Solution for Statistical analysis of network data – Sheet 3

1. Is the graph C_4 the cycle graph on 4 nodes strictly balanced?

   * Define $\bar{d} = 2e/n$. Define the maximum average degree of a graph F as

   $$m(F) = \max_{H \subset F} \{\bar{d}(H)\}.$$

   H is strictly balanced if $F \subset H$ and $d(F) = m(H)$ imply that $F = H$. The subgraphs of C_4 is P_2, P_3 and C_4 itself. $\bar{d} = 24/4 = 2$. On the other hand $\bar{d}$ for these subgraphs is $2/2 = 1$, $4/3 = 1.333...$. and C_4 is $8/4 = 2$. Equality implies the latter case, hence $\square$.

2. Is the graph of a tadpole of a cycle with 5 nodes joined to a 3-node tail, e.g. a graph on 5+3-1 vertices strictly balanced?

   * Define $\bar{d} = 2e/n = 14/7 = 2$. Subgraphs are S_3, P_4, P_7.... $\bar{d}$ is $6/4 = 3/2$, $6/4$ and $12/7 < 2$... Thus it follows that it is not strictly balanced.

3. Count the number of triangles in the graph below, excluding the large connected components of the biggest connected component.

   * The edges have nil triangles. The triangles have one each. K_4 has $4 \times 3 \times 2/6 = 4$. K_5 has $5 \times 4 \times 3/6 = 10$. The lollipop graph has the same as K_4 has $4 \times 3 \times 2/6 = 4$.

4. Assume that G is an Erdos-Renyi graph on n nodes. Determine the large sample approximation to the distribution of $X_{C_4}(G)$. Chose the success probability of the Erdos-Renyi graph so that a known limit arises.

   * Suppose that H is a fixed strictly balanced graph with k vertices and $l \geq 2$ edges, and its automorphism group has a members. Let $c > 0$ be a constant and set $p = cn^{-k/l}$. For G generated as an Erdos–Renyi graph with success probability p denote by $X_H(G)$ the number of copies of H in G. Then $X_H(G)$ asymptotically becomes a Poisson random variable with mean c^l/a. Here we recognise $l = 4$.

5. Assume that G is a Stochastic Blockmodel with 2 groups on n nodes. Determine the large sample approximation to the distribution of $X_{C_4}(G)$.

   We use the Theorem provided in the lecture. We denote the four probabilities θ_{11}, θ_{12}, θ_{21}, θ_{22}. Then take

   $$\lambda = \mathrm{E}X_H(G_n) = \binom{n}{4} \frac{4!}{8} \mu(H),$$

   with

   $$\mu(H) = \sum_{c_1, c_2}^{Q} h_{c_1} h_{c_2} \prod_{uv} \theta_{c_u c_v}.$$

   It has been shown by Coulson et al (2015) that the distribution of $X_H(G_n)$ becomes Poisson with mean λ.

6. What is the automorphosm group of K_4? How large it is?

   The automorphism group on K_n symmetric group S_n of degree n is the group of all permutations on n symbols.