CHOOSE INDEPENDENTLY TWO NUMBERS B AND C AT RANDOM FROM THE INTERVAL [0, 1] WITH UNIFORM DENSITY. PROVE THAT B AND C ARE PROPER PROBABILITY DISTRIBUTIONS. NOTE THAT THE POINT (B,C) IS THEN CHOSEN AT RANDOM IN THE UNIT SQUARE.

<u>FIND THE PROBABILITY THAT</u>:

(a) $B + C < 1/2$

SINCE B AND C ARE INDEPENDENT, $f(B,C) = f(x,y) = f(x)f(y)$, WHERE $B = x$ $C = y$:

$$f(x,y) = 1$$

$$P(B+C < 1/2) = P(x+y < 1/2)$$

$$y < 1/2 - x$$

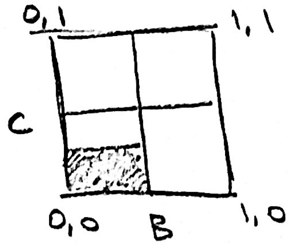
<u>WE CAN USE DOUBLE INTEGRATION</u>

$$\int_0^{1/2}\int_0^{1/2-x} 1\,dx\,dy \rightarrow \int_0^{1/2} (1/2 - x) - 0\,dx \rightarrow \int_0^{1/2} (1/2 - x)\,dx$$

$$\rightarrow 1/2 \int_0^{1/2} -x\,dx \rightarrow (1/2) - \frac{x^2}{2} \rightarrow (1/2)(1/2) - \frac{(1/2)^2}{2}$$

$$\rightarrow |1/4 - 1/8| = \frac{1}{8}$$

$$\boxed{\text{THE PROBABILITY } B+C < 1/2 \text{ IS EQUAL TO } 0.125}$$



UNIT SQUARE

FIND THE PROBABILITY THAT:

(b) $BC < 1/2$

SINCE B AND C ARE INDEPENDENT, $f(B,C) = f(x,y) = f(x)f(y)$ WHERE $B = x$ $C = y$ AND:

$$f(x,y) = 1$$

$$P(BC < 1/2) = P(xy < 1/2)$$

$$y < \frac{1}{2x} \qquad x < \frac{1}{2y}$$

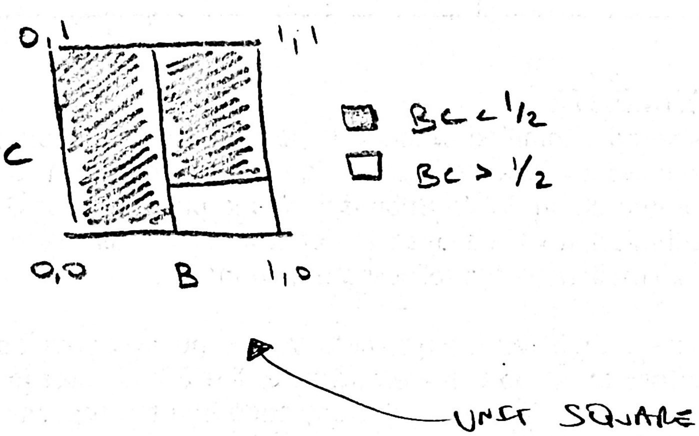
BECAUSE THESE EQUATIONS ARE THE SAME, DOUBLE INTEGRATION WILL NOT BE NEEDED

$$\int_{1/2}^{1}\int_{0}^{1/2x} f(x,y)\,dx\,dy \longrightarrow \frac{1}{2}\int_{1/2}^{1}\frac{1}{2x}\,dx \longrightarrow \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\ln|x|$$

$$\rightarrow \left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)\ln|1| - \ln\left|\tfrac{1}{2}\right| \longrightarrow \frac{1}{2} + \frac{1}{2}\left(0 - \ln\left|\tfrac{1}{2}\right|\right)$$

$$\rightarrow \frac{1}{2} + \frac{1}{2}\left(\ln\left|\tfrac{1}{2}\right|\right) \approx 0.847$$

$$\boxed{\text{THE PROBABILITY } BC < 1/2 \text{ IS EQUAL TO } \approx 0.847}$$



1 FIND THE PROBABILITY THAT:

(c) $|B - C| < \frac{1}{2}$

SINCE B AND C ARE INDEPENDENT, $f(B,C) = f(x,y) = f(x)f(y)$ WHERE $B = x$, $C = y$ AND $f(x,y) = 1$.

