mathbf{x}_1 - \boldsymbol{\mu}_1\right)' \Psi_{11} \left(\mathbf{x}_1 - \boldsymbol{\mu}_1\right) - \left(\mathbf{x}_1 - \boldsymbol{\mu}_1\right)' \Psi_{12} \left(\mathbf{x}_2 - \boldsymbol{\mu}_2\right)\right) \\ &\overset{\mathbf{x}_1}{\propto} \exp\left(-\frac{1}{2}\mathbf{x}_1' \Psi_{11} \mathbf{x}_1 + \mathbf{x}_1' \left(\Psi_{11}\boldsymbol{\mu}_1 - \Psi_{12}\left(\mathbf{x}_2 - \boldsymbol{\mu}_2\right)\right)\right), \end{aligned}$$

   which is proportional to a multivariate gaussian density. By identification, the precision matrix is Ψ_{11} and the mean is

$$\boldsymbol{\mu}_{1|2} = \boldsymbol{\mu}_1 - \Psi_{11}^{-1} \Psi_{12} \left(\mathbf{x}_2 - \boldsymbol{\mu}_2\right)$$

   Using the results of exercise 3, we obtain

$$\begin{aligned} \boldsymbol{\mu}_{1|2} &= \boldsymbol{\mu}_1 + \Sigma_{12} \Sigma_{22}^{-1} \left(\mathbf{x}_2 - \boldsymbol{\mu}_2\right) \\ \Sigma_{1|2} &= \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}. \end{aligned}$$

