HOMEWORK ASSIGNMENT # 8

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CHAPTER 7.2 EXERCISE # 11

A COMPANY BUYS 100 LIGHTBULBS, EACH OF WHICH HAS AN EXPONENTIAL LIFETIME OF 1000 HOURS. WHAT IS THE EXPECTED TIME FOR THE FIRST OF THESE BULBS TO BURN OUT?

SINCE WE APPROACH THIS WITH AN EXPONENTIAL LIFETIME, WE CAN DETERMINE λ FROM THE FOLLOWING:

$$E[X_i] = \frac{1}{\lambda_i}, \text{ WHERE } \lambda_i \text{ IS } \frac{1}{1000}$$

WITH THIS IN MIND, WE CAN SOLVE FOR 100 LIGHT BULBS:

$$\sum_{i=1}^{100} \lambda_i = 100 \cdot \frac{1}{1000} = \frac{1}{10}$$

$$\text{AND } E[X_i] = \frac{1}{\frac{1}{10}} = 10 \rightarrow$$

ANSWER: THE EXPECTED TIME FOR THE FIRST OF THESE LIGHT BULBS TO BURN OUT IS 10 HOURS

CHAPTER 7.2 EXERCISE # 14

ASSUME THAT X_1 AND X_2 ARE INDEPENDENT RANDOM VARIABLES, EACH HAVING AN EXPONENTIAL DENSITY WITH PARAMETER λ. SHOW THAT $Z = X_1 - X_2$ HAS DENSITY

$$f_Z(z) = \left(\frac{1}{2}\right)\lambda e^{-\lambda|z|}$$

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WE CAN USE THE FOLLOWING CONVOLUTION FORMULA

$$W = X + Y$$

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(w-x)\, dx$$

ALTHOUGH THIS SHOWS THE SUM, AND WE ARE LOOKING TO SUBTRACT, WE CAN ADJUST THIS TO:

$$Z = X_1 + (-X_2)$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X_1}(x)\, f_{-X_2}(z-x)\, dx \quad \text{AND}$$

$$f_{-X_2}(z-x) = f_{X_2}(x-z) \quad \text{SO}$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X_1}(x)\, f_{X_2}(x-z)\, dx$$

EXPONENTIAL DENSITY

$$f_X(x) = \begin{cases} 0, & x < 0 \\ \lambda e^{-\lambda x}, & x \geq 0 \end{cases}$$

FOR $z < 0$ AND USING THE EXPONENTIAL DENSITY FUNCTION:

$$\int_0^{\infty} \lambda e^{-\lambda x}\, \lambda e^{-\lambda(x-z)}\, dx$$

$$\lambda e^{\lambda x z} \int_0^{\infty} \lambda e^{-2\lambda x}\, dx$$

$$\lambda e^{\lambda x z}\left(-\frac{1}{2} e^{-2\lambda x}\right) = -\frac{1}{2} \cdot \frac{-e^{-2\lambda x}}{1} = -\frac{1}{2} \cdot \frac{1}{-e^{2\lambda x}} = \frac{1}{2e^{2\lambda x}}$$

$$\frac{\lambda e^{\lambda x z}}{2e^{2\lambda x}} = \frac{\lambda}{2} \cdot \frac{1^{\lambda x z}}{e^{2\lambda x}} = \frac{\lambda}{2} \cdot \frac{1}{e^{-\lambda z}} = \frac{\lambda e^{\lambda z}}{2}$$

THEREFORE, $f_Z(z) = \frac{\lambda}{2} e^{\lambda z}$ WHEN $z < 0$

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FOR $z \geq 0$, SINCE WE KNOW THAT X_1 AND X_2 ARE INDEPENDENT RANDOM VARIABLES AND:

$Z = X_1 - X_2$ AND $-Z = X_2 - X_1$, THESE HAVE THE SAME DISTRIBUTION AND WILL BE SYMMETRIC AROUND ZERO.

THEREFORE, $f_Z(z) = f_Z(-z)$ AND:

$$f_Z(z) \begin{cases} f_Z(z) = \frac{\lambda}{2} e^{\lambda z}, & z < 0 \\ f_Z(z) = \frac{\lambda}{2} e^{-\lambda z}, & z \geq 0 \end{cases}$$

AND THUS PROVING THAT $f_Z(z) = \frac{\lambda}{2} e^{-\lambda |z|}$

CHAPTER 8.2 EXERCISE #1

LET X BE A CONTINUOUS RANDOM VARIABLE WITH MEAN $\mu = 10$ AND VARIANCE $\sigma^2 = 100/3$. USING CHEBYSHEV'S INEQUALITY, FIND AN UPPER BOUND FOR THE FOLLOWING PROBABILITIES: $\sigma = \sqrt{\frac{100}{3}} = \frac{10}{\sqrt{3}}$ $\quad \varepsilon = k\sigma = k$

(a) $P(|X - 10| \geq 2)$

$$P(|X-\mu| \geq k\sigma) \leq \frac{\sigma^2}{k^2\sigma^2} \rightarrow \frac{1}{k^2}$$

$k\sigma = 2$ AND $k \cdot \frac{10}{\sqrt{3}} = 2 \longrightarrow k = \frac{2\sqrt{3}}{10}$

$$\frac{1}{\left(\frac{2\sqrt{3}}{10}\right)^2} = \frac{1}{\left(\frac{4 \cdot 3}{100}\right)} = \frac{1}{\frac{12}{100}} = 8.333$$

HOWEVER, SINCE THE PROBABILITY CANNOT BE GREATER THAN 1, THE UPPER BOUND = 1.

b) $P(|X-10| \geq 5)$

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

$k\sigma = 5$ AND $k \cdot \frac{10}{\sqrt{3}} = 5 \longrightarrow k = \frac{5\sqrt{3}}{10}$

$$\frac{1}{\left(\frac{5\sqrt{3}}{10}\right)^2} = \frac{1}{\frac{25 \cdot 3}{100}} = \frac{1}{\frac{75}{100}} = 1.333$$

AGAIN, SINCE THE PROBABILITY CANNOT BE GREATER THAN 1, THE UPPER BOUND = 1.

c) $P(|X-10| \geq 9)$

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

$k\sigma = 9$ AND $k \cdot \frac{10}{\sqrt{3}} = 9 \longrightarrow k = \frac{9\sqrt{3}}{10}$

$$\frac{1}{\left(\frac{9\sqrt{3}}{10}\right)^2} = \frac{1}{\frac{81 \cdot 3}{100}} = \frac{1}{\frac{243}{100}} = 0.412$$

THE UPPER BOUND FOR $P(|X-10| \geq 9) = 0.412$

d) $P(|X-10| \geq 20)$

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

$k\sigma = 20$ AND $k \cdot \frac{10}{\sqrt{3}} = 20 \longrightarrow k = \frac{20\sqrt{3}}{10}$

$$\frac{1}{\left(\frac{20\sqrt{3}}{10}\right)^2} = \frac{1}{\frac{400 \cdot 3}{100}} = \frac{1}{\frac{1200}{100}} = \frac{1}{12} = 0.0833$$

THE UPPER BOUND FOR $P(|X-10| \geq 20) = \frac{1}{12}$ OR 0.0833