DUE TO THE DEFINITION OF ABSOLUTE VALUE,

$C = B + 1/2$ OR $C = B - 1/2$

$y = x + 1/2$ OR $y = x - 1/2$

$y - x = 1/2$ OR $y - x = -1/2$

THROUGH INTEGRATION:

$$\int_0^{-1/2} x - \frac{1}{2}\, dx \longrightarrow \frac{1}{2}\,\frac{\left(-\frac{1}{2}\right)^2}{2} = 0.375$$

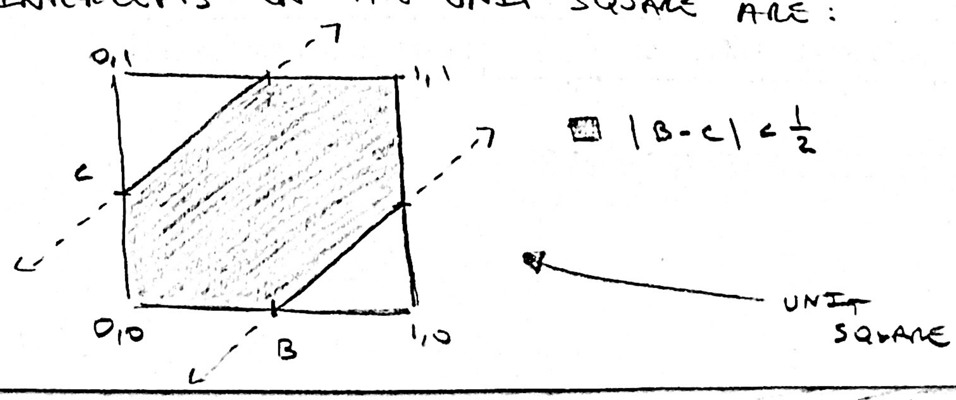
$$\int_0^{1/2} x + \frac{1}{2}\, dx \longrightarrow \frac{1}{2}\,\frac{\left(\frac{1}{2}\right)^2}{2} = 0.375$$

THEN, WE ADD THESE TWO PROBABILITIES TOGETHER:

$0.375 + 0.375 = 0.75$

$\boxed{\text{THE PROBABILITY } |B - C| < \tfrac{1}{2} \text{ IS EQUAL TO } 0.75}$

WE ALSO KNOW FROM OUR EQUATIONS $y = x + 1/2$ AND $y = x - 1/2$, THE INTERCEPTS ON THE UNIT SQUARE ARE:



FIND THE PROBABILITY THAT:

(d) $\max\{B, C\} < \frac{1}{2}$

SINCE WE ALREADY KNOW FROM PAST CALCULATIONS THAT IF THE MAX OF THE PRODUCT OF BC IS LESS THAN $\frac{1}{2}$, THEN WE CAN ASSUME THAT B AND C ARE GOING TO BE LESS THAN $\frac{1}{2}$.

BECAUSE $P(B \leq \frac{1}{2}) = \frac{1}{2}$ AND $P(C \leq \frac{1}{2}) = \frac{1}{2}$, WE KNOW THAT THE PROBABILITY OF BOTH BEING LESS THAN $\frac{1}{2}$ IS:

$$\left(\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right) = \frac{1}{4} = 0.25$$

$$\boxed{\text{THE PROBABILITY OF } \max\{B, C\} < \frac{1}{2} \text{ IS } 0.25}$$

FIND THE PROBABILITY THAT:

(e) $\min\{B, C\} < \frac{1}{2}$

SIMILAR LOGIC TO PART D, WE KNOW THAT:

$$P\left(B \leq \frac{1}{2}, C \leq \frac{1}{2}\right) = \frac{1}{4} \quad \text{AND,}$$

$$P\left(B \geq \frac{1}{2}, C \geq \frac{1}{2}\right) = 1 - \left[1 - P\left(B \leq \frac{1}{2}\right)\right]\left[1 - P\left(C \leq \frac{1}{2}\right)\right]$$

$$P\left(B \geq \frac{1}{2}, C \geq \frac{1}{2}\right) = 1 - \left[1 - \frac{1}{2}\right]\left[1 - \frac{1}{2}\right]$$

$$P\left(B \geq \frac{1}{2}, C \geq \frac{1}{2}\right) = 1 - \frac{1}{4} = 0.75$$

$$\boxed{\text{THE PROBABILITY OF } \min\{B, C\} < \frac{1}{2} \text{ IS EQUAL TO } 0.75}$$