ASSIGNMENT #3

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PROBLEM SET 1

① WHAT IS THE RANK OF THE MATRIX A?

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 5 & 4 & -2 & -3 \end{bmatrix}$$

TO FIND THE RANK, WE'LL FIRST NEED TO PUT IT IN ROW ECHELON FORM:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 5 & 4 & -2 & -3 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 7 \\ 0 & 1 & -2 & 1 \\ 5 & 4 & -2 & -3 \end{bmatrix} \xrightarrow{2R_3 - R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 7 \\ 0 & 0 & -8 & -5 \\ 5 & 4 & -2 & -3 \end{bmatrix}$$

$$\xrightarrow{1/5 R_4 - R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 7 \\ 0 & 0 & -8 & -5 \\ 0 & -6/5 & -\frac{17}{5} & -\frac{23}{5} \end{bmatrix} \xrightarrow{-5R_4} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 7 \\ 0 & 0 & -8 & -5 \\ 0 & -6 & -17 & -23 \end{bmatrix} \xrightarrow{1/3 R_4 + R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 7 \\ 0 & 0 & -8 & -5 \\ 0 & 0 & -5/3 & -\frac{2}{3} \end{bmatrix}$$

$$\xrightarrow{-3R_4} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 7 \\ 0 & 0 & -8 & -5 \\ 0 & 0 & 5 & 2 \end{bmatrix} \xrightarrow{8R_4 + 5R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 7 \\ 0 & 0 & -8 & -5 \\ 0 & 0 & 0 & -9 \end{bmatrix} \xrightarrow[\substack{-1/8 R_3 \\ -1/9 R_4}]{1/2 R_2} \begin{bmatrix} \underline{1} & 2 & 3 & 4 \\ 0 & \underline{1} & 2 & 7/2 \\ 0 & 0 & \underline{1} & 5/8 \\ 0 & 0 & 0 & \underline{1} \end{bmatrix}$$

SINCE WE CAN SEE THAT THERE ARE 4 PIVOTS FOR MATRIX A, WE KNOW THAT THE $\boxed{\text{RANK} = 4}$ ⟵ ANSWER

② GIVEN AN $m \times n$ MATRIX WHERE $m > n$, WHAT CAN BE THE MAXIMUM RANK? THE MINIMUM RANK, ASSUMING THAT THE MATRIX IS NON-ZERO?

BECAUSE THE RANK HAS TO BE NO GREATER THAN THE SMALLER OF THE ROW OR COLUMN DIMENSION, AND GIVEN $m > n$,

$\boxed{\text{THE MAXIMUM RANK} = n}$

ASSUMING THAT THE MATRIX IS NON-ZERO:

$\boxed{\text{THE MINIMUM RANK} = 1}$

③ WHAT IS THE RANK OF MATRIX B?

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 2 & 4 & 2 \end{bmatrix}$$

AGAIN, WE'LL PUT THIS INTO ROW ECHELON FORM:

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 2 & 4 & 2 \end{bmatrix} \xrightarrow{-3R_1 + R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 2 & 4 & 2 \end{bmatrix} \xrightarrow{-2R_1 + R_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

SINCE WE CAN SEE THAT THERE IS 1 PIVOT FOR MATRIX B, WE KNOW THAT THE $\boxed{\text{RANK} = 1}$

PROBLEM SET 2

COMPUTE THE EIGENVALUES AND EIGENVECTORS OF THE MATRIX A. YOU'LL NEED TO SHOW YOUR WORK. YOU'LL NEED TO WRITE OUT THE CHARACTERISTIC POLYNOMIAL AND SHOW YOUR SOLUTION.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \left[\begin{matrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{matrix}\right] \begin{matrix} 1-\lambda & 2 \\ 0 & 4-\lambda \\ 0 & 0 \end{matrix}$$

$$(1-\lambda)(4-\lambda)(6-\lambda) - 0 - 0 - 0 - 0 - 0$$

$$(1-\lambda)(4-\lambda)(6-\lambda)$$

$$(1-\lambda)(24-10\lambda+\lambda^2)$$

$$24-10\lambda+\lambda^2-24\lambda+10\lambda^2-\lambda^3$$

$$24-34\lambda+11\lambda^2-\lambda^3$$

$$\boxed{-\lambda^3+11\lambda^2-34\lambda+24=0} \leftarrow \text{CHARACTERISTIC POLYNOMIAL}$$

$$(1-\lambda)(4-\lambda)(6-\lambda)=0$$

$$\boxed{\lambda_1=1 \quad \lambda_2=4 \quad \lambda_3=6} \leftarrow \text{EIGENVALUES}$$

