CHAPTER 9.3 EXERCISE # 11

THE PRICE OF ONE SHARE OF STOCK IN THE PILSDORFF BEER COMPANY IS GIVEN Y_n ON THE nTH DAY OF THE YEAR. FINN OBSERVES THAT THE DIFFERENCES $X_n = Y_{n+1} - Y_n$ APPEAR TO BE INDEPENDENT RANDOM VARIABLES WITH A COMMON DISTRIBUTION HAVING MEAN $\mu = 0$ AND VARIANCE $\sigma^2 = \frac{1}{4}$. IF $Y_1 = 100$, ESTIMATE THE PROBABILITY THAT Y_{365} IS:

SINCE X_n IS AN INDEPENDENT RANDOM VARIABLE, THEN ITS SUM S_n IS NORMALLY DISTRIBUTED:

$$S_n = X_1 + X_2 + X_3 + \dots + X_n$$

$$\mu \text{ of } S_n = n \cdot \mu_X = 0$$

$$\text{VARIANCE OF } S_n: \sigma^2_{S_n} = \sigma^2_{X_1} + \sigma^2_{X_2} + \sigma^2_{X_3} \dots + \sigma^2_{X_n} = \frac{n}{4}$$

$$\sigma \text{ of } S_n: \sqrt{\frac{n}{2}}$$

IF $n = 364$, THEN $S_{364} = Y_{365} - 100$, OR $Y_{365} = S_{364} + 100$

$$\sigma_{S_{364}} = \sqrt{\frac{364}{4}} = \sqrt{91}$$

(a) $\underline{P(Y_{365} \geq 100)} = P(S_{364} + 100 \geq 100) = P(S_{364} \geq 0)$

BECAUSE S_n IS NORMALLY DISTRIBUTED WITH MEAN 0 AND THE NORMAL DISTRIBUTION IS SYMMETRIC, EXACTLY 1/2 OF VALUES WILL BE GREATER THAN $\mu = 0$.

$$\boxed{P(Y_{365} \geq 100) = 0.5}$$

ANSWER

(b) $\underline{P(Y_{365} \geq 110)} = P(S_{364} + 100 \geq 110)$

$\downarrow$

$P(S_{364} \geq 10) = P(S^*_{364} \geq 10/\sqrt{91})$

PUTTING THIS INTO THE PNORM FUNCTION

IN R: PNORM$\left(\frac{10}{\sqrt{91}}\right.$, LOWER.TAIL = FALSE$\left.\right)$ $\boxed{= 0.1473}$

ANSWER

(c) $\underline{P(Y_{365} \geq 120)} = P(S_{364} + 100 \geq 120)$

$\downarrow$

$P(S_{364} \geq 20) = P(S^*_{364} \geq 20/\sqrt{91})$

PUTTING THIS INTO THE PNORM FUNCTION

IN R: PNORM$\left(\frac{20}{\sqrt{91}}\right.$, LOWER.TAIL = FALSE$\left.\right)$ $\boxed{= 0.0180}$

ANSWER

$\boxed{\text{PROBLEM \#2}}$

CALCULATE THE EXPECTED VALUE AND VARIANCE OF THE BINOMIAL DISTRIBUTION USING THE MOMENT GENERATING FUNCTION.

FOR BINOMIAL DISTRIBUTION, $P(X=k) = \binom{n}{k} p^k q^{n-k}$, WHERE $q = 1-p$.

MOMENT GENERATING FUNCTION IS $M_X(t) = (q + pe^t)^n$

CONTINUED ON NEXT PAGE

SINCE THE FIRST MOMENT IS $M_X'(t) = n(q+pe^t)^{n-1} pe^t$

TO FIND THE EXPECTED VALUE, WE CAN EVALUATE AT $t=0$:

$$E(X) = M_X'(0) = n(q+pe^0)^{n-1} pe^0$$
$$= n(q+p)^{n-1} p$$
$$= np(1-p+p)^{n-1}$$
$$= np(1)^{n-1}$$
$$= np \qquad E(X) = np$$

CONTINUING, THE SECOND MOMENT IS:

$$M_X''(t) = n(n-1)(q+pe^t)^{n-2} p^2 e^{2t} + n(q+pe^t)^{n-1} pe^t$$

THEN, THE SECOND MOMENT AT $t=0$:

$$E(X^2) = M_X''(0) = n(n-1)(q+pe^0)^{n-2} p^2 e^0 + n(q+pe^0)^{n-1} pe^0$$
$$= n(n-1)(1-p+p)^{n-2} p^2 + n(1-p+p)^{n-1} p$$
$$= n(n-1)p^2 + np$$

$$E(X^2) = n(n-1)p^2 + np$$

TO CALCULATE THE VARIANCE $\rightarrow$ $V(X) = E(X^2) - E(X)^2$

$$V(X) = n(n-1)p^2 + np - n^2p^2$$
$$= np((n-1)p + 1 - np)$$
$$= np(np - p + 1 - np)$$
$$= np(1-p) \longrightarrow 1-p = q$$
$$= npq \qquad V(X) = npq$$

CONTINUED ON NEXT PAGE

ANSWER: $E(X) = np$ AND $V(X) = npq$ ← ANSWER

PROBLEM # 3

CALCULATE THE EXPECTED VALUE AND VARIANCE OF THE EXPONENTIAL DISTRIBUTION USING THE MOMENT GENERATING FUNCTION:

FOR EXPONENTIAL DISTRIBUTIONS → $f(x) = \lambda e^{-\lambda x}$

$$g(t) = \int_0^\infty e^{tx} \lambda e^{-\lambda x}\, dx$$

$$\int_0^\infty \frac{\lambda e^{(t-\lambda)x}}{t-\lambda} = \frac{\lambda}{\lambda - t}$$

↓

$$g'(t) = \frac{\lambda}{(\lambda-t)^2} \text{ AND } g'(0) = \frac{\lambda}{(\lambda-0)^2} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$g''(t) = \frac{2\lambda}{(\lambda-t)^3} \text{ AND } g''(0) = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$$

WITH THE MEAN $(\mu) = g'(0) = \lambda^{-1}$ AND

VARIANCE $\mu_2 - \mu_1 = \sigma^2 = g''(0) - g'(0)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \lambda^{-2} = \frac{1}{\lambda^2}$

ANSWER: $E(X) = \frac{1}{\lambda}$ AND $V(X) = \frac{1}{\lambda^2}$ ← ANSWER