2. The marginal density can be computed by integrating the joint density and completing the squares. Indeed,

$$
\begin{aligned}
&f(\mathbf{x}_1) = \int f(\mathbf{x}_1, \mathbf{x}_2) d\mathbf{x}_2 \\
&\propto \int \exp\left\{-\frac{1}{2}(\mathbf{x}_1-\boldsymbol{\mu}_1)'\Psi_{11}(\mathbf{x}_1-\boldsymbol{\mu}_1) - \frac{1}{2}(\mathbf{x}_2-\boldsymbol{\mu}_2)'\Psi_{22}(\mathbf{x}_2-\boldsymbol{\mu}_2) - (\mathbf{x}_2-\boldsymbol{\mu}_2)'\Psi_{21}(\mathbf{x}_1-\boldsymbol{\mu}_1)\right\} d\mathbf{x}_2 \\
&\propto \exp\left\{-\frac{1}{2}(\mathbf{x}_1-\boldsymbol{\mu}_1)'\Psi_{11}(\mathbf{x}_1-\boldsymbol{\mu}_1) + \boldsymbol{\mu}_2'\Psi_{21}(\mathbf{x}_1-\boldsymbol{\mu}_1)\right\} \times \int \exp\left\{-\frac{1}{2}\mathbf{x}_2'\Psi_{22}\mathbf{x}_2 + \mathbf{x}_2'\Psi_{22}\boldsymbol{\mu}_2 - \mathbf{x}_2'\Psi_{21}(\mathbf{x}_1-\boldsymbol{\mu}_1)\right\} d\mathbf{x}_2 \\
&\propto \exp\left\{-\frac{1}{2}(\mathbf{x}_1-\boldsymbol{\mu}_1)'\Psi_{11}(\mathbf{x}_1-\boldsymbol{\mu}_1) + \boldsymbol{\mu}_2'\Psi_{21}(\mathbf{x}_1-\boldsymbol{\mu}_1)\right\} \times \int \exp\left\{-\frac{1}{2}\mathbf{x}_2'\Psi_{22}\mathbf{x}_2 + \mathbf{x}_2'\Psi_{22}\underbrace{\left[\boldsymbol{\mu}_2 - \Psi_{22}^{-1}\Psi_{21}(\mathbf{x}_1-\boldsymbol{\mu}_1)\right]}_{:=\boldsymbol{\nu}}\right\} d\mathbf{x}_2 \\
&\propto \exp\left\{-\frac{1}{2}(\mathbf{x}_1-\boldsymbol{\mu}_1)'\Psi_{11}(\mathbf{x}_1-\boldsymbol{\mu}_1) + \boldsymbol{\mu}_2'\Psi_{21}(\mathbf{x}_1-\boldsymbol{\mu}_1)\right\} \times \int \exp\left\{-\frac{1}{2}\mathbf{x}_2'\Psi_{22}\mathbf{x}_2 + \mathbf{x}_2'\Psi_{22}\boldsymbol{\nu} - \frac{1}{2}\boldsymbol{\nu}'\Psi_{22}\boldsymbol{\nu} + \frac{1}{2}\boldsymbol{\nu}'\Psi_{22}\boldsymbol{\nu}\right\} d\mathbf{x}_2 \\
&\propto \exp\left\{-\frac{1}{2}(\mathbf{x}_1-\boldsymbol{\mu}_1)'\Psi_{11}(\mathbf{x}_1-\boldsymbol{\mu}_1) + \boldsymbol{\mu}_2'\Psi_{21}(\mathbf{x}_1-\boldsymbol{\mu}_1) + \frac{1}{2}\boldsymbol{\nu}'\Psi_{22}\boldsymbol{\nu}\right\} \times \int \exp\left\{-\frac{1}{2}\mathbf{x}_2'\Psi_{22}\mathbf{x}_2 + \mathbf{x}_2'\Psi_{22}\boldsymbol{\nu} - \frac{1}{2}\boldsymbol{\nu}'\Psi_{22}\boldsymbol{\nu}\right\} d\mathbf{x}_2 \\
&\propto \exp\left\{-\frac{1}{2}(\mathbf{x}_1-\boldsymbol{\mu}_1)'\Psi_{11}(\mathbf{x}_1-\boldsymbol{\mu}_1) + \boldsymbol{\mu}_2'\Psi_{21}(\mathbf{x}_1-\boldsymbol{\mu}_1) + \frac{1}{2}\boldsymbol{\nu}'\Psi_{22}\boldsymbol{\nu}\right\} \times \underbrace{\int \exp\left\{-\frac{1}{2}(\mathbf{x}_2-\boldsymbol{\nu})'\Psi_{22}(\mathbf{x}_2-\boldsymbol{\nu})\right\} d\mathbf{x}_2}_{:=C} \\
&\propto \exp\left\{-\frac{1}{2}(\mathbf{x}_1-\boldsymbol{\mu}_1)'\Psi_{11}(\mathbf{x}_1-\boldsymbol{\mu}_1) + \boldsymbol{\mu}_2'\Psi_{21}(\mathbf{x}_1-\boldsymbol{\mu}_1) + \frac{1}{2}\left[\boldsymbol{\mu}_2 - \Psi_{22}^{-1}\Psi_{21}(\mathbf{x}_1-\boldsymbol{\mu}_1)\right]'\Psi_{22}\left[\boldsymbol{\mu}_2 - \Psi_{22}^{-1}\Psi_{21}(\mathbf{x}_1-\boldsymbol{\mu}_1)\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2}(\mathbf{x}_1-\boldsymbol{\mu}_1)'\Psi_{11}(\mathbf{x}_1-\boldsymbol{\mu}_1) + \boldsymbol{\mu}_2'\Psi_{21}(\mathbf{x}_1-\boldsymbol{\mu}_1) + \frac{1}{2}(\mathbf{x}_1-\boldsymbol{\mu}_1)'\Psi_{12}\Psi_{22}^{-1}\Psi_{21}(\mathbf{x}_1-\boldsymbol{\mu}_1) - \frac{1}{2}\boldsymbol{\mu}_2'\Psi_{21}(\mathbf{x}_1-\boldsymbol{\mu}_1)\right\} \\
&\propto \exp\left\{-\frac{1}{2}(\mathbf{x}_1-\boldsymbol{\mu}_1)'(\Psi_{11} - \Psi_{12}\Psi_{22}^{-1}\Psi_{21})(\mathbf{x}_1-\boldsymbol{\mu}_1)\right\}
\end{aligned}
$$