EIGENVECTOR FOR $\lambda_1 = 1$

$$\begin{bmatrix} 1-1 & 2 & 3 \\ 0 & 4-1 & 5 \\ 0 & 0 & 6-1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 0 & 1 & 3/2 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\frac{9}{2} - \frac{10}{2} = \frac{-1}{2}$$

$$\downarrow 3R_1 - R_2$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xleftarrow{2R_2} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \xleftarrow{10R_2 - R_3} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1/2 \\ 0 & 0 & 5 \end{bmatrix} \xleftarrow{3R_2 - R_1} \begin{bmatrix} 0 & 1 & 3/2 \\ 0 & 0 & -1/2 \\ 0 & 0 & 5 \end{bmatrix}$$

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$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{aligned} V &= t \\ V_2 &= 0 \\ V_3 &= 0 \end{aligned}$$

$$\boxed{E_{\lambda=1} = \text{SPAN}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)}$$

EIGENVECTOR FOR $\lambda_1 = 1$

EIGENVECTOR FOR $\lambda = 4$

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix} \longrightarrow \begin{bmatrix} -3 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow[\frac{1}{2}R_3]{\frac{1}{5}R_2} \begin{bmatrix} -3 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 - R_3} \begin{bmatrix} -3 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_1} \begin{bmatrix} 1 & -\frac{2}{3} & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} V_1 - \tfrac{2}{3}V_2 &= 0 \\ V_1 &= \tfrac{2}{3}V_2 \\ V_3 &= 0 \end{aligned}$$

EIGENVECTOR FOR $\lambda_2 = 4$

$$\boxed{E_{\lambda=4} = \text{SPAN}\left(\begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix}\right)}$$

EIGENVECTOR FOR $\lambda = 6$

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix} \longrightarrow \begin{bmatrix} -5 & 2 & 3 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{5}R_1} \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{3}{5} \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{3}{5} \\ 0 & 1 & -\frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{2}{5}R_2 + R_1} \begin{bmatrix} 1 & 0 & -\frac{8}{5} \\ 0 & 1 & -\frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & -8/5 \\ 0 & 1 & -5/2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$V_1 = \frac{8}{5}t$
$V_2 = \frac{5}{2}t$
$V_3 = t$

$$V_1 - 8/5 V_3 = 0$$
$$V_1 = \frac{8}{5} V_3$$
$$V_2 - 5/2 V_3 = 0$$
$$V_2 = \frac{5}{2} V_3$$

$$\boxed{E_{\lambda=6} = \text{SPAN}\left(\begin{bmatrix} 8/5 \\ 5/2 \\ 1 \end{bmatrix}\right)}$$

EIGENVECTOR FOR $\lambda_3 = 6$

TO SUMMARIZE PROBLEM SET 2

CHARACTERISTIC POLYNOMIAL: $-\lambda^3 + 11\lambda^2 - 34\lambda + 24 = 0$

EIGENVALUES: $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = 6$

EIGENVECTORS:

$\lambda_1 = 1$

$\downarrow$

$$E_{\lambda=1} = \text{SPAN}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

$\lambda_2 = 4$

$\downarrow$

$$E_{\lambda=4} = \text{SPAN}\left(\begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix}\right)$$

$\lambda_3 = 6$

$\downarrow$

$$E_{\lambda=6} = \text{SPAN}\left(\begin{bmatrix} 8/5 \\ 5/2 \\ 1 \end{bmatrix}\right)$$