DATA 605 - Homework #14

Zach Alexander

5/9/2020

This week, we'll work out some Taylor Series expansions of popular functions. For each function, only consider its valid ranges as indicated in the notes when you are computing the Taylor Series expansion.

For Taylor expansions, we'll use the following formula for a function $f(x)$:

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 ...$$

And when $a = 0$, the series is called a Maclaurin series:

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n = f(0) + \frac{f'(0)}{1!}(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 ...$$

Therefore, we can use these functions to solve:

$f(x) = \frac{1}{1-x}$

Function and Derivative	Function at $x = 0$
$f(x) = \frac{1}{1-x}$	$f(0) = \frac{1}{1} = 1$
$f(x)' = \frac{1}{(1-x)^2}$	$f(0) = \frac{1}{(1)^2} = 1$
$f(x)'' = \frac{2}{(1-x)^3}$	$f(0) = \frac{2}{(1)^3} = 2$
$f(x)''' = \frac{6}{(1-x)^4}$	$f(0) = \frac{6}{(1)^4} = 6$
$f(x)^4 = \frac{24}{(1-x)^5}$	$f(0) = \frac{24}{(1)^5} = 24$
$f(x)^5 = \frac{120}{(1-x)^6}$	$f(0) = \frac{120}{(1)^6} = 120$
$f(x)^6 = \frac{720}{(1-x)^7}$	$f(0) = \frac{720}{(1)^7} = 720$
$f(x)^7 = \frac{5400}{(1-x)^8}$	$f(0) = \frac{5400}{(1)^8} = 5400$
$f(x)^8 = \frac{43000}{(1-x)^9}$	$f(0) = \frac{43000}{(1)^9} = 43000$

We can therefore use the Maclaurin series (the Taylor series centered at zero), in order to find the series representation for $f(x) = \frac{1}{1-x}$:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n = f(0) + \frac{f'(0)}{1!}(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 ... + \frac{fn(0)}{n!}(x)^n$$

And from our table computations above, we can substitute the values:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n = 1+\frac{1}{1!}(x)+\frac{2}{2!}(x)^2+\frac{6}{3!}(x)^3+\frac{24}{4!}(x)^4+\frac{120}{5!}(x)^5++\frac{720}{6!}(x)^6+\frac{5400}{7!}(x)^7+\frac{43000}{8!}(x)^8+...$$

Which can then be simplified to:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n = 1 + x + x^2 + x^3 + x^4 + x^5 + +x^6 + x^7 + x^8 + ... + x^n$$

And therefore, the series representation for function $f(x) = \frac{1}{1-x}$ will only converge when:

Solution:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n for|x| < 1$$

so the valid range is (-1, 1).

```
# check using taylor function
f <- function(x) 1/(1-x)
taylor(f, 0, 8)
```

```
## [1] 1.022551 1.007014 1.001710 1.000293 1.000029 1.000003 1.000000 1.000000
## [9] 1.000000
```

We can see the convergence to 1. This is confirmed.

$f(x) = e^x$

Similar to the first function, we'll find the first few derivatives and compute at $x = 0$:

Function and Derivative	Function at $x = 0$
$f(x) = e^x$	$f(0) = e^0 = 1$
$f(x)' = e^x$	$f(0) = e^0 = 1$
$f(x)'' = e^x$	$f(0) = e^0 = 1$
$f(x)''' = e^x$	$f(0) = e^0 = 1$
$f(x)^4 = e^x$	$f(0) = e^0 = 1$
$f(x)^5 = e^x$	$f(0) = e^0 = 1$
$f(x)^6 = e^x$	$f(0) = e^0 = 1$
$f(x)^7 = e^x$	$f(0) = e^0 = 1$
$f(x)^8 = e^x$	$f(0) = e^0 = 1$

Again, we can use the Maclaurin series (the Taylor series centered at zero), in order to find the series representation for $f(x) = e^x$:

$$e^x = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n = f(0) + \frac{f'(0)}{1!}(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 ... + \frac{fn(0)}{n!}(x)^n$$

And substituting our values from our table above:

$$e^x = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n = 1 + \frac{1}{1!}(x) + \frac{1}{2!}(x)^2 + \frac{1}{3!}(x)^3 + \frac{1}{4!}(x)^4 + \frac{1}{5!}(x)^5 + +\frac{1}{6!}(x)^6 + \frac{1}{7!}(x)^7 + \frac{1}{8!}(x)^8 + ...$$

Which simplifies to:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + ... + \frac{x^n}{n!}$$

This function does not converge, and thus, the series representation for function $f(x) = e^x$ is:

Solution:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

for all real numbers.

$f(x) = ln(1 + x)$

And we'll follow the same pattern for this final function:

Function and Derivative	Function at $x = 0$
$f(x) = ln(1 + x)$	$f(0) = ln(1 + 0) = 1$
$f(x)' = \frac{1}{1+x}$	$f(0)' = \frac{1}{1+0} = 1$
$f(x)'' = \frac{-1}{(1+x)^2}$	$f(0)'' = \frac{-1}{(1+0)^2} = -1$
$f(x)''' = \frac{2}{(1+x)^3}$	$f(0)''' = \frac{2}{(1+0)^3} = 2$
$f(x)^4 = \frac{-6}{(1+x)^4}$	$f(0)^4 = \frac{-6}{(1+0)^4} = -6$
$f(x)^5 = \frac{24}{(1+x)^5}$	$f(0)^5 = \frac{24}{(1+0)^5} = 24$
$f(x)^6 = \frac{-120}{(1+x)^6}$	$f(0)^6 = \frac{-120}{(1+0)^6} = -120$
$f(x)^7 = \frac{720}{(1+x)^7}$	$f(0)^7 = \frac{720}{(1+0)^7} = 720$
$f(x)^8 = \frac{-5040}{(1+x)^8}$	$f(0)^8 = \frac{-5040}{(1+0)^8} = -5040$

Then using the Maclaurin series again:

$$e^x = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n = f(0) + \frac{f'(0)}{1!}(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 ... + \frac{fn(0)}{n!}(x)^n$$

And substituting with the values calculated above:

$$ln(1+x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n = 0+\frac{1}{1!}(x)+\frac{-1}{2!}(x)^2+\frac{2}{3!}(x)^3+\frac{-6}{4!}(x)^4+\frac{24}{5!}(x)^5++\frac{-120}{6!}(x)^6+\frac{720}{7!}(x)^7+\frac{5400}{8!}(x)^8+...$$

Which simplifies to:

$$ln(1 + x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + ...$$

And therefore, the series representation for function $f(x) = \frac{1}{1-x}$ will only converge when:

Solution:

$$\frac{1}{1-x} = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} for|x| < 1$$

so the valid range is (-1, 1).