The step written in blue corresponds to adding and removing the same term in order to *complete the squares*; Indeed, this allows us to obtain the density function of a Gaussian random vector with mean $\boldsymbol{\nu}$ et precision matrix Ψ_{22} inside the integral. We notice that we have omitted the normalisation constant $C = 1/\sqrt{2\pi\|\Psi_{22}^{-1}\|}^{p-k}$. Since for the density function f, $\int f(\mathbf{x}_2)d\mathbf{x}_2 = 1$, we can replace the unknown integral above by the constant C.
In conclusion, the marginal density is proportional to

$$\exp\left\{-\frac{1}{2}(\mathbf{x}_1-\boldsymbol{\mu}_1)'(\Psi_{11} - \Psi_{12}\Psi_{22}^{-1}\Psi_{21})(\mathbf{x}_1-\boldsymbol{\mu}_1)\right\}$$

so we can identify it as a Gaussian density with precision matrix $\Psi_{11} - \Psi_{12}\Psi_{22}^{-1}\Psi_{21}$ (which is equal to Σ_{11}^{-1}, see Ex.4, Serie 1) and mean equal to $\boldsymbol{\mu}_1$.

Exercise 2 We consider $\mathbf{X} \sim N_p(\boldsymbol{\mu}_X, \Sigma_X)$ and $\mathbf{Y} \mid \mathbf{X} = \mathbf{x} \sim N_q(\boldsymbol{\alpha} + \beta\mathbf{x}, \Sigma)$ where $\boldsymbol{\mu} \in \mathbb{R}^p$, $\Sigma_X \in R^{p\times p}$, $\boldsymbol{\alpha} \in \mathbb{R}^q$, $\beta \in \mathbb{R}^{q\times p}$ and $\Sigma \in \mathbb{R}^{q\times q}$.

1. Prove that $(\mathbf{X}', \mathbf{Y}')' \sim N_{p+q}$, compute its mean and show that

$$\mathrm{V}\,(\mathbf{X}', \mathbf{Y}')' = \begin{pmatrix} \Sigma_X & \Sigma_X \beta' \\ \beta\Sigma_X & \Sigma + \beta\Sigma_X\beta' \end{pmatrix}$$

Hint : Start by setting $\mathbf{U} = \mathbf{Y} - \boldsymbol{\alpha} - \beta\mathbf{X}$ and showing that $\mathbf{X}$ and $\mathbf{U}$ are independent, then find a matrix A and a vector $\mathbf{c}$ such that $(\mathbf{X}', \mathbf{Y}')' = A(\mathbf{X}', \mathbf{U}')' + \mathbf{c}$.

2. Show that the conditional distribution of $\mathbf{X} \mid \mathbf{Y} = \mathbf{y}$ is Gaussian with

$$\begin{aligned} \mathrm{E}\,[\mathbf{X} \mid \mathbf{Y} = \mathbf{y}] &= \boldsymbol{\mu}_X + \beta\Sigma_X(\Sigma + \beta\Sigma_{\mathbf{X}}\beta')^{-1}(\mathbf{y} - \boldsymbol{\alpha} - \boldsymbol{\mu}_X\beta), \\ \mathrm{V}\,[\mathbf{X} \mid \mathbf{Y} = \mathbf{y}] &= \Sigma_X - \Sigma_X\beta'(\Sigma + \beta\Sigma_X\beta')^{-1}\beta\Sigma_X, \end{aligned}$$

assuming that the matrices $\Sigma, \Sigma_X, (\Sigma + \beta\Sigma_X\beta')$ are invertible.

Solution 2

1. We set $\mathbf{U} = \mathbf{Y} - \boldsymbol{\alpha} - \beta\mathbf{X}$. We will show that $\mathbf{X}$ and $\mathbf{U}$ are independent using the characteristic function.

$$\begin{aligned}
\Phi_{(\mathbf{X}',\mathbf{U}')'}\begin{pmatrix}\mathbf{t}_1\\ \mathbf{t}_2\end{pmatrix} &= \mathrm{E}\left[\exp\left(\mathrm{i}\begin{pmatrix}\mathbf{t}_1 & \mathbf{t}_2\end{pmatrix}\begin{pmatrix}\mathbf{X}\\ \mathbf{U}\end{pmatrix}\right)\right] && (1)\\
&= \int\int \exp\left[\mathrm{i}\mathbf{t}_1\mathbf{x} + \mathrm{i}\mathbf{t}_2\mathbf{u}\right] f_{X,U}(\mathbf{x},\mathbf{u})d\mathbf{x}d\mathbf{u} && (2)\\
&= \int\int \exp\left[\mathrm{i}\mathbf{t}_1\mathbf{x} + \mathrm{i}\mathbf{t}_2\mathbf{u}\right] f_{X,Y}(\mathbf{x},\mathbf{u} + \beta\mathbf{x} + \boldsymbol{\alpha})d\mathbf{x}d\mathbf{u} && (3)\\
&= \int\int \exp\left[\mathrm{i}\mathbf{t}_1\mathbf{x} + \mathrm{i}\mathbf{t}_2\mathbf{u}\right]) f_{Y|X=\mathbf{x}}(\mathbf{u} + \boldsymbol{\alpha} + \beta\mathbf{x}) f_X(\mathbf{x})d\mathbf{x}d\mathbf{u} && (4)\\
&\propto \int\int \exp\left[\mathrm{i}\mathbf{t}_1\mathbf{x} + \mathrm{i}\mathbf{t}_2\mathbf{u}\right]\exp\left(-\frac{1}{2}\mathbf{u}'\Sigma^{-1}\mathbf{u}\right)\exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_X)'\Sigma_X^{-1}(\mathbf{x} - \boldsymbol{\mu}_X)\right)d\mathbf{x}d\mathbf{u} && (5)\\
&\propto \int \exp\left[\mathrm{i}\mathbf{t}_1\mathbf{x} - \frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_X)'\Sigma_X^{-1}(\mathbf{x} - \boldsymbol{\mu}_X)\right]d\mathbf{x}\int \exp\left[\mathrm{i}\mathbf{t}_2\mathbf{u} - \frac{1}{2}\mathbf{u}'\Sigma^{-1}\mathbf{u}\right]d\mathbf{u} && (6)
\end{aligned}$$

From line (2) to (3), we considered the transformation

$$\begin{pmatrix}\mathbf{X}\\ \mathbf{U}\end{pmatrix} = \begin{pmatrix} I_p & 0_{p\times q}\\ -\beta & I_q\end{pmatrix}\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix} + \begin{pmatrix}0_p\\ -\boldsymbol{\alpha}\end{pmatrix},$$

for which the determinant of the Jacobian is 1. We then obtain

$$\Phi_{(\mathbf{X}',\mathbf{U}')'}\begin{pmatrix}\mathbf{t}_1\\ \mathbf{t}_2\end{pmatrix} = \Phi_{\mathbf{X}}(\mathbf{t_1})\Phi_{\mathbf{U}}(\mathbf{t_2})$$

So $\mathbf{U}$ and $\mathbf{X}$ are independent and the variance of $(\mathbf{X}', \mathbf{U}')'$ is

$$\mathrm{V}\begin{pmatrix}\mathbf{X}\\ \mathbf{U}\end{pmatrix} = \begin{pmatrix}\Sigma_X & 0_{p\times q}\\ 0_{q\times p} & \Sigma\end{pmatrix}.$$

Now, we write

$$\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix} = \begin{pmatrix} I_p & 0_{p\times q}\\ \beta & I_q\end{pmatrix}\begin{pmatrix}\mathbf{X}\\ \mathbf{U}\end{pmatrix} + \begin{pmatrix}0_p\\ \boldsymbol{\alpha}\end{pmatrix}.$$

Hence

$$\mathrm{E}\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix} = \begin{pmatrix}\boldsymbol{\mu}_X\\ \boldsymbol{\alpha} + \beta\boldsymbol{\mu}_X\end{pmatrix}$$

and

$$\mathrm{V}\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix} = \begin{pmatrix}\Sigma_X & \Sigma_X\beta'\\ \beta\Sigma_X & \Sigma + \beta\Sigma_X\beta'\end{pmatrix}$$

2. We can directly apply the results of question 1 in Exercise 1, where we consider $\mathbf{X}_1 = \mathbf{X}$ and $\mathbf{X}_2 = \mathbf{Y}$.

Exercise 3 Let $\mathbf{X}_1, \cdots, \mathbf{X}_n$ be i.i.d. random vectors in $\mathbb{R}^p$ with mean $\boldsymbol{\mu}$ and covariance matrix Σ. Consider the sample mean $\bar{\mathbf{X}}$ and the sample variance-covariance matrix S and show that $T_0^2 = n(\bar{\mathbf{X}} - \boldsymbol{\mu})' S^{-1}(\bar{\mathbf{X}} - \boldsymbol{\mu})$ is invariant under all the afine transformations $\mathbf{Y}_i = A\mathbf{X}_i + \mathbf{b}$, where A is an invertible matrix of constants and $\mathbf{b}$ is a vector of constants.

Solution 3

$$
\begin{aligned}
T_0^2(\mathbf{Y}) &= n(\bar{\mathbf{Y}} - \boldsymbol{\mu}_Y)' S_Y^{-1}(\bar{\mathbf{Y}} - \boldsymbol{\mu}_Y) \\
&= n(A\bar{\mathbf{X}} - A\boldsymbol{\mu}_X)' (AS_X A')^{-1}(A\bar{\mathbf{X}} - A\boldsymbol{\mu}_X) \\
&= n(\bar{\mathbf{X}} - \boldsymbol{\mu}_X)' S_X^{-1}(\bar{\mathbf{X}} - \boldsymbol{\mu}_X) = T_0^2(\mathbf{X}).
\end{aligned}
$$

Exercise 4 (Simulation of Gaussian random vectors in `R`)
We consider the random vector $\mathbf{X}$ in $\mathbb{R}^2$ from a standard gaussian distribution $N_2(\mathbf{0}, \mathbf{1})$.

1. Generate $N = 1000$ independent random replicates $\mathbf{X}_1, \ldots, \mathbf{X}_n$ of $\mathbf{X}$. *Hint :* The function `rnorm(n)` generates n independent replicates of a univariate random distribution.
   a) Estimate the sample mean and covariance matrix.
   b) plot the sample and draw diagnostic plots of the marginal distributions.

2. We define another Gaussian vector $\mathbf{Y}$ satisfying : $\mathbb{E}[Y_1] = 5$, $\mathbb{E}[Y_2] = 5$, $\mathrm{Var}(Y_1) = 1$, $\mathrm{Var}(Y_2) = 1$ and $\mathrm{Cov}(Y_1, Y_2) = 0.9$.
   a) Generate $N = 1000$ independent replicates $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ starting from $\mathbf{X}_1, \ldots, \mathbf{X}_n$.
   Hint : The function `chol` computes the Choleski factorization of Σ and returns $\mathbf{A}$ such that $\mathbf{A}'\mathbf{A} = \Sigma$.
   b) plot the sample and draw visual diagnostic plots of the marginal distributions.

Solution 4 See `R